But these Markov controls are only conditionally optimal. They tell us which control to use after a disturbance has occurred, but they are optimal only in the absence of further disturbances.

If we stipulate the probability of future disturbances and then want to optimize the expected value of the objective functional, this gives a stochastic control problem, in which optimal Markov controls are determined by a different set of necessary conditions.

Jumps in State Variables

So far we have assumed that the control functions are piecewise continuous, and the state variables are continuous. In certain applications (e.g. in the theory of investment), the optimum may require sudden jumps in the state variables. See e.g. Seierstad and Sydsæter (1987), Chapter 3.

DIFFERENCE EQUATIONS

He (an economist) must study the present in the light of the past for the purpose of the future.
—J. N. Keynes

Many of the quantities economists study (such as income, consumption, and savinc recorded at fixed time intervals (for example, each day, week, quarter, or year). tions that relate such quantities at different discrete moments of time are called **diffe equations**. For example, such an equation might relate the amount of national income period to the national income in one or more previous periods. In fact difference equatio be viewed as the discrete time counterparts of the differential equations in continuous tim were studied in Chapters 5–7.

11.1 First-Order Difference Equations

Let $t = 0, 1, 2, \ldots$ denote different discrete time periods or moments of time. We u call $t = 0$ the *initial period*. If $x(t)$ is a function defined for $t = 0, 1, 2, \ldots$, we oft $x_0, x_1, x_2, \ldots$ to denote $x(0), x(1), x(2), \ldots$, and in general, we write x_t for $x(t)$.

Let $f(t, x)$ be a function defined for all positive integers t and all real number first-order difference equation in x_t can usually be written in the form

$$x_{t+1} = f(t, x_t), \qquad t = 0, 1, 2, \ldots$$

This is a first-order equation because it relates the value of a function in period $t + 1$ value of the same function in the previous period t only.[1]

[1] It would be more appropriate to call (1) a "recurrence relation", and to reserve the term " ence equation" for an equation of the form $\Delta x_t = \tilde{f}(t, x_t)$, where Δx_t denotes the difference x_t However, it is obvious how to transform a difference equation into an equivalent recurrence re and *vice versa*, so we make no distinction between the two kinds of equation.

Suppose x_0 is given. Then repeated application of equation (1) yields

$$x_1 = f(0, x_0)$$
$$x_2 = f(1, x_1) = f(1, f(0, x_0))$$
$$x_3 = f(2, x_2) = f(2, f(1, f(0, x_0)))$$

and so on. *For a given value of x_0, we can compute x_t for any value of t.* We call this the "insertion method" of solving (1).

Sometimes we can find a simple formula for x_t, but often this is not possible. A **general solution** of (1) is a function of the form $x_t = g(t; A)$ that satisfies (1) for every value of A, where A is an arbitrary constant. For each choice of x_0 there is usually one value of A such that $g(0, A) = x_0$.

EXAMPLE 1 A simple case of equation (1) is

$$x_{t+1} = 2x_t, \qquad t = 0, 1, \ldots \tag{$*$}$$

Suppose x_0 is given. Repeatedly applying (1) gives $x_1 = 2x_0$, $x_2 = 2x_1 = 2 \cdot 2x_0 = 2^2 x_0$, $x_3 = 2x_2 = 2 \cdot 2^2 x_0 = 2^3 x_0$ and so on. In general,

$$x_t = 2^t x_0, \qquad t = 0, 1, \ldots \tag{$**$}$$

The function $x_t = 2^t x_0$ satisfies ($*$) for all t, as can be verified directly. For the given value of x_0, there is clearly no other function that satisfies the equation.

In general, for each choice of x_0, there is a corresponding unique solution of (1). Consequently, there are infinitely many solutions. When x_0 is given, the successive values of x_t can be computed for any natural number t. Does this not tell us the whole story?

In fact, we often need to know more. In economic applications, we are usually interested in establishing qualitative results. For example, we might be interested in the behaviour of the solution when t becomes very large, or in how the solution depends on some parameters that might influence the difference equation. Such questions are difficult or impossible to handle if we rely only on the above insertion method.

Actually, the insertion method suffers from another defect as a numerical procedure. For example, suppose that we have a difference equation like (1), and we want to compute x_{100}. A time-consuming process of successive insertions will finally yield an expression for x_{100}. However, computational errors can easily occur, and if we work with approximate numbers (as we are usually forced to do in serious applications), the approximation error might well explode and in the end give an entirely misleading answer. So there really is a need for a more systematic theory of difference equations. Ideally, the solutions should be expressed in terms of elementary functions. Unfortunately, this is possible only for rather restricted classes of equations.

A Simple First-Order Equation

Consider the difference equation

$$x_{t+1} = ax_t + b_t, \qquad t = 0, 1, \ldots$$

where a is a constant. The equation in Example 1 is a special case with $a = 2$, $b_t =$ Starting with a given x_0, we can calculate x_t algebraically for small t. Indeed

$$x_1 = ax_0 + b_0$$
$$x_2 = ax_1 + b_1 = a(ax_0 + b_0) + b_1 = a^2 x_0 + ab_0 + b_1$$
$$x_3 = ax_2 + b_2 = a(a^2 x_0 + ab_0 + b_1) + b_2 = a^3 x_0 + a^2 b_0 + ab_1 + b_2$$

and so on. This makes the pattern clear. In each case, the formula for x_t begins with t term $a^t x_0$, and then adds the terms $a^{t-1} b_0, a^{t-2} b_1, \ldots, ab_{t-2}, b_{t-1}$ in turn. We thus arri at the following general result:

$$x_{t+1} = ax_t + b_t \iff x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_{k-1}, \quad t = 0, 1, 2, \ldots \qquad ($$

(Note that $a^{t-k} = a^0 = 1$ when $k = t$.) Indeed, to check that we have really found solution to (2), substitute the expression suggested by (3) for x_t into the right-hand side (2). This yields

$$ax_t + b_t = a\Big(a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_{k-1}\Big) + b_t = a^{t+1} x_0 + \sum_{k=1}^{t} a^{t+1-k} b_{k-1} + b_t$$
$$= a^{t+1} x_0 + \sum_{k=1}^{t+1} a^{t+1-k} b_{k-1}$$

This matches our expression for x_{t+1}, so (3) does solve the difference equation.

Consider the special case when $b_k = b$ for all $k = 0, 1, 2, \ldots$. Then

$$\sum_{k=1}^{t} a^{t-k} b_{k-1} = b \sum_{k=1}^{t} a^{t-k} = b(a^{t-1} + a^{t-2} + \cdots + a + 1)$$

According to the summation formula for a geometric series, $1 + a + a^2 + \cdots + a^{t-1} =$ $(1 - a^t)/(1 - a)$, for $a \neq 1$. Thus, for $t = 0, 1, 2, \ldots,$

$$x_{t+1} = ax_t + b \iff x_t = a^t\Big(x_0 - \frac{b}{1-a}\Big) + \frac{b}{1-a} \quad (a \neq 1) \qquad (4$$

For $a = 1$, we have $1 + a + \cdots + a^{t-1} = t$ and $x_t = x_0 + tb$ for $t = 1, 2, \ldots$.

EXAMPLE 2 Solve the following difference equations:

$$\text{(a)}\quad x_{t+1} = \tfrac{1}{2}x_t + 3, \qquad \text{(b)}\quad x_{t+1} = -3x_t + 4$$

Solution: (a) Using (4) we obtain the solution

$$x_t = \left(\tfrac{1}{2}\right)^t (x_0 - 6) + 6$$

(b) In this case, (4) gives

$$x_t = (-3)^t (x_0 - 1) + 1$$

EXAMPLE 3 **(A Multiplier–Accelerator Model of Growth)** Let Y_t denote national income, I_t total investment, and S_t total saving—all in period t. Suppose that savings are proportional to national income, and that investment is proportional to the change in income from period t to $t+1$. Then, for $t = 0, 1, 2, \dots$,

$$S_t = \alpha Y_t$$
$$I_{t+1} = \beta(Y_{t+1} - Y_t)$$
$$S_t = I_t$$

The last equation is the familiar equilibrium condition that saving equals investment in each period. Here α and β are positive constants, and we assume that $\beta > \alpha > 0$. Deduce a difference equation determining the path of Y_t, given Y_0, and solve it.

Solution: From the first and third equations, $I_t = \alpha Y_t$, and so $I_{t+1} = \alpha Y_{t+1}$. Inserting these into the second equation yields $\alpha Y_{t+1} = \beta(Y_{t+1} - Y_t)$, or $(\alpha - \beta)Y_{t+1} = -\beta Y_t$. Thus,

$$Y_{t+1} = \frac{\beta}{\beta - \alpha} Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right) Y_t, \qquad t = 0, 1, 2, \dots \qquad (*)$$

Using (4) gives the solution

$$Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right)^t Y_0, \qquad t = 0, 1, 2, \dots$$

The difference equation $(*)$ constitutes an instance of the equation

$$Y_{t+1} = (1 + g)Y_t, \qquad t = 0, 1, 2, \dots$$

which describes growth at the constant proportional rate g each period. The solution of the equation is $Y_t = (1+g)^t Y_0$. Note that $g = (Y_{t+1} - Y_t)/Y_t$.

Equilibrium States and Stability

Consider the solution of $x_{t+1} = ax_t + b$ given in (4). If $x_0 = b/(1-a)$, then $x_t = b/(1-a)$ for all t. In fact, if $x_s = b/(1-a)$ for any $s \geq 0$, then $x_{s+1} = a(b/(1-a)) + b = b/(1-a)$, and again $x_{s+2} = b/(1-a)$, and so on. We conclude that if x_s ever becomes equal to $b/(1-a)$ at some time s, then x_t will remain at this constant level for each $t \geq s$. The constant $x^* = b/(1-a)$ is called an **equilibrium** (or **stationary**) state for $x_{t+1} = ax_t + b$.

NOTE 1 An alternative way of finding an equilibrium state x^* is to seek a solutio $x_{t+1} = ax_t + b$ with $x_t = x^*$ for all t. Such a solution must satisfy $x_{t+1} = x_t = x^*$ an $x^* = ax^* + b$. Therefore, for $a \neq 1$, we get $x^* = b/(1-a)$ as before.

Suppose the constant a in (4) is less than 1 in absolute value—that is, $-1 < a < 1$. T $a^t \to 0$ as $t \to \infty$, so (4) implies that

$$x_t \to x^* = \frac{b}{1-a} \qquad \text{as} \qquad t \to \infty$$

Hence, if $|a| < 1$, the solution converges to the equilibrium state as $t \to \infty$. The equa is then called **globally asymptotically stable**. Two kinds of stability are shown in Fig (a) and (b). In the first case, x_t converges monotonically down to the equilibrium s In the second case, x_t exhibits decreasing fluctuations or **damped oscillations** around equilibrium state.

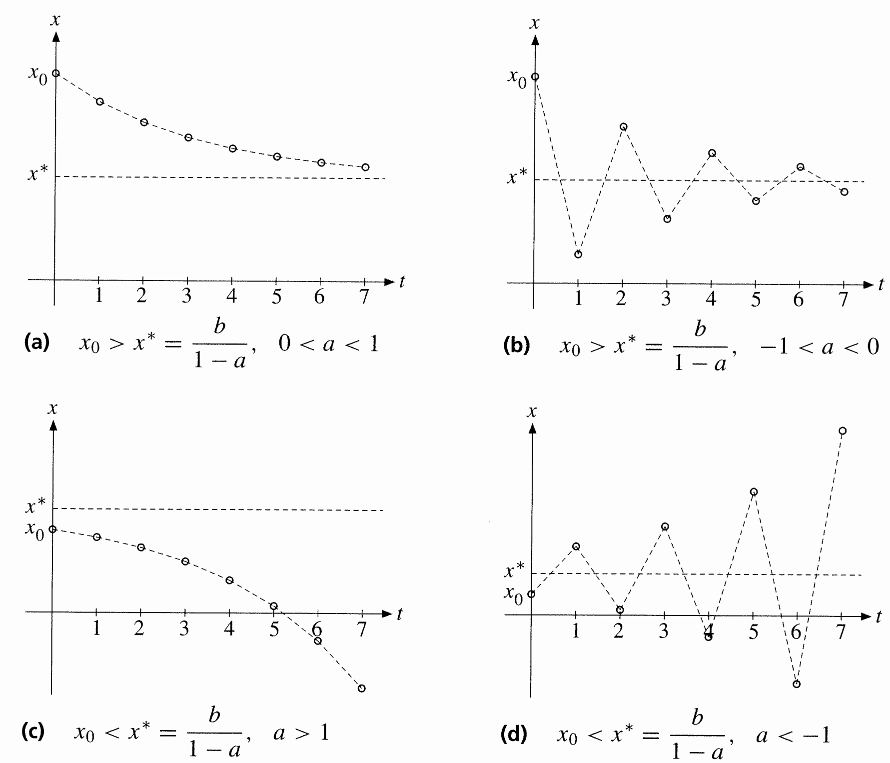

Figure 1

If $|a| > 1$, then the absolute value of a^t tends to ∞ as $t \to \infty$. From (4), it follows tha moves farther and farther away from the equilibrium state, except when $x_0 = b/(1 -$ Two versions of this phenomenon are illustrated in Figs. 1 (c) and (d). In the first ca x_t tends to $-\infty$, and in the second case, x_t exhibits increasing fluctuations or **explos oscillations** around the equilibrium state.

EXAMPLE 4 Equation (a) in Example 2 is stable because $a = 1/2$. The equilibrium state is $b/(1-a) = 3/(1-1/2) = 6$. We see from the solution given in that example that $x_t \to 6$ as $t \to \infty$.

Equation (b) in Example 2 is not stable because $|a| = |-3| = 3 > 1$. The solution does *not* converge to the equilibrium state $x^* = 1$ as $t \to \infty$, except if $x_0 = 1$—in fact, there are explosive oscillations.

EXAMPLE 5 **(The Hog Cycle: A Cobweb Model)** Assume that the total cost of raising q pigs is $C(q) = \alpha q + \beta q^2$. Suppose there are N identical pig farms. Let the demand function for pigs be given by $D(p) = \gamma - \delta p$, as a function of the price p, where the constants α, β, γ, and δ are all positive. Suppose, too, that each farmer behaves competitively, taking the price p as given and maximizing profits $\pi(q) = pq - C(q) = pq - \alpha q - \beta q^2$.

The quantity $q > 0$ maximizes profits only if

$$\pi'(q) = p - \alpha - 2\beta q = 0 \quad \text{and so} \quad q = (p-\alpha)/2\beta$$

It follows that $\pi'(q) > 0$ for $q < (p-\alpha)/2\beta$, and $\pi'(q) < 0$ for $q > (p-\alpha)/2\beta$. Thus, $q = (p-\alpha)/2\beta$ maximizes profits provided $p > \alpha$. In aggregate, the total supply of pigs from all N farms is the function

$$S = N(p-\alpha)/2\beta \qquad (p > \alpha)$$

of the price p. Now, suppose it takes one period to raise each pig, and that when choosing the number of pigs to raise for sale at time $t+1$, each farmer remembers the price p_t at time t and expects p_{t+1} to be the same as p_t. Then the aggregate supply at time $t+1$ will be $S(p_t) = N(p_t - \alpha)/2\beta$.

Equilibrium of supply and demand in all periods requires that $S(p_t) = D(p_{t+1})$, which implies that $N(p_t - \alpha)/2\beta = \gamma - \delta p_{t+1}$, $t = 0, 1, \ldots$. Solving for p_{t+1} in terms of p_t and the parameters gives the difference equation

$$p_{t+1} = -\frac{N}{2\beta\delta} p_t + \frac{\alpha N + 2\beta\gamma}{2\beta\delta}, \qquad t = 1, 2, \ldots \qquad (*)$$

The equilibrium price p^* with $p_{t+1} = p_t$ occurs at $p^* = (\alpha N + 2\beta\gamma)/(2\beta\delta + N)$. The solution of $(*)$ can be expressed as

$$p_t = p^* + (-a)^t (p_0 - p^*) \qquad (a = N/2\beta\delta)$$

Equation $(*)$ is stable if $|-a| < 1$, which happens when $N < 2\beta\delta$. In this case, $p_t \to p^*$ as $t \to \infty$. The solution in this case is illustrated in Fig. 2. Here, q_0 is the supply of pigs at time 0. The price at which all these can be sold is p_0. This determines the supply q_1 one period later. The resulting price at which they sell is p_1, and so on.

The resulting price cycles are damped, and both price and quantity converge to a steady-state equilibrium at (q^*, p^*). This is also an equilibrium of supply and demand. If $N > 2\beta\delta$, however, then the oscillations explode, and eventually p_t becomes less than α. Then the pig farms go out of business, and the solution has to be described in a different way. There is no convergence to a steady state in this case. A third, intermediate, case occurs when $N = 2\beta\delta$ and $a = 1$. Then the pair (q_t, p_t) oscillates perpetually between the two values $(\gamma - \delta p_0, p_0)$ and $(\delta(p_0 - \alpha), \alpha + \gamma/\delta - p_0)$ in even- and odd-numbered periods, respectively.

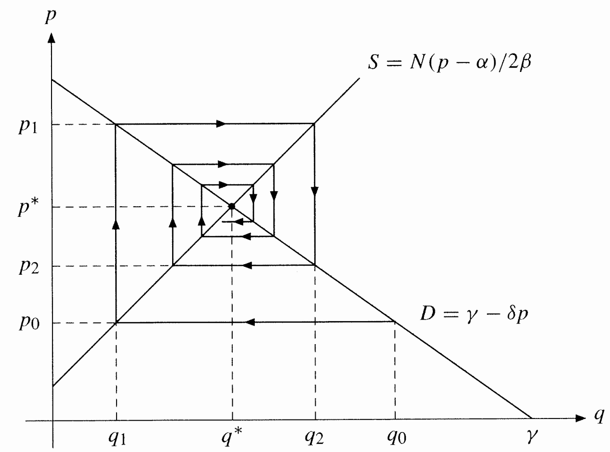

Figure 2 The cobweb model in Example 5—the stable case.

PROBLEMS FOR SECTION 11.1

1. Find the solutions of the following difference equations with the given values of x_0:

(a) $x_{t+1} = 2x_t + 4, \quad x_0 = 1$ (b) $3x_{t+1} = x_t + 2, \quad x_0 = 2$

(c) $2x_{t+1} + 3x_t + 2 = 0, \quad x_0 = -1$ (d) $x_{t+1} - x_t + 3 = 0, \quad x_0 = 3$

2. Consider the difference equation $x_{t+1} = ax_t + b$ in (4) and explain how its solutic behaves in each of the following cases, with $x^* = b/(1-a)$ (for $a \neq 1$):

(a) $0 < a < 1, \ x_0 < x^*$ (b) $-1 < a < 0, \ x_0 < x^*$ (c) $a > 1, \ x_0 > x^*$

(d) $a < -1, \ x_0 > x^*$ (e) $a \neq 1, \ x_0 = x^*$ (f) $a = -1, \ x_0 \neq x^*$

(g) $a = 1, \ b > 0$ (h) $a = 1, \ b < 0$ (i) $a = 1, \ b = 0$

3. (a) Consider the difference equation

$$y_{t+1}(a + by_t) = cy_t, \qquad t = 0, 1, \ldots$$

where a, b, and c are positive constants, and $y_0 > 0$. Show that $y_t > 0$ for all t.

(b) Define a new function x_t by $x_t = 1/y_t$. Show that by using this substitution, tl new difference equation is of the type in (4). Next solve the difference equatic $y_{t+1}(2 + 3y_t) = 4y_t$, assuming that $y_0 = 1/2$. What is the limit of y_t as $t \to \infty$

4. By substituting $y_t = x_t - b/(1-a)$ transform equation (2) into a simple differen equation in y_t. Solve it and find a new confirmation of (4).

5. Consider the difference equation $x_t = \sqrt{x_{t-1} - 1}$ with $x_0 = 5$. Compute x_1, x_2, ar x_3. What about x_4? (This problem illustrates the need to take care if the domain of tl function f in (1) is restricted in any way.)

11.2 Present Discounted Values

The theory in the previous section can be applied to describe the changes over time in a savings account whose balance is subject to compound interest. Let a_t denote the value of the assets held in the account at the end of period t. Further, let c_t be the amount withdrawn for consumption and y_t the amount deposited as income during period t. If the interest rate per period is a constant r, the relevant difference equation is

$$a_{t+1} = (1+r)a_t + y_{t+1} - c_{t+1}, \qquad t = 0, 1, 2, \dots \tag{1}$$

The result in (11.1.3) implies that the solution of (1) is

$$a_t = (1+r)^t a_0 + \sum_{k=1}^{t} (1+r)^{t-k}(y_k - c_k), \qquad t = 1, 2, \dots \tag{2}$$

Let us multiply each term in (2) by $(1+r)^{-t}$, which is a factor of sufficient economic importance to have earned a standard name, namely the **discount factor**. The result is

$$(1+r)^{-t} a_t = a_0 + \sum_{k=1}^{t} (1+r)^{-k}(y_k - c_k) \tag{3}$$

If time 0 is now, then the left-hand side is the **present discounted value** (PDV) of the assets in the account at time t. Equation (3) says that this is equal to

(a) initial assets a_0

(b) plus the total PDV of all future deposits, $\sum_{k=1}^{t}(1+r)^{-k} y_k$

(c) *minus* the total PDV of all future withdrawals $\sum_{k=1}^{t}(1+r)^{-k} c_k$

If time t is now, the formula for a_t in (2) can be interpreted as follows: *Current assets a_t reflect the interest earned on initial assets a_0, with adjustments for the interest earned on all later deposits, or foregone because of later withdrawals.*

EXAMPLE 1 **(Mortgage Repayments)** A particular case of the difference equation (1) occurs when a family borrows an amount K at time 0 as a home mortgage. Suppose there is a fixed interest rate r per period (usually a month rather than a year). Suppose, too, that there are equal repayments of amount a each period, until the mortgage is paid off after n periods (for example, 360 months = 30 years). The outstanding balance or *principal* b_t on the loan in period t satisfies the difference equation $b_{t+1} = (1+r)b_t - a$, with $b_0 = K$ and $b_n = 0$. This difference equation can be solved by using (11.1.4), which gives

$$b_t = (1+r)^t \left(K - \frac{a}{r}\right) + \frac{a}{r}$$

But $b_t = 0$ when $t = n$, so $0 = (1+r)^n\left(K - a/r\right) + a/r$. Solving for K yields

$$K = \frac{a}{r}[1 - (1+r)^{-n}] = a\sum_{t=1}^{n}(1+r)^{-t} \tag{$*$}$$

The original loan, therefore, is equal to the PDV of n equal repayments of amount a period, starting in period 1. Solving for a instead yields

$$a = \frac{rK}{1-(1+r)^{-n}} = \frac{rK(1+r)^n}{(1+r)^n - 1}$$

Formulas ($*$) and ($**$) are the same as those derived by a more direct argument in EM Chapter 10.

PROBLEMS FOR SECTION 11.2

1. Find the solution of (1) for $r = 0.2$, $a_0 = 1000$, $y_t = 100$, and $c_t = 50$.

2. Suppose that at time $t = 0$, you borrow \$100 000 at the fixed interest rate $r = 0.0$ year. You are supposed to repay the loan in 30 equal annual repayments so that $n = 30$ years, the mortgage is paid off. How much is each repayment?

3. A loan of amount \$$L$ is taken out on January 1 of year 0. Instalment payments fo principal and interest are paid annually, commencing on January 1 of year 1. Le interest rate be $r < 2$, so that the interest amounts to rL for the first payment. contract states that the principal share of the repayment will be half the size of interest share.

   (a) Show that the debt after January 1 of year n is $(1 - r/2)^n L$.

   (b) Find r when it is known that exactly half the original loan is paid after 10 yea

   (c) What will the remaining payments be each year if the contract is not changed?

11.3 Second-Order Difference Equations

So far this chapter has considered first-order difference equations, in which each value of a function is related to the value x_t of the function in the previous period only. Next present a typical example from economics where it is necessary to consider second-o difference equations.

EXAMPLE 1 **(A Multiplier–Accelerator Growth Model)** Let Y_t denote national income, C_t t consumption, and I_t total investment in a country at time t. Assume that for $t = 0, 1, \dots$

$$\text{(i)}\ Y_t = C_t + I_t \qquad \text{(ii)}\ C_{t+1} = aY_t + b \qquad \text{(iii)}\ I_{t+1} = c(C_{t+1} - C_t)$$

where a, b, and c are positive constants.

Equation (i) simply states that national income is divided between consumption and investment. Equation (ii) expresses the assumption that consumption in period $t+1$ is a linear function of national income in the previous period. This is the "multiplier" part of the model. Finally, equation (iii) states that investment in period $t+1$ is proportional to the change in consumption from the previous period. The idea is that the existing capital stock provides enough capacity for production to meet current consumption. So investment is only needed when consumption increases. This is the "accelerator" part of the model. The combined "multiplier–accelerator" model has been studied by several economists, notably P. A. Samuelson.

Assume that consumption C_0 and investment I_0 are known in the initial period $t=0$. Then by (i), $Y_0 = C_0 + I_0$, and by (ii), $C_1 = aY_0 + b$. From (iii), we obtain $I_1 = c(C_1 - C_0)$, and then (i) in turn gives $Y_1 = C_1 + I_1$. Hence, Y_1, C_1, and I_1 are all known. Turning to (ii) again, we find C_2, then (iii) gives us the value of I_2, and (i) in turn produces the value of Y_2. Obviously, in this way, we can obtain expressions for C_t, Y_t, and I_t for all t in terms of C_0, Y_0, and the constants a, b, and c. However, the expressions derived get increasingly complicated.

Another method of studying the system is usually more enlightening. It consists of eliminating two of the unknown functions so as to end up with one difference equation in one unknown. Here we use this method to end up with a difference equation in Y_t. To do so, note that equations (i) to (iii) are valid for all $t = 0, 1, \ldots$. Replace t with $t+1$ in (ii) and (iii), and t with $t+2$ in (i) to obtain

$$\text{(iv) } C_{t+2} = aY_{t+1} + b \qquad \text{(v) } I_{t+2} = c(C_{t+2} - C_{t+1}) \qquad \text{(vi) } Y_{t+2} = C_{t+2} + I_{t+2}$$

Inserting (iv) and (ii) into (v) yields $I_{t+2} = ac(Y_{t+1} - Y_t)$. Inserting this result and (iv) into (vi) gives $Y_{t+2} = aY_{t+1} + b + ac(Y_{t+1} - Y_t)$. Rearranging gives

$$Y_{t+2} - a(1+c)Y_{t+1} + acY_t = b, \qquad t = 0, 1, \ldots \tag{vii}$$

This is a second-order linear difference equation with Y_t as the unknown function. The next section sets out a general method for solving such equations. (See Problem 11.4.3.)

The typical second-order difference equation can be written in the form

$$x_{t+2} = f(t, x_t, x_{t+1}), \qquad t = 0, 1, \ldots \tag{1}$$

Suppose that f is defined for all possible values of the variables (t, x_t, x_{t+1}). Suppose x_0 and x_1 have fixed values. Letting $t=0$ in (1), we see that $x_2 = f(0, x_0, x_1)$. Letting $t=1$ yields $x_3 = f(1, x_1, f(0, x_0, x_1))$. By successively inserting $t=2$, $t=3$, ... into (1), the values of x_t for *all* t are uniquely determined in terms of x_0 and x_1. Note in particular that there are infinitely many solutions, and that the solution of the equation is uniquely determined by its values in the first two periods. By definition, a **general** solution of (1) is a function of the form

$$x_t = g(t; A, B) \tag{2}$$

that satisfies (1) and has the property that every solution of (1) can be obtained from (2) by choosing appropriate values of A and B.

Linear Equations

The general second-order linear difference equation is

$$x_{t+2} + a_t x_{t+1} + b_t x_t = c_t \tag{3}$$

where a_t, b_t, and c_t are given functions of t. The associated **homogeneous** equation

$$x_{t+2} + a_t x_{t+1} + b_t x_t = 0 \tag{4}$$

is obtained from (3) by replacing c_t with 0. Compare these equations with the linear differential equations (6.2.1) and (6.2.2). By arguments which are much the same as for differential equations (but simpler), the following results are easy to establish:

THEOREM 11.3.1

The homogeneous difference equation

$$x_{t+2} + a_t x_{t+1} + b_t x_t = 0$$

has the **general solution**

$$x_t = Au_t^{(1)} + Bu_t^{(2)}$$

where $u_t^{(1)}$ and $u_t^{(2)}$ are any two linearly independent solutions, and A and B are arbitrary constants.

THEOREM 11.3.2

The nonhomogeneous difference equation

$$x_{t+2} + a_t x_{t+1} + b_t x_t = c_t$$

has the **general solution**

$$x_t = Au_t^{(1)} + Bu_t^{(2)} + u_t^*$$

where $Au_t^{(1)} + Bu_t^{(2)}$ is the general solution of the associated homogeneous equation (with c_t replaced by zero), and u_t^* is any particular solution of the nonhomogeneous equation.

NOTE 1 In order to use these theorems, we need to know when two solutions of (4) are linearly independent. The following necessary and sufficient condition is easy to apply (and generalizes easily to the case of n functions):

$$\begin{vmatrix} u_0^{(1)} & u_0^{(2)} \\ u_1^{(1)} & u_1^{(2)} \end{vmatrix} \neq 0 \iff u_t^{(1)} \text{ and } u_t^{(2)} \text{ are linearly independent} \tag{5}$$

See Problem 5 for a proof.

A General Solution

There is no universally applicable method of discovering the two linearly independent solutions of (4) that we need in order to find the general solution of the equation. But if we know two linearly independent solutions of (4) and thereby its general solution, then it is always possible to find the general solution of (3).

Consider the equation

$$x_{t+2} + a_t x_{t+1} + b_t x_t = c_t, \qquad t = 0, 1, 2, \ldots \tag{6}$$

Suppose $u_t^{(1)}$ and $u_t^{(2)}$ are linearly independent solutions of the corresponding homogeneous equation and define

$$D_t = u_t^{(1)} u_{t+1}^{(2)} - u_{t+1}^{(1)} u_t^{(2)}$$

Then, if $D_t \neq 0$ for all $t = 1, 2, \ldots$, the general solution of (6) is given by

$$x_t = Au_t^{(1)} + Bu_t^{(2)} - u_t^{(1)} \sum_{k=1}^{t} \frac{c_{k-1} u_k^{(2)}}{D_k} + u_t^{(2)} \sum_{k=1}^{t} \frac{c_{k-1} u_k^{(1)}}{D_k} \tag{7}$$

where A and B are arbitrary constants. (See Hildebrand (1968).)

When the coefficients a_t and b_t in (4) are constants independent of t, then it is always possible to find a simple formula for the general solution of (4). The next section shows how to do this.

PROBLEMS FOR SECTION 11.3

1. Prove by direct substitution that the following functions of t are solutions of the associated difference equation (A and B are constants):

   (a) $x_t = A + B\,2^t, \qquad x_{t+2} - 3x_{t+1} + 2x_t = 0$

   (b) $x_t = A\,3^t + B\,4^t, \qquad x_{t+2} - 7x_{t+1} + 12x_t = 0$

2. Prove that $x_t = A + B\,t$ is the general solution of $x_{t+2} - 2x_{t+1} + x_t = 0$.

3. Prove that $x_t = A\,3^t + B\,4^t$ is the general solution of $x_{t+2} - 7x_{t+1} + 12x_t = 0$.

4. Prove that $x_t = A\,2^t + B\,t2^t + 1$ is the general solution of $x_{t+2} - 4x_{t+1} + 4x_t = 1$.

5. Prove the equivalence in (5). (*Hint:* If the determinant is zero, then the two columns are linearly dependent, and since both $u_t^{(1)}$ and $u_t^{(2)}$ are solutions of equation (4), this dependency will propagate to $u_t^{(1)}$ and $u_t^{(2)}$ for all t.)

11.4 Constant Coefficients

Consider the homogeneous equation

$$x_{t+2} + ax_{t+1} + bx_t = 0$$

where a and b are arbitrary constants, $b \neq 0$, and x_t is the unknown function. Accorc to Theorem 11.3.1, finding the general solution of (1) requires us to discover two soluti $u_t^{(1)}$ and $u_t^{(2)}$ that are linearly independent. On the basis of experience gained in some of previous problems, it should come as no surprise that we try to find solutions to (1) of form $x_t = m^t$. Then $x_{t+1} = m^{t+1} = m \cdot m^t$ and $x_{t+2} = m^{t+2} = m^2 \cdot m^t$. So inserting tl expressions into (1) yields $m^t(m^2 + am + b) = 0$. If $m \neq 0$, then m^t satisfies (1) provi that

$$m^2 + am + b = 0$$

This is the **characteristic equation** of the difference equation. Its solutions are

$$m_1 = -\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2 - 4b}, \qquad m_2 = -\tfrac{1}{2}a - \tfrac{1}{2}\sqrt{a^2 - 4b}$$

There are three different cases, which are summed up in the following theorem:

THEOREM 11.4.1

The **general solution** of

$$x_{t+2} + ax_{t+1} + bx_t = 0 \qquad (b \neq 0)$$

is as follows:

(I) If $a^2 - 4b > 0$ (the characteristic equation has two distinct real roots),

$$x_t = Am_1^t + Bm_2^t, \qquad m_{1,2} = -\tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{a^2 - 4b}$$

(II) If $a^2 - 4b = 0$ (the characteristic equation has one real double root),

$$x_t = (A + Bt)m^t, \qquad m = -\tfrac{1}{2}a$$

(III) If $a^2 - 4b < 0$ (the characteristic equation has no real roots),

$$x_t = r^t(A\cos\theta t + B\sin\theta t), \qquad r = \sqrt{b}, \;\; \cos\theta = -\frac{a}{2\sqrt{b}}, \quad \theta \in [0, \pi]$$

NOTE 1 If x_0 and x_1 are given numbers, then in all three cases the constants A an are uniquely determined. For instance, in case (I), A and B are uniquely determined by equations $x_0 = A + B$ and $x_1 = Am_1 + Bm_2$.

NOTE 2 The solution in case (III) can be expressed as

$$x_t = Cr^t\cos(\theta t + \omega)$$

where ω and C are arbitrary constants. (See the corresponding case for differential equat in Section 6.3.)

Proof of Theorem 11.4.1: (I): The case $a^2 - 4b > 0$ is the simplest. Then m_1 and m_2 are real and different, and m_1^t and m_2^t are both solutions of (1). The determinant in (11.3.5) has the value $m_2 - m_1 \neq 0$, so the two solutions are linearly independent, and the general solution is consequently as given in (I).

(II): If $a^2 - 4b = 0$, then $m = -\frac{1}{2}a$ is a double root of (2). This means that $m^2 + am + b = (m + \frac{1}{2}a)^2$. In addition to m^t, the function tm^t also satisfies (1) (see Problem 6). Moreover, these two functions are linearly independent because the determinant in (11.3.5) is equal to $m = -\frac{1}{2}a$. (Note that $a \neq 0$ because $b \neq 0$.) The general solution is, therefore, as indicated in (II).

(III): If $a^2 - 4b < 0$, the roots of (2) are complex. The two functions $u_t^{(1)} = r^t \cos\theta t$ and $u_t^{(2)} = r^t \sin\theta t$ are linearly independent. Indeed, the determinant in (11.3.5) is

$$\begin{vmatrix} 1 & 0 \\ r\cos\theta & r\sin\theta \end{vmatrix} = r\sin\theta = \sqrt{b}\sqrt{1-\cos^2\theta} = \sqrt{b}\,\sqrt{1-a^2/4b} = \tfrac{1}{2}\sqrt{4b-a^2} > 0$$

Moreover, direct substitution shows that both these functions satisfy (1).

Indeed, let us show that $u_t^{(1)} = r^t \cos\theta t$ satisfies (1). We find that $u_{t+1}^{(1)} = r^{t+1}\cos\theta(t+1)$ and $u_{t+2}^{(1)} = r^{t+2}\cos\theta(t+2)$. Hence, using the formula (B.1.8) for the cosine of a sum, we get

$$\begin{aligned} u_{t+2}^{(1)} + au_{t+1}^{(1)} + bu_t^{(1)} &= r^{t+2}\cos\theta(t+2) + ar^{t+1}\cos\theta(t+1) + br^t\cos\theta t \\ &= r^t[r^2(\cos\theta t\cos 2\theta - \sin\theta t\sin 2\theta) + ar(\cos\theta t\cos\theta - \sin\theta t\sin\theta) + b\cos\theta t] \\ &= r^t[(r^2\cos 2\theta + ar\cos\theta + b)\cos\theta t - (r^2\sin 2\theta + ar\sin\theta)\sin\theta t] \end{aligned}$$

Here the coefficients of $\cos\theta t$ and $\sin\theta t$ are both equal to 0 because $r^2\cos 2\theta + ar\cos\theta + b = r^2(2\cos^2\theta - 1) + ar\cos\theta + b = b(2a^2/4b - 1) + a\sqrt{b}(-a/2\sqrt{b}) + b = 0$, and likewise $r^2\sin 2\theta + ar\sin\theta = 2r^2\sin\theta\cos\theta + ar\sin\theta = 2r^2(-a/2r)\sin\theta + ar\sin\theta = 0$. This shows that $u_t^{(1)} = r^t\cos\theta t$ satisfies equation (1), and a similar argument shows that so does $u_t^{(2)} = r^t\sin\theta$. ∎

NOTE 3 An alternative argument for the solution in (III) relies on properties of the complex exponential function. In trigonometric form the roots in (3) are $m_1 = \alpha + i\beta = r(\cos\theta + i\sin\theta)$ and $m_2 = \alpha - i\beta = r(\cos\theta - i\sin\theta)$, with $\theta \in [0, \pi]$, $r = \sqrt{\alpha^2+\beta^2} = \sqrt{b}$, $\cos\theta = \alpha/r = -a/2\sqrt{b}$, and $\sin\theta = \beta/r = (\sqrt{b - a^2/4})/\sqrt{b}$.

By de Moivre's formula, (B.3.8), $m_1^t = r^t(\cos\theta t + i\sin\theta t)$ and $m_2^t = r^t(\cos\theta t - i\sin\theta t)$. The complex functions m_1^t and m_2^t both satisfy (1), and so does every linear combination of them. In particular, $\frac{1}{2}(m_1^t + m_2^t) = r^t\cos\theta t$ and $\frac{1}{2i}(m_1^t - m_2^t) = r^t\sin\theta t$ both satisfy (1). The general solution of (1) is therefore as given in case (III).

We see that when the characteristic equation has complex roots, the solution of (1) involves oscillations. The number r is the **growth factor**. Note that when $|r| < 1$, then $|Ar^t| \to 0$ as $t \to \infty$ and the oscillations are **damped**. If $|r| > 1$, the oscillations are **explosive**, and in the case $|r| = 1$, we have undamped oscillations.

Let us now consider some examples of difference equations of the form (1).

EXAMPLE 1 Find the general solutions of

(a) $x_{t+2} - 3.9x_{t+1} + 3.78x_t = 0$ (b) $x_{t+2} - 6x_{t+1} + 9x_t = 0$ (c) $x_{t+2} - x_{t+1} + x_t = 0$

Solution: (a) The characteristic equation is $m^2 - 3.9m + 3.78 = 0$, whose roots $m_1 = 1.8$ and $m_2 = 2.1$, so the general solution is

$$x_t = A(1.8)^t + B(2.1)^t$$

(b) The characteristic equation is $m^2 - 6m + 9 = (m-3)^2 = 0$, so $m = 3$ is a double r
The general solution is

$$x_t = (A + Bt)3^t$$

(c) The characteristic equation is $m^2 - m + 1 = 0$, with complex roots $m_1 = \frac{1}{2}(1 + i$ and $m_2 = \frac{1}{2}(1 - i\sqrt{3})$. Here $r = \sqrt{b} = 1$ and $\cos\theta = 1/2$, so $\theta = \frac{1}{3}\pi$. The ger solution is

$$x_t = A\cos\frac{\pi}{3}t + B\sin\frac{\pi}{3}t$$

The frequency is $(\pi/3)/(2\pi) = 1/6$ and the growth factor is $\sqrt{b} = 1$, so the oscillat are undamped.

The Nonhomogeneous Case

Now consider the nonhomogeneous equation

$$x_{t+2} + ax_{t+1} + bx_t = c_t \qquad (b \neq 0)$$

According to Theorem 11.3.2, its general solution is

$$x_t = Au_t^{(1)} + Bu_t^{(2)} + u_t^*$$

where $Au_t^{(1)} + Bu_t^{(2)}$ is the general solution of the associated homogeneous equatior and u_t^* is a particular solution of (5). Theorem 11.4.1 tells us how to find $Au_t^{(1)} + B$ How do we find u_t^*? The general formula in (11.3.7) gives one answer, but it involves of work, even when c_t is a simple function.

In some cases it is much easier. For example, suppose $c_t = c$, where c is a cons Then (5) takes the form

$$x_{t+2} + ax_{t+1} + bx_t = c \qquad (c \text{ is a constant})$$

We look for a solution of the form $x_t = C$, where C is a constant. Then $x_{t+1} = x_{t+2}$ so inserting $x_t = C$ into (7) gives $C + aC + bC = c$, that is, $C = c/(1 + a + b)$. He

$$u_t^* = \frac{c}{1+a+b} \qquad \text{is a particular solution of (7) when } 1 + a + b \neq 0$$

(If $1 + a + b = 0$, no constant function satisfies (7). To handle this case, see Problem

Consider more generally the case in which c_t in (5) is a linear combination of terr the form

$$a^t, \quad t^m, \quad \cos qt, \quad \text{or} \quad \sin qt$$

or products of such terms. Then the method of undetermined coefficients can be used to obtain a particular solution of (5). (If the function c_t in (5) happens to satisfy the homogeneous equation, the procedures described below must be modified.)[2]

EXAMPLE 2 Solve the equation $x_{t+2} - 5x_{t+1} + 6x_t = 4^t + t^2 + 3$.

Solution: The associated homogeneous equation has $m^2 - 5m + 6 = 0$ as its characteristic equation, with the two roots $m_1 = 2$ and $m_2 = 3$. Its general solution is, therefore, $A\,2^t + B\,3^t$. To find a particular solution we look for constants C, D, E, and F such that

$$u_t^* = C\,4^t + D\,t^2 + E\,t + F$$

is a solution. (You cannot put $E = 0$.) This requires that

$$C4^{t+2} + D(t+2)^2 + E(t+2) + F - 5[C4^{t+1} + D(t+1)^2 + E(t+1) + F] + 6(C4^t + Dt^2 + Et + F) = 4^t + t^2 + 3$$

Expanding then rearranging yields $2C4^t + 2Dt^2 + (-6D + 2E)t + (-D - 3E + 2F) = 4^t + t^2 + 3$. For this to hold for all $t = 0, 1, \ldots$ one must have $2C = 1$, $2D = 1$, $-6D + 2E = 0$, and $-D - 3E + 2F = 3$. It follows that $C = 1/2$, $D = 1/2$, $E = 3/2$, and $F = 4$. The general solution of the equation is, therefore,

$$x_t = A\,2^t + B\,3^t + \tfrac{1}{2}4^t + \tfrac{1}{2}t^2 + \tfrac{3}{2}t + 4$$

Stability

Suppose an economy evolves according to some difference equation (or system of difference equations). If the right number of initial conditions are imposed, the system has a unique solution. Also, if one or more initial conditions are changed, the solution changes. An important question is this: Will small changes in the initial conditions have any effect on the long-run behaviour of the solution, or will the effect die out as $t \to \infty$? In the latter case, the system is called **stable**. On the other hand, if small changes in the initial conditions might lead to significant differences in the long run behaviour of the solution, then the system is **unstable**. Because an initial state cannot be pinpointed exactly, but only approximately, stability in the sense indicated above is sometimes a minimum requirement for a model to be economically meaningful.

Consider in particular the second-order nonhomogeneous difference equation (5) whose general solution is of the form $x_t = Au_t^{(1)} + Bu_t^{(2)} + u_t^*$. Equation (5) is called **globally asymptotically stable** if the general solution $Au_t^{(1)} + Bu_t^{(2)}$ of the associated homogeneous equation tends to 0 as $t \to \infty$, for all values of A and B. So the effect of the initial conditions dies out as $t \to \infty$.

If $Au_t^{(1)} + Bu_t^{(2)}$ tends to 0 as $t \to \infty$, for all values of A and B, then in particular $u_t^{(1)} \to 0$ as $t \to \infty$ (choose $A = 1$, $B = 0$), and $u_t^{(2)} \to 0$ as $t \to \infty$ (choose $A = 0$, $B = 1$). On the other hand, these two conditions are obviously sufficient for $Au_t^{(1)} + Bu_t^{(2)}$ to approach 0 as $t \to \infty$.

[2] For more details, we refer to Goldberg (1958) or Gandolfo (1980).

For the remainder of this section, $u_t^{(1)}$ and $u_t^{(2)}$ will denote the particular solutions c that were used in the proof of Theorem 11.4.1.

We claim that $u_t^{(1)} \to 0$ and $u_t^{(2)} \to 0$ as $t \to \infty$ if and only if the moduli of the of $m^2 + am + b = 0$ are both less than 1.[3]

First, in the case when the characteristic polynomial has two distinct real roots, $m_1 \neq$ the two solutions are $u_t^{(1)} = m_1^t$ and $u_t^{(2)} = m_2^t$. In this case, we see that $u_t^{(1)} \to$ ($u_t^{(2)} \to 0$ as $t \to \infty$ if and only if $|m_1| < 1$ and $|m_2| < 1$.

Second, when the characteristic polynomial has a double root, $m = -a/2$, then the linearly independent solutions are m^t and tm^t. Again, $|m| < 1$ is a necessary and suffi condition for these two solutions to approach 0 as $t \to \infty$.

Third, suppose the characteristic polynomial has complex roots $m = \alpha \pm i\beta$. Then $-\frac{1}{2}a$ and $\beta = \frac{1}{2}\sqrt{4b - a^2}$. So the modulus of either root is equal to $|m| = \sqrt{\alpha^2 + \beta^2} =$ We argued before that the two solutions $r^t \cos\theta t$ and $r^t \sin\theta t$ tend to 0 as t tends to in if and only if $r = \sqrt{b} < 1$—that is, if and only if $b < 1$.

To summarize, we have the following result:

THEOREM 11.4.2

The equation

$$x_{t+2} + ax_{t+1} + bx_t = c_t$$

is globally asymptotically stable if and only if the following two equivalent conditions are satisfied:

(A) The roots of the characteristic equation $m^2 + am + b = 0$ have moduli strictly less than 1.

(B) $|a| < 1 + b$ and $b < 1$

It remains to prove that (B) is equivalent to (A). *Assume first that* $b > a^2/4$. The characteristic equation has complex roots $m_{1,2} = \alpha \pm i\beta$ and $|m_1| = |m_2| = \sqrt{b}$, a (B) obviously implies (A). On the other hand, since $f(m) = m^2 + am + b$ is never and since $f(0) = b$ is positive, the Intermediate Value Theorem tells us that $f(m)$ be positive for all m. In particular $f(1) = 1 + a + b > 0$ and $f(-1) = 1 - a + b$ But these conditions together are equivalent to $|a| < 1 + b$, so (A) implies (B) are necessary. Problem 11 asks you to analyse the case of real roots.

EXAMPLE 3 Investigate the stability of the equation $x_{t+2} - \frac{1}{6}x_{t+1} - \frac{1}{6}x_t = c_t$.

Solution: In this case $a = -1/6$ and $b = -1/6$, so $|a| = 1/6$ and $1 + b = 5/6$. according to Theorem 11.4.2, the equation is stable. This conclusion can be confirm looking at the general solution of the associated homogeneous equation, which is $A(1/2)^t + B(-1/3)^t$. Clearly, $x_t \to 0$ irrespective of the values of A and B, so the equation is globally asymptotically stable.

[3] See Section B.3. Note that if m is a real number, the modulus of m equals the absolute of m.

EXAMPLE 4 Investigate the stability of equation (vii) in Example 11.3.1, where a and c are positive,

Solution: From Theorem 11.4.2 (B) it follows that the equation is stable if and only if $a(1+c) < 1 + ac$ and $ac < 1$—that is, if and only if $a < 1$ and $ac < 1$. (See also Problem 3.)

PROBLEMS FOR SECTION 11.4

Find the general solutions of the difference equations in Problems 1 and 2.

1. (a) $x_{t+2} - 6x_{t+1} + 8x_t = 0$ (b) $x_{t+2} - 8x_{t+1} + 16x_t = 0$

   (c) $x_{t+2} + 2x_{t+1} + 3x_t = 0$ (d) $3x_{t+2} + 2x_t = 4$

2. (a) $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$ (b) $x_{t+2} - 3x_{t+1} + 2x_t = 3 \cdot 5^t + \sin(\frac{1}{2}\pi t)$

3. Consider the difference equation (vii) in Example 11.3.1, with $a > 0$, $c > 0$, and $a \neq 1$.

   (a) Find a special solution of the equation.

   (b) Find the characteristic equation of the associated homogeneous equation and determine when it has two different real roots, or a double real root, or two complex roots.

4. Consider equation (7) and assume that $1 + a + b = 0$. If $a \neq -2$, find a constant D such that Dt satisfies (7). If $a = -2$, find a constant D such that Dt^2 satisfies (7).

5. A model of location uses the difference equation

$$D_{n+2} - 4(ab+1)D_{n+1} + 4a^2b^2D_n = 0, \qquad n = 0, 1, \ldots$$

   where a and b are constants, and D_n is the unknown function. Find the solution of this equation assuming that $1 + 2ab > 0$.

6. Consider equation (1) assuming that $\frac{1}{4}a^2 - b = 0$, so that the characteristic equation has a real double root $m = -a/2$. Let $x_t = u_t(-a/2)^t$ and prove that x_t satisfies (1) provided that u_t satisfies the equation $u_{t+2} - 2u_{t+1} + u_t = 0$. Use the result in Problem 11.3.2 to find x_t.

7. Investigate the global asymptotic stability of the following equations:

   (a) $x_{t+2} - \frac{1}{3}x_t = \sin t$ (b) $x_{t+2} - x_{t+1} - x_t = 0$ (c) $x_{t+2} - \frac{1}{8}x_{t+1} + \frac{1}{6}x_t = t^2e^t$

8. (a) A model due to B. J. Ball and E. Smolensky is based on the following system:

$$C_t = cY_{t-1}, \qquad K_t = \sigma Y_{t-1}, \qquad Y_t = C_t + K_t - K_{t-1}$$

   Here C_t denotes consumption, K_t capital stock, Y_t net national product, where and σ are positive constants. Give an economic interpretation of the equations

   (b) Derive a difference equation of the second order for Y_t. Find necessary and suffic conditions for the solution of this equation to have explosive oscillations.

9. (a) A model by J. R. Hicks uses the following difference equation:

$$Y_{t+2} - (b+k)Y_{t+1} + kY_t = a(1+g)^t, \qquad t = 0, 1, \ldots$$

   where a, b, g, and k are constants. Find a special solution Y_t^* of the equation.

   (b) Give conditions for the characteristic equation to have two complex roots.

   (c) Find the growth factor r of the oscillations when the conditions obtained in part are satisfied, and determine when the oscillations are damped.

10. The authors Frisch, Haavelmo, Nørregaard-Rasmussen, and Zeuthen, in their stud the "wage-price spiral" of inflation, considered the following system for $t = 0, 1,$

$$\text{(i)} \quad \frac{W_{t+2} - W_{t+1}}{W_{t+1}} = \frac{P_{t+1} - P_t}{P_t} \qquad \text{(ii)} \quad P_t = \gamma + \beta W_t$$

   Here W_t denotes the wage level and P_t the price index at time t, whereas γ an are constants. The first equation states that the proportional increase in wages is e to the proportional increase in the price index one period earlier, whereas the sec equation relates prices to current wages.

   (a) Deduce from (i) and (ii) the following equation for W_t:

$$\frac{W_{t+2}}{\gamma + \beta W_{t+1}} = \frac{W_{t+1}}{\gamma + \beta W_t}, \qquad t = 0, 1, \ldots$$

   (b) Use (iii) to prove that $W_{t+1} = c(\gamma + \beta W_t)$, $t = 0, 1, \ldots$, where $c = W_1/$ and find a general expression for W_t when $c\beta \neq 1$. Under what conditions the equation be globally asymptotically stable, and what is then the limit of W $t \to \infty$?

HARDER PROBLEMS

11. Prove that the conditions in (B) in Theorem 11.4.2 are equivalent to the condition in for the case when the characteristic polynomial has real roots, by studying the paral $f(m) = m^2 + am + b$. (Consider the values of $f(-1)$, $f(1)$, $f'(-1)$, and $f'(1)$.)

11.5 Higher-Order Equations

In this section we briefly record some results for nth order difference equations,

$$x_{t+n} = f(t, x_t, x_{t+1}, \dots, x_{t+n-1}), \qquad t = 0, 1, \dots \tag{1}$$

Suppose f is defined for all values of the variables. If we require that $x_0, x_1, \dots, x_{n-1}$ have given fixed values by substituting $t = 0$ into (1), we find that $x_n = f(0, x_0, x_1, \dots, x_{n-1})$ is uniquely determined. Then substituting $t = 1$ into (1) yields $x_{n+1} = f(1, x_1, x_2, \dots, x_n) = f(1, x_1, x_2, \dots, f(0, x_0, x_1, \dots, x_{n-1}))$. And so on. Thus the solution of equation (1) is uniquely determined by the values x_t takes in the first n periods, $0, 1, \dots, n-1$.

The **general solution** of (1) is a function $x_t = g(t; C_1, \dots, C_n)$ depending on n arbitrary constants, $C_1, \dots, C_n$, that satisfies (1) and has the property that every solution of (1) can be obtained by giving $C_1, \dots, C_n$ appropriate values.

Linear Equations

The general theory for second-order linear difference equations is easily generalized to nth order equations.

THEOREM 11.5.1

The general solution of the homogeneous difference equation

$$x_{t+n} + a_1(t)x_{t+n-1} + \cdots + a_{n-1}(t)x_{t+1} + a_n(t)x_t = 0$$

where $a_n(t) \neq 0$, is given by

$$x_t = C_1 u_t^{(1)} + \cdots + C_n u_t^{(n)}$$

where $u_t^{(1)}, \dots, u_t^{(n)}$ are n linearly independent solutions of the equation and $C_1, \dots, C_n$ are arbitrary constants.

THEOREM 11.5.2

The general solution of the nonhomogeneous difference equation

$$x_{t+n} + a_1(t)x_{t+n-1} + \cdots + a_{n-1}(t)x_{t+1} + a_n(t)x_t = b_t$$

where $a_n(t) \neq 0$, is given by

$$x_t = C_1 u_t^{(1)} + \cdots + C_n u_t^{(n)} + u_t^*$$

where $C_1 u_t^{(1)} + \cdots + C_n u_t^{(n)}$ is the general solution of the corresponding homogeneous equation, and u_t^* is a particular solution of the nonhomogeneous equation.

NOTE 1 In using the theorem we need the following generalization of (11.3.5): If $u_t^{(1)}$, . $u_t^{(n)}$ are solutions of the homogeneous difference equation in Theorem 11.5.1, then

$$\begin{vmatrix} u_0^{(1)} & \dots & u_0^{(n)} \\ u_1^{(1)} & \dots & u_1^{(n)} \\ \dots & \dots & \dots \\ u_{n-1}^{(1)} & \dots & u_{n-1}^{(n)} \end{vmatrix} \neq 0 \iff u_t^{(1)}, \dots, u_t^{(n)} \text{ are linearly independent}$$

Constant Coefficients

The general linear difference equation of nth order with constant coefficients takes the fc

$$x_{t+n} + a_1 x_{t+n-1} + \cdots + a_{n-1}x_{t+1} + a_n x_t = b_t, \qquad t = 0, 1, \dots$$

The corresponding homogeneous equation is

$$x_{t+n} + a_1 x_{t+n-1} + \cdots + a_{n-1}x_{t+1} + a_n x_t = 0, \qquad t = 0, 1, \dots$$

We try to find solutions to (4) of the form $x_t = m^t$. Inserting this solution and cancell the common factor m^t yields the **characteristic equation**

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1}m + a_n = 0$$

According to the fundamental theorem of algebra, this equation has exactly n roots, wl each is counted according to its multiplicity.

Suppose first that equation (5) has n different real roots $m_1, m_2, \dots, m_n$. Then m_1^t, ı $\dots, m_n^t$ all satisfy (4). These functions are moreover linearly independent, so the gene solution of (4) in this case is

$$x_t = C_1 m_1^t + C_2 m_2^t + \cdots + C_n m_n^t$$

This is *not* the general solution of (4) if equation (5) has multiple roots and/or comp roots. The general method for finding n linearly independent solutions of (4) is as follo

Find the roots of equation (5) together with their multiplicity.

(A) A real root m_i with multiplicity 1 gives the solution m_i^t.

(B) A real root m_j with multiplicity $p > 1$ gives the solutions $m_j^t, tm_j^t, \dots, t^{p-1}m_j^t$.

(C) A pair of complex roots $\alpha \pm i\beta$, each with multiplicity 1, gives the solutions r^t cos $r^t \sin\theta t$, where $r = \sqrt{\alpha^2 + \beta^2}$, and $\theta \in [0, \pi]$ satisfies $\cos\theta = \alpha/r$, $\sin\theta = \beta/r$.

(D) A pair of complex roots $\alpha \pm i\beta$, each with multiplicity $q > 1$, gives the solutions u $tu, tv, \dots, t^{q-1}u, t^{q-1}v$, with $u = r^t\cos\theta t$ and $v = r^t \sin\theta t$, where $r = \sqrt{\alpha^2 +$ and $\theta \in [0, \pi]$ satisfies $\cos\theta = \alpha/r$ and $\sin\theta = \beta/r$.

In order to find the general solution of the nonhomogeneous equation (3), it remains to f a particular solution u_t^* of (3). If b_t is a linear combination of products of terms of the fc $a^t, t^m, \cos qt$ and $\sin qt$, as in Section 11.4, the method of undetermined coefficients ag can be used.

Stability

Equation (3) is **globally asymptotically stable** if the general solution $C_1 u_t^{(1)} + \cdots + C_n u_t^{(n)}$ of the associated homogeneous equation (4) tends to 0 as $t \to \infty$, for all values of the constants $C_1, \ldots, C_n$. Then the effect of the initial conditions "dies out" as $t \to \infty$.

As in the case $n = 2$, equation (3) is globally asymptotically stable if and only if $u_i(t) \to 0$ as $t \to \infty$ for all $i = 1, \ldots, n$. Each u_i corresponds to a root, m_i, of the characteristic polynomial. Again, $u_i(t) \to 0$ as $t \to \infty$ if and only if modulus of the corresponding solution of the characteristic equation is < 1.

THEOREM 11.5.3

A necessary and sufficient condition for (3) to be global asymptotically stable is that all roots of the characteristic polynomial of the equation have moduli strictly less than 1.

The following result gives a stability condition based directly on the coefficients of the characteristic equation. (The dashed lines have been included to make it easier to see the structure of the determinants.) See Chipman (1950) for a discussion of this theorem.

THEOREM 11.5.4 (SCHUR)

Let

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n$$

be a polynomial of degree n with real coefficients. A necessary and sufficient condition for all roots of the polynomial to have moduli less than 1 is that

$$\left|\begin{array}{c:c} 1 & a_n \\ \hdashline a_n & 1 \end{array}\right| > 0, \quad \left|\begin{array}{cc:cc} 1 & 0 & a_n & a_{n-1} \\ a_1 & 1 & 0 & a_n \\ \hdashline a_n & 0 & 1 & a_1 \\ a_{n-1} & a_n & 0 & 1 \end{array}\right| > 0, \quad \ldots,$$

$$\left|\begin{array}{cccc:cccc} 1 & 0 & \ldots & 0 & a_n & a_{n-1} & \ldots & a_1 \\ a_1 & 1 & \ldots & 0 & 0 & a_n & \ldots & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \ldots & 1 & 0 & 0 & \ldots & a_n \\ \hdashline a_n & 0 & \ldots & 0 & 1 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & \ldots & 0 & 0 & 1 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_n & 0 & 0 & \ldots & 1 \end{array}\right| > 0$$

Let us see what the theorem tell us in the case $n = 1$. When $n = 1$, Theorem 11.5.4 that $m + a_1 = 0$ has a root with modulus < 1 if and only if $\begin{vmatrix} 1 & a_1 \\ a_1 & 1 \end{vmatrix} > 0$, i.e. if and or $a_1^2 < 1$. (Of course, this is clear without using the theorem.) Now, $a_1^2 < 1 \Leftrightarrow |a_1| < 1$

$$x_{t+1} + a_1 x_t = c_t \text{ is globally asymptotically stable } \iff |a_1| < 1$$

When $n = 2$, Theorem 11.5.4 says that both roots of $m^2 + a_1 m + a_2 = 0$ have mc < 1 if and only if

$$D_1 = \left|\begin{array}{c:c} 1 & a_2 \\ \hdashline a_2 & 1 \end{array}\right| > 0 \quad \text{and} \quad D_2 = \left|\begin{array}{cc:cc} 1 & 0 & a_2 & a_1 \\ a_1 & 1 & 0 & a_2 \\ \hdashline a_2 & 0 & 1 & a_1 \\ a_1 & a_2 & 0 & 1 \end{array}\right| > 0$$

Evaluating the determinants yields

$$D_1 = 1 - a_2^2 \quad \text{and} \quad D_2 = (1 - a_2)^2(1 + a_1 + a_2)(1 - a_1 + a_2)$$

Here $D_1 > 0 \iff |a_2| < 1$. If $D_1 > 0$, then in particular $1 - a_2 \neq 0$, so $D_2 > 0 \iff (1 + a_1 + a_2)(1 - a_1 + a_2) > 0 \iff |a_1| < 1 + a_2$. The pro $(1 + a_1 + a_2)(1 - a_1 + a_2)$ is positive if and only if either both factors are positive or l are negative. If both are negative, $1 + a_1 + a_2 < 0$ and $1 - a_1 + a_2 < 0$. Adding tl inequalities yields $2 + 2a_2 < 0$, i.e. $1 + a_2 < 0$, which contradicts $D_1 > 0$. Henc $D_1 > 0$ and $D_2 > 0$, then

$$1 + a_1 + a_2 > 0 \quad \text{and} \quad 1 - a_1 + a_2 > 0 \quad \text{and} \quad 1 - a_2 > 0 \qquad ($$

On the other hand, if these inequalities are satisfied, then adding the first two implies $2 + 2a_2 > 0$, i.e. $1 + a_2 > 0$. But then we see that (**) implies that D_1 and D_2 defi by (*) are both positive. Thus the conditions in (*) are equivalent to the conditions in (Since $1 + a_1 + a_2 > 0$ and $1 - a_1 + a_2 > 0$ are equivalent to $|a_1| < 1 + a_2$, we see Theorem 11.4.2 is the particular case of Theorem 11.5.4 that holds when $n = 2$.

PROBLEMS FOR SECTION 11.5

1. Solve the following difference equations

   (a) $x_{t+3} - 3x_{t+1} + 2x_t = 0$ (b) $x_{t+4} + 2x_{t+2} + x_t = 8$

2. Examine the stability of the following difference equations:

   (a) $x_{t+2} - \frac{1}{3}x_t = \sin t$ (b) $x_{t+2} - x_{t+1} - x_t = 0$

   (c) $x_{t+2} - \frac{1}{8}x_{t+1} + \frac{1}{6}x_t = t^2 e^t$ (d) $x_{t+2} + 3x_{t+1} - 4x_t = t - 1$

3. In the $a_1 a_2$-plane, describe the domain defined by the inequalities (**).

4. Examine when the equation in problem 11.4.9 is globally asymptotically stable, assuming $k > 0$ and $b > 0$.

5. A paper by Akerlof and Stiglitz studies the equation

$$K_{t+2} + \left(\frac{\sigma\beta}{\alpha} - 2\right)K_{t+1} + (1 - \sigma\beta)K_t = d$$

where the constants α, β, and σ are positive.

(a) Find a condition for both roots of the characteristic polynomial to be complex.

(b) Find a necessary and sufficient condition for stability.

11.6 Systems of Difference Equations

A first-order system of difference equations can usually be expressed in the **normal form**:[4]

$$\begin{aligned} x_1(t+1) &= f_1(t, x_1(t), \dots, x_n(t)) \\ &\dots\dots\dots\dots, \qquad t = 0, 1, \dots \\ x_n(t+1) &= f_n(t, x_1(t), \dots, x_n(t)) \end{aligned} \tag{1}$$

If $x_1(0), \dots, x_n(0)$ are specified, then $x_1(1), \dots, x_n(1)$ are found by substituting $t = 0$ in (1), next $x_1(2), \dots, x_n(2)$ are found by substituting $t = 1$, etc. Thus the values of $x_1(t)$, $\dots, x_n(t)$ are uniquely determined for all t (assuming that $f_1, \dots, f_n$ are defined for all values of the variables). Thus the solution of (1) is uniquely determined by the values of $x_1(0), \dots, x_n(0)$.

The **general solution** of (1) is given by n functions

$$x_1 = g_1(t; C_1, \dots, C_n), \dots, x_n = g_n(t; C_1, \dots, C_n) \tag{**}$$

with the property that an arbitrary solution $(x_1(t), \dots, x_n(t))$ is obtained from (**) by giving $C_1, \dots, C_n$ appropriate values.

Of course, there are no general methods that lead to explicit solutions of (1) in "closed" form. Only in some special cases can we find closed form solutions.

EXAMPLE 1 Find the general solution of the system

$$\text{(i)}\quad x_{t+1} = \tfrac{1}{2}x_t + \tfrac{1}{3}y_t, \qquad \text{(ii)}\quad y_{t+1} = \tfrac{1}{2}x_t + \tfrac{2}{3}y_t, \qquad t = 0, 1, \dots$$

[4] In this section, the argument t is often included in parentheses, when subscripts are needed to indicate different variables in the system.

Solution: Guided by the method we used to solve systems of two differential equat in Section 6.5, we try to derive a second-order difference equation with x_t as the unknown. From (i) we obtain (iii) $y_t = 3x_{t+1} - \frac{3}{2}x_t$, which inserted into (ii) yields $y_{t+1} = 2x_{t+1} - \frac{1}{2}x_t$. Replacing t by $t+1$ in (i), we obtain (v) $x_{t+2} = \frac{1}{2}x_{t+1} + \frac{1}{3}$ Inserting (iv) into (v), then rearranging, one obtains

$$x_{t+2} - \tfrac{7}{6}x_{t+1} + \tfrac{1}{6}x_t = 0$$

The characteristic equation is $m^2 - \frac{7}{6}m + \frac{1}{6} = 0$, with the roots $m_1 = 1$, $m_2 = \frac{1}{6}$. general solution is then easily found. In turn, (iii) is used to find y_t. The result is

$$x_t = A + B\left(\tfrac{1}{6}\right)^t, \qquad y_t = \tfrac{3}{2}A - B\left(\tfrac{1}{6}\right)^t$$

Matrix Formulation of Linear Systems

If the functions $f_1, \dots, f_n$ in (1) are linear, we obtain the system

$$\begin{aligned} x_1(t+1) &= a_{11}(t)x_1(t) + \dots + a_{1n}(t)x_n(t) + b_1(t) \\ &\dots\dots\dots\dots, \qquad t = 0, 1, \dots \\ x_n(t+1) &= a_{n1}(t)x_1(t) + \dots + a_{nn}(t)x_n(t) + b_n(t) \end{aligned}$$

Suppose we define

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

Then (2) is equivalent to the matrix equation

$$\mathbf{x}(t+1) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t), \qquad t = 0, 1, \dots$$

The method suggested in Example 1 allows one, in general, to derive a linear nth o difference equation in one of the unknowns, say x_1. When all the coefficients $a_{ij}(t)$ constants, $a_{ij}(t) = a_{ij}$, this method will lead to a linear difference equation with cons coefficients.

Alternatively, if $\mathbf{A}(t)$ is a constant matrix $\mathbf{A}$, then (3) reduces to

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t), \qquad t = 0, 1, \dots$$

Inserting $t = 0, 1, \dots$, we get successively $\mathbf{x}(1) = \mathbf{A}\mathbf{x}(0) + \mathbf{b}(0)$, $\mathbf{x}(2) = \mathbf{A}\mathbf{x}(1) + \mathbf{b}(1$ $\mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{b}(0) + \mathbf{b}(1)$, $\mathbf{x}(3) = \mathbf{A}\mathbf{x}(2) + \mathbf{b}(2) = \mathbf{A}^3\mathbf{x}(0) + \mathbf{A}^2\mathbf{b}(0) + \mathbf{A}\mathbf{b}(1) + \mathbf{b}(2)$, in general,

$$\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0) + \mathbf{A}^{t-1}\mathbf{b}(0) + \mathbf{A}^{t-2}\mathbf{b}(1) + \dots + \mathbf{b}(t-1)$$

If $\mathbf{b}(t) = \mathbf{0}$ for all t, then (with $\mathbf{A}^0 = \mathbf{I}$ as the identity matrix)

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) \iff \mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0), \qquad t = 0, 1, \dots$$

Stability of Linear Systems

The linear system (4) is said to be **globally asymptotically stable** if, no matter what the initial conditions, the general solution of the corresponding homogeneous system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$ tends to $\mathbf{0}$ as t tends to infinity. According to (6), the homogeneous system has the solution $\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0)$. Hence we see that (4) is globally asymptotically stable if and only if $\mathbf{A}^t\mathbf{x}(0)$ tends to $\mathbf{0}$ as $t \to \infty$, for each choice of initial vector $\mathbf{x}(0) = \mathbf{x}_0$. From linear algebra it is known that

$$\mathbf{A}^t\mathbf{x}_0 \underset{t\to\infty}{\longrightarrow} \mathbf{0} \quad \text{for all } \mathbf{x}_0 \text{ in } \mathbb{R}^n \iff \mathbf{A}^t \underset{t\to\infty}{\longrightarrow} \mathbf{0} \tag{7}$$

in the sense that every component of the $n \times n$ matrix $\mathbf{A}^t$ tends to 0. A necessary and sufficient condition for this is:

$$\mathbf{A}^t \underset{t\to\infty}{\longrightarrow} \mathbf{0} \iff \text{all the eigenvalues of } \mathbf{A} \text{ have moduli less than } 1 \tag{8}$$

The following result follows immediately:

THEOREM 11.6.1

A necessary and sufficient condition for system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$ to be globally asymptotically stable is that all the eigenvalues of the matrix $\mathbf{A}$ have moduli (strictly) less than 1.

Suppose in particular that the vector $\mathbf{b}(t)$ is independent of t, $\mathbf{b}(t) = \mathbf{b}$. According to (5) the solution of the system is

$$\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0) + (\mathbf{A}^{t-1} + \mathbf{A}^{t-2} + \cdots + \mathbf{A} + \mathbf{I})\mathbf{b} \tag{9}$$

Suppose that the system is globally asymptotically stable so that all the eigenvalues of $\mathbf{A}$ have moduli less than 1. Now,

$$(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{t-1})(\mathbf{I} - \mathbf{A}) = \mathbf{I} - \mathbf{A}^t \tag{10}$$

(Verify this by expanding the left-hand side.) Since $\lambda = 1$ is not an eigenvalue for $\mathbf{A}$ (it has modulus *equal to* 1), the determinant $|\mathbf{A} - \mathbf{I}|$ is not 0. But then $|\mathbf{I} - \mathbf{A}| \neq 0$, so $(\mathbf{I} - \mathbf{A})^{-1}$ exists. Multiplying (10) on the right by $(\mathbf{I} - \mathbf{A})^{-1}$ yields

$$\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{t-1} = (\mathbf{I} - \mathbf{A}^t)(\mathbf{I} - \mathbf{A})^{-1}$$

As $t \to \infty$, it follows from (8) that $\mathbf{A}^t \to 0$, and we conclude that

$$\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{t-1} \to (\mathbf{I} - \mathbf{A})^{-1} \quad \text{as } t \to \infty \tag{11}$$

We obtain therefore the following conclusion:

THEOREM 11.6.2

If all the eigenvalues of $\mathbf{A} = (a_{ij})_{n\times n}$ have moduli (strictly) less than 1, the difference equation

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}, \quad t = 0, 1, \ldots$$

is globally asymptotically stable, and every solution $\mathbf{x}(t)$ of the equation converges to the constant vector $(\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$.

The following theorem can often be used to show that a matrix has only eigenvalue[s] moduli less than 1:

THEOREM 11.6.3

Let $\mathbf{A} = (a_{ij})$ be an arbitrary $n \times n$ matrix and suppose that

$$\sum_{j=1}^{n} |a_{ij}| < 1 \quad \text{for all } i = 1, \ldots, n$$

Then all the eigenvalues of $\mathbf{A}$ have moduli less than 1.

PROBLEMS FOR SECTION 11.6

1. Find the solutions of the following systems of difference equations with the given conditions (in each case $t = 0, 1, \ldots$):

(a) $\begin{aligned} x_{t+1} &= 2y_t \\ y_{t+1} &= \tfrac{1}{2}x_t \end{aligned}$, $\quad x_0 = y_0 = 1$ $\qquad$ (b) $\begin{aligned} x_{t+1} &= -y_t - z_t + 1 \\ y_{t+1} &= -x_t - z_t + t \\ z_{t+1} &= -x_t - y_t + 2t \end{aligned}$, $\quad x_0 = y_0$ $\quad z_0 = 1$

2. Find the general solutions of the systems.

(a) $\begin{aligned} x_{t+1} &= ay_t \\ y_{t+1} &= bx_t \end{aligned}$ $\qquad$ (b) $\begin{aligned} x_{t+1} &= ay_t + ck^t \\ y_{t+1} &= bx_t + dk^t \end{aligned}$ $\quad (k^2 \neq ab)$

3. A study of the US economy by R.J. Ball and E. Smolensky uses the system

$$y_t = 0.49y_{t-1} + 0.68i_{t-1}, \quad i_t = 0.032y_{t-1} + 0.43i_{t-1}$$

where y_t denotes production and i_t denotes investment at time t.

(a) Derive a difference equation of order 2 for y_t, and find its characteristic equ[ation]

(b) Find approximate solutions of the characteristic equation, and indicate the g[eneral] solution of the system.

11.7 Stability of Nonlinear Difference Equations

Stability of an equilibrium state for a first-order linear difference equation with constant coefficients was considered in Section 11.1. In the present section we take a brief look at the nonlinear case, and also the possibility of cycles of order 2.

Consider an autonomous first-order difference equation

$$x_{t+1} = f(x_t) \tag{1}$$

where $f : I \to I$ is defined on an interval I in $\mathbb{R}$. An **equilibrium** or **stationary state** for (1) is a number x^* such that $x^* = f(x^*)$, i.e. the constant function $x_t = x^*$ is a solution of (1). In the language of Chapter 14, x^* is a fixed point of f. As in the case of differential equations, equilibrium states for (1) may be stable or unstable.

An equilibrium state x^* for (1) is called **locally asymptotically stable** if every solution that starts close enough to x^* converges to x^*—i.e. there exists an $\varepsilon > 0$ such that if $|x_0 - x^*| < \varepsilon$, then $\lim_{t\to\infty} x_t = x^*$. The equilibrium state x^* is **locally unstable** if a solution that starts close to x^* tends to move away from x^*, at least to begin with. More precisely, x^* is locally unstable if there exists an $\varepsilon > 0$ such that for every x with $0 < |x - x^*| < \varepsilon$ one has $|f(x) - x^*| > |x - x^*|$.

The following result, analogous to (5.7.2), is an easy consequence of the mean-value theorem.

THEOREM 11.7.1

Let x^* be an equilibrium state for the difference equation (1), and suppose that f is C^1 in an open interval around x^*.

(a) If $|f'(x^*)| < 1$, then x^* is locally asymptotically stable.

(b) If $|f'(x^*)| > 1$, then x^* is locally unstable.

Proof: (a) Since f' is continuous and $|f'(x^*)| < 1$, there exist an $\varepsilon > 0$ and a positive number $k < 1$ such that $|f'(x)| \le k$ for all x in $(x^* - \varepsilon, x^* + \varepsilon)$. Then, provided that $|x_0 - x^*| < \varepsilon$, the mean-value theorem tells us that

$$|x_1 - x^*| = |f(x_0) - f(x^*)| = |f'(c)(x_0 - x^*)| \le k|x_0 - x^*|$$

for some c between x_0 and x^*. By induction on t, it follows that $|x_t - x^*| \le k^t|x_0 - x^*|$ for all $t \ge 0$, and so $x_t \to x^*$ as $t \to \infty$.

(b) Now suppose that $|f'(x^*)| > 1$. By continuity there exist an $\varepsilon > 0$ and a $K > 1$ such that $|f'(x)| > K$ for all x in $(x^* - \varepsilon, x^* + \varepsilon)$. Hence if $x_t \in (x^* - \varepsilon, x^* + \varepsilon)$, then

$$|x_{t+1} - x^*| = |f(x_t) - f(x^*)| \ge K|x_t - x^*|$$

Thus if x_t is close to but not equal to x^*, the distance between the solution x and the equilibrium x^* is magnified by a factor K or more at each step as long as x_t remains in $(x^* - \varepsilon, x^* + \varepsilon)$. ∎

NOTE 1 If $|f'(x)| < 1$ for all x in I, then x^* is actually globally asymptotically stab[le in] the obvious sense.

An equilibrium state x^* of equation (1) corresponds to a point (x^*, x^*) where the g[raph] $y = f(x)$ of f intersects the straight line $y = x$. Figures 1 and 2 show two poss[ible] configurations around a stable equilibrium. In Fig. 1, $f'(x^*)$ is positive and the sequ[ence] $x_0, x_1, \dots$ converges monotonically to x^*, whereas in Fig. 2, $f'(x^*)$ is negative and we [get] a cobweb-like behaviour with x_t alternating between values less than and greater than [the] equilibrium state $x^* = \lim_{t\to\infty} x_t$. In both cases the sequence of points $P_t = (x_t, x_{t+1}) =$ $(x_t, f(x_t))$, $t = 0, 1, 2, \dots$, on the graph of f converges towards the point (x^*, x^*).

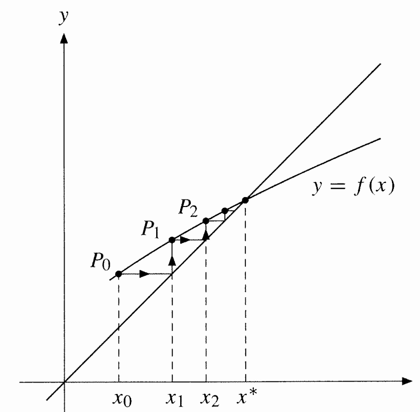

Figure 1 x^* stable, $f'(x^*) \in (0, 1)$.

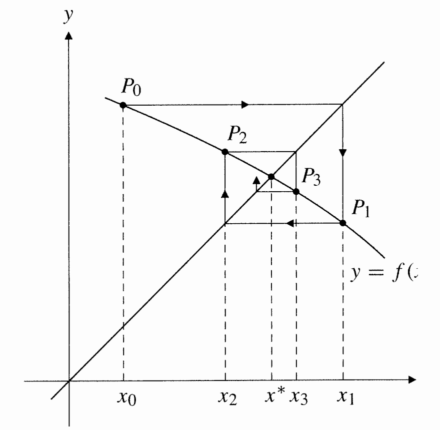

Figure 2 x^* stable, $f'(x^*) \in (-1, 0)$.

In Fig. 3, the graph of f near the equilibrium is too steep for convergence. Figure 4 sh[ows] that an equation of the form (1) may have solutions that exhibit cyclic behaviour, in [this] case a cycle of period 2. This is the topic of the next subsection.

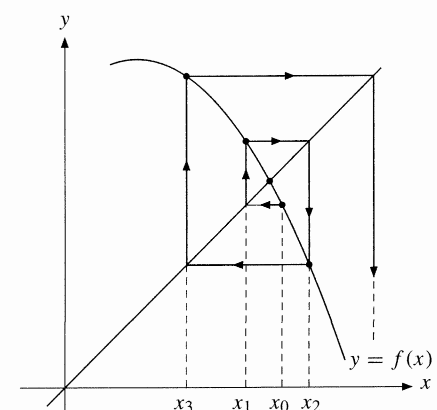

Figure 3 x^* unstable, $|f'(x^*)| > 1$.

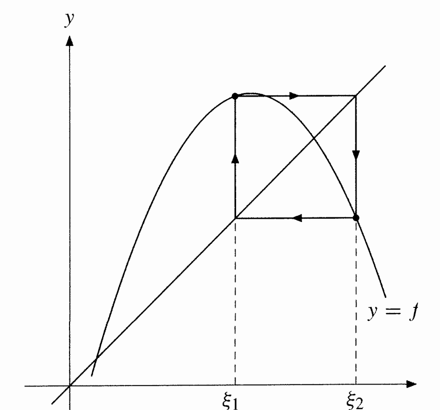

Figure 4 A cycle of period 2.

Cycles of Period 2

A **cycle** or **periodic solution** of (1) with period 2 is a solution x_t for which $x_{t+2} = x_t$ for all t, but $x_{t+1} \neq x_t$. In other words, $x_1 \neq x_0$, but $x_0 = x_2 = x_4 = \cdots$ and $x_1 = x_3 = x_5 = \cdots$.

Thus equation (1) admits a cycle of period 2 if and only if there exist distinct numbers ξ_1 and ξ_2 such that $f(\xi_1) = \xi_2$ and $f(\xi_2) = \xi_1$. If we let $F = f \circ f$, it is clear that ξ_1 and ξ_2 must be fixed points of F, i.e. they are equilibria of the difference equation

$$y_{t+1} = F(y_t) \equiv f(f(y_t)) \tag{2}$$

Such a cycle is said to be **locally asymptotically stable** if every solution of (1) that comes close to ξ_1 or ξ_2 converges to the cycle. Thus the cycle is locally asymptotically stable if and only if ξ_1 is a locally asymptotically stable equilibrium of equation (2), or equivalently, if and only if ξ_2 is such an equilibrium. The cycle is **locally unstable** if ξ_1 and ξ_2 are locally unstable equilibria of $f \circ f$. By the chain rule, $F'(x) = f'(f(x))f'(x)$, and so

$$F'(\xi_1) = f'(\xi_2)f'(\xi_1) = F'(\xi_2)$$

Theorem 11.7.2 implies the following.

If equation (1) admits a cycle of period 2, alternating between the values ξ_1 and ξ_2, then:

(a) If $|f'(\xi_1)f'(\xi_2)| < 1$, the cycle is locally asymptotically stable.

(b) If $|f'(\xi_1)f'(\xi_2)| > 1$, the cycle is locally unstable.

(3)

The Quadratic Case

A linear difference equation $x_{t+1} = ax_t + b$ with constant coefficients has no interesting cycles. The simplest nonlinear case is the case of a quadratic polynomial. So let $f(x) = ax^2 + bx + c$ (with $a \neq 0$) and consider the difference equation

$$x_{t+1} = f(x_t) = ax_t^2 + bx_t + c \tag{4}$$

The equilibrium states of (4), if any, are the solutions

$$x_1 = \frac{1 - b + \sqrt{(b-1)^2 - 4ac}}{2a}, \qquad x_2 = \frac{1 - b - \sqrt{(b-1)^2 - 4ac}}{2a}$$

of the quadratic equation $x = f(x)$, i.e. $ax^2 + (b-1)x + c = 0$. These solutions exist if and only if $(b-1)^2 \geq 4ac$, and they are distinct if and only if $(b-1)^2 > 4ac$. The values of f' at these points are

$$f'(x_{1,2}) = 2ax_{1,2} + b = 1 \pm \sqrt{(b-1)^2 - 4ac}$$

It follows that if the equilibrium points exist and are distinct, then x_1 is always unstable, x_2 is locally asymptotically stable if $(b-1)^2 - 4ac < 4$, and unstable if $(b-1)^2 - 4a$
(If $(b-1)^2 - 4ac = 4$, then x_2 is "locally asymptotically stable on one side" and un on the other side.)

Equation (4) admits a cycle of period 2 if there exist distinct numbers ξ_1 and ξ_2 suc $f(\xi_1) = \xi_2$ and $f(\xi_2) = \xi_1$. These numbers must be solutions of the equation $x = f($ Since $f(f(x))$ is a polynomial of degree 4, it seems at first sight that we have to a rather difficult equation in order to find ξ_1 and ξ_2. Fortunately the equation sim because any solution of $x = f(x)$ is also a solution of $x = f(f(x))$, so $x - f$ a factor of the polynomial $x - f(f(x))$. A simple but tedious computation show $x - f(f(x)) = (x - f(x))g(x)$, where

$$g(x) = a^2x^2 + a(b+1)x + ac + b + 1$$

The cycle points are the roots of the equation $g(x) = 0$, which are

$$\xi_1 = \frac{-(b+1) + \sqrt{(b-1)^2 - 4ac - 4}}{2a}, \qquad \xi_2 = \frac{-(b+1) - \sqrt{(b-1)^2 - 4ac -}}{2a}$$

These roots exist and are distinct if and only if $(b-1)^2 > 4ac + 4$. Hence, if the cycle of period 2, the equilibrium points x_1 and x_2 also exist, and are both unstable. also Problem 1.)

Because $f'(\xi) = 2a\xi + b$, while $\xi_1 + \xi_2 = -(b+1)/a$ and $\xi_1\xi_2 = (ac + b + 1)$ simple calculation shows that $f'(\xi_1)f'(\xi_2) = 4ac - (b-1)^2 + 5$. Then

$$|f'(\xi_1)f'(\xi_2)| < 1 \iff 4 < (b-1)^2 - 4ac < 6$$

It follows that if the inequalities on the right are satisfied, then equation (4) admits a cycle of period 2. (The first inequality on the right is precisely the necessary and suff condition for a period 2 cycle to exist.)

PROBLEMS FOR SECTION 11.7

1. Show that if $f : I \to I$ is continuous and the difference equation $x_{t+1} = f(x_t)$ a a cycle ξ_1, ξ_2 of period 2, it also has at least one equilibrium solution between ξ ξ_2. (*Hint:* Consider the function $f(x) - x$ over the interval with endpoints ξ_1 an

2. A solution x^* of the equation $x = f(x)$ can be viewed as an equilibrium soluti the difference equation

$$x_{t+1} = f(x_t)$$

If this equilibrium is stable and x_0 is a sufficiently good approximation to x^*, th solution $x_0, x_1, x_2, \ldots$ of $(*)$ starting from x_0 will converge to x^*.

(a) Use this technique to determine the negative solution of $x = e^x - 3$ to at least decimal places.

(b) The equation $x = e^x - 3$ also has a positive solution, but this is an unstable equilibrium of $x_{t+1} = e^{x_t} - 3$. Explain how nevertheless we can find the positive solution by rewriting the equation and using the same technique as above.

3. The function f in Fig. 4 is given by $f(x) = -x^2 + 4x - 4/5$. Find the values of the cycle points ξ_1 and ξ_2, and use (5) to determine whether the cycle is stable. It is clear from the figure that the difference equation $x_{t+1} = f(x_t)$ has two equilibrium states. Find these equilibria, show that they are both unstable, and verify the result in Problem 1.

12

DISCRETE TIME OPTIMIZATION

In science, what is capable of proof
must not be believed without a proof.[1]
—R. Dedekind (1887)

This chapter gives a brief introduction to *discrete time dynamic optimization problems* term *dynamic* refers to the fact that the problems involve systems evolving over time. is here measured by the number of whole periods (say weeks, quarters, or years) that passed since time 0. So we speak of *discrete* time. In this case it is natural to study dyr systems whose development is governed by difference equations.

If the horizon is finite, then such dynamic problems can be solved, in principle, using cla calculus methods. There are, however, special solution techniques described in the pr chapter that take advantage of the special structure of discrete dynamic optimization prob

Most of the chapter is concerned with dynamic programming. This is a general me for solving discrete time optimization problems that was formalized by R. Bellman in the 1950s. There is also a brief introduction to discrete time control theory. The last two sec cover stochastic dynamic programming. (This is the only part of the book that relies on knowledge of probability theory, though at a basic level.)

12.1 Dynamic Programming

Consider a system that is observed at times $t = 0, 1, \ldots, T$. Suppose the **state** of the sy at time t is characterized by a real number x_t. For example, x_t might be the quantity of that is stockpiled at time t. Assume that the initial state x_0 is historically given, anc from then on the system evolves through time under the influence of a sequence of **con** u_t, which can be chosen freely from a given set U, called the **control region**. For exar u_t might be the fraction of grain removed from the stock x_t during period t. The cor

[1] There is no ideal English translation of the German original: "Was beweisbar ist, soll i Wissenschaft nicht ohne Beweis geglaubt werden."