Bruno de Finetti's Objectivity

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Bruno de Finetti (1906–1985) in 1979

Objectivism vs. objectivity

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Objectivism?

- No!

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- . . . and aim to be a "good" description.

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  - finitely additive probability

Some quotes

(1962) Though maintaining the subjectivist idea that no fact can prove or disprove belief, I find no difficulty in admitting that any form of comparison between probability evaluations. . . and actual events may be an element influencing my further judgment.

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(1962) Though maintaining the subjectivist idea that no fact can prove or disprove belief, I find no difficulty in admitting that any form of comparison between probability evaluations... and actual events may be an element influencing my further judgment.

(1974) Subjectivists believe that every evaluation of probability is based on available information, including objective data.

(1974) Every probability evaluation essentially depends on two components:

(1). the objective component, consisting of the evidence of known data and facts; and

(2). the subjective component, consisting of the opinion concerning unknown facts based on known evidence.

Definition or evaluation?

The evaluation of probability should take into account all available evidence, including frequencies and symmetries. However, it would be a mistake to put these elements, which are useful ingredients of the evaluation of probability, at the basis of the definition of probability.

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- How to interpret?

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- How to quantify?

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- How to interpret?
- How to quantify?
- How to assess?

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- Infinite sequence of (binary) variables $X_1, X_2, \ldots$.
- Often natural to assume indifference to the order in which presented.
- Any joint distribution with this property is a mixture of Bernoulli sequences:

$$P(x_1, x_2, \ldots, x_n) = \int_0^1 \theta^r (1-\theta)^{n-r} dF(\theta) \qquad \left(r = \sum_{i=1}^{n} x_i \right)$$

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- With P-probability 1,

$$P(X_{n+1} = 1 \mid X_1, X_2, \ldots, X_n) - (R/n) \to 0$$

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  - $\phi(u_i, v_j)$???
- Thomas's failure rate on his previous attempts at Statistics?

Data array

?	0	1	0	1	0	1	0	0	1
1	0	1	0	1	1	0	1	0	0
0	0	0	1	1	0	0	0	1	1
1	1	0	1	0	1	1	0	1	0
0	0	1	1	1	0	1	0	0	1
1	0	1	0	0	1	0	1	1	0
0	0	1	0	1	1	0	1	0	1
0	1	0	0	0	0	1	0	1	1
1	0	1	1	0	1	0	1	0	1

Local pattern

?	**0**	**1**	0	1	0	1	0	0	1
1	**0**	**1**	0	1	1	0	1	0	0
0	**0**	**0**	1	1	0	0	0	1	1
1	**1**	**0**	1	0	1	1	0	1	0
0	**0**	**1**	1	1	0	1	0	0	1
1	0	1	0	1	1	0	1	1	0
0	0	1	0	1	1	0	1	0	1
0	1	0	0	0	0	1	0	1	1
1	0	1	1	0	1	0	1	0	1

Repeat pattern

?	**0**	**1**	0	1	0	1	0	0	1
1	**0**	**1**	0	1	1	0	1	0	0
0	**0**	**0**	1	1	0	0	0	1	1
1	**1**	**0**	1	0	1	1	0	1	0
0	**0**	**1**	**1**	**1**	0	1	0	0	1
1	0	1	**0**	**1**	1	0	1	1	0
0	0	**1**	**0**	**1**	1	0	1	0	1
0	1	**0**	**0**	**0**	0	1	0	1	1
1	0	**1**	**1**	**0**	1	0	1	0	1

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- S is termed a proper scoring rule if my expected score $S(P, Q)$ is minimised in Q at $Q = P$; and strictly proper if $S(P, Q) > S(P, P)$ for $Q \neq P$.
- When S is proper, honesty is the best policy: If I believe $X \sim P$, I will minimise my expected score by quoting $Q = P$.

Probability score table (Brier)

A occurs	A does not occur	q
25.0	25.0	.500
22.6	27.6	.525
20.2	30.2	.550
18.1	33.1	.575
16.0	36.0	.600
14.1	39.1	.625
12.2	42.2	.650
10.6	45.6	.675
9.0	49.0	.700
7.6	52.6	.750
6.2	56.2	.725
5.1	60.1	.775
4.0	64.0	.800
3.1	68.1	.825
2.2	72.2	.850
1.6	76.6	.875
1.0	81.0	.900
0.6	85.6	.925
0.2	90.2	.950
0.1	95.1	.975
0.0	100.0	1 000

Probability score table (Brier)—reduced

A occurs	*A does not occur*
25.0	**25.0**
22.6	**27.6**
20.2	**30.2**
18.1	**33.1**
16.0	**36.0**
14.1	**39.1**
12.2	**42.2**
10.6	**45.6**
9.0	**49.0**
7.6	**52.6**
6.2	**56.2**
5.1	**60.1**
4.0	**64.0**
3.1	**68.1**
2.2	**72.2**
1.6	**76.6**
1.0	**81.0**
0.6	**85.6**
0.2	**90.2**
0.1	**95.1**
0.0	**100.0**

Probability assessment quiz

Mark in Column 1 the answer you consider correct. In Column 2, attach a number between 0.5 and 1 to indicate your personal probability that this is the correct answer. When the answers are revealed, indicate in Column 3 whether you were right or wrong, and insert the corresponding penalty score (from Table 1) in Column 4.

		1	2	3	4
1.	**Crystals of common salt are** a) octahedral b) cubical				
2.	**Shi'ism is a branch of** a) Islam b) Confucianism				
3.	**Which has the larger area?** a) France b) The Iberian Peninsula (Spain and Portugal)				
4.	**The blood in the pulmonary artery flows** a) from heart to lungs b) from lungs to heart				
5.	**An ibex is** a) a bird b) a goat				
6.	**Buddhism had its origins in** a) India b) China				
7.	**Sea-water freezes at** a) –3°C b) +2°C				
8.	**In Heraldry, "gules" refers to the colour** a) red b) blue				
9.	**An anticyclone is a region of** a) low pressure b) high pressure				
10	**The Venetian gondola is propelled by** a) a pole b) an oar				
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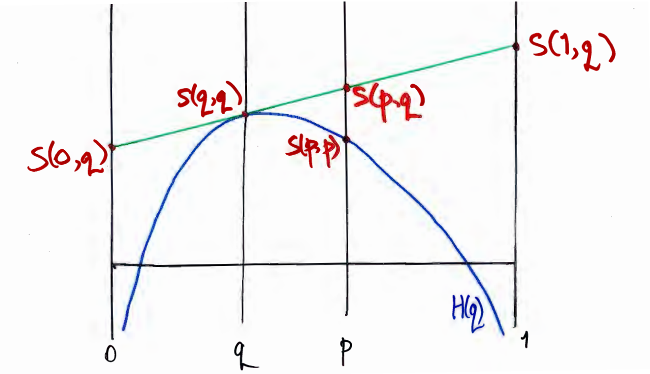

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log score (Good)

$$\begin{aligned} S(x, q) &= -\{x \log q + (1 - x) \log(1 - q)\} \\ &\quad \text{log likelihood} \\ H(p) &= -\{p \log p + (1 - p) \log(1 - p)\} \\ &\quad \text{Shannon entropy} \\ D(p, q) &= p \log(p/q) + (1 - p) \log\{(1 - p)/(1 - q)\} \\ &\quad \text{KL divergence} \end{aligned}$$

Independence of scoring rule

Suppose my quoted probability for an event E is:

- q_1 under scoring rule S_1

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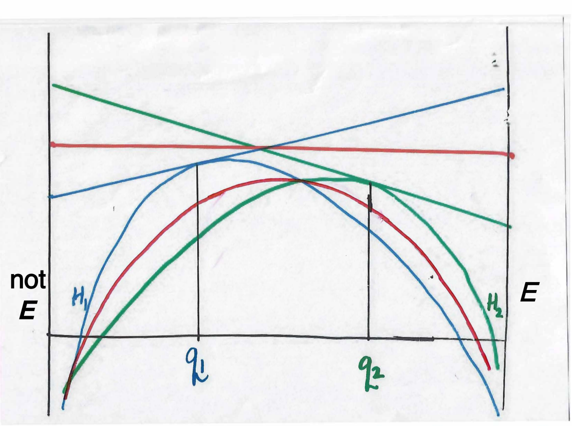
If $q_1 \neq q_2$, I could have done better, however E turns out.



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(1-q_1)^2 + q_0^2 & \text{if } E \text{ occurs} \\
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q_1^2 + (1-q_0)^2 & \text{if } E \text{ does not occur} \\
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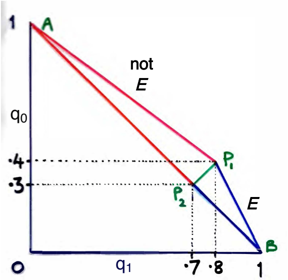

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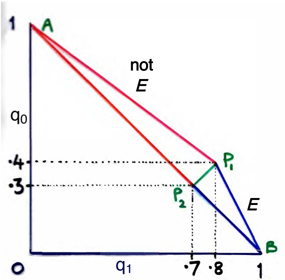

If I do not have $q_i \geq 0$, $q_1 + q_0 = 1$, I could have done better (*e.g.*, with $P_2 = (0.7, 0.3)$), however E turns out.

Compound probability

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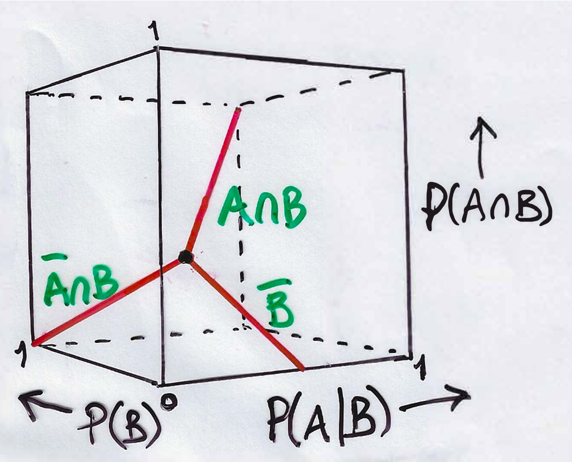
Unless red lines are coplanar—which requires $P(A \cap B) = P(A \mid B)\, P(B)$—I could have done better in all circumstances.



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Outcome	0	0	1	0	1	0	1	1	1	0	1	0	0	1

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Group by assigned probability:

Probability π	0.2	0.3	0.4	0.5	0.6
Instances n	2	3	3	2	4
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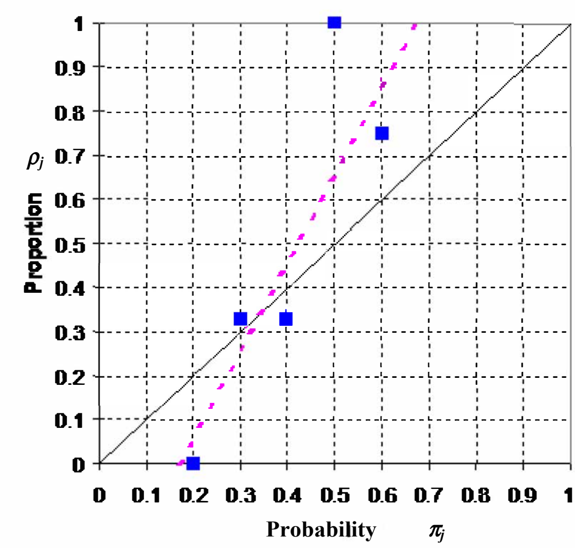
Refinement: ρ_j's close to 0 or 1

Calibration plot



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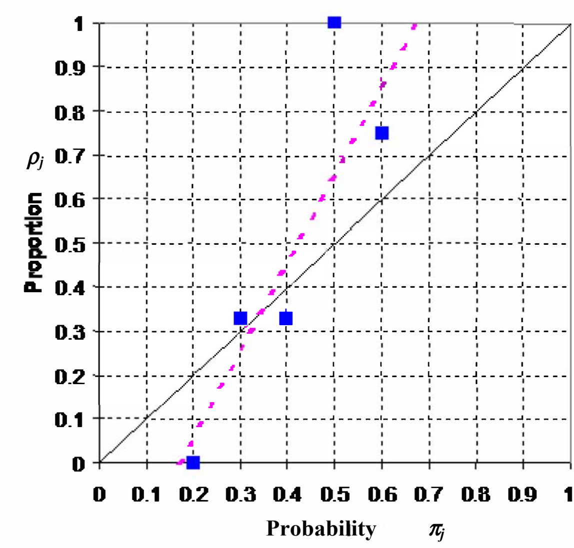

If forecasts are "good", the points should lie near the diagonal.

Scoring rule decomposition

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Similar decomposition for other proper scoring rules

Real weather forecasts

- Forecasts of precipitation occurrence have been routinely expressed in probabilistic terms on a nationwide basis in the US since 1965.
- A subjective precipitation probability forecast expresses the forecaster's "degree of belief" that a measurable amount of precipitation (≥ 0.01 in.) will occur during a specified period (generally 12 h) at a particular point in the forecast area (generally the official rain gauge).
- A typical forecast might state that "the precipitation probability for Denver (Colorado) today is 30 per cent".
- Millions of such forecasts have been formulated and issued to the public by the National Weather Service (NWS).

Reliability diagram

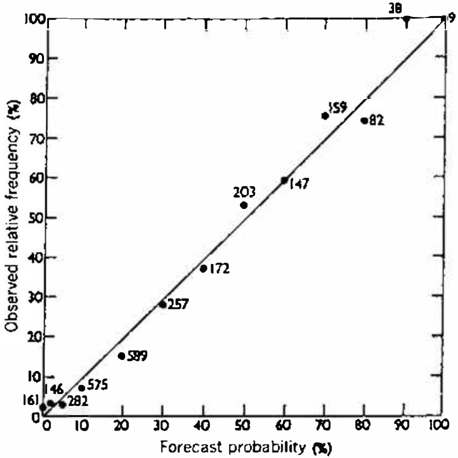

Figure: The reliability diagram for all of the precipitation probability forecasts formulated by NWS forecasters at Chicago during the period from July 1972 to June 1976.

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If the forecasts are conditional probabilities based on a joint distribution P, then this property holds almost surely under P.

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Deterministic Chaos?

Lessons?

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- Subjective?

Probability assessment quiz: Solutions

Mark in Column 1 the answer you consider correct. In Column 2, attach a number between 0.5 and 1 to indicate your personal probability that this is the correct answer. When the answers are revealed, indicate in Column 3 whether you were right or wrong, and insert the corresponding penalty score (from Table 1) in Column 4.

		1	2	3	4
1.	**Crystals of common salt are** **a)** octahedral **b)** cubical	b			
2.	**Shi'ism is a branch of** **a)** Islam **b)** Confucianism	a			
3.	**Which has the larger area?** **a)** France **b)** The Iberian Peninsula (Spain and Portugal)	a			
4.	**The blood in the pulmonary artery flows** **a)** from heart to lungs **b)** from lungs to heart	a			
5.	**An ibex is** **a)** a bird **b)** a goat	b			
6.	**Buddhism had its origins in** **a)** India **b)** China	a			
7.	**Sea-water freezes at** **a)** –3°C **b)** +2°C	a			
8.	**In Heraldry, "gules" refers to the colour** **a)** red **b)** blue	a			
9.	**An anticyclone is a region of** **a)** low pressure **b)** high pressure	b			
10	**The Venetian gondola is propelled by** **a)** a pole **b)** an oar	b			
	TOTAL SCORE				

THANK YOU!

Acknowledgment

I wish to express my special appreciation to Maria Carla Galavotti, Emeritus Professor of Philosophy at the University of Bologna, for sharing her expert knowledge and understanding of de Finetti and his works.

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