Nonlinear Optimization Exercises

1. Prove that the solution set S of an arbitrary (possibly infinite) system $a_\alpha' x \leq b_\alpha$, $\alpha \in \mathcal{A}$ of linear inequalities for $x \in \mathbb{R}^n$ is convex.

2. Prove that unit balls of norms on $\mathbb{R}^n$ are exactly the same as convex sets V in $\mathbb{R}^n$ satisfying the following three properties:

   (a) V is symmetric with respect to the origin;

   (b) V is bounded and closed;

   (c) V contains a neighborhood of the origin.

   A set V satisfying the outlined properties is the unit ball of the norm $\|x\| = \inf\{t \geq 0 : x/t \in V\}$.

3. Prove that if M is a convex set in $\mathbb{R}^n$ and $\epsilon > 0$, then for every norm on $\mathbb{R}^n$, the ϵ-neighborhood of M, i.e., the set

   $$M_\epsilon = \{y \in \mathbb{R}^n : \inf_{x \in M} \|y - x\| \leq \epsilon\}$$

   is convex.

4. Prove that a set M in $\mathbb{R}^n$ is convex if and only of it is closed with respect to taking all convex combinations of its elements.

5. Prove that for nonempty $M \subset \mathbb{R}^n$, the convex hull $\text{conv}(M)$, defined as the intersection of all convex sets containing M, is the set of all convex combinations of vectors from M.

6. Prove that a nonempty subset M of $\mathbb{R}^n$ is a cone if and only of it is conic (i.e., $x \in M$ and $t \geq 0$ implies $tx \in M$) and closed with respect to addition (i.e., $x, y \in M$ implies $x + y \in M$).

7. Prove that the following operations preserve convexity of sets:

   (a) Intersection: if $(M_\alpha)_{\alpha \in \mathcal{A}}$ are convex sets, so is $\bigcap_{\alpha \in \mathcal{A}} M_\alpha$.

   (b) Direct product: if $M_1 \subset \mathbb{R}^{n_1}$ and $M_2 \subset \mathbb{R}^{n_2}$ are convex, so is

   $$M_1 \times M_2 = \{y = (y_1, y_2) : y_1 \in M_1, y_2 \in M_2\}.$$

   (c) Linear combination: if $\lambda_1, \ldots, \lambda_k$ are arbitrary reals, and $M_1, \ldots, M_k$ are convex sets in $\mathbb{R}^n$, so is

   $$\lambda_1 M_1 + \cdots + \lambda_k M_k = \{\lambda_1 x_1 + \cdots \lambda_k x_k : x_i \in M_i, i = 1, \ldots, k\}.$$

   (d) Taking the image under affine mapping: if $M \subset \mathbb{R}^n$ is convex and $x \mapsto \mathcal{A}(x) = Ax + b$ is an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

   $$\mathcal{A}(M) = \{y = \mathcal{A}(x) : x \in M\}$$

   is a convex set in $\mathbb{R}^m$.

(e) Taking the inverse image under affine mapping: if $M \subset \mathbb{R}^n$ is convex and $y \mapsto \mathcal{A}(y) = Ay + b$ is an affine mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$\mathcal{A}^{-1}(M) = \{y : \mathcal{A}(y) \in M\}$$

is a convex set in $\mathbb{R}^m$.

8. Prove that the relative interior of a simplex with vertices $y_0, \ldots, y_m$ is exactly the set $\{x = \sum_{i=0}^m \lambda_i y_i : \lambda_i > 0, i = 0, \ldots, m; \sum_{i=0}^m \lambda_i = 1\}$.

9. Let $S_1, \ldots, S_N$ be a family of N convex sets in $\mathbb{R}^n$, and let m be the affine dimension of $\mathrm{aff}(S_1 \cup \cdots \cup S_N)$. Assume that every $m+1$ sets from the family have a point in common. Prove that all sets from the family have a point in common.

10. Let S and T be nonempty convex sets in $\mathbb{R}^n$. Prove that a linear form a separates S and T if and only if

$$\sup_{x \in S} a'x \leq \inf_{y \in T} a'y, \qquad \inf_{x \in S} a'x < \sup_{y \in T} a'y.$$

The separation is strong if and only if $\sup_{x \in S} a'x < \inf_{y \in T} a'y$.

11. Let $S = \{x = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 > 0, \xi_2 \geq 1/\xi_1\}$ and $T = \{x \in \mathbb{R}^2 : \xi_1 < 0, \xi_2 \geq -1/\xi_1\}$. Can S and T be separated? Can they be strongly separated?

12. Use the Separation Theorem to prove that every closed convex set in $\mathbb{R}^n$ is the solution of a (perhaps infinite) system of non-strict linear inequalities.

13. Provide an alternative proof of the Separation Theorem using the following approach. Let S be a non-empty closed convex set in $\mathbb{R}^n$.

(a) Prove that if x is a point in $\mathbb{R}^n$, then there is a unique $\Pi_S(x)$ in S (the projection of x on S), so that $\min\{\|x - y\|_2 : y \in S\} = \|x - \Pi_S(x)\|_2$.

(b) Next, show that if $x \notin S$, then the linear form $e = x - \Pi_S(x)$ strongly separates $\{x\}$ and S:

$$\max_{y \in S} e'y = e'\Pi_S(x) = e'x - e'e < e'x,$$

thus getting a direct proof of the possibility to strongly separate a non-empty closed convex set and a point outside this set.

(c) From this, derive the Separation Theorem.

14. For a non-empty subset M of $\mathbb{R}^n$, let

$$\mathrm{polar}(M) = \{a : \sup_{x \in M} \langle a, x \rangle \leq 1\}$$

be the polar (set) of M. Prove that

(a) $\mathrm{polar}(M)$ is a closed convex set containing the origin.

(b) $\mathrm{polar}(M) = \mathrm{polar}(\mathrm{cl}(M))$.

(c) M is bounded if and only if $0 \in \mathrm{int}(\mathrm{polar}(M))$.

(d) If M is a cone or if $\mathrm{polar}(M)$ is a cone, then

$$\mathrm{polar}(M) = \{a : \sup_{x \in M} \langle a, x \rangle \leq 0\}.$$

15. Let L be a linear subspace in $\mathbb{R}^n$. Show that $\text{polar}(L) = L^\perp$, the orthogonal complement of L.

16. For an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$, its *dual norm* $\|\cdot\|^*$ is defined is

$$\|a\|^* = \sup_{x:\|x\|\leq 1} |\langle a, x\rangle| = \sup_{x\neq 0} \frac{|\langle a, x\rangle|}{\|x\|}.$$

For $1 \leq p < \infty$, the p-norm on $\mathbb{R}^n$ is $\|x\|_p = \left(\sum_i |\xi_i|^p\right)^{1/p}$; $\|x\|_\infty = \max_i |\xi_i|$.

   (a) Use Hölder's inequality to show that for $1 \leq p \leq \infty$, $\|\cdot\|_p^* = \|\cdot\|_q$, where $1/p + 1/q = 1$.

   (b) Let A be a regular $n \times n$ matrix and $\|x\| = \|Ax\|_p$. Find $\|a\|^*$.

   (c) For $\gamma > 0$, let $M = \{x : \|x\| \leq \gamma\}$. Find $\text{polar}(M)$.

   (d) Let S be a symmetric positive definite matrix and $M = \{x : x'Sx \leq \gamma\}$. Find $\text{polar}(M)$.

17. Let M be a convex set containing the origin. Prove that

   (a) $\text{int}(M) \neq \emptyset$ if and only $\text{polar}(M)$ does not contain straight lines.

   (b) If M is closed, it is a cone if and only if $\text{polar}(M)$ is a (closed) cone.

18. For a non-empty subset M of $\mathbb{R}^n$, the polar and dual cones M° and M^* are given by

$$M^\circ = \{a : \sup_{x\in M} \langle a, x\rangle \leq 0\}, \qquad M^* = \{a : \inf_{x\in M} \langle a, x\rangle \geq 0\} = -M^\circ,$$

respectively. Show that both M° and M^* are closed convex cones.

19. Let M be a convex cone in $\mathbb{R}^n$, and M^* its dual cone. Prove that M is *solid* (i.e, has non-empty interior) if and only if M^* is *pointed* (i.e., does not contain lines), and that

$$\text{int}(M) = \{x : \langle a, x\rangle > 0 \text{ for all non-zero } a \in M^*\}.$$

From this, conclude that M is a pointed solid closed convex cone if and only if its dual has the same properties.

20. Let M be a convex set in $\mathbb{R}^n$ and $x \in M$. Prove that

   (a) x is extreme if and only if $x \pm h \in M$ implies that h is the zero vector;

   (b) x is extreme if and only if $M \setminus \{x\}$ is convex.

21. Let M be a convex set in $\mathbb{R}^n$. The set $\text{rec}(M) = \{h : M + h \subseteq M\}$ is called the recession cone of M. Show that

   (a) $\text{rec}(M)$ is indeed a cone;

   (b) If M is closed, $\text{rec}(M)$ is closed;

   (c) If M is closed, $\text{rec}(M)$ is the set of all h for which $x + \text{cone}(h) \subseteq M$ for *some* $x \in M$;

   (d) $M + \text{rec}(M) = M$.

22. Let M be a closed non-empty convex set in $\mathbb{R}^n$. Provide that $\text{rec}(M) \neq \{0\}$ if and only if M is unbounded.

23. Consider the space $\mathbb{R}^{n\times n}$ of square $n \times n$ matrices with real entries, and its subsets $\mathbb{S}^n$, the set of all symmetric matrices (for which $A = A'$), and $\mathbb{J}^n$, the set of all skew-symmetric matrices (for which $A = -A'$).

   (a) Show that $\mathbb{S}^n$ and $\mathbb{J}^n$ are linear subspaces of $\mathbb{R}^{n\times n}$.

   (b) Find the dimension of $\mathbb{S}^n$, and a basis for it.

   (c) Find the dimension of $\mathbb{J}^n$, and a basis for it.

   (d) What are the sum and the intersection of $\mathbb{S}^n$ and $\mathbb{J}^n$?

24. Let $L = \{x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n : \sum_i \xi_i = 0\}$. What is the orthogonal complement of L?

25. Let L_1 and L_2 be linear subspaces of $\mathbb{R}^n$. Show that

$$(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp, \qquad (L_1 \cap L_2)^\perp = L_1^\perp + L_2^\perp.$$

26. Consider the space $\mathbb{R}^{m\times n}$ of real-valued $m \times n$ matrices equipped with the "standard" inner product (Frobenius product)

$$\langle A, B\rangle = \sum_{i,j} \alpha_{ij}\beta_{ij}.$$

   (a) Verify that the Frobenius product can be written as $\langle A, B\rangle = \mathrm{tr}(A'B) = \mathrm{tr}(AB')$, where "tr" is the trace (the sum of the diagonal elements) of a square matrix.

   (b) Build an orthonormal basis of the linear subspace $\mathbb{S}^n$ of symmetric matrices in $\mathbb{R}^{n\times n}$.

   (c) What is the orthogonal complement of $\mathbb{S}^n$ in $\mathbb{R}^{n\times n}$?

   (d) Find the orthogonal decomposition of $A = [1, 2; 3, 4]$ with respect to $\mathbb{S}^2$.

27. Let M be a convex cone. Prove that the largest linear subspace contained in M is $M \cap (-M)$, and that the smallest linear subspace containing M is $M - M = \mathrm{aff}(M)$.

28. Let $K_1, \dots, K_m$ be cones. Show that $(K_1 + \cdots + K_m)^\circ = K_1^\circ \cap \cdots \cap K_m^0$.

29. Find the dual cones of $\mathrm{cone}(\{a_1, \dots, a_m\})$ and $\{y = Ax : x \geq 0\}$, where $a_1, \dots, a_m \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n\times m}$, respectively.

30. Let $K = \{x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n : 0 \leq \xi_1 \leq \cdots \leq \xi_n\}$. Show that K is a cone, and determine its polar cone K°.

31. Let S be a subset of $\mathbb{R}^n$, and $K = \bigcap_{s\in S}\{x : \langle s, x\rangle \geq 0\}$. Show that K is a closed convex cone with dual $K^* = \mathrm{cl}(\mathrm{cone}(S))$.

32. Let $\mathbb{S}^n_+$ be the set of all positive semidefinite $n \times n$ matrices. Show that $\mathbb{S}^n_+$ is a self-dual closed convex cone in $\mathbb{S}^n$.

33. Let K be a closed convex cone. Show that K is self-dual iff $\langle x, y\rangle \geq 0$ for all $x, y \in K$ and $\langle u, v\rangle \geq 0$ for all $u, v \in K^*$.

34. Let $\mathbb{D}^n$ be the set of all doubly stochastic $n \times n$ matrices A, i.e., the matrices with non-negative entries and all row and column sums one. Show that $\mathbb{D}^n$ is convex and all permutation matrices (i.e., binary doubly stochastic matrices) are extreme points of $\mathbb{D}^n$.

35. A matrix $A \in \mathbb{S}^n$ is *co-positive* if $x'Ax \geq 0$ for all $x \geq 0$. Show that the set K of all co-positive matrices is a solid pointed closed convex cone in $\mathbb{S}^n$, and find its dual cone.

    Hint. Express K as an intersection of convex cones.

36. A symmetric $n \times n$ matrix $A =$ is a *Euclidean distance matrix* if it has the form $A = [\|y_i - y_j\|_2^2]_{1 \leq i,j \leq n}$ for some $y_1, \ldots, y_n \in \mathbb{R}^k$ and some k. One can show that A is a Euclidean distance matrix if and only if it has zero diagonal and $x'Ax \leq 0$ for all x with $1'x = 0$ (where 1 denotes the vector of all ones). Show that the set K of Euclidean distance matrices is a convex cone in $\mathbb{S}^n$, and find its dual cone.

    Hint. Express K as an intersection of convex cones.

37. Let K_1 and K_2 be closed convex cones. Show that $K_1 \subseteq K_2$ if and only of $K_2^* \subseteq K_1^*$. What does this give for $K_1 = \mathrm{cone}(\{a\})$ and $K_2 = \mathrm{cone}(\{a_1, \ldots, a_m\})$ with $a, a_1, \ldots, a_m \in \mathbb{R}^n$?

38. Let K be a closed convex cone in $\mathbb{R}^n$. Prove that every $x \in \mathbb{R}^n$ can be represented as

$$x = \Pi_K(x) + \Pi_{K^\circ}(x), \qquad \langle \Pi_K(x), \Pi_{K^\circ}(x) \rangle = 0.$$

39. Let K be a convex cone, and define the relation $\preceq_K$ by $x \preceq_K y$ iff $y - x \in K$.

    (a) Show that $\preceq_K$ is reflexive and transitive, and preserved under taking conic combinations (i.e., if $x_i \preceq_K y_i$ and $\lambda_i \geq 0$ for all i, then $\sum_i \lambda_i x_i \preceq_K \sum_i \lambda_i y_i$).

    (b) Show that if K is pointed, $\preceq_K$ is anti-symmetric and hence an partial order relation ("generalized inequality relation"). When is it a linear order relation?

    (c) Show that if K is closed, $\preceq_K$ is preserved under limits (i.e, if $x_i \preceq_K y_i$ for all i and $x = \lim_i x_i$ and $y = \lim_i y_i$, then $x \preceq_K y$).

    (d) Show that if K is closed, then $x \preceq_K y$ iff $\langle a, x \rangle \leq \langle a, y \rangle$ for all $a \succeq_{K^*} 0$.

    (If K is solid, one can also define a generalized strict inequality relation by $x \prec_K y$ iff $y - x \in \mathrm{int}(K)$. Thus, one typically considers *proper* (closed convex solid and pointed) cones for definining generalized inequality relations.)

40. Let C be a closed convex set containing the origin. Show that

$$\mathrm{rec}(C) = \bigcap\nolimits_{\epsilon > 0} \epsilon C = \{h : h/\epsilon \in C \text{ for all } \epsilon > 0\}$$

    and conclude that $\mathrm{rec}(C)$ is the largest closed convex cone contained in C.

41. Let C and C° be a polar pair of closed convex sets containing the origin. Prove that the recession cone of C and the closure of $\mathrm{cone}(C^\circ)$ are polar to each other, and the lineality space of C (the set of all y such that for all $x \in C$, the line through x in the direction of y is contained in C) and $\mathrm{lin}(C^\circ)$ are orthogonally complementary to each other Dually, also, with C and C° interchanged.

42. Suppose f is defined on $S \subseteq \mathbb{R}^n$ with values in the extended reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let

$$\mathrm{epi}(f) = \{(x, \tau) : x \in S, \tau \in \mathbb{R}, f(x) \leq \tau\}$$

be the epigraph of f and

$$\mathrm{dom}(f) = \{x : \exists \tau \text{ such that } (x, \tau) \in \mathrm{epi}(f)\} = \{x : x \in S, f(x) < \infty\}$$

be the (essential) domain of f. We say that f is convex on S iff $\mathrm{epi}(f)$ is a convex set. Show that this implies that $\mathrm{dom}(f)$ is a convex set.

43. Suppose a (convex) function $f : S \to \overline{\mathbb{R}}$ is called *proper* iff $\mathrm{epi}(f)$ is non-empty and contains no vertical lines. Show that this is equivalent to $f(x) > -\infty$ for all $x \in S$ and $f(x) < \infty$ for some $x \in S$.

44. Show that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is lower semicontinuous iff $\mathrm{epi}(f)$ is closed.

45. Let $f(x) = \|\cdot\|$ be a norm on $\mathbb{R}^n$. Show that its epigraph K is a cone in $\mathbb{R}^{n+1}$, and that its polar cone has the form

$$K^\circ = \{(a, \mu) : a \in R^n, \mu \in \mathbb{R}, \mu \leq -\|a\|_*\}.$$

46. Let Z be a set in $\mathbb{R}^n$ and $\|\cdot\|$ be a norm in $\mathbb{R}^n$. Show that $x \mapsto \sup_{z \in Z} \|x - z\|$ is convex, and that if Z is convex, $x \mapsto \inf_{z \in Z} \|x - z\|$ is convex.

47. The convex hull of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$g(x) = \inf(\{\tau : (x, \tau) \in \mathrm{conv}(\mathrm{epi}(f))\}).$$

Show that g is the greatest convex minorant of f, i.e., that g is convex and that $h \leq g$ for all convex $h \leq f$.

48. For a vector $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ we define $\xi_{[j]}, j = 1, \ldots, n$ as its ordered coordinates:

$$\xi_{[1]} \geq \cdots \geq \xi_{[n]}.$$

Prove that for all $1 \leq k \leq n$ the function $f_k(x) = \sum_{j=1}^k \xi_{[j]}$ is convex, and calculate its subdifferential.

49. Let Y be a compact convex polyhedron in $\mathbb{R}^m$ and A be an $n \times m$ matrix. Prove that the function $F : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x) = \max_{y \in Y} \langle x, Ay \rangle$$

is convex, and calculate its subdifferential.

50. Let C be a closed convex set in $\mathbb{R}^n$ and δ_C its *indicator function*, i.e., $\delta_C(x) = 0$ if $x \in C$, and ∞ otherwise. Show that the subdifferential of δ_C at $x \in C$ is given by $\partial \delta_C(x) = (\mathrm{cone}(C - x))^\circ$.

51. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Show that the subdifferential of $\|\cdot\|$ at x is given by

$$\partial\|x\| = \{a \in \mathbb{R}^n : \|a\|_* \leq 1, \langle a, x \rangle = \|x\|\}.$$

In particular, what are the subdifferentials of the 1-norm and ∞-norm at $x = 0$?

52. Show that a function f on $\mathbb{R}^n$ is positively homogeneous if and only if its epigraph $\mathrm{epi}(f)$ is a cone.

53. Prove that a convex function f is continuous at a point $x \in \mathrm{dom}(f)$ if and only if $(x, f(x)+\epsilon) \in \mathrm{int}(\mathrm{epi}(f))$ for some (all) $\epsilon > 0$.

54. Show that for any convex function f, $\mathrm{ri}(\mathrm{epi}(f))$ consists of the points (x, τ) for which $x \in \mathrm{ri}(\mathrm{dom}(f))$ and $f(x) < \tau < \infty$.

55. For a non-empty subset Z of $\mathbb{R}^n$, let

$$K_Z(x) = \mathrm{cone}(Z - x)$$

be the radial cone of Z at x. Show that if Z is convex, $K_Z(x)$ is the convex cone of all feasible directions h for which $x + \epsilon h \in Z$ for all ϵ sufficiently small. Determine $K_Z(x)$ for a closed convex cone Z and $x \in Z$.

56. For $f : \mathbb{R}^n \to (-\infty, \infty]$ and $x \in \mathrm{dom}(f)$, determine the normal cone of $\mathrm{epi}(f)$ at $(x, f(x))$.

57. Show that the subdifferential is a closed convex set.

58. Show that if f is a convex function and $x \in \mathrm{ri}(\mathrm{dom}(f))$, then the subdifferential $\partial f(x)$ is non-empty.

59. Let f be a convex function, and $x \in \mathrm{dom}(f)$. Show that for each y, the function $\lambda \mapsto (f(x+\lambda y) - f(x))/\lambda$ is non-decreasing for $\lambda > 0$, so that the one-sided directional derivative

$$f'(x; y) = \lim_{\lambda \to 0+} \frac{f(x+\lambda y) - f(x)}{\lambda}$$

exists and is given by

$$f'(x; y) = \inf_{\lambda > 0} \frac{f(x+\lambda y) - f(x)}{\lambda}.$$

Moreover, $y \mapsto f'(x, y)$ is a positively homogeneous convex function with $f'(x; 0) = 0$ and $-f'(x; -y) \leq f'(x; y)$.

60. Let f be a proper convex function. Show that the recession cone $\mathrm{rec}(\mathrm{epi}(f))$ is the epigraph of some function g, the *recession function* $\mathrm{rec}(f)$ of f. Prove that $\mathrm{rec}(f)$ is a positively homogeneous proper convex function, and that for all y,

$$\mathrm{rec}(f)(y) = \sup\{f(x+y) - f(x) : x \in \mathrm{dom}(f)\}.$$

Finally, show that if f is closed, $\mathrm{rec}(f)$ is closed too, and for any $x \in \mathrm{dom}(f)$,

$$\mathrm{rec}(f)(y) = \sup_{\lambda > 0} \frac{f(x+\lambda y) - f(x)}{\lambda} = \lim_{\lambda \to \infty} \frac{f(x+\lambda y) - f(x)}{\lambda}.$$

61. Use the fact that a closed convex set is the intersection of all closed halfspaces containing it to show that a closed convex function f is the pointwise supremum of the family of all affine functions h for which $h \leq f$.

62. The *support function* σ_Z of a set Z in $\mathbb{R}^n$ is defined as $\sigma_Z(x) = \sup_{z\in Z} \langle x, z\rangle$. Show that for arbitrary Z, σ_Z is convex and satisfies $\sigma_Z = \sigma_{\mathrm{conv}(Z)}$, and that for arbitrary Z and Y, $\sigma_{Z+Y} = \sigma_Z + \sigma_Y$ and $\sigma_{Z\cup Y} = \max(\sigma_Z, \sigma_Y)$.

63. Show that for arbitrary sets Z in $\mathbb{R}^n$, the conjugate of the indicator function of Z is the support function of Z, i.e., $\delta_Z^* = \sigma_Z$.

64. Let K be a convex cone in $\mathbb{R}^n$. Show that the conjugate δ_K^* of its indicator function is the indicator function δ_{K° of its polar cone.

65. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and B its unit ball. Show that the conjugate δ_B^* of the indicator function of B is the dual norm $\|\cdot\|_*$, and that the conjugate of the norm is the indicator function δ_{B_*} of the dual unit ball.

66. Show that if f is a positively homogeneous function on $\mathbb{R}^n$ (i.e., $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$), its conjugate f^* is the indicator function δ_Y of the set $Y = \{y : \langle y, x\rangle \leq f(x) \text{ for all } x \in \mathbb{R}^n\}$. Use this to determine the conjugate σ_C^* of the support function of a non-empty closed convex set C.

67. The *gauge* (or Minkowski) function of a set Z in $\mathbb{R}^n$ is defined as $\gamma_Z(x) = \inf(\{\tau > 0 : x/\tau \in Z\})$. Show that γ_Z is positively homogeneous and that, for convex Z, γ_Z is convex, and determine the conjugate γ_Z^* of γ_Z.

68. Let $A \in \mathbb{S}^n_{++}$ be a positive definite symmetric $n \times n$ matrix. Calculate the conjugate function to $f_A(x) = \langle x, Ax\rangle /2$.

69. Let Z be a closed convex set in $\mathbb{R}^n$ and let

$$f(x) = \|x\|_2^2 - \min_{z\in Z} \|x - z\|_2^2.$$

Show that f is convex.

Hint. Represent f using a conjugate of a convex function.

70. Let Z be a closed convex set in $\mathbb{R}^n$ and let

$$f(x) = \mathrm{dist}(x, Z) = \min_{z\in Z} \|x - z\|$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$. Prove that the conjugate function f^* is the sum of the support function σ_Z of Z (given by $\sigma_Z(a) = \sup_{z\in Z} \langle a, z\rangle$) and the indicator function δ_{B_*} of the dual unit ball B_*. Use this result to calculate the subdifferential of f. In particular, what is the subdifferential of $f(x) = \min_{z\in Z} \|x - z\|_2$ at $x \notin Z$?