10

CONTROL THEORY WITH MANY VARIABLES

To be sure, mathematics can be extended to any branch of knowledge, including economics, provided the concepts are so clearly defined as to permit accurate symbolic representation. That is only another way of saying that in some branches of discourse it is desirable to know what you are talking about.
—D. MacDouglas (1956)

This chapter begins by extending the optimal control theory developed in the previous chap to problems with several state and control variables. In the first section the main emphe is on appropriate generalizations of results from Chapter 9. There is not too much discussi because the essential motivation was given in Chapter 9. However, we give a proof of the Arr sufficiency theorem in the case of several state and control variables.

Section 10.2 deals with examples illustrating the theory.

Section 10.3 extends the infinite horizon theory of Section 9.11. In fact, the majority of t control models that appear in economics literature assume an infinite horizon.

Section 10.4 begins with a discussion of the existence of optimal controls. Then we prese precise sensitivity results which are seldom spelled out except in specialized literature.

Section 10.5 offers a heuristic proof of the maximum principle, which, at least in the case a free end, is close to a proper proof. In economics literature necessary conditions for optima are often obtained by using the "Lagrangian method". This consists of introducing a suita Lagrangian and equating its "derivatives" to 0. There is no justification for this method, bu might serve as a mnemonic device.

The chapter concludes with a short discussion of control problems with mixed constraints the type $\mathbf{h}(t, \mathbf{x}, \mathbf{u}) \geq \mathbf{0}$, as well as pure state constraints of the type $\mathbf{h}(t, \mathbf{x}) \geq \mathbf{0}$. Many of t control problems that economists have considered involve additional constraints of these typ

10.1 Several Control and State Variables

Chapter 9 studied control problems with only one state and one control variable. In tl section most of the results from Chapter 9 are generalized to control problems with arbitrary number of state and control variables.

The **standard end constrained problem** is to find, for fixed values of t_0 and t_1, a pair of vector functions $(\mathbf{x}(t), \mathbf{u}(t)) = (x_1(t), \dots, x_n(t), u_1(t), \dots, u_r(t))$ defined on $[t_0, t_1]$ which

$$\text{maximizes} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \tag{1}$$

subject to the dynamic constraints

$$\left.\begin{aligned} \frac{dx_1(t)}{dt} &= g_1(t, \mathbf{x}(t), \mathbf{u}(t)) \\ &\cdots\cdots\cdots\cdots \\ \frac{dx_n(t)}{dt} &= g_n(t, \mathbf{x}(t), \mathbf{u}(t)) \end{aligned}\right. \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \tag{2}$$

the initial conditions

$$x_i(t_0) = x_i^0, \quad i = 1, \dots, n \qquad (\mathbf{x}^0 = (x_1^0, \dots, x_n^0) \text{ is a given point in } \mathbb{R}^n) \tag{3}$$

the terminal conditions

$$\begin{aligned} &\text{(a)} \quad x_i(t_1) = x_i^1, && i = 1, \dots, l \\ &\text{(b)} \quad x_i(t_1) \geq x_i^1, && i = l+1, \dots, m \\ &\text{(c)} \quad x_i(t_1) \text{ free}, && i = m+1, \dots, n \end{aligned} \tag{4}$$

and the control variable restrictions

$$\mathbf{u}(t) = (u_1(t), \dots, u_r(t)) \in U \subseteq \mathbb{R}^r, \qquad (U \text{ is a given set in } \mathbb{R}^r) \tag{5}$$

In (2) the system of differential equations is also written as a vector differential equation, where $\dot{\mathbf{x}} = (dx_1/dt, dx_2/dt, \dots, dx_n/dt)$, and $\mathbf{g} = (g_1, g_2, \dots, g_n)$ is a vector function.

The pair $(\mathbf{x}(t), \mathbf{u}(t))$ is **admissible** if $u_1(t), \dots, u_r(t)$ are all piecewise continuous, $\mathbf{u}(t)$ takes values in U and $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ is the corresponding continuous and piecewise differentiable vector function that satisfies (2), (3), and (4). The functions f and $g_1, \dots, g_n$ and their partial derivatives w.r.t. the x_i's are assumed to be continuous in all the $n + r + 1$ variables.

There are n differential equations in (2) describing the rate of growth of each of the n state variables. By analogy with the single variable problem in Section 9.4, associate n adjoint functions $p_1(t), \dots, p_n(t)$ with the n differential equations. The Hamiltonian $H = H(t, \mathbf{x}, \mathbf{u}, \mathbf{p})$, with $\mathbf{p} = (p_1, \dots, p_n)$, is then defined by

$$H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) = p_0 f(t, \mathbf{x}, \mathbf{u}) + \mathbf{p} \cdot \mathbf{g}(t, \mathbf{x}, \mathbf{u}) = p_0 f(t, \mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} p_i g_i(t, \mathbf{x}, \mathbf{u}) \tag{6}$$

The Maximum Principle

The maximum principle for the problem gives *necessary* conditions for optimality, but the conditions are far from sufficient. For a proof see Fleming and Rishel (1975).

THEOREM 10.1.1 (THE MAXIMUM PRINCIPLE. STANDARD END CONSTRAINTS)

Suppose that $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is an optimal pair for the standard end constrained problem (1)–(5). Then there exist a constant p_0, with $p_0 = 0$ or $p_0 = 1$, and a continuous and piecewise differentiable function $\mathbf{p}(t) = (p_1(t), \dots, p_n(t))$ such that for all t in $[t_0, t_1]$, $(p_0, \mathbf{p}(t)) \neq (0, \mathbf{0})$, and:

(A) The control function $\mathbf{u}^*(t)$ maximizes the Hamiltonian $H(t, \mathbf{x}^*(t), \mathbf{u}, \mathbf{p}(t))$ for $\mathbf{u} \in U$, i.e.

$$H(t, \mathbf{x}^*(t), \mathbf{u}, \mathbf{p}(t)) \leq H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t)) \quad \text{for all } \mathbf{u} \text{ in } U$$

(B) Wherever $\mathbf{u}^*(t)$ is continuous, the adjoint variables satisfy

$$\dot{p}_i(t) = -\frac{\partial H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t))}{\partial x_i}, \quad i = 1, \dots, n$$

(C) Corresponding to the terminal conditions (a), (b), and (c) in (4), one has the transversality conditions:

$$\begin{aligned} &\text{(a}'\text{)} \quad p_i(t_1) \text{ no condition}, && i = 1, \dots, l \\ &\text{(b}'\text{)} \quad p_i(t_1) \geq 0 \ \ (p_i(t_1) = 0 \text{ if } x_i^*(t_1) > x_i^1), && i = l+1, \dots, m \\ &\text{(c}'\text{)} \quad p_i(t_1) = 0, && i = m+1, \dots, n \end{aligned}$$

NOTE 1 If some of the inequalities in (4) (b) are reversed, the corresponding inequali in (9)(b′) are reversed as well.

NOTE 2 One can show the following additional properties:

(a) The Hamiltonian

$$H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t)) \quad \text{is continuous for all } t \qquad ($$

(b) If the partial derivatives $\partial f/\partial t$ and $\partial g_i/\partial t$, $i = 1, \dots, n$, exist and are continuous, tl

$$\frac{d}{dt} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t)) = \frac{\partial H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t))}{\partial t} \qquad ($$

at all points of continuity of $\mathbf{u}^*(t)$. (See Problem 2.)

(c) Moreover,

$$U \text{ convex and } H \text{ strictly concave in } \mathbf{u} \Rightarrow \mathbf{u}^*(t) \text{ continuous for all } t \qquad ($$

NOTE 3 Suppose that the terminal condition is that $x_i(t_1)$ is free for $i = 1, \dots, n$. Tl (9)(c′) yields $\mathbf{p}(t_1) = \mathbf{0}$, and then $p_0 = 1$.

NOTE 4 The adjoint variables in Theorem 10.1.1 can be given price interpretations c responding to the price interpretations in Section 9.6 for the case $n = r = 1$. Indeed,

$\mathbf{x}^1 = (x_1^1, \dots, x_m^1)$ and define the value function V associated with the standard problem as

$$V(\mathbf{x}^0, \mathbf{x}^1, t_0, t_1) = \max \left\{ \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \; : \; (\mathbf{x}(t), \mathbf{u}(t)) \text{ admissible} \right\} \tag{13}$$

Then for $i = 1, 2, \dots, n$ (for precise assumptions, see Section 10.4),

$$\frac{\partial V}{\partial x_i^0} = p_i(t_0), \quad \frac{\partial V}{\partial x_i^1} = -p_i(t_1), \quad \frac{\partial V}{\partial t_0} = -H^*(t_0), \quad \frac{\partial V}{\partial t_1} = H^*(t_1) \tag{14}$$

Here H^* denotes the Hamiltonian evaluated along the optimal path.

Sufficient Conditions

The simplest general sufficiency theorem is the following:

THEOREM 10.1.2 (MANGASARIAN)

Consider the standard end constrained problem (1)–(5) with U convex, and suppose that the partial derivatives $\partial f/\partial u_j$ and $\partial g_i/\partial u_j$ all exist and are continuous. If the pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ satisfies all the conditions in Theorem 10.1.1 with $p_0 = 1$, and if

$$H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t)) \text{ is concave in } (\mathbf{x}, \mathbf{u}) \text{ for all } t \text{ in } [t_0, t_1] \tag{15}$$

then $(\mathbf{x}^*(t), \mathbf{u}^*(t)$ solves the problem.

If the function $H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$ is strictly concave in $(\mathbf{x}, \mathbf{u})$, then $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is the unique solution to the problem.

NOTE 5 Because a sum of concave functions is concave, the concavity condition (15) is satisfied if f and $p_1(t)g_1, \dots, p_n(t)g_n$ are all concave in $(\mathbf{x}, \mathbf{u})$.

At this point the reader might want to study Example 10.2.1 and then do problem 10.2.1.

The proof of Theorem 10.1.2 is very similar to the proof of Theorem 9.6.1, so we skip it. Instead, we take a closer look at Arrow's proposed generalization of the Mangasarian theorem. (See Arrow and Kurz (1970).) Define

$$\widehat{H}(t, \mathbf{x}, \mathbf{p}) = \max_{\mathbf{u} \in U} H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{16}$$

assuming that the maximum value is attained. Then the appropriate generalization of Theorem 9.7.2 is this:

THEOREM 10.1.3 (ARROW)

Suppose that $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is an admissible pair in the standard end constrained problem (1)–(5) that, together with the continuous and piecewise differentiable adjoint (vector) function $\mathbf{p}(t)$, satisfies all the conditions in Theorem 10.1.1 with $p_0 = 1$. Suppose further that

$$\widehat{H}(t, \mathbf{x}, \mathbf{p}(t)) \text{ is concave in } \mathbf{x} \text{ for all } t \text{ in } [t_0, t_1]$$

Then $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ solves the problem.

Proof: Let $(\mathbf{x}, \mathbf{u}) = (\mathbf{x}(t), \mathbf{u}(t))$ be an arbitrary admissible pair. We must show that $\int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))\, dt - \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \geq 0$. Let us simplify the notation by letti denote $f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))$, f denote $f(t, \mathbf{x}(t), \mathbf{u}(t))$, H^* denote $H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t))$, etc. the proof of Theorem 9.7.1, it is easy to see that

$$D_{\mathbf{u}} = \int_{t_0}^{t_1} (H^* - H)\, dt + \int_{t_0}^{t_1} \mathbf{p(t)} \cdot (\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}^*(t))\, dt$$

Integration by parts yields

$$\int_{t_0}^{t_1} \mathbf{p}(t) \cdot (\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}^*(t))\, dt = \Big|_{t_0}^{t_1} \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) - \int_{t_0}^{t_1} \dot{\mathbf{p}}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t))\, dt$$
$$\geq - \int_{t_0}^{t_1} \dot{\mathbf{p}}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t))\, dt$$

To explain the last inequality, note first that because $\mathbf{x}(t_0) = \mathbf{x}^*(t_0)$ we get

$$\Big|_{t_0}^{t_1} \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) = \mathbf{p}(t_1) \cdot (\mathbf{x}(t_1) - \mathbf{x}^*(t_1)) = \sum_{i=1}^{n} p_i(t_1)(x_i(t_1) - x_i^*(t_1))$$

We claim that this sum is ≥ 0, which will imply the inequality in (ii). In fact, for $i = 1, 2,$ we have $x_i(t_1) = x_i^*(t_1) = x_i^1$, so the corresponding terms are 0. Also for $i = m+1, \dots,$ corresponding terms in the sum in (iii) are 0 because by (9)(c′), $p_i(t_1) = 0$. If $i = l+1, \dots,$ $x_i^*(t_1) > x_i^1$, the corresponding terms are 0 because by (9)(b′), $p_i(t_1) = 0$. Finally, if $x_i^*(t_1)$ then $x_i(t_1) - x_i^*(t_1) \geq 0$ and by (9)(b′), $p_i(t_1) \geq 0$ so the corresponding terms are ≥ 0. All this proves that the sum in (iii) is ≥ 0.

To proceed, note that by the definition of $\widehat{H}$,

$$H^* = \widehat{H}^* \quad \text{and} \quad H \leq \widehat{H}$$

It follows from (i)–(iv) that

$$D_{\mathbf{u}} \geq \int_{t_0}^{t_1} [\widehat{H}^* - \widehat{H} - \dot{\mathbf{p}}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t))]\, dt$$

But (8) implies that $-\dot{\mathbf{p}}(t)$ is the (partial) gradient vector $\nabla_{\mathbf{x}} H^*$, which must equal $\nabla_{\mathbf{x}} \widehat{H}^*$ b envelope Theorem 3.1.6. Because $\widehat{H}$ is concave w.r.t. $\mathbf{x}$, it follows from Theorem 2.4.1 that

$$\widehat{H} - \widehat{H}^* \leq -\dot{\mathbf{p}}(t)(\mathbf{x}(t) - \mathbf{x}^*(t)), \quad \text{or} \quad \widehat{H}^* - \widehat{H} \geq \dot{\mathbf{p}}(t)(\mathbf{x}(t) - \mathbf{x}^*(t))$$

This means that the integral on the right hand of (v) is nonnegative for all t in $[t_0, t_1]$, so $D_{\mathbf{u}} \geq$ required.

NOTE 6 The result in Problem 3 shows that condition (15) implies (17). Thus Theorem 10.1.3 generalizes Theorem 10.1.2.

NOTE 7 Suppose that in the standard end constrained problem (1)–(5) one requires that $\mathbf{x}(t) \in A(t)$ for all t, where $A(t)$ for each t is a given convex set in $\mathbb{R}^n$. Suppose also that $\mathbf{x}^*(t)$ is an interior point of $A(t)$ for each t. Theorem 10.1.3 is then valid, and $\mathbf{x} \mapsto \widehat{H}(t, \mathbf{x}, \mathbf{p}(t))$ need only be concave in the set $A(t)$.

Variable Final Time

Consider problem (1)–(5) with variable final time t_1. The problem is among all control functions $\mathbf{u}(t)$ that during the time interval $[t_0, t_1]$ steer the system from $\mathbf{x}^0$ along a time path satisfying (2) to a point where the boundary conditions in (4) are satisfied, to find one which maximizes the integral in (1). The time t_1 at which the process stops is not fixed, because the different admissible control functions can be defined on different time intervals. Theorem 9.8.1 has then the following immediate generalization:

THEOREM 10.1.4 (THE MAXIMUM PRINCIPLE. VARIABLE FINAL TIME)

Suppose that $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is an admissible pair defined on $[t_0, t_1^*]$ that solves problem (1)–(5) with t_1 free, ($t_1 \in (t_0, \infty)$). Then all the conditions in the maximum principle (Theorem 10.1.1) are satisfied on $[t_0, t_1^*]$, and, in addition,

$$H(t_1^*, \mathbf{x}^*(t_1^*), \mathbf{u}^*(t_1^*), \mathbf{p}(t_1^*)) = 0 \tag{18}$$

For a proof, see Hestenes (1966). Neither the Mangasarian nor the Arrow theorems apply to variable final time problems. For sufficiency results, see Seierstad and Sydsæter (1987).

Current Value Formulations with Scrap Values

The theorems in Section 9.10 on current value formulations of optimal control problems with scrap value functions can easily be generalized to the following problem involving several state and control variables.

$$\max_{\mathbf{u} \in U \subseteq \mathbb{R}^r} \left\{ \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u}) e^{-rt}\, dt + S(\mathbf{x}(t_1)) e^{-rt_1} \right\}, \quad \dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{u}), \quad \mathbf{x}(t_0) = \mathbf{x}^0 \tag{19}$$

$$\begin{aligned} &\text{(a)} \quad x_i(t_1) = x_i^1, && i = 1, \dots, l \\ &\text{(b)} \quad x_i(t_1) \geq x_i^1, && i = l+1, \dots, m \\ &\text{(c)} \quad x_i(t_1) \text{ free}, && i = m+1, \dots, n \end{aligned} \tag{20}$$

Here r denotes a discount factor (or an interest rate). The current value Hamiltonian is by definition

$$H^c(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = \lambda_0 f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda} \cdot \mathbf{g}(t, \mathbf{x}, \mathbf{u}) \tag{21}$$

and the maximum principle is as follows:

THEOREM 10.1.5 (THE MAXIMUM PRINCIPLE)

Suppose that $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is an optimal pair for the problem (19)–(20). Then there exist a continuous and piecewise continuously differentiable vector function $\boldsymbol{\lambda}(t) = (\lambda_1(t), \dots, \lambda_n(t))$ and a constant λ_0, with $\lambda_0 = 0$ or $\lambda_0 = 1$, such that $(\lambda_0, \boldsymbol{\lambda}(t)) \neq (0, \mathbf{0})$ for all t in $[t_0, t_1]$, and such that:

(A) For all t in $[t_0, t_1]$,

$$\mathbf{u} = \mathbf{u}^*(t) \text{ maximizes } H^c(t, \mathbf{x}^*(t), \mathbf{u}, \boldsymbol{\lambda}(t)) \text{ for } \mathbf{u} \in U$$

(B) Wherever $\mathbf{u}^*(t)$ is continuous,

$$\dot{\lambda}_i(t) - r\lambda_i(t) = -\frac{\partial H^c(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t))}{\partial x_i}, \qquad i = 1, \dots, n$$

(C) Finally, corresponding to the terminal conditions (20) (a), (b), and (c), one has the transversality conditions:

(a′) $\lambda_i(t_1)$ no condition, $\qquad i = 1, \dots, l$

(b′) $\lambda_i(t_1) \geq \lambda_0 \dfrac{\partial S^*(x^*(t_1))}{\partial x_i}$ (with $=$ if $x_i^*(t_1) > x_i^1$), $\qquad i = l+1, \dots, m$

(c′) $\lambda_i(t_1) = \lambda_0 \dfrac{\partial S^*(x^*(t_1))}{\partial x_i}$, $\qquad i = m+1, \dots, n$

THEOREM 10.1.6 (SUFFICIENT CONDITIONS. ARROW)

The conditions in Theorem 10.1.5 are sufficient (with $\lambda_0 = 1$) if

$$\widehat{H}^c(t, \mathbf{x}, \boldsymbol{\lambda}(t)) = \max_{\mathbf{u} \in U} H^c(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}(t)) \text{ is concave in } \mathbf{x}$$

and

$$S(\mathbf{x}) \text{ is concave in } \mathbf{x}.$$

The problems for this section are of a theoretical nature. Non-theoretical exercises are fo[und] at the end of the next section.

PROBLEMS FOR SECTION 10.1

1. Consider the variational problem with an integral constraint

$$\max \int_{t_0}^{t_1} F(t, x, \dot{x})\, dt, \quad x(t_0) = x^0, \; x(t_1) = x^1, \; \int_{t_0}^{t_1} G(t, x, \dot{x})\, dt = K$$

Transform the problem to a control problem with one control variable ($u = \dot{x}$) and t[wo] state variables $x = x(t)$ and $y(t) = \int_{t_0}^{t} G(\tau, x(\tau), \dot{x}(\tau))\, d\tau$.

2. Prove (11) assuming that $\mathbf{u}^*(t)$ is differentiable and $\mathbf{u}^*(t)$ belongs to the interior of U. (*Hint:* Differentiate $H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t))$ totally w.r.t. t.)

3. Let S and U be convex sets in $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively, and let $F(\mathbf{x}, \mathbf{u})$ be a real-valued concave function of $(\mathbf{x}, \mathbf{u})$, $\mathbf{x} \in S$, $\mathbf{u} \in U$. Define

$$f(\mathbf{x}) = \max_{\mathbf{u} \in U} F(\mathbf{x}, \mathbf{u}) \qquad (*)$$

where we assume that the maximum value exists for each $\mathbf{x} \in S$. Prove that f is concave in S. (*Hint:* Let $\mathbf{x}_1, \mathbf{x}_2 \in S$, $\lambda \in (0, 1)$ and choose $\mathbf{u}_1, \mathbf{u}_2 \in U$ such that $f(\mathbf{x}_1) = F(\mathbf{x}_1, \mathbf{u}_1)$, $f(\mathbf{x}_2) = F(\mathbf{x}_2, \mathbf{u}_2)$.) Let B be a convex set in $\mathbb{R}^n \times \mathbb{R}^r$ and define the set $U_\mathbf{x} = \{\mathbf{u} : (\mathbf{x}, \mathbf{u}) \in B\}$. Prove that $g(\mathbf{x}) = \max_{\mathbf{u} \in U_\mathbf{x}} F(\mathbf{x}, \mathbf{u})$ is concave.

4. Rewrite the following problem as one of the type (1)–(5),

$$\max \int_{t_0}^{t_1} f(t, x, u)\, dt,\ \dot{x} = g(t, x, u),\ x(t_0) = x^0, u \in U,\ \int_{t_0}^{t_1} h(t, x, u) dt = K$$

Here t_0, t_1, x^0, and K are given numbers, f, g, and h are given functions, and U is a subset of $\mathbb{R}$.

10.2 Some Examples

In this section the theory from the previous section is used to solve some multidimensional control problems. The first is intended to be simple enough for you to be able to make a real effort to solve it before looking at the suggested solution.

EXAMPLE 1 Solve the problem

$$\max_{u(t) \in \mathbb{R}} \int_0^T \left(x(t) + y(t) - \tfrac{1}{2}(u(t))^2\right) dt, \quad \begin{cases} \dot{x}(t) = y(t), & x(0) = 0, & x(T) \text{ is free} \\ \dot{y}(t) = u(t), & y(0) = 0, & y(T) \text{ is free} \end{cases}$$

Verify that the last equality in (14) is satisfied.

Solution: Suppose that $(x^*(t), y^*(t), u^*(t))$ solves the problem. With the two adjoint variables p_1 and p_2, the Hamiltonian is $H = x + y - \frac{1}{2}u^2 + p_1 y + p_2 u$, which is clearly concave in (x, y, u). (Because $x(T)$ and $y(T)$ are free, Note 10.1.3 implies $p_0 = 1$.) We see that $H'_x = 1$, $H'_y = 1 + p_1$, and $H'_u = -u + p_2$.

The differential equations for p_1 and p_2 are $\dot{p}_1(t) = -1$ with $p_1(T) = 0$, and $\dot{p}_2(t) = -1 - p_1(t)$ with $p_2(T) = 0$. It follows that $p_1(t) = T - t$. Hence, $\dot{p}_2(t) = -1 + t - T$ and therefore $p_2(t) = -t + \frac{1}{2}t^2 - Tt + A$. The requirement $p_2(T) = 0$ implies that $A = \frac{1}{2}T^2 + T$. Thus

$$p_1(t) = T - t, \qquad p_2(t) = \tfrac{1}{2}(T - t)^2 + T - t$$

H is concave in u and $u \in \mathbb{R}$, so H has its maximum when $H'_u = 0$. This gives $u^*(t)$
$p_2(t) = \frac{1}{2}(T-t)^2+T-t$. Since $\dot{y}^*(t) = u^*(t) = \frac{1}{2}(T-t)^2+T-t$, we find by integrat
that $y^*(t) = -\frac{1}{6}(T-t)^3 + Tt - \frac{1}{2}t^2 + B$. The initial condition $y^*(0) = 0$ gives $B = \frac{1}{6}$
From $\dot{x}^*(t) = y^*(t)$ we get $x^*(t) = \frac{1}{24}(T-t)^4 + \frac{1}{2}Tt^2 - \frac{1}{6}t^3 + \frac{1}{6}T^3t + C$. The requirem
$x^*(0) = 0$ gives $C = -\frac{1}{24}T^4$. Hence the optimal choices for x^* and y^* are

$$x^*(t) = \tfrac{1}{24}(T-t)^4 + \tfrac{1}{2}Tt^2 - \tfrac{1}{6}t^3 + \tfrac{1}{6}T^3t - \tfrac{1}{24}T^4, \quad y^*(t) = -\tfrac{1}{6}(T-t)^3 + Tt - \tfrac{1}{2}t^2 + \tfrac{1}{6}$$

Mangasarian's theorem shows that we have found the optimal solution.

The value function is $V(T) = \int_0^T \left(x^*(t) + y^*(t) - \frac{1}{2}(u^*(t))^2\right) dt$, and a rather tedi
computation (using Leibniz's formula) yields that $V'(T) = \frac{1}{2}T^2 + \frac{1}{2}T^3 + \frac{1}{8}T^4$. On
other hand, $H^*(T) = x^*(T) + y^*(T) - \frac{1}{2}(u^*(T))^2 + p_1(T)y^*(T) + p_2(T)u^*(T)$ is eas
seen to equal $\frac{1}{2}T^2 + \frac{1}{2}T^3 + \frac{1}{8}T^4$, so confirming (10.1.14).

EXAMPLE 2 **(Two-sector Model)** This model is related to a model of Mahalanobis.) Consider
economy which is divided into two sectors. Sector 1 produces investment goods, wh
sector 2 produces consumption goods. Define

$x_i(t) =$ output in sector i per unit of time, $i = 1, 2$

$u(t) =$ the fraction of investment allocated to sector 1

Assume that $\dot{x}_1 = aux_1$ and $\dot{x}_2 = a(1 - u)x_1$, where a is a positive constant, so t
the increase in production per unit of time in each sector is proportional to the fractior
investment allocated to that sector. By definition, $0 \le u(t) \le 1$, and if the planning per
starts at time $t = 0$, then $x_1(0)$ and $x_2(0)$ are historically given.

We consider the problem of maximizing total consumption in a given planning per
$[0, T]$. The problem is then, with a, T, x_1^0, and x_2^0 as positive constants:

$$\max_{u(t) \in [0,1]} \int_0^T x_2(t)\, dt, \quad \begin{cases} \dot{x}_1(t) = au(t)x_1(t), & x_1(0) = x_1^0, & x_1(T) \text{ is free} \\ \dot{x}_2(t) = a(1 - u(t))x_1(t), & x_2(0) = x_2^0, & x_2(T) \text{ is free} \end{cases}$$

The Hamiltonian is $H = x_2 + p_1aux_1 + p_2a(1 - u)x_1$, where p_1 and p_2 are the adj
variables associated with the two differential equations. (Because both terminal stocks
free, Note 10.1.3 implies $p_0 = 1$.)

Suppose that $(x_1^*(t), x_2^*(t))$ and $u^*(t)$ solve the problem. According to Theorem 10.
there exists a continuous vector function $(p_1(t), p_2(t))$ such that for all t in $[0, T]$, u^*
is the value of u in $[0, 1]$ which maximizes $x_2^*(t) + p_1(t)aux_1^*(t) + p_2(t)a(1 - u)x_1^*$
Collecting the terms in H which depend on u, note that $u^*(t)$ must be chosen as that va
of u in $[0, 1]$ which maximizes $a(p_1(t) - p_2(t))x_1^*(t)u$. Now, $x_1^*(0) = x_1^0 > 0$, and beca
$\dot{x}_1^*(t) = au^*(t)x_1^*(t)$, it follows that $x_1^*(t) > 0$ for all t. The maximum condition theref
implies that $u^*(t)$ should be chosen as

$$u^*(t) = \begin{cases} 1 & \text{if } p_1(t) > p_2(t) \\ 0 & \text{if } p_1(t) < p_2(t) \end{cases}$$

The function $p_2(t)$ satisfies $\dot{p}_2(t) = -\partial H^*/\partial x_2 = -1$ with $p_2(T) = 0$. Hence

$$p_2(t) = T - t$$

The function $p_1(t)$ satisfies $\dot{p}_1(t) = -\partial H^*/\partial x_1 = -p_1(t)au^*(t) - p_2(t)a(1-u^*(t))$, with $p_1(T) = 0$. Because $p_1(T) = p_2(T) = 0$, one has $\dot{p}_1(T) = 0$. From $\dot{p}_2(t) = -1$, it follows that $p_1(t) < p_2(t)$ in an interval to the left of T. (See Fig. 1.) Let t^* be the largest value of t in $[0, T]$ for which $p_1(t) \geq p_2(t) = T - t$. (Possibly, $t^* = 0$.) Using (i) it follows that $u^*(t) = 0$ in (t^*, T). Hence $\dot{p}_1(t) = -ap_2(t) = -a(T-t)$ in (t^*, T). Integration yields $p_1(t) = -aTt + \frac{1}{2}at^2 + C_1$. But $p_1(T) = 0$, so $C_1 = \frac{1}{2}aT^2$ and hence

$$p_1(t) = -aTt + \tfrac{1}{2}at^2 + \tfrac{1}{2}aT^2 = \tfrac{1}{2}a(T-t)^2, \qquad t \in [t^*, T]$$

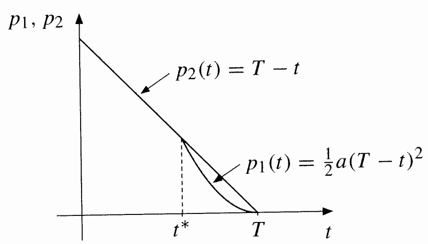

Figure 1 The behaviour of p_1 and p_2.

Unless $p_1(t) < p_2(t)$ for all t in $[0, T]$, the number t^* is determined by the requirement $p_1(t^*) = p_2(t^*)$. Using the expressions found for $p_1(t)$ and $p_2(t)$, it follows that

$$t^* = T - 2/a \text{ if } T > 2/a, \quad \text{otherwise } t^* = 0$$

Consider the case when $T > 2/a$, so $t^* > 0$. How does $p_1(t)$ behave in the interval $[0, t^*)$? Note first that

$$\dot{p}_1(t) = \begin{cases} -ap_1(t) & \text{if } p_1(t) > p_2(t) \\ -ap_2(t) & \text{if } p_1(t) \leq p_2(t) \end{cases}$$

If $p_1(t) > p_2(t)$, then $-ap_1(t) < -ap_2(t)$. Whatever is the relationship between $p_1(t)$ and $p_2(t)$, we always have

$$\dot{p}_1(t) \leq -ap_2(t) = a(t - T)$$

In particular, if $t < t^*$, then $\dot{p}_1(t) \leq a(t-T) < a(t^* - T) = -2$. Because $\dot{p}_2(t) = -1$ for all t and $p_1(t^*) = p_2(t^*)$, we conclude that $p_1(t) > p_2(t)$ for $t < t^*$. Hence, $u^*(t) = 1$ for t in $[0, t^*]$. The maximum principle therefore yields the following candidate for an optimal control, in the case when $T > 2/a$:

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, T - 2/a] \\ 0 & \text{if } t \in (T - 2/a, T] \end{cases} \qquad (T > 2/a) \qquad \text{(ii)}$$

In $[0, T - 2/a]$, $u^*(t) = 1$ and so $\dot{p}_1 = -ap_1$, i.e. $p_1(t) = Ce^{-at}$. Because $p_1(t^*) = p_2(t^*) = T - t^* = 2/a$, this yields

$$p_1(t) = (2/a)e^{-a(t-T+2/a)}, \qquad t \in [0, T - 2/a)$$

It is easy to find explicit expressions for $x_1^*(t)$ and $x_2^*(t)$. (See Problem 2.)

In the other case, when $T \leq 2/a$, one has $t^* = 0$, so the candidate for an optim control is

$$u^*(t) = 0 \quad \text{for all } t \text{ in } [0, T] \qquad (T \leq 2/a)$$

In this example the maximum principle yields only one candidate for an optimal control (i each of the cases $T > 2/a$ and $T \leq 2/a$).

The Hamiltonian is not concave in (x_1, x_2, u) (because of the product ux_1). Thus th Mangasarian theorem does not apply. The maximized Hamiltonian $\widehat{H}$ defined in (10.1.1 is for $x_1 \geq 0, x_2 \geq 0$,

$$\widehat{H}(t, x_1, x_2, p_1, p_2) = \begin{cases} x_2 + ap_1x_1 & \text{if } p_1 > p_2 \\ x_2 + ap_2x_1 & \text{if } p_1 \leq p_2 \end{cases}$$

For each t in $[0, T]$, the function $\widehat{H}$ is linear in (x_1, x_2). It is therefore concave in the s $A = \{(x_1, x_2) : x_1 \geq 0, \ x_2 \geq 0\}$. According to Theorem 10.1.3 and Note 10.1.7, th solution to the two-sector problem has been found.

PROBLEMS FOR SECTION 10.2

1. Solve the problem

$$\max_{u \in [-1,1]} \int_0^4 (10 - x_1 + u)\,dt, \quad \begin{cases} \dot{x}_1(t) = x_2(t), & x_1(0) = 2, \quad x_1(4) \text{ is free} \\ \dot{x}_2(t) = u(t), & x_2(0) = 4, \quad x_2(4) \text{ is free} \end{cases}$$

2. In Example 2, for the case when $T > 2/a$, find the functions $x_1^*(t)$ and $x_2^*(t)$ corre ponding to the control function given in (ii).

3. (a) Solve the problem

$$\max \int_0^T (\tfrac{1}{2}x_1 + \tfrac{1}{5}x_2 - u_1 - u_2)\,dt, \quad \begin{cases} \dot{x}_1(t) = u_1(t), & x_1(0) = 0, \quad x_1(T) \text{ is fre} \\ \dot{x}_2(t) = u_2(t), & x_2(0) = 0, \quad x_2(T) \text{ is fre} \end{cases}$$

with $0 \leq u_1(t) \leq 1$, $0 \leq u_2(t) \leq 1$, and with T as a fixed number greater than 5

(b) Replace the objective functional by $\int_0^T (\frac{1}{2}x_1 + \frac{1}{5}x_2 - u_1 - u_2)\,dt + 3x_1(T) + 2x_2(T$ and find the solution in this case.

4. Solve the problem

$$\max \int_0^T (x_2(t) + c(1 - u_1 - u_2))\,dt$$

$$\dot{x}_1(t) = au_1(t), \qquad x_1(0) = x_1^0, \quad x_1(T) \text{ free}$$

$$\dot{x}_2(t) = au_2(t) + bx_1(t), \qquad x_2(0) = x_2^0, \quad x_2(T) \text{ free}$$

$$0 \leq u_1, \quad 0 \leq u_2, \quad u_1 + u_2 \leq 1$$

where T, a, b, and c are positive constants and $T - c/a > T - 2/b > 0$. (Compared with Example 2, an extra flow of income amounting to one unit (say 1 billion per year) can be divided between extra capital investment in either the investment or consumption goods sectors, or consumed directly.)

5. Solve the problem

$$\max_{u \in [0, u^0]} \int_0^T (x_1 - cx_2 + u^0 - u)\, dt, \qquad \begin{cases} \dot{x}_1 = u, & x_1(0) = x_1^0, \quad x_1(t) \text{ is free} \\ \dot{x}_2 = bx_1, & x_2(0) = x_2^0, \quad x_1(t) \text{ is free} \end{cases}$$

where T, b, c, and u^0 are positive constants. (Economic interpretation: Oil is produced at the rate of u^0 per unit of time. The proceeds can be used to increase the capacity x_1 in the sector producing consumption goods. By adjusting the physical units, assume $\dot{x}_1 = u$. The production of consumption goods is proportional to x_1, and by adjusting the time unit, the constant of proportionality is chosen as 1. The production of consumption goods increases the stock of pollution, x_2, at a constant rate per unit. This subtracts cx_2 from utility per unit of time.)

6. Consider the problem:

$$\max \int_0^T U(c(t))e^{-rt}\, dt, \qquad \begin{cases} \dot{K}(t) = f(K(t), u(t)) - c(t), & K(0) = K_0, \quad K(T) = K_T \\ \dot{x}(t) = -u(t), & x(0) = x_0, \quad x(T) = 0 \end{cases}$$

where $u(t) \geq 0$, $c(t) \geq 0$. Here $K(t)$ denotes capital stock, $x(t)$ is the stock of a natural resource, $c(t)$ is consumption, and $u(t)$ is the rate of extraction. Moreover, U is a utility function and f is the production function. The constants T, K_0, K_T, and x_0 are positive. Assume that $U' > 0$, $U'' \leq 0$, $f'_K > 0$, $f'_u > 0$, and that $f(K, u)$ is concave in (K, u). This problem has two state variables (K and x) and two control variables (u and c).

(a) Write down the conditions in Theorem 10.1.1, assuming that $u(t) > 0$ and $c(t) > 0$ at the optimum.

(b) Derive from these conditions that

$$\frac{\dot{c}}{c} = \frac{r - f'_K(K, u)}{\check{\omega}}, \qquad \frac{d}{dt}(f'_u(K, u)) = f'_K(K, u) f'_u(K, u)$$

where $\check{\omega}$ is the elasticity of the marginal utility. See Section 8.4.

7. Solve the problem

$$\max_{u \in [0,1]} \int_0^2 (x - \tfrac{1}{2}u)\, dt, \qquad \begin{cases} \dot{x} = u, & x(0) = 1, \quad x(2) \text{ is free} \\ \dot{y} = u, & y(0) = 0, \quad y(2) \leq 1 \end{cases}$$

10.3 Infinite Horizon

Infinite horizon control problems were introduced in Section 9.11. This section extends analysis in several directions. Consider as a point of departure the problem

$$\max_{\mathbf{u}(t) \in U} \int_{t_0}^{\infty} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt, \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}^0, \quad \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}^1$$

where $\mathbf{x}^1$ is a fixed vector in $\mathbb{R}^n$. Suppose the integral converges whenever $(\mathbf{x}(t), \mathbf{u}($ satisfies the differential equation and $\mathbf{x}(t)$ tends to the limit $\mathbf{x}^1$ as t tends to ∞. For t problem the maximum principle holds. If we replace the condition $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^1$ w $\lim_{t\to\infty} \mathbf{x}(t) \geq \mathbf{x}^1$ or $\lim_{t\to\infty} \mathbf{x}(t)$ free, then the maximum principle again holds, except the transversality conditions.

When the integral in (1) does not converge for all admissible pairs, what is a reasona optimality criterion? Suppose $(\mathbf{x}(t), \mathbf{u}(t))$ is an arbitrary admissible pair, and $(\mathbf{x}^*(t), \mathbf{u}^*($ is a pair we wish to test for optimality. Define

$$D_{\mathbf{u}}(t) = \int_{t_0}^{t} f(\tau, \mathbf{x}^*(\tau), \mathbf{u}^*(\tau))\, d\tau - \int_{t_0}^{t} f_0(\tau, \mathbf{x}(\tau), \mathbf{u}(\tau))\, d\tau$$

There are several optimality criteria in economics literature which differ in how $D_{\mathbf{u}}$ behaves for large values of t. The simplest of these criteria is:

OVERTAKING OPTIMAL

The pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is **OT optimal** if for each admissible pair $(\mathbf{x}(t), \mathbf{u}(t))$ there exists a number $T_{\mathbf{u}}$ such that $D_{\mathbf{u}}(t) \geq 0$ for all $t \geq T_{\mathbf{u}}$.

More important than overtaking optimality is the next criterion:

CATCHING-UP OPTIMAL

The pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is **CU optimal** if for each admissible pair $(\mathbf{x}(t), \mathbf{u}(t))$ and every $\varepsilon > 0$ there exists a number $T_{\mathbf{u},\varepsilon}$ such that $D_{\mathbf{u}}(t) \geq -\varepsilon$ whenever $t \geq T_{\mathbf{u},\varepsilon}$.

NOTE 1 In general, let $f(t)$ be a function defined for all $t \geq t_0$. Define the functi $F(t) = \inf\{ f(\tau) : \tau \geq t \}$. Then $F(t)$ is an increasing function of t, and we define

$$\underline{\lim}_{t\to\infty} f(t) = \lim_{t\to\infty} F(t) = \lim_{t\to\infty} \left(\inf\{ f(\tau) : \tau \geq t \} \right)$$

Here we allow $\lim_{t\to\infty} F(t) = \infty$. The following characterization is useful and qu straightforward to prove.

$$\underline{\lim}_{t\to\infty} f(t) \geq a \iff \begin{cases} \text{For each } \varepsilon > 0 \text{ there exists a } t' \\ \text{such that } f(t) \geq a - \varepsilon \text{ for all } t \geq t' \end{cases}$$

With this definition the requirement in (4) can be formulated as:

$$(\mathbf{x}^*(t), \mathbf{u}^*(t)) \text{ is CU optimal} \iff \underline{\lim}_{t\to\infty} D_{\mathbf{u}}(t) \geq 0 \text{ for all admissible pairs } (\mathbf{x}(t), \mathbf{u}(t))$$

We turn next to the behaviour of $\mathbf{x}(t)$ as t approaches infinity. The requirement that $\mathbf{x}(t)$ tends to a limit as t approaches infinity is often too restrictive. So is the alternative requirement that $\lim_{t\to\infty} \mathbf{x}(t) \geqq \mathbf{x}^1$ because it excludes paths where $\mathbf{x}(t)$ oscillates indefinitely. Among many possible terminal conditions consider the following:

$$\lim_{t\to\infty} x_i(t) \text{ exists and is equal to } x_i^1, \qquad i = 1, \dots, l \tag{7a}$$

$$\underline{\lim}_{t\to\infty} x_i(t) \geq x_i^1, \qquad i = l+1, \dots, m \tag{7b}$$

$$\text{no conditions imposed on } x_i(t) \text{ as } t \to \infty, \qquad i = m+1, \dots, n \tag{7c}$$

One can show the following theorem (Halkin (1974)):

THEOREM 10.3.1 (THE MAXIMUM PRINCIPLE. INFINITE HORIZON)

Suppose the pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ satisfies the differential equation in (1), the initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$, and the terminal conditions (7). If this pair is OT or CU optimal, then it must satisfy all the conditions in Theorem 10.1.1 except the transversality condition.

The problem with this theorem is that when $l < n$ it gives too many solution candidates, because it includes no transversality condition.

Here is a result that gives sufficient conditions for CU optimality.

THEOREM 10.3.2 (SUFFICIENT CONDITIONS FOR AN INFINITE HORIZON)

Consider problem (1) and (7) and suppose that U is convex. If $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ for $p_0 = 1$ and for all $t \geq t_0$ satisfies the conditions in Theorem 10.1.1, except the transversality conditions, and if moreover

$$H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t)) \text{ is concave in } (\mathbf{x}, \mathbf{u}) \tag{8}$$

and

$$\underline{\lim}_{t\to\infty} [\mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t))] \geq 0 \text{ for all admissible } \mathbf{x}(t) \tag{9}$$

then the pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is CU optimal.

Proof: Applying the arguments in the proof of Theorem 9.7.1 and putting $t_1 = t$, we obtain $D_{\mathbf{u}}(t) \geq \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t))$. Taking $\underline{\lim}$ on both sides, it follows that $\underline{\lim}_{t\to\infty} D_{\mathbf{u}}(t) \geq 0$. ■

The following conditions are *sufficient* for (9) to hold (see Seierstad and Sydsæter (1987 Section 3.7, Note 16). For all admissible $\mathbf{x}(t)$:

$\underline{\lim}_{t\to\infty} [p_i(t)(x_i^1 - x_i^*(t))] \geq 0$ $\qquad i = 1, \dots, m$ (10

The exists a constant M such that $|p_i(t)| \leq M$ for all $t \geq t_0$ $\qquad i = 1, \dots, m$ (10

Either there exists a number $t' \geq t_0$ such that $p_i(t) \geq 0$ for all $t \geq t'$, or there exists a number P such that $|x_i(t)| \leq P$ for all $t \geq t_0$ and $\underline{\lim}_{t\to\infty} p_i(t) \geq 0$ $\qquad i = l+1, \dots, m$ (10

There exists a number Q such that $|x_i(t)| < Q$ for all $t \geq t_0$, and $\lim_{t\to\infty} p_i(t) = 0$ $\qquad i = m+1, \dots, n$ (10

NOTE 2 (**Malinvaud's transversality conditions**) If the terminal conditions 7(a)–(c) a replaced by the conditions $x_i(t) \geq x_i^1$ for all t and all $i = 1, \dots, n$, then the inequaliti $\mathbf{p}(t) \geqq \mathbf{0}$ for all $t \geq t_0$ and 10(a) are sufficient for (9) to hold.

PROBLEMS FOR SECTION 10.3

1. Given $r \in (0, 1)$, solve the problem

$$\max \int_0^\infty (x - u)e^{-rt}\, dt, \quad \dot{x} = ue^{-t}, \quad x(0) = x_0 \geq 0, \quad u \in [0, 1]$$

2. (a) Solve the following problem when $r > a > 0$:

$$\max_{u\in[0,1]} \int_0^\infty x_2 e^{-rt}\, dt, \quad \begin{cases} \dot{x}_1 = aux_1, & x_1(0) = x_1^0 \geq 0 \\ \dot{x}_2 = a(1-u)x_1, & x_2(0) = x_2^0 = 0 \end{cases}$$

(b) Show that the problem has no solution when $r < a$.

10.4 Existence Theorems and Sensitivity

We mentioned at the end of Section 9.3 the role played by existence theorems in optimal control theory. Not every control problem has an optimal solution. For example, in most control problems in economics it is easy to impose requirements on the final state that are entirely unattainable. These are trivial examples of problems without optimal solutions. Moreover, when the control region U is open or unbounded, it is frequently the case that no optimal solution exists. Even if U is compact and there exist admissible pairs, there is no guarantee that an optimal pair exists.

As a practical control problem without an optimal solution, think of trying to keep a pan of boiling water at the constant temperature of 100°C for one hour when it is being heated on an electric burner whose only control is an on/off switch. If we disregard the cost of switching, there is no limit to the number of times we should turn the burner on and off.

In applications one often sees the argument that practical physical or economic considerations strongly suggest the existence of an optimum. Such considerations may be useful as heuristic arguments, but they can never replace a proper mathematical existence proof. In general, a necessary condition for a mathematical optimization problem to give a realistic representation of physical or economic reality is that the problem has a solution. If a practical problem appears to have no solution, the fault may lie with the mathematical description used to model it.

Consider the standard end constrained problem (10.1.1)–(10.1.5). For every $(t, \mathbf{x})$ in $\mathbb{R}^{n+1}$, define the set

$$N(t, \mathbf{x}) = \left\{ \left(f(t, \mathbf{x}, \mathbf{u}) + \gamma,\ g_1(t, \mathbf{x}, \mathbf{u}),\ \dots,\ g_n(t, \mathbf{x}, \mathbf{u})\right) : \gamma \leq 0,\ \mathbf{u} \in U \right\} \tag{1}$$

This is a set in $\mathbb{R}^{n+1}$ generated by letting γ take all values ≤ 0, while $\mathbf{u}$ varies in the control region U.

The next theorem requires the set $N(t, \mathbf{x})$ to be convex. This implies that if the system starts in position $\mathbf{x}$ at time t and can be driven at either of the two velocity vectors $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$, then it can also be driven at any velocity vector which is a convex combination of $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$. The "gain" obtained (measured in terms of the value of f) is no smaller than the convex combination of the gains associated with $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$. (For a proof of the theorem see Cesari (1983).)

THEOREM 10.4.1 (FILIPPOV—CESARI'S EXISTENCE THEOREM)

Consider the standard end constrained problem (10.1.1)–(10.1.5). Suppose that there exists an admissible pair, and suppose further that:

(a) $N(t, \mathbf{x})$ in (1) is convex for every $(t, \mathbf{x})$.

(b) U is compact.

(c) There exists a number $b > 0$ such that $\|(\mathbf{x}(t)\| \leq b$ for all t in $[t_0, t_1]$ and all admissible pairs $(\mathbf{x}(t), \mathbf{u}(t))$.

Then there exists an optimal pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ (where the control function $\mathbf{u}^*(t)$ is measurable).

NOTE 1 The condition (a) in Theorem 10.4.1 can be dropped if all the functions g_i ar the form $g_i(t, \mathbf{x}, \mathbf{u}) = h_i(t, \mathbf{x}) + k_i(t, \mathbf{u})$, where the h_i functions are linear in $\mathbf{x}$.

NOTE 2 Condition (c) in the theorem is implied by the following sufficient conditio

There exist continuous functions $a(t)$ and $b(t)$ such that
$\|\mathbf{g}(t, \mathbf{x}, \mathbf{u})\| \leq a(t)\|\mathbf{x}\| + b(t)$ for all $(\mathbf{x}, \mathbf{u})$, $\mathbf{u} \in U$

NOTE 3 For an existence theorem for infinite horizon problems, see Seierstad and Syds (1987), Section 3.7, Theorem 15.

NOTE 4 Consider problem (10.1.1)–(10.1.5) where t_1 is free to take values in an inte $[T_1, T_2]$ with $T_1 \geq t_0$. Then Theorem 10.4.1 is still valid if the requirements are satisfie all t in $[t_0, T_2]$.

There is a technical problem with the Filippov–Cesari existence theorem which is sugge by the word "measurable". In order to ensure the existence of an optimal control, the clas admissible control functions must be enlarged to include "measurable" functions. These be much "more discontinuous" than piecewise continuous functions. (For a brief sur see Lee and Markus (1967), p. 55–56.) In almost all control problems encountere applications one can assume that if there is a measurable control that solves the probl then there exists a piecewise continuous control that solves the problem.

EXAMPLE 1 Consider the problem $\max \int_0^1 x^2 dt$, $\dot{x} = 1 - u^2$, $x(0) = x(1) = 4$, $u \in [-1, 2] =$ The Hamiltonian $H = x^2 + p(1 - u^2)$ is not concave in (x, u) and Arrow's sufficie condition also fails. In Problem 3 you are asked to find a unique solution candidate by u the maximum principle. Use Theorem 10.4.1 to prove that this candidate is optimal.

Solution: Note first that $(x(t), u(t)) \equiv (4, 1)$ is an admissible pair. Also,

$$N(t, x) = \{(x^2 + \gamma, 1 - u^2) : \gamma \leq 0,\ u \in [-1, 2]\}$$

which does not depend on t. As u varies in $[-1, 2]$, the second coordinate takes all va between 1 and -3. For fixed x, the first coordinate takes all values less than or equal t The set $N(t, x)$ is therefore as illustrated in Fig. 1.

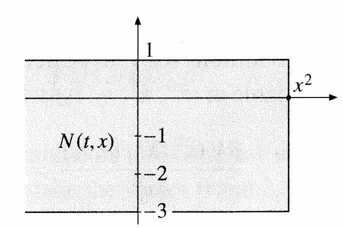

Figure 1 The set $N(t, x)$ in Example 1 is convex.

Obviously, $N(t, x)$ is convex as an "infinite rectangle", so (a) is satisfied. The set U $[-1, 2]$ is compact. Since $|\dot{x}(t)| = |1 - u^2(t)| \leq 3$ for all admissible $u(t)$, any admiss $x(t)$ satisfies $1 \leq x(t) \leq 7$ for all t in $[0, 1]$, which takes care of (c). We conclude that unique pair satisfying the conditions in the maximum principle is optimal.

EXAMPLE 2 Show the existence of an optimal control for Example 10.2.2. (*Hint:* Use Note 2.)

Solution: Clearly, $u(t) \equiv 0$ gives an admissible solution, and the set $U = [0, 1]$ is compact. The set $N = N(t, \mathbf{x})$ is here

$$N(t, x_1, x_2) = \{ (x_2 + \gamma,\ aux_1,\ a(1-u)x_1) \ :\ \gamma \le 0,\ u \in [0,1] \}$$

This is the set of points (ξ_1, ξ_2, ξ_3) in $\mathbb{R}^3$ with $\xi_1 \le x_2$ and (ξ_2, ξ_3) lying on the line segment that joins $(0, ax_1)$ to $(ax_1, 0)$ in $\mathbb{R}^2$. Hence N is convex.

The inequality in (2) is also satisfied because

$$\begin{aligned}\|\mathbf{g}(t, x_1, x_2, u)\| &= \|(aux_1, a(1-u)x_1)\| = \sqrt{(aux_1)^2 + (a(1-u)x_1)^2} \\ &= a|x_1|\sqrt{2u^2 - 2u + 1} \le a|x_1| = a\sqrt{x_1^2} \le a\sqrt{x_1^2 + x_2^2} = a\|(x_1, x_2)\|\end{aligned}$$

using the fact that $2u^2 - 2u + 1 = 2u(u-1) + 1 \le 1$ for all u in $[0, 1]$. The existence of a (measurable) optimal control follows from Theorem 10.4.1.

Precise Sensitivity Results

We want to discuss briefly precise conditions for the sensitivity results in (10.1.14) to hold. Consider the standard end constrained problem (10.1.1)–(10.1.5) and assume that admissible pairs exist. Suppose one could compute the value of the objective functional in (10.1.1) for all admissible pairs $(\mathbf{x}(t), \mathbf{u}(t))$. Let $\mathbf{x}^1 = (x_1^1, \dots, x_m^1)$ and define

$$V(\mathbf{x}^0, \mathbf{x}^1, t_0, t_1) = \sup\left\{ \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\,dt \ :\ (\mathbf{x}(t), \mathbf{u}(t)) \text{ admissible} \right\} \tag{3}$$

(If $m = 0$, the right end point is free and V will not have $\mathbf{x}^1$ as an argument.) The function V is called the (optimal) **value function** of the problem. It is defined only for those $(\mathbf{x}^0, \mathbf{x}^1, t_0, t_1)$ for which admissible pairs exist. If for a given $(\mathbf{x}^0, \mathbf{x}^1, t_0, t_1)$ an *optimal* pair exists, then V is finite and equal to the integral in (10.1.1) evaluated along the optimal pair. (This was the case studied in Section 9.6). If the set in (3) is not bounded above, then $V = \infty$.)

Suppose that $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ solves problem (10.1.1)–(10.1.5) with $\mathbf{x}^0 = \bar{\mathbf{x}}^0$, $\mathbf{x}^1 = \bar{\mathbf{x}}^1$, $t_0 = \bar{t}_0$, $t_1 = \bar{t}_1$ for $p_0 = 1$, with corresponding adjoint function $\mathbf{p}(t)$. The next theorem gives sufficient conditions for V to be defined in a neighbourhood of $(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1, \bar{t}_0, \bar{t}_1)$, and for V to be differentiable at $(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1, \bar{t}_0, \bar{t}_1)$ with the following partial derivatives:

$$\frac{\partial V(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1, \bar{t}_0, \bar{t}_1)}{\partial x_i^0} = p_i(\bar{t}_0), \qquad i = 1, \dots, n \tag{4}$$

$$\frac{\partial V(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1, \bar{t}_0, \bar{t}_1)}{\partial x_i^1} = -p_i(\bar{t}_1), \qquad i = 1, \dots, n \tag{5}$$

$$\frac{\partial V(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1, \bar{t}_0, \bar{t}_1)}{\partial t_0} = -H(\bar{t}_0, \mathbf{x}^*(\bar{t}_0), \mathbf{u}^*(\bar{t}_0), \mathbf{p}(\bar{t}_0)) \tag{6}$$

$$\frac{\partial V(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1, \bar{t}_0, \bar{t}_1)}{\partial t_1} = H(\bar{t}_1, \mathbf{x}^*(\bar{t}_1), \mathbf{u}^*(\bar{t}_1), \mathbf{p}(\bar{t}_1)) \tag{7}$$

THEOREM 10.4.2

Consider the standard end constrained problem (10.1.1)–(10.1.5) with a compact control region U. Suppose that

(a) $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is a unique optimal solution.

(b) $\mathbf{p}(t)$ is uniquely determined by the necessary conditions given $\mathbf{x}^*(t)$, $\mathbf{u}^*(t)$, and $p_0 = 1$.

(c) There exist continuous functions $a(t)$ and $b(t)$ such that

$$\|f(t, \mathbf{x}, \mathbf{u})\| \le a(t)\|\mathbf{x}\| + b(t) \quad \text{for all } (\mathbf{x}, \mathbf{u}) \text{ with } \mathbf{u} \in U$$

(d) The set $N(t, \mathbf{x})$ in (1) is convex for each $(t, \mathbf{x})$.

Then (4)–(7) are all valid.

For a proof of this theorem see Clarke (1983).

NOTE 5 Assume in this note that the uniqueness condition in (b) is replaced by the con that the function $\mathbf{x} \mapsto \widehat{H}(t, \mathbf{x}, \mathbf{p}(t))$ is concave. Then the function V is defined for t_0 $t_1 = \bar{t}_0$, and $(\mathbf{x}^0, \mathbf{x}^1)$ in a neighbourhood of $(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1)$, and the partial derivatives are by (4) and (5) at $(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}^1)$. If $l = n$ (and so the end point is fixed), or if $\mathbf{x} \mapsto \widehat{H}(t, \mathbf{x},$ is *strictly* concave, then *all* the partial derivatives (including those in (6) and (7)) exis further details see Seierstad and Sydsæter (1987).

PROBLEMS FOR SECTION 10.4

1. Show the existence of a optimal control and draw a picture of the set $N(t, x)$ fo problem

$$\max \int_0^1 x(t)\,dt, \qquad \dot{x}(t) = x(t) + u(t),\ \ x(0) = 0,\ \ x(1) \ge 1,\ \ u \in [-1, 1$$

2. Solve the problem: $\displaystyle\max_{u \in [0,1]} \int_0^1 (1-u)x^2\,dt, \quad \dot{x} = ux, \quad x(0) = x_0 > 0, \quad x(1)$

3. Find the unique solution candidate in Example 1 using the maximum principle. (Argue why $u^*(t)$ can only take the values 0 and 2, and why any admissible $x(t)$ is in [0, 1].)

10.5 A Heuristic Proof of the Maximum Principle

A full proof of the general maximum principle is quite demanding and draws on several advanced results in the theory of differential equations which are not in the toolkit of most economists. The heuristic arguments for the main results given below, although not precise, give a good indication of why the maximum principle is correct. We restrict our attention to problems with one state and one control variable.

Consider the following control problem with two alternative terminal conditions

$$\max_{u\in U} \int_{t_0}^{t_1} f(t,x,u)\,dt, \quad \dot{x} = g(t,x,u), \; x(t_0) = x_0, \quad \begin{cases} x(t_1) \text{ free} \\ x(t_1) = x_1 \end{cases} \tag{i}$$

Think of $x = x(t)$ as a firm's capital stock and $\int_{t_0}^{t_1} f(t,x,u)\,dt$ as the total profit over the planning period $[t_0, t_1]$, in line with our general economic interpretation in Section 9.6. Define the value function by

$$V(t,x) = \max_{u\in U} \left\{ \int_t^{t_1} f(s, x(s), u(s))\,ds : \dot{x}(s) = g(s, x(s), u(s)),\; x(t) = x,\; \begin{cases} x(t_1) \text{ free} \\ x(t_1) = x_1 \end{cases} \right\} \tag{ii}$$

Thus $V(t,x)$ is the maximum profit obtainable if we start at time t with the capital stock x. Suppose the problem in (ii) has a unique solution, which we denote by $\tilde{u}(s;t,x), \tilde{x}(s;t,x)$, for $t_0 \le t \le s \le t_1$. Then, by definition, $\tilde{x}(t;t,x) = x$.

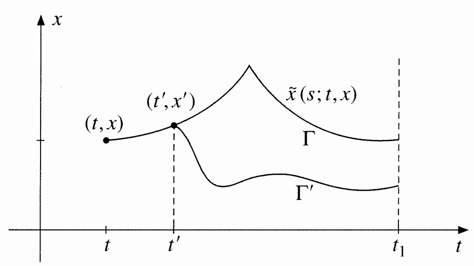

Figure 1 The case of $x(t_1)$ free.

Consider any starting point (t', x') which lies on the optimal path Γ defined by the original solution $\tilde{x}(s;t,x)$. If there were a better path Γ' starting at (t', x'), it would have been optimal for the solution starting at (t,x) to follow this improved path over the time interval $[t', t_1]$.[1] (See Fig. 1, which deals with the case when $x(t_1)$ is free.) For this reason, an optimal solution starting at (t,x) is automatically an optimal solution from (t', x') as well: *The "tail" of an optimal solution is optimal.* Using the uniqueness of $(\tilde{x}(s;t,x), \tilde{u}(s;t,x))$ for all (t,x), this implies the relations

$$\tilde{u}(s;t',x') = \tilde{u}(s;t,x), \quad \tilde{x}(s;t',x') = \tilde{x}(s;t,x)$$

whenever $t' \in [t,s]$ and $x' = \tilde{x}(t';t,x)$. Hence,

$$V(t', \tilde{x}(t';t,x)) = \int_{t'}^{t_1} f(s, \tilde{x}(s;t,x), \tilde{u}(s;t,x))\,ds$$

Differentiate this equation w.r.t. t' at $t' = t$. Because $d\tilde{x}(t';t,x)/dt' = g(t', \tilde{x}(t';t,x), \tilde{u}(t';t,x))$, we have

$$V_t'(t,x) + V_x'(t,x)g(t,x,\tilde{u}(t;t,x)) = -f(t,x,\tilde{u}(t;t,x)) \tag{iii}$$

[1] "Better path" Γ' is intuitive language. It means that there exists an admissible pair $(x(s), u(s))$ (with corresponding path Γ') that gives a higher value to the integral of f over $[t', t_1]$ when $(x(s), u(s))$ is inserted, as compared with the value resulting from $(\tilde{x}(s;t,x), \tilde{u}(s;t,x))$.

Hence, if we define

$$\bar{p}(t,x) = V_x'(t,x)$$

and introduce the Hamiltonian function $H(t,x,u,p) = f(t,x,u) + p\,g(t,x,u)$, then equatio can be written in the form

$$V_t'(t,x) + H(t,x,\tilde{u}(t;t,x), \bar{p}(t,x)) = 0$$

Starting at the point (t,x), consider an alternative control which is a constant v on an in $[t, t+\Delta t]$ and optimal thereafter. Let the corresponding state variable be $x^v(s)$ for s in $[t, t +$ Then

$$V(t,x) \ge \int_t^{t+\Delta t} f(s, x^v(s), v)\,ds + V(t+\Delta t, x^v(t+\Delta t))$$

and so

$$V(t+\Delta t, x^v(t+\Delta t)) - V(t,x) + \int_t^{t+\Delta t} f(s, x^v(s), v)\,ds \le 0$$

Dividing this inequality by Δt and letting $\Delta t \to 0$, we get $\frac{d}{dt}V(t, x^v(t)) + f(t,x,v) \le 0$. $\frac{d}{dt}V(t,x) = V_t'(t,x(t)) + V_x'(t,x)\dot{x}^v$. Since $V_x'(t,x) = \bar{p}(t,x)$ and $\dot{x}^v(t) = g(t,x,v)$, we have

$$V_t'(t,x) + \bar{p}(t,x)g(t,x,v) + f(t,x,v) \le 0$$

Thus for all v in U,

$$V_t'(t,x) + H(t,x,v,\bar{p}(t,x)) \le 0$$

Because of (iv), this implies that the optimal control $\tilde{u}(t;t,x)$ must maximize $H(t,x,u,\bar{p}($ w.r.t. $u \in U$. In addition,

$$V_t'(t,x) + \max_{u\in U} H(t,x,u,V_x'(t,x)) = 0$$

This is called the **Hamilton–Jacobi–Bellman equation**.

Next, define $x^*(t) = \tilde{x}(t;t_0,x_0)$ and $u^*(t) = \tilde{u}(t;t_0,x_0)$. These functions give the op solution to the original problem. Also, let $p(t) = \bar{p}(t, x^*(t))$. Then $\tilde{u}(t;t,x^*(t)) = u^*(t)$ therefore

$$u = u^*(t) \text{ maximizes } H(t, x^*(t), u, p(t)) \text{ w.r.t. } u \in U$$

Finally, differentiating (iv) w.r.t. x and using the envelope theorem (see Section 3.8), we get

$$V_{tx}'' + H_x' + H_p'\bar{p}_x' = 0$$

Because $p = V_x'$ and $H_p' = g$, this can be written as $\bar{p}_t' + \bar{p}_x' g = -H_x'$, where g is eval at $(t,x,\tilde{u}(t;t,x))$. If we let $x = x^*(t)$ and use $u^*(t) = \tilde{u}(t;t,x)$, then $\dot{p} = \bar{p}_t' + \bar{p}_x'\dot{x} =$ $\bar{p}_x' g(t,x,u^*(t))$, so

$$\dot{p}(t) = -H_x'(t, x^*(t), u^*(t), p(t))$$

By definition of V, if $x(t_1)$ is free, then $V(t_1, x) = 0$ for all x. Thus $\bar{p}(t_1, x) = 0$, and so we the transversality condition

$$p(t_1) = 0$$

Conditions (vii) to (ix) are the necessary conditions in the maximum principle (with t_1 fixe $x(t_1)$ free). If $x(t_1)$ is fixed, condition (ix) is not valid (and not needed).

We have shown that

$$V_{t_0}' = -H^*(t_0), \quad V_{x_0}' = p(t_0)$$

In fact, the first equality follows from (iv) and the second one from the definitions of the functi and p. These are two of the formulas in (10.1.14). Reversing time gives the other two relatio (10.1.14):

$$V_{t_1}' = H^*(t_1), \quad V_{x_1}' = -p(t_1)$$

Variable Final Time Problems

Consider problem (i) with t_1 free. Suppose $(x^*(t), u^*(t))$ is an optimal solution defined on $[t_0, t_1^*]$. Then conditions (vi)–(viii) must be valid on the interval $[t_0, t_1^*]$, because $(x^*(t), u^*(t))$ must be an optimal pair for the corresponding fixed time problem with $t_1 = t_1^*$. Moreover, at the terminal time t_1^* the value function's derivative w.r.t. t_1 must be 0. (As a function of t_1 it has a maximum at t_1^*.) Because of (xi), this means that

$$H^*(t_1^*) = H(t_1^*, x^*(t_1^*), u^*(t_1^*), p(t_1^*)) = V'_{t_1}(t_1^*, x^*(t_1^*)) = 0 \qquad \text{(xi)}$$

This equation gives an extra condition for determining t_1^*, and is precisely condition (9.8.2).

NOTE 1 In the above heuristic "proof" of the maximum principle, differentiability of the function V was assumed without proof.

10.6 Mixed Constraints

This section describes control problems where the admissible pairs $(\mathbf{x}, \mathbf{u})$ are required to satisfy additional constraints of the form $\mathbf{h}(t, \mathbf{x}, \mathbf{u}) \geqq \mathbf{0}$. Such restrictions often occur in economic models. If the control variable **u** as well as the state vector **x** appear in the function **h**, the restriction is often referred to as a *"mixed constraint"*, while restrictions of the type $\mathbf{h}(t, \mathbf{x}) \geqq \mathbf{0}$ are called *"pure state constraints"*.

Whether or not mixed constraints are present in a given control problem is partly a question of the form in which the problem is stated. Consider the following problem.

EXAMPLE 1 Consider the growth problem

$$\max_u \int_0^T U((1-u)f(K))\,dt, \quad \dot{K} = u,\ K(0) = K_0,\ K(T) = K_T,\ u \geq 0,\ f(K) - u \geq 0$$

Here there are two constraints for each t—namely, $h_1(t, K, u) = u \geq 0$ and $h_2(t, K, u) = f(K) - u \geq 0$. However, if we specify a control variable v so that $\dot{K} = vf(K)$, then the simple restriction $0 \leq v \leq 1$ replaces the mixed constraints. (If we require $f(K) - u \geq k > 0$, this trick does not work.)

We consider the **mixed constraints problem**

$$\max_{\mathbf{u}} \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\,dt, \qquad \dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}^0 \qquad (1)$$

$$\mathbf{h}(t, \mathbf{x}, \mathbf{u}) \geqq \mathbf{0} \quad \text{for all } t \qquad (2)$$

with the terminal conditions

$$\begin{aligned} &\text{(a)} \quad x_i(t_1) = x_i^1, && i = 1, \dots, l \\ &\text{(b)} \quad x_i(t_1) \geq x_i^1, && i = l+1, \dots, m \qquad (3) \\ &\text{(c)} \quad x_i(t_1) \text{ free}, && i = m+1, \dots, n \end{aligned}$$

As usual, **x** is n-dimensional and **u** is r-dimensional, while **h** is an s-dimensional ve function, so that the inequality $\mathbf{h}(t, \mathbf{x}, \mathbf{u}) \geqq \mathbf{0}$ represents the s inequalities

$$h_k(t, \mathbf{x}(t), \mathbf{u}(t)) \geq 0, \qquad k = 1, \dots, s$$

All the restrictions on $\mathbf{u}(t)$ are assumed to have been incorporated into (2). Thus, no ditional requirement of the form $\mathbf{u} \in U$ is imposed. In addition to the usual requirem on f and **g**, it is assumed that **h** is a C^1 function in $(t, \mathbf{x}, \mathbf{u})$. The pair $(\mathbf{x}(t), \mathbf{u}(t))$ is **missible** if $u_1(t), \dots, u_r(t)$ are all piecewise continuous, and $\mathbf{x}(t) = (x_1(t), \dots, x_n$ is the corresponding continuous and piecewise differentiable vector function that sati $\dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$, $\mathbf{x}(t_0) = \mathbf{x}^0$, (2), and (3). The theorem below gives sufficient condition the solution of the mixed constraints problem (1)–(3). To economists, it will come a surprise that we associate multipliers $q_1(t), \dots, q_s(t)$ with the constraints (2) and define Lagrangian function, with $\mathbf{q} = (q_1, \dots, q_s)$, as

$$\mathcal{L}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}, \mathbf{q}) = H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) + \sum_{k=1}^{s} q_k h_k(t, \mathbf{x}, \mathbf{u})$$

where the Hamiltonian is as before $H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) = f(t, \mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} p_i g_i(t, \mathbf{x}, \mathbf{u})$ ($p_0 = 1$).

In the following theorem $\mathcal{L}^*$ denotes evaluation of $\mathcal{L}$ at $(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t))$.

THEOREM 10.6.1 (SUFFICIENT CONDITIONS)

Suppose $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is an admissible pair in the mixed constraints problem (1)–(3). Suppose further that there exist functions $\mathbf{p}(t) = (p_1(t), \dots, p_n(t))$ and $\mathbf{q}(t) = (q_1(t), \dots, q_s(t))$, where $\mathbf{p}(t)$ is continuous, while $\dot{\mathbf{p}}(t)$ and $\mathbf{q}(t)$ are piecewise continuous, such that the following requirements are satisfied:

$$\frac{\partial \mathcal{L}^*}{\partial u_j} = 0, \qquad j = 1, \dots, r$$

$$q_k(t) \geq 0 \quad (q_k(t) = 0 \text{ if } h_k(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) > 0), \qquad k = 1, \dots, s$$

$$\dot{p}_i(t) = -\frac{\partial \mathcal{L}^*}{\partial x_i} \text{ at all continuity points of } \mathbf{u}^*(t), \qquad i = 1, \dots, n$$

No conditions on $p_i(t_1)$, $\quad i = 1, \dots, l$

$$p_i(t_1) \geq 0 \quad (p_i(t_1) = 0 \text{ if } x_i^*(t_1) > x_i^1), \qquad i = l+1, \dots, m$$

$$p_i(t_1) = 0, \qquad i = m+1, \dots, n$$

$H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$ is concave in $(\mathbf{x}, \mathbf{u})$

$h_k(t, \mathbf{x}, \mathbf{u})$ is quasiconcave in $(\mathbf{x}, \mathbf{u})$, $\quad k = 1, \dots, s$

Then $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ solves the problem.

A proof of this theorem is given in Seierstad and Sydsæter (1987), Section 4.3, which discusses necessary conditions, generalizations, and examples, and has further reference other literature. A simpler treatment can be found in Léonard and Long (1992), Chapter 6. Note that as in nonlinear programming a constraint qualification is often needed to be able to find a pair $(x(t), u(t))$ of the type occurring in the theorem. The constraint qualification, more or less, requires that the control u appears in each constraint.

EXAMPLE 2 Solve the mixed constraints problem

$$\max \int_0^T u\,dt, \quad \dot{x} = ax - u, \quad x(0) = x^0, \quad x(T) \text{ free}, \quad \begin{cases} h_1(t, x, u) = u - c \geq 0 \\ h_2(t, x, u) = ax - u \geq 0 \end{cases}$$

Here x is the capital stock, u is consumption, and c is a subsistence level. The constants T, a, c, and x^0 are positive, with $T > 1/a$ and $ax^0 > c$.

Solution: The Hamiltonian and the Lagrangian are

$$H = u + p(ax - u), \qquad \mathcal{L} = H + q_1(u - c) + q_2(ax - u)$$

Here H as well as h_1 and h_2 are linear and hence concave in (x, u). The following conditions from Theorem 10.6.1 are therefore sufficient for optimality:

$$\frac{\partial \mathcal{L}^*}{\partial u} = 1 - p(t) + q_1(t) - q_2(t) = 0 \qquad \text{(i)}$$

$$q_1(t) \geq 0 \qquad (q_1(t) = 0 \text{ if } u^*(t) > c) \qquad \text{(ii)}$$

$$q_2(t) \geq 0 \qquad (q_2(t) = 0 \text{ if } ax^*(t) > u^*(t)) \qquad \text{(iii)}$$

$$\dot{p}(t) = -\frac{\partial \mathcal{L}^*}{\partial x} = -ap(t) - aq_2(t), \qquad p(T) = 0 \qquad \text{(iv)}$$

$$u^*(t) \geq c, \qquad ax^*(t) - u^*(t) \geq 0 \qquad \text{(v)}$$

Because $x^*(0) = x^0 > 0$ and $\dot{x}^*(t) = ax^*(t) - u^*(t) \geq 0$ for all t, one has $x^*(t) \geq x^0$ for all t. If $u^*(t) = c$, then $ax^*(t) - u^*(t) = ax^*(t) - c \geq ax^0 - c > 0$, and then from (iii), $q_2(t) = 0$. If $u^*(t) > c$, then (ii) implies $q_1(t) = 0$. Hence, for all t in $[0, T]$, either $q_1(t)$ or $q_2(t)$ is 0.

For $t < T$ close to T, because $p(T) = 0$, it follows that $p(t) < 1$. Define t^* as the latest time t in $[0, T]$ such that $p(t) \geq 1$, with $t^* = 0$ in case $p(t) < 1$ for all t in $[0, T]$. Then for all t in $(t^*, T]$, (i) implies that $q_2(t) = 1 - p(t) + q_1(t) > q_1(t)$ so $q_2(t) > 0$ and $\dot{p}(t) = -ap(t) - aq_2(t) = -ap(t) - a(1 - p(t) + q_1(t)) = -a$, since $q_1(t) = 0$. It follows that $p(t) = a(T - t)$ and so $q_2(t) = 1 - p(t) = 1 - a(T - t)$ for all t in $(t^*, T]$. Also, from (iii), $u^*(t) = ax^*(t)$ for all t in this interval.

From Problem 5.4.9 we see that the solution to (iv) is $p(t) = \int_t^T aq_2(\tau)e^{-a(t-\tau)}\,d\tau$, which is clearly ≥ 0. Moreover, $p(t)$ is continuous, so $p(t^*) = 1$ unless $t^* = 0$, and $\dot{p}(t^*) = -a - aq_2(t^*) < 0$. It follows from (iv) that $\dot{p}(t) \leq 0$, so that $p(t) > 1$ in $[0, t^*)$ and $p(t) < 1$ in $(t^*, T]$. Because $p(t) = a(T - t)$ for all t in $(t^*, T]$, unless $t^* = 0$, one has $1 = p(t^*) = a(T - t^*)$ and so $t^* = T - 1/a$. Then, because of the hypothesis that $T > 1/a$, the case $t^* = 0$ cannot arise.

In the interval $[0, t^*)$ one has $p(t) > 1$, so (i) implies that $q_1(t) > q_2(t)$ and, either $q_1(t)$ or $q_2(t)$ is 0, in fact $q_2(t) = 0$. Then from (ii), $u^*(t) = c$ so that $ax^*(t) - c$, with $x^*(0) = x^0$. Solving this linear differential equation yields $(x^0 - c/a)e^{at} + c/a$. The differential equation for $p(t)$ is $\dot{p} = -ap$ because $q_2 = 0$ $p(t) = Ae^{-at}$ with $p(t^*) = 1$, so $p(t) = e^{-a(t-t^*)}$.

Since $x^*(t)$ is continuous also at t^*, and $\dot{x}^*(t) \equiv 0$ in $(t^*, T]$, $x^*(t)$ has the value $(x^0 - c/a)e^{at^*} + c/a$ in $(t^*, T]$.

We have found the following candidate for an optimal solution, with $t^* = T -$

	$u^*(t)$	$x^*(t)$	$p(t)$	$q_1(t)$	$q_2(t)$
$[0, t^*]$	c	$(x^0 - c/a)e^{at} + c/a$	$e^{-a(t-t^*)}$	$e^{-a(t-t^*)} - 1$	0
$(t^*, T]$	$ax^*(t)$	$(x^0 - c/a)e^{at^*} + c/a$	$a(T - t)$	0	$1 - a(T -$

Mangasarian's theorem implies that this candidate is optimal. Note that in this exan multipliers $q_1(t)$ and $q_2(t)$ are continuous.

PROBLEMS FOR SECTION 10.6

1. (a) Write down the conditions in Theorem 10.6.1 for the problem

$$\max \int_0^2 (-\tfrac{1}{2}u^2 - x)\,dt, \quad \dot{x} = -u, \quad x(0) = 1, \quad x(2) \text{ free}, \quad x \geq u$$

(b) Solve the problem. (*Hint:* Guess that $u^*(t) = x^*(t)$ on some interval $[0, t$ $u^*(t) < x^*(t)$ on $(t^*, 2]$. Then $q(t^{*+}) = 0$, and $u^*(t^{*-}) = x^*(t^*) \geq u$ We can use the following argument[2] to show that $q(t^{*-}) = 0$: From $\partial \mathcal{L}^*/$ we get $q(t) = -p(t) - u^*(t)$. In particular, $q(t^{*-}) = -p(t^*) - u^*(t$ $-p(t^*) - u^*(t^{*+}) = q(t^{*+}) = 0$.)

2. Solve the problem

$$\max \int_0^2 (x - \tfrac{1}{2}u^2)\,dt, \quad \dot{x} = u, \quad x(0) = 1, \quad x(2) \text{ free}, \quad x \geq u$$

(*Hint:* Guess that $u^*(t) = x^*(t)$ on some interval $[0, t^*]$, and $u^*(t) < x^*(t)$ on As in Problem 1, $q(t^{*-}) = 0$.)

3. Solve the following variant of Example 2.

$$\max \int_0^T u\,dt, \quad \dot{x} = ax - u, \quad x(0) = x^0 > 0, \quad x(T) \geq x_T, \quad c \leq u \leq a$$

where $a > 0$, $c > 0$, $T > 1/a$, $ax^0 > c$, and $x^0 \leq x_T < (x^0 - c/a)e^{aT} + c/a$. model can be interpreted as a simple growth model with a subsistence level c.)

[2] The same argument is useful in other problems also, for example in Problem 2.

4. Solve the problem

$$\max \int_0^1 x\,dt, \quad \dot{x} = x + u, \quad x(0) = 0, \quad x(1) \text{ free}, \quad \begin{cases} h_1(t, x, u) = 1 - u \ge 0 \\ h_2(t, x, u) = 1 + u \ge 0 \\ h_3(t, x, u) = 2 - x - u \ge 0 \end{cases}$$

(*Hint:* See the solution to Example 9.4.1. Try with $u^*(t) = 1$, $x^*(t) = e^t - 1$ in the beginning.

10.7 Pure State Constraints

This section briefly discusses a result giving sufficient conditions for a pure state constrained problem. It gives an indication of the type of results that can be proved, but we refer to literature for proofs, examples, and generalizations.

Consider the following **pure state constrained problem,**

$$\max \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\,dt, \quad \dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{u}), \quad \mathbf{x}(t_0) = \mathbf{x}^0, \quad \mathbf{u}(t) \in U \subseteq \mathbb{R}^r \tag{1}$$

$$\mathbf{h}(t, \mathbf{x}) \ge \mathbf{0} \quad \text{for all } t \tag{2}$$

with the terminal conditions

$$\begin{aligned} &\text{(a)} \quad x_i(t_1) = x_i^1, && i = 1, \dots, l \\ &\text{(b)} \quad x_i(t_1) \ge x_i^1, && i = l + 1, \dots, m \\ &\text{(c)} \quad x_i(t_1) \text{ free}, && i = m + 1, \dots, n \end{aligned} \tag{3}$$

Note that in contrast to the mixed constraints case, we now allow a restriction of the form $u \in U$. The vector function $\mathbf{h}$ is s-dimensional, and the pure state constraint (2) can be written

$$h_k(t, \mathbf{x}(t)) \ge 0, \qquad k = 1, \dots, s \tag{4}$$

The sufficient conditions given in the next theorem are somewhat more complicated than those in Theorem 10.6.1. In particular, the adjoint functions may have jumps at the terminal time.

The Lagrangian associated with this problem is

$$\mathcal{L}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}, \mathbf{q}) = H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) + \sum_{k=1}^{s} q_k h_k(t, \mathbf{x}) \tag{5}$$

with $H(t, \mathbf{x}, \mathbf{u}, \mathbf{p})$ as the usual Hamiltonian (with $p_0 = 1$).

THEOREM 10.7.1 (SUFFICIENT CONDITIONS)

Suppose $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is admissible in problem (1)–(3), and that there exist vector functions $\mathbf{p}(t)$ and $\mathbf{q}(t)$, where $\mathbf{p}(t)$ is continuous and $\dot{\mathbf{p}}(t)$ and $\mathbf{q}(t)$ are piecewise continuous in $[t_0, t_1)$, and numbers β_k, $k = 1, \dots, s$, such that the following conditions are satisfied with $p_0 = 1$:

$\mathbf{u} = \mathbf{u}^*(t)$ maximizes $H(t, \mathbf{x}^*(t), \mathbf{u}, \mathbf{p}(t))$ for $\mathbf{u}$ in U.

$q_k(t) \ge 0 \quad (q_k(t) = 0 \text{ if } h_k(t, \mathbf{x}^*(t)) > 0), \qquad k = 1, \dots, s$

$\dot{p}_i(t) = -\dfrac{\partial \mathcal{L}^*}{\partial x_i}$ at all continuity points of $\mathbf{u}^*(t)$, $\qquad i = 1, \dots, n$

At t_1, $p_i(t)$ can have a jump discontinuity, in which case

$$p_i(t_1^-) - p_i(t_1) = \sum_{k=1}^{s} \beta_k \frac{\partial h_k(t_1, \mathbf{x}^*(t_1))}{\partial x_i}, \qquad i = 1, \dots, n$$

$\beta_k \ge 0 \quad (\beta_k = 0 \text{ if } h_k(t_1, \mathbf{x}^*(t_1)) > 0), \qquad k = 1, \dots, s$

No conditions on $p_i(t_1)$, $\qquad i = 1, \dots, l$

$p_i(t_1) \ge 0 \quad (p_i(t_1) = 0 \text{ if } x_i^*(t_1) > x_i^1), \qquad i = l + 1, \dots, m$

$p_i(t_1) = 0, \qquad i = m + 1, \dots, n$

$\widehat{H}(t, \mathbf{x}, \mathbf{p}(t)) = \max_{\mathbf{u} \in U} H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$ is concave in $\mathbf{x}$.

$h_k(t, \mathbf{x})$ is quasiconcave in $\mathbf{x}$, $\qquad k = 1, \dots, s$

Then $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ solves the problem.

Here $\mathbf{p}(t) = (p_1(t), \dots, p_n(t))$ and $\mathbf{q}(t) = (q_1(t), \dots, q_s(t))$, while $\mathcal{L}^*$ denotes eval of $\mathcal{L}$ at $(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t))$.

NOTE 1 The conditions in this theorem are somewhat restrictive. In particular, some one must allow $\mathbf{p}(t)$ to have discontinuities at interior points of $[t_0, t_1]$. For details proof, see Seierstad and Sydsaeter (1987).

EXAMPLE 1 Solve the problem

$$\max \int_0^4 (x - (u - 2)^2)\,dt, \quad \dot{x} = u \in \mathbb{R}, \quad x(0) = 0, \quad x(4) \text{ free}, \quad x(t) \le 1$$

Solution: The Lagrangian is $\mathcal{L} = H + q(1 - x) = x - (u - 2)^2 + pu + q(1$ Here H is concave in (x, u) and $h(t, x) = 1 - x$ is quasiconcave, so the conditions (i below are therefore sufficient for optimality. Equation (i) results from the observatio H is concave in u and $u \in \mathbb{R}$, so condition (6) is equivalent to the condition $\partial H^*/\partial u$

$$u^*(t) = \tfrac{1}{2} p(t) + 2$$

$$q(t) \geq 0 \quad (q(t) = 0 \text{ if } x^*(t) < 1) \tag{ii}$$

$$\dot{p}(t) = -\frac{\partial \mathcal{L}^*}{\partial x} = -1 + q(t), \quad p(4) = 0 \tag{iii}$$

$$p(4^-) - p(4) = -\beta \leq 0 \quad (\beta = 0 \text{ if } x^*(4) < 1) \tag{iv}$$

We can make guesses as to the behaviour of the solution as long as we verify that all the conditions in the theorem are satisfied. We guess that $x^*(t) < 1$ in an interval $[0, t^*)$ and that $x^*(t) = 1$ in $(t^*, 4]$. Then in $(t^*, 4)$, $u^*(t) = \dot{x}^*(t) = 0$, and from (i), $p(t) = -4$. But then from (iii) and (iv), $\beta = p(4) - p(4^-) = 4$. On $[0, t^*)$, from (ii) and (iii), $\dot{p}(t) = -1$. Since $p(t)$ is continuous at t^*, $p(t^{*-}) = -4$. Hence $p(t) = -4 + (t^* - t)$, and from (i), $u^*(t) = \frac{1}{2}(t^* - t)$. Integrating $\dot{x}^*(t) = \frac{1}{2}(t^* - t)$ yields $x^*(t) = -\frac{1}{4}(t^* - t)^2 + C$ on $[0, t^*)$. Since $x^*(t^*) = 1$, we get $x^*(t^*) = C = 1$. But $x^*(0) = 0$, so $t^* = 2$. Our suggestion is therefore:

In $[0, 2]$: $u^*(t) = 1 - \frac{1}{2}t$, $x^*(t) = 1 - \frac{1}{4}(2-t)^2$, $p(t) = -t - 2$, and $q(t) = 0$.

In $(2, 4]$: $u^*(t) = 0$, $x^*(t) = 1$, $p(t) = -4$ (except that $p(4) = 0$), and $q(t) = 1$ with $\beta = 4$.

You should now verify that all the conditions (i)–(iv) are satisfied. Note that $p(t)$ has a jump at $t = 4$, from -4 to 0.

PROBLEMS FOR SECTION 10.7

1. Solve the problem

$$\min \int_0^5 (u + x)\,dt, \quad \dot{x} = u - t, \quad x(0) = 1, \quad x(5) \text{ free}, \quad x \geq 0, \quad u \geq 0$$

(*Hint:* See if it pays to keep $x(t)$ as low as possible all the time.)

2. Solve the problem

$$\max \int_0^2 (1 - x)\,dt, \quad \dot{x} = u, \quad x(0) = 1, \quad x(2) \text{ free}, \quad x \geq 0, \quad u \in [-1, 1]$$

(*Hint:* Start by reducing $x(t)$ as much as possible until $x(t) = 0$.)

3. Solve the problem

$$\max \int_0^{10} (-u^2 - x)\,dt, \quad \dot{x} = u, \quad x(0) = 1, \quad x(10) \text{ free}, \quad x \geq 0, \quad u \in \mathbb{R}$$

4. Consider the problem

$$\max \int_0^3 (4 - t)u\,dt, \quad \dot{x} = u \in [0, 2], \quad x(0) = 1, \quad x(3) = 3, \quad t + 1 -$$

(a) Solve the problem when the constraint $t + 1 - x \geq 0$ is not imposed.

(b) Solve problem (∗).

10.8 Generalizations

In Chapter 9 and the previous sections of this chapter we have discussed som optimal control theory. Many important economic problems cannot be treatec methods described in this book.

More General Terminal Conditions

In some dynamical optimization problems the standard terminal conditions are r the requirement that $\mathbf{x}(t)$ at time t_1 hits a **target** defined as a certain curve or sur

The optimal path in such a problem must end at some point $\mathbf{x}^1$ and therefore, in will solve the corresponding control problem where *all* the admissible paths The conditions in Theorem 10.1.1 must therefore still be valid, except the tra conditions, which must be adjusted. See e.g. Seierstad and Sydsæter (1984), Ch

Markov Controls

The optimal solutions we have been looking for have been functions of time, $\mathbf{x}^*(t)$. Such control functions are called "open-loop controls". Faced with the p steering an economic system optimally, such open-loop controls are often inade problem is that "disturbances" of many types will almost always occur, which wil system from the optimal path initially computed. If one still uses the "old" con one can end up with a development of the economy which is far from optimal, does not necessarily bring the economy to a desirable final state.

This problem is partly resolved if we are able to "synthesize" the optimal con sense of expressing the optimal control as a function of the present time s and t state $\mathbf{y}$. In this case, for each time s and each point $\mathbf{y}$ in the state space, we s optimal control $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_{s,\mathbf{y}}$ to use. Such controls are called **closed-loop** or **Marko** We can find such Markov controls by solving the control problem with an arbi point $(s, \mathbf{y})$, $s \in [t_0, t_1)$. The controls $\mathbf{u}^*(t)$ obtained will depend on the starting p $\mathbf{u}^*(t) = \mathbf{u}^*_{s,\mathbf{y}}(t)$. Of course, at time s, the control $\tilde{\mathbf{u}}(s, \mathbf{y}) = \mathbf{u}^*_{s,\mathbf{y}}(s)$ is used. Then the required Markov control.

But these Markov controls are only conditionally optimal. They tell us which control to use after a disturbance has occurred, but they are optimal only in the absence of further disturbances.

If we stipulate the probability of future disturbances and then want to optimize the expected value of the objective functional, this gives a stochastic control problem, in which optimal Markov controls are determined by a different set of necessary conditions.

Jumps in State Variables

So far we have assumed that the control functions are piecewise continuous, and the state variables are continuous. In certain applications (e.g. in the theory of investment), the optimum may require sudden jumps in the state variables. See e.g. Seierstad and Sydsæter (1987), Chapter 3.

DIFFERENCE EQUATIONS

He (an economist) must study the present in the light of the past for the purpose of the future.
—J. N. Keynes

Many of the quantities economists study (such as income, consumption, and savings) recorded at fixed time intervals (for example, each day, week, quarter, or year). E tions that relate such quantities at different discrete moments of time are called **differe equations**. For example, such an equation might relate the amount of national income in period to the national income in one or more previous periods. In fact difference equations be viewed as the discrete time counterparts of the differential equations in continuous time were studied in Chapters 5–7.

11.1 First-Order Difference Equations

Let $t = 0, 1, 2, \ldots$ denote different discrete time periods or moments of time. We usu call $t = 0$ the *initial period*. If $x(t)$ is a function defined for $t = 0, 1, 2, \ldots$, we often $x_0, x_1, x_2, \ldots$ to denote $x(0), x(1), x(2), \ldots$, and in general, we write x_t for $x(t)$.

Let $f(t, x)$ be a function defined for all positive integers t and all real numbers first-order difference equation in x_t can usually be written in the form

$$x_{t+1} = f(t, x_t), \qquad t = 0, 1, 2, \ldots$$

This is a first-order equation because it relates the value of a function in period $t + 1$ t value of the same function in the previous period t only.[1]

[1] It would be more appropriate to call (1) a "recurrence relation", and to reserve the term "d ence equation" for an equation of the form $\Delta x_t = \tilde{f}(t, x_t)$, where Δx_t denotes the difference x_{t+1} However, it is obvious how to transform a difference equation into an equivalent recurrence rela and *vice versa*, so we make no distinction between the two kinds of equation.