The Integrand Depends on Higher Order Derivatives

Consider a variational problem where the integrand depends on higher order derivatives of the unknown function. With appropriate requirements on F, the problem is to maximize or minimize

$$\int_{t_0}^{t_1} F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dt^n}\right) dt \tag{2}$$

where $x(t)$ and its first $n-1$ derivatives have given values at t_0 and t_1. One can show that a necessary condition for $x^* = x^*(t)$ to solve this problem is that it satisfies the following **generalized Euler equation**

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{x}}\right) - \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial F}{\partial x^{(n)}}\right) = 0 \tag{3}$$

We refer to Gelfand and Fomin (1963) and Hestenes (1966) for further details.

The Unknown Function Depends on Two Variables

Suppose the variable function has two arguments t and s. With appropriate requirements on F, the problem is to maximize or minimize

$$\iint_R F\left(t, s, x, \frac{\partial x}{\partial t}, \frac{\partial x}{\partial s}\right) dt\, ds \tag{4}$$

where R is a closed domain in the plane and $x = x(t, s)$ is the unknown function. In addition, require that $x(t, s)$ takes prescribed values on the boundary of R. One can then prove that a necessary condition for $x^* = x^*(t, s)$ to solve the problem is that it satisfies the following partial differential equation (see Gelfand and Fomin (1963)):

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x'_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial x'_s}\right) = 0 \tag{5}$$

PROBLEMS FOR SECTION 8.6

1. Consider the problem of maximizing $J(x)$, subject to the given conditions.

$$J(x) = \int_0^{\pi/2} (\ddot{x}^2 - x^2 + t^2)\, dt, \qquad \begin{matrix} x(0) = 1, & x(\pi/2) = 0, \\ \dot{x}(0) = 0, & \dot{x}(\pi/2) = -1 \end{matrix}$$

Find the associated Euler equation and its solution.

2. Consider the problem of maximizing $J(y)$ w.r.t $y(x)$, subject to the given conditions.

$$J(y) = \int_{-1}^{1} (\tfrac{1}{2}\mu y''^2 + \rho y)\, dx, \qquad y(-1) = 0,\ \ y'(-1) = 0,\ \ y(1) = 0,\ \ y'(1) = 0$$

where μ and ρ are constants. Find the associated Euler equation and its solution.

9

CONTROL THEORY: BASIC TECHNIQUES

A person who insists on understanding every tiny step before going to the next is liable to concentrate so much on looking at his feet that he fails to realize he is walking in the wrong direction.
—I. Stewart (1975)

Optimal control theory is a modern extension of the classical calculus of variations. Wher[e] the Euler equation, the main result of the latter theory, dates back to 1744, the m[ain] result in optimal control theory, called the **maximum principle**, was developed in the 195[0s] by a group of Russian mathematicians. (See Pontryagin et al. (1962).) The maximum princi[ple] gives necessary conditions for optimality in a wide range of dynamic optimization probler[ns]. It includes all the necessary conditions that emerge from the classical theory, but can also [be] applied to a significantly wider range of problems.

Since 1960, thousands of papers in economics literature have used control theory. It has be[en] applied to, for instance, economic growth, inventory control, taxation, extraction of natu[ral] resources, irrigation, and the theory of regulation under asymmetric information.

This chapter contains some important results based on reasoning that appears widely [in] economics literature. ("What every economist should know about optimal control theory.") [It] concentrates on the case where there is a single control variable and a single state variable.

9.1 The Basic Problem

Consider a system whose state at time t is characterized by a number $x(t)$, the **state variab[le]**. The process that causes $x(t)$ to change can be controlled, at least partially, by a **contr[ol] function** $u(t)$. We assume that the rate of change of $x(t)$ depends on t, $x(t)$, and $u(t)$. T[he] state at some initial point t_0 is typically known, $x(t_0) = x_0$. Hence the evolution of $x(t)$ [is] described by a controlled differential equation

$$\dot{x}(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0 \tag{1}$$

Suppose we choose some control function $u(t)$ defined for $t \geq t_0$. Inserting this functi[on] into (1) gives a first-order differential equation for $x(t)$ alone. Because the initial point [is] fixed, a unique solution of (1) is usually obtained.

By choosing different control functions $u(t)$, the system can be steered along many different paths, not all of which are equally desirable. As usual in economic analysis, assume that it is possible to measure the benefits associated with each path. More specifically, assume that the benefits can be measured by means of the integral

$$J = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \tag{2}$$

where f is a given function. Here, J is called the **objective function** or the **criterion function**. Certain restrictions are often placed on the final state $x(t_1)$. Moreover, the time t_1 at which the process stops is not necessarily fixed. The fundamental problem that we study is:

Among all pairs $(x(t), u(t))$ that obey the differential equation in (1) with $x(t_0) = x_0$ and that satisfy the constraints imposed on $x(t_1)$, find one that maximizes (2).

EXAMPLE 1 **(Economic Growth)** Consider the control problem

$$\max \int_0^T (1 - s) f(k)\, dt, \quad \dot{k} = sf(k), \quad k(0) = k_0, \quad k(T) \geq k_T, \quad 0 \leq s \leq 1$$

Here $k = k(t)$ is the real capital stock of a country and $f(k)$ is its production function. Moreover, $s = s(t)$, the control variable, is the rate of investment, and it is natural to require that $s \in [0, 1]$. The quantity $(1-s)f(k)$ is the flow of consumption per unit of time. We wish to maximize the integral of this quantity over $[0, T]$, i.e. to maximize total consumption over the period $[0, T]$. The constant k_0 is the initial capital stock, and the condition $k(T) \geq k_T$ means that we wish to leave a capital stock of at least k_T to those who live after time T. (Example 9.6.3(b) studies a special case of this model.)

EXAMPLE 2 **(Oil Extraction)** Let $x(t)$ denote the amount of oil in a reservoir at time t. Assume that at $t = 0$ the field contains K barrels of oil, so that $x(0) = K$. If $u(t)$ is the rate of extraction, then[1]

$$\dot{x}(t) = -u(t), \quad x(0) = K \tag{$*$}$$

Suppose that the market price of oil at time t is known to be $q(t)$, so that the sales revenue per unit of time at t is $q(t)u(t)$. Assume further that the cost C per unit of time depends on t, x and u, so that $C = C(t, x, u)$. The instantaneous rate of profit at time t is then

$$\pi(t, x(t), u(t)) = q(t)u(t) - C(t, x(t), u(t))$$

If the discount rate is r, the total discounted profit over the interval $[0, T]$ is

$$\int_0^T \big[q(t)u(t) - C(t, x(t), u(t))\big] e^{-rt}\, dt \tag{$**$}$$

[1] Integrating each side of $(*)$ yields $x(t) - x(0) = -\int_0^t u(\tau)\, d\tau$, or $x(t) = K - \int_0^t u(\tau)\, d\tau$. This equation just says that the amount of oil left at time t is equal to the initial amount K, minus the total amount that has been extracted during the time span $[0, t]$, namely $\int_0^t u(\tau)\, d\tau$.

It is natural to assume that $u(t) \geq 0$, and that $x(T) \geq 0$.

Problem I: Find the rate of extraction $u(t) \geq 0$ that maximizes $(**)$ subject to $(*$
$x(T) \geq 0$ over a fixed extraction period $[0, T]$.

Problem II: Find the rate of extraction $u(t) \geq 0$ and also the optimal terminal time T
maximizes $(**)$ subject to $(*)$ and $x(T) \geq 0$.

These two problems are *optimal control problems*. Problem I has a fixed terminal tim
whereas Problem II is referred to as a free terminal time problem. See Example 9.8.1.

9.2 A Simple Case

We begin by studying a control problem with no restrictions on the control variable an
restrictions on the terminal state—that is, no restrictions are imposed on the value of
at $t = t_1$. Given the fixed times t_0 and t_1, our problem is

$$\text{maximize} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt, \quad u(t) \in (-\infty, \infty)$$

subject to

$$\dot{x}(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0, \quad x_0 \text{ fixed}, \quad x(t_1) \text{ free}$$

Given any control function $u(t)$ defined on $[t_0, t_1]$, the associated solution of the differer
equation in (2) with $x(t_0) = x_0$ will usually be uniquely determined on the whole of $[t_0,$
A pair $(x(t), u(t))$ that satisfies (2) is called an **admissible pair**. Among all admissible p
we search for an **optimal pair**, i.e. a pair of functions that maximizes the integral in (1

Notice that the problem is to maximize an objective function (or integral) w.r.t. u sub
to the constraint (2). Because this constraint is a differential equation on the interval $[t_0,$
it can be regarded as an infinite number of equality constraints, one for each time t in $[t_0,$

Economists usually incorporate equality constraints in their optimization problems
forming a Lagrangian function, with a Lagrange multiplier corresponding to each constra
Here, by an analogy, the necessary conditions for the problem associate a number $p(t)$ w
the constraint (2) for each t in $[t_0, t_1]$. The resulting function $p = p(t)$ is called the **adjo**
function (or **co-state variable**) associated with the differential equation. Correspond
to the Lagrangian function in the present problem is the **Hamiltonian** H. For each tim
in $[t_0, t_1]$ and each possible triple (x, u, p), of the state, control, and adjoint variables,
Hamiltonian is defined by

$$H(t, x, u, p) = f(t, x, u) + pg(t, x, u)$$

A set of necessary conditions for optimality is given in the following theorem. (So
regularity conditions required are discussed in the next section.)

THEOREM 9.2.1 (THE MAXIMUM PRINCIPLE)

Suppose that $(x^*(t), u^*(t))$ is an optimal pair for problem (1)–(2). Then there exists a continuous function $p(t)$ such that, for all t in $[t_0, t_1]$,

$$u = u^*(t) \text{ maximizes } H(t, x^*(t), u, p(t)) \text{ for } u \in (-\infty, \infty) \tag{4}$$

$$\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)), \quad p(t_1) = 0 \tag{5}$$

NOTE 1 The requirement that $p(t_1) = 0$ in (5) is called a **transversality condition**. So condition (5) tells us that in the case where $x(t_1)$ is free, the adjoint variable vanishes at t_1.

The conditions in Theorem 9.2.1 are necessary, but not sufficient for optimality. The following theorem gives sufficient conditions.

THEOREM 9.2.2 (MANGASARIAN)

If the requirement

$$H(t, x, u, p(t)) \quad \text{is concave in } (x, u) \text{ for each } t \text{ in } [t_0, t_1] \tag{6}$$

is added to the requirements in Theorem 9.2.1, then we obtain *sufficient* conditions. Thus, if we find a triple $(x^*(t), u^*(t), p(t))$ that satisfies (2), (4), (5), and (6), then $(x^*(t), u^*(t))$ is optimal.

NOTE 2 Changing $u(t)$ on a small interval causes $f(t, x, u)$ to change immediately. Moreover, at the end of this interval $x(t)$ has changed and this change is transmitted throughout the remaining time interval. In order to steer the process optimally, the choice of $u(t)$ at each instant of time must anticipate the future changes in $x(t)$. In short, we have to plan ahead. In a certain sense, the adjoint function $p(t)$ takes care of this need for forward planning. Equation (5) implies that $p(t) = \int_t^{t_1} H'_x(s, x^*(s), u^*(s), p^*(s))\, ds$.

NOTE 3 If the problem is to minimize the objective in (1), then we can rewrite the problem as one of maximizing the negative of the original objective function. Alternatively, we could reformulate the maximum principle for the minimization problem: An optimal control will minimize the Hamiltonian, and convexity of $H(t, x, u, p(t))$ w.r.t. (x, u) is the relevant sufficient condition.

Since the control region is $(-\infty, \infty)$, a *necessary* condition for (4) is that

$$H'_u(t, x^*(t), u^*(t), p(t)) = 0 \tag{7}$$

If $H(t, x(t), u, p(t))$ is concave in u, condition (7) is also sufficient for the maximum condition (4) to hold, because we recall that an interior stationary point for a concave function is (globally) optimal.

It is helpful to see how these conditions allow some simple examples to be solved.

EXAMPLE 1 Solve the problem

$$\max \int_0^T \left[1 - tx(t) - u(t)^2\right] dt, \qquad \dot{x}(t) = u(t), \; x(0) = x_0, \; x(T) \text{ free}, \; u \in \mathbb{R}$$

where x_0 and T are positive constants.

Solution: The Hamiltonian is $H(t, x, u, p) = 1 - tx - u^2 + pu$, and the cor
$u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$ w.r.t. u only if it satisfies $H'_u = -2u + p(t) =$
Thus $u^*(t) = \frac{1}{2}p(t)$. Because $H'_x = -t$, the conditions in (5) reduce to $\dot{p}(t) = t$
$p(T) = 0$. Integrating gives $p(t) = \frac{1}{2}t^2 + C$ with $\frac{1}{2}T^2 + C = 0$, so

$$p(t) = -\tfrac{1}{2}(T^2 - t^2) \quad \text{and then} \quad u^*(t) = -\tfrac{1}{4}(T^2 - t^2)$$

Because $\dot{x}^*(t) = u^*(t) = -\frac{1}{4}(T^2 - t^2)$, integrating $\dot{x}^*(t) = u^*(t)$ and imposing $x^*(0) =$
gives

$$x^*(t) = x_0 - \tfrac{1}{4}T^2 t + \tfrac{1}{12}t^3$$

Thus, there is only one pair $(x^*(t), u^*(t))$ that, together with $p(t)$, satisfies both necess
conditions (4) and (5). We have therefore found the only possible pair which could sc
the problem. Because $H(t, x, u, p) = 1 - tx - u^2 + pu$ is concave in (x, u) (it is a sun
concave functions), $(x^*(t), u^*(t))$ is indeed optimal.

EXAMPLE 2 **(A Macroeconomic Control Problem)** Consider once again the macroecono
model of Example 8.2.2. If we drop the terminal constraint at the end of the plann
period, we face the following control problem

$$\min_{u(t)} \int_0^T [x(t)^2 + cu(t)^2]\, dt, \quad \dot{x}(t) = u(t), \; x(0) = x_0, \; x(T) \text{ free}$$

where $u(t) \in \mathbb{R}$ and $c > 0$. Use the maximum principle to solve the problem.

Solution: We maximize $-\int_0^T [x(t)^2 + cu(t)^2]\, dt$. The Hamiltonian is

$$H(t, x, u, p) = -x^2 - cu^2 + pu$$

So $H'_x = -2x$ and $H'_u = -2cu + p$. A necessary condition for $u^*(t)$ to maximize
Hamiltonian is that $H'_u = 0$ at $u = u^*(t)$, or that $-2cu^*(t) + p(t) = 0$. Therefore $u^*(t)$
$p(t)/2c$. The differential equation for $p(t)$ is

$$\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 2x^*(t) \quad ($$

From $\dot{x}^*(t) = u^*(t)$ and $u^*(t) = p(t)/2c$, we have

$$\dot{x}^*(t) = p(t)/2c \quad (*$$

The two first-order differential equations (∗) and (∗∗) can be used to determine the functions p and x^*. Differentiate (∗) w.r.t. t and then use (∗∗) to obtain $\ddot{p}(t) = 2\dot{x}^*(t) = p(t)/c$, whose general solution is

$$p(t) = Ae^{rt} + Be^{-rt}, \quad \text{where } r = 1/\sqrt{c}$$

Imposing the boundary conditions $p(T) = 0$ and $\dot{p}(0) = 2x^*(0) = 2x_0$ implies that $Ae^{rT} + Be^{-rT} = 0$ and $r(A - B) = 2x_0$. These two equations determine A and B, which must be $A = 2x_0 e^{-rT}/[r(e^{rT} + e^{-rT})]$ and $B = -2x_0 e^{rT}/[r(e^{rT} + e^{-rT})]$. Therefore

$$p(t) = \frac{2x_0}{r\left(e^{rT} + e^{-rT}\right)}\left[e^{-r(T-t)} - e^{r(T-t)}\right] \quad \text{and} \quad x^*(t) = \tfrac{1}{2}\dot{p}(t) = x_0 \frac{e^{r(T-t)} + e^{-r(T-t)}}{e^{rT} + e^{-rT}}$$

The Hamiltonian $H = -x^2 - cu^2 + pu$ is concave in (x, u), which confirms that this is the solution to the problem. (The same result was obtained in Example 8.5.2.)

PROBLEMS FOR SECTION 9.2

Solve the control problems 1–5:

1. $$\max_{u(t)\in(-\infty,\infty)} \int_0^2 [e^t x(t) - u(t)^2]\,dt, \quad \dot{x}(t) = -u(t), \quad x(0) = 0, \quad x(2) \text{ free}$$

2. $$\max_{u(t)\in(-\infty,\infty)} \int_0^1 [1 - u(t)^2]\,dt, \quad \dot{x}(t) = x(t) + u(t), \quad x(0) = 1, \quad x(1) \text{ free}$$

3. $$\min_{u(t)\in(-\infty,\infty)} \int_0^1 [x(t) + u(t)^2]\,dt, \quad \dot{x}(t) = -u(t), \quad x(0) = 0, \quad x(1) \text{ free}$$

4. $$\max_{u\in(-\infty,\infty)} \int_0^{10} \left[1 - 4x(t) - 2u(t)^2\right]dt, \quad \dot{x}(t) = u(t), \quad x(0) = 0, \quad x(10) \text{ free}$$

5. $$\max_{u(t)\in(-\infty,\infty)} \int_0^T (x - u^2)\,dt, \quad \dot{x} = x + u, \quad x(0) = 0, \quad x(T) \text{ free}$$

6. (a) Write down conditions (7) and (5) for the problem

$$\max_{I\in(-\infty,\infty)} \int_0^T [qf(K) - c(I)]\,dt, \quad \dot{K} = I - \delta K, \quad K(0) = K_0, \quad K(T) \text{ free}$$

($K = K(t)$ denotes the capital stock of a firm, $f(K)$ is the production function, q is the price per unit of output, $I = I(t)$ is investment, $c(I)$ is the cost of investment, δ is the rate of depreciation of capital, K_0 is the initial capital stock, and T is the planning horizon.)

(b) Let $f(K) = K - 0.03K^2$, $q = 1$, $c(I) = I^2$, $\delta = 0.1$, $K_0 = 10$, and $T = 10$. Derive a second-order differential equation for K, and explain how to find the solution.

9.3 Regularity Conditions

In most applications of control theory to economics, the control functions are explicitly or implicitly restricted in various ways. For instance, in the oil extraction problem of Section 9.1, $u(t) \geq 0$ was a natural restriction, because it means that you cannot pump oil back into the reservoir.

In general, assume that $u(t)$ takes values in a fixed subset U of the reals, called the **control region**. In the oil extraction problem, then, $U = [0, \infty)$, and $u(t)$ can take the value 0. Actually, an important aspect of control theory is that the control region can be closed, so that $u(t)$ can take values at the boundary of U. (In the classical calculus of variation, by contrast, one usually considered open control regions, although developments in the theory around 1930–1940 paved the way for the modern theory.)

What regularity conditions is it natural to impose on the control function $u(t)$? Among the many papers in economics literature that use control theory, the majority assume implicitly or explicitly that the control functions are continuous. Consequently, many of our examples and problems will deal with continuous controls. Yet in some applications, continuity is too restrictive. For example, the control variable $u(t)$ could be the fraction of investment in one plant, with the remaining fraction $1 - u(t)$ allocated to a second plant. Then it is natural to allow control functions that suddenly switch all the investment from one plant to the other. Because they alternate between extremes, such functions are often called **bang-bang** controls. A simple example of such a control is

$$u(t) = \begin{cases} 1 & \text{for } t \text{ in } [t_0, t'] \\ 0 & \text{for } t \text{ in } (t', t_1] \end{cases}$$

which involves a single shift at time t'. In this case $u(t)$ is *piecewise continuous*, with a jump discontinuity at $t = t'$.

By definition, a function has a **finite jump** at a point of discontinuity if it has (finite) one-sided limits at the point. A function is **piecewise continuous** if it has at most a finite number of discontinuities on each finite interval, with finite jumps at each point of discontinuity. (The value of a control $u(t)$ at a point of discontinuity will not be of any importance, but let us agree to choose the value of $u(t)$ at a point of discontinuity t' as the left-hand limit of $u(t)$ at t'. Then $u(t)$ will be **left-continuous** as illustrated in Fig. 1.) Moreover, if the control problem concerns the time interval $[t_0, t_1]$, we shall assume that $u(t)$ is continuous at both end points of this interval.

What is meant by a "solution" of $\dot{x} = g(t, x, u)$ when $u = u(t)$ has discontinuities? A **solution** is a **continuous** function $x(t)$ that has a derivative that satisfies the equation, except at points where $u(t)$ is discontinuous. The graph of $x(t)$ will, in general, have "kinks" at the points of discontinuity of $u(t)$, and it will usually not be differentiable at these kinks. It is, however, still continuous at the kinks.

For the oil extraction problem in Example 9.1.2, Fig. 1 shows one possible control function, whereas Fig. 2 shows the corresponding development of the state variable. The rate of extraction is initially a constant u_0 on the interval $[0, t']$, then a different constant u_1 (with $u_1 < u_0$) on $(t', t'']$. Finally, on $(t'', T]$, the rate of extraction $u(t)$ gradually declines from a level lower than u_1 until the field is exhausted at time T. Observe that the graph of $x(t)$ is connected, but has kinks at t' and t''.

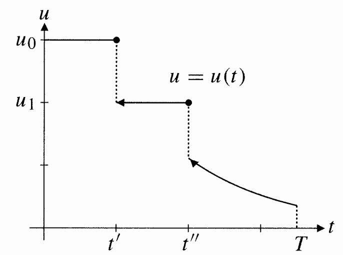

Figure 1

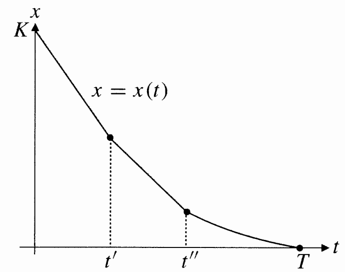

Figure 2

So far no restrictions have been placed on the functions $g(t, x, u)$ and $f(t, x, u)$. For the analysis presented in this chapter, it suffices to assume that f, g, and their first-order partial derivatives w.r.t. x and u are continuous in (t, x, u). These continuity assumptions will be implicitly assumed from now on.

Necessary Conditions, Sufficient Conditions, and Existence

In static optimization theory there are three main types of result that can be used to find possible global solutions: Theorems giving necessary conditions for optimality (typically, first-order conditions), theorems giving sufficient conditions (typically, first-order conditions supplemented by appropriate concavity/convexity requirements), and finally existence theorems (typically, the extreme value theorem).

In control theory the situation is similar. The maximum principle, in different versions, gives *necessary* conditions for optimality, i.e. conditions which a possible optimal control *must* satisfy. These conditions do not guarantee that the maximization problem has a solution.

The second type of theorem consists of sufficiency results, of the kind originally developed by Mangasarian. Theorems of this type impose certain concavity/convexity requirements on the functions involved. If a control function $u^*(t)$ (with corresponding state variable $x^*(t)$ and adjoint variable $p(t)$) satisfies the stated sufficient conditions, then $(x^*(t), u^*(t))$ solves the maximization problem. But these sufficient conditions are rather demanding, and in many problems there are optimal solutions although the sufficient conditions are not satisfied.

Existence theorems give conditions which ensure that an optimal solution of the problem really exists. The conditions needed for existence are less stringent than the sufficient conditions. Existence theorems are used (in principle) in the following way: One finds, by using the necessary conditions, all the "candidates" for a solution of the problem. If the existence of an optimal solution is assured, then an optimal solution can be found by simply examining which of the candidates gives the largest values of the objective function. (This direct comparison of different candidates is unnecessary if we use sufficient conditions.)

9.4 The Standard Problem

Section 9.2 studied a control problem with no restriction on the control function at ar time, and also no restriction on the state variable at the terminal time; $x(t_1)$ was free. The: features are unrealistic in many economic models, as has already been pointed out.

This section considers the **"standard end constrained problem"**

$$\max \int_{t_0}^{t_1} f(t, x, u)\, dt, \quad u \in U \subseteq \mathbb{R} \qquad ($$

$$\dot{x}(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0 \qquad (:$$

with one of the following terminal conditions imposed

$$\text{(a) } x(t_1) = x_1 \quad \text{(b) } x(t_1) \geq x_1 \quad \text{or} \quad \text{(c) } x(t_1) \text{ free} \qquad (:$$

Again, t_0, t_1, x_0, and x_1 are fixed numbers and U is the fixed control region. A pair $(x(t), u(t$ that satisfies (2) and (3) is called an **admissible pair**. Among all admissible pairs we see an **optimal pair**, i.e. a pair of functions that maximizes the integral in (1).

In order to formulate correct necessary conditions, we need to define the Hamiltonian :

$$H(t, x, u, p) = p_0 f(t, x, u) + p g(t, x, u) \qquad (4$$

The new feature is the constant number p_0 in front of $f(t, x, u)$. If $p_0 \neq 0$, we can divic by p_0 to get a new Hamiltonian in which $p_0 = 1$, in effect. But if $p_0 = 0$, this normalizatic is impossible.[2]

THEOREM 9.4.1 (THE MAXIMUM PRINCIPLE. STANDARD END CONSTRAINTS)

Suppose that $(x^*(t), u^*(t))$ is an optimal pair for the standard end constrained problem (1)–(3). Then there exists a continuous function $p(t)$ and a number p_0, which is either 0 or 1, such that for all t in $[t_0, t_1]$ we have $(p_0, p(t)) \neq (0, 0)$ and, moreover:

(A) The control $u^*(t)$ maximizes the Hamiltonian $H(t, x^*(t), u, p(t))$ w.r.t. $u \in U$, i.e.

$$H(t, x^*(t), u, p(t)) \leq H(t, x^*(t), u^*(t), p(t)) \text{ for all } u \text{ in } U \qquad (5$$

(B) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t))$ (6

(C) Corresponding to each of the terminal conditions (b) and (c) in (3) there is a **transversality condition** on $p(t_1)$:

(b′) $p(t_1) \geq 0$ (with $p(t_1) = 0$ if $x^*(t_1) > x_1$) (7

(c′) $p(t_1) = 0$

[In case (a) there is no condition on $p(t_1)$.]

[2] For a proof see Fleming and Rishel (1975).

NOTE 1 In some "bizarre" problems the conditions in the theorem are only satisfied with $p_0 = 0$. (See Problem 10.) Note that in this case the conditions in the maximum principle do not change at all if f is replaced by any arbitrary function. In fact, when $p_0 = 0$, then (5) takes the form $pg(t, x^*(t), u, p(t)) \le pg(t, x^*(t), u^*(t), p(t))$ for all u in U.

In the examples and problems to follow we shall assume without proof that $p_0 = 1$, except in Example 4 where we show the type of argument needed to prove that $p_0 = 1$. (Almost all papers in economic literature using control theory assume that the problem is "normal" in the sense that $p_0 = 1$.)

If $x(t_1)$ is free, then according to (7)(c′), $p(t_1) = 0$. Since $(p_0, p(t_1))$ cannot be $(0, 0)$, we conclude that in this case $p_0 = 1$ and Theorem 9.2.1 is correct as stated.

NOTE 2 If the inequality sign in (3)(b) is reversed, so are the inequality signs in (7)(b′).

NOTE 3 The derivative $\dot{p}(t)$ in (6) does not necessarily exist at the discontinuity points of $u^*(t)$, and (6) need hold only wherever $u^*(t)$ is continuous.

NOTE 4 If U is a convex set and the function H is strictly concave in u, one can show that an optimal control $u^*(t)$ must be continuous.

The conditions in the maximum principle are necessary, but generally not sufficient for optimality. The following theorem gives sufficient conditions.

THEOREM 9.4.2 (MANGASARIAN)

Suppose that $(x^*(t), u^*(t))$ is an admissible pair with corresponding adjoint function $p(t)$ such that the conditions (A)–(C) in Theorem 9.4.1 are satisfied with $p_0 = 1$. Suppose further that the control region U is convex and that $H(t, x, u, p(t))$ is concave in (x, u) for every t in $[t_0, t_1]$. Then $(x^*(t), u^*(t))$ is an optimal pair.

In general, it is not easy to apply Theorems 9.4.1 and 9.4.2. In principle one can use the following approach:

(a) For each triple (t, x, p), maximize $H(t, x, u, p)$ w.r.t. $u \in U$. In many cases, this maximization occurs at a unique maximum point $u = \hat{u}(t, x, p)$.

(b) Insert this function into the differential equations (2) and (6) to obtain

$$\dot{x}(t) = g(t, x(t), \hat{u}(t, x(t), p(t))) \quad \text{and} \quad \dot{p}(t) = -H'_x(t, x(t), \hat{u}(t, x(t), p(t)), p(t))$$

This gives two differential equations to determine the functions $x(t)$ and $p(t)$.

(c) The constants in the general solution $(x(t), p(t))$ of these differential equations are determined by combining the initial condition $x(t_0) = x_0$ with the terminal conditions and the transversality conditions (7). The state variable obtained in this way is denoted by $x^*(t)$, and the corresponding control variable is $u^*(t) = \hat{u}(t, x^*(t), p(t))$. The pair $(x^*(t), u^*(t))$ is then a candidate for optimality.

This sketch suggests that the maximum principle may contain enough information to g only one or perhaps a few solution candidates, and in fact the procedure (a)–(c) is usefu many problems.

EXAMPLE 1 Solve the problem

$$\max \int_0^1 x(t)\,dt, \quad \dot{x}(t) = x(t) + u(t), \quad x(0) = 0, \quad x(1) \text{ free}, \quad u \in [-1, 1]$$

Solution: Looking at the objective function, we see that it pays to have $x(t)$ as large possible all the time, and from the differential equation it follows that this is obtained having u as large as possible all the time, i.e. $u(t) = 1$ for all t. So this must be the optin control. Let us confirm this by using the maximum principle.

The Hamiltonian function with $p_0 = 1$ is $H(t, x, u, p) = x + px + pu$, which is line and hence concave in (x, u), so Theorem 9.4.2 applies. The differential equation (6) togeth with $p(1) = 0$ (see (7)(c′)) gives

$$\dot{p} = -1 - p, \qquad p(1) = 0$$

This differential equation is especially simple because it is linear with constant coefficien According to (5.4.3), the general solution is $p(t) = Ae^{-t} - 1$, where A is determined l $0 = p(1) = Ae^{-1} - 1$, which gives $A = e$. Hence, $p(t) = e^{1-t} - 1$, and we see th $p(t) > 0$ for all t in $[0, 1)$. Since the optimal control should maximize $H(t, x^*(t), u, p(t$ we see from the expression for H that we must have $u^*(t) = 1$ for all t in $[0, 1]$. Tl corresponding path $x^*(t)$ for the state variable x satisfies the equation $\dot{x}^*(t) = x^*(t) +$ with general solution $x^*(t) = Be^t - 1$. Since $x^*(0) = 0$, we obtain $B = 1$, and so

$$x^*(t) = e^t - 1$$

We see now that $u^*(t)$, $x^*(t)$, and $p(t)$ satisfy all the requirements in Theorem 9.4.2. W conclude that we have found the solution to the problem.

EXAMPLE 2 **(Optimal Consumption)** Consider a consumer who expects to live from the prese time, when $t = 0$, until time T. Let $c(t)$ denote his consumption expenditure at time t an $y(t)$ his predicted income. Let $w(t)$ denote his wealth at time t. Then

$$\dot{w}(t) = r(t)w(t) + y(t) - c(t) \qquad (*$$

where $r(t)$ is the instantaneous rate of interest at time t. Suppose the consumer wants t maximize the "lifetime intertemporal utility function"

$$\int_0^T e^{-\alpha t} u\big(c(t)\big)\,dt$$

where $\alpha > 0$, and $u'(c) > 0$, $u''(c) < 0$ for all $c > 0$. The dynamic constraint is $(*)$ above. In addition, $w(0) = w_0$ is given, and there is the terminal constraint $w(T) \geq 0$ preventing the consumer from dying in debt.

This is an optimal control problem with $w(t)$ as the state variable and $c(t)$ as the control variable. We assume that $c(t) > 0$ so that the control region is $(0, \infty)$. We will try to characterize the optimal consumption path, and look at some special cases.

The Hamiltonian for this problem is $H(t, w, c, p) = e^{-\alpha t}u(c) + p[r(t)w + y - c]$, with $p_0 = 1$ and with $p = p(t)$ as the adjoint function. Let $c^* = c^*(t)$ be an optimal solution. Then $H'_c = 0$ at c^*, i.e.

$$e^{-\alpha t}u'(c^*(t)) = p(t) \tag{i}$$

Hence, the adjoint variable is equal to the discounted value of marginal utility. Also,

$$\dot{p}(t) = -H'_w = -p(t)r(t) \tag{ii}$$

so that the adjoint variable decreases at a proportional rate equal to the rate of interest. Notice that (ii) is a separable differential equation whose solution is (see Example 5.3.6)

$$p(t) = p(0)\exp\left[-\int_0^t r(s)\,ds\right] \tag{iii}$$

A more explicit formula is not possible, except in special cases. One such is when $r(t) = r$, independent of time, and $r = \alpha$. Then (iii) reduces to $p(t) = p(0)e^{-rt}$, and (i) becomes $e^{-rt}u'(c^*(t)) = p(0)e^{-rt}$, or $u'(c^*(t)) = p(0)$. It follows that $c^*(t)$ is a constant, $c^*(t) = \bar{c}$, independent of time. Then $(*)$ becomes $\dot{w} = rw + y(t) - \bar{c}$, whose solution is

$$w^*(t) = e^{rt}\left[w_0 + \int_0^t e^{-rs}y(s)\,ds - \frac{\bar{c}}{r}(1 - e^{-rt})\right] \tag{iv}$$

Because of (7)(b′), the terminal constraint $w^*(T) \geq 0$ implies that

$$p(T) \geq 0 \text{ (with } p(T) = 0 \text{ if } w^*(T) > 0)$$

It follows that if $w^*(T) > 0$, then $p(T) = 0$, which contradicts (i). Thus $w^*(T) = 0$, so it is optimal for the consumer to leave no legacy after time T. The condition $w^*(T) = 0$ determines the optimal level of $\bar{c}$, which is [3]

$$\bar{c} = \frac{r}{1 - e^{-rT}}\left[w_0 + \int_0^T e^{-rs}y(s)\,ds\right]$$

It is interesting to consider the special cases where the utility function u is

$$u(c) = \frac{(c - \underline{c})^{1-\varepsilon}}{1 - \varepsilon} \quad (\varepsilon > 0;\ \varepsilon \neq 1) \qquad \text{or} \qquad u(c) = \ln(c - \underline{c}) \tag{v}$$

[3] This is the same answer as that derived in Example 2.4.2, equation (iii).

Then $u'(c) = (c - \underline{c})^{-\varepsilon}$ in both cases, with $\varepsilon = 1$ when $u(c) = \ln(c - \underline{c})$. Note that wh $\underline{c} = 0$, the *elasticity of marginal utility* is $\text{El}_c u'(c) = cu''(c)/u'(c) = -\varepsilon$.

When $\underline{c} > 0$, the level $\underline{c}$ of consumption can be regarded as minimum subsistence, bel which consumption should never be allowed to fall, if possible. With utility given by (equation (i) can be solved explicitly for $c^*(t)$. In fact

$$c^*(t) = \underline{c} + \left[e^{\alpha t}p(t)\right]^{-1/\varepsilon} \qquad ($$

In order to keep the algebra manageable, restrict attention once again to the case wh $r(t) = r$, independent of time, but now $r \neq \alpha$ is allowed. Still, $p(t) = p(0)e^{-rt}$ and so (implies that

$$c^*(t) = \underline{c} + \left[e^{(\alpha - r)t}p(0)\right]^{-1/\varepsilon} = \underline{c} + Ae^{\gamma t}$$

where $A = p(0)^{-1/\varepsilon}$ and $\gamma = (r - \alpha)/\varepsilon$. Then $(*)$ becomes

$$\dot{w} = rw + y - \underline{c} - Ae^{\gamma t}$$

Multiplying this first-order equation by the integrating factor e^{-rt} leads to

$$\frac{d}{dt}(e^{-rt}w) = e^{-rt}(\dot{w} - rw) = e^{-rt}(y - \underline{c} - Ae^{\gamma t})$$

Integrating each side from 0 to t gives

$$e^{-rt}w(t) - w_0 = \int_0^t e^{-rs}y(s)\,ds - \frac{\underline{c}}{r}(1 - e^{-rt}) - \frac{A}{r - \gamma}[1 - e^{-(r-\gamma)t}]$$

In particular,

$$w(T) = e^{rT}w_0 + \int_0^T e^{r(T-t)}y(t)\,dt - \frac{\underline{c}}{r}(e^{rT} - 1) - \frac{A}{r - \gamma}(e^{rT} - e^{\gamma T})$$

Again $p(T) > 0$ and thus $w^*(T) = 0$, so the optimal path involves choosing $p(0)$ such th $A = p(0)^{-1/\varepsilon}$ has the value

$$A = \frac{r - \gamma}{e^{rT} - e^{\gamma T}}\left[e^{rT}w_0 + \int_0^T e^{r(T-t)}y(t)\,dt - \frac{\underline{c}}{r}(e^{rT} - 1)\right]$$

There are two significantly different cases involved here. The first is when $r > \alpha$ and s $\gamma > 0$. Then consumption grows over time starting from the level $\underline{c} + A$. But if $r < \alpha$ an so $\gamma < 0$, then optimal consumption shrinks over time. This makes sense because $r < \alpha$ i the case when the agent discounts future utility at a rate α that exceeds the rate of interest

The previous case with constant consumption is when $\gamma = 0$. The same solution emerge in the limit as $\varepsilon \to \infty$, which represents the case when the consumer is extremely avers to fluctuations in consumption.

In the next example the optimal control is bang-bang.

EXAMPLE 3 Solve the following control problem:

$$\max \int_0^1 (2x - x^2)\,dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(1) = 0, \quad u \in [-1, 1]$$

Solution: The Hamiltonian is $H = 2x - x^2 + pu$, which is concave in (x, u). The optimal control $u^*(t)$ must maximize $2x^*(t) - (x^*(t))^2 + p(t)u$ subject to $u \in [-1, 1]$. Only the term $p(t)u$ depends on u, so

$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > 0 \\ -1 & \text{if } p(t) < 0 \end{cases} \tag{$*$}$$

The differential equation for $p(t)$ is

$$\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 2x^*(t) - 2 = 2(x^*(t) - 1) \tag{$**$}$$

Note that $\dot{x}^*(t) = u^*(t) \le 1$. Because $x^*(0) = 0$, it follows that $x^*(t) < 1$ for all t in $[0, 1)$. Then $(**)$ implies that $p(t)$ is strictly decreasing in $[0, 1]$.

Suppose there could be a solution with $p(1) \ge 0$. Because $p(t)$ is strictly decreasing in $[0, 1]$, one would have $p(t) > 0$ in $[0, 1)$, and then $(*)$ would imply that $u^*(t) = 1$ for all t. In this case, $\dot{x}^*(t) = 1$ for all t in $[0, 1]$. With $x^*(0) = 0$ we get $x^*(t) \equiv t$ and thus $x^*(1) = 1$, which is incompatible with the terminal condition $x^*(1) = 0$. Thus any solution must satisfy $p(1) < 0$. Suppose $p(t) < 0$ for all t in $(0, 1]$. Then from $(*)$, $u^*(t) = -1$ for all such t, so $x^*(t) \equiv -t$ with $x^*(1) = -1$, violating the terminal condition. Hence, for some t^* in $(0, 1)$, the function $p(t)$ switches from being positive to being negative, with $p(t^*) = 0$. A possible path for $p(t)$ is shown in Fig. 1.

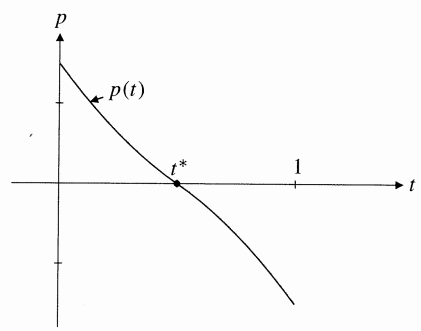

Figure 1

It follows that $u^*(t) = 1$ in $[0, t^*]$[4] and $u^*(t) = -1$ in $(t^*, 1]$. On $[0, t^*)$, therefore, $\dot{x}^*(t) = 1$, and with $x^*(0) = 0$ this yields $x^*(t) = t$. Since $x^*(t)$ is required to be continuous at t^*, $x^*(t^*) = x^*(t^{*-}) = t^*$. In $(t^*, 1]$, $\dot{x}^*(t) = -1$ so $x^*(t) = -t + C$ for some constant C.

[4] Recall our convention to let $u^*(t)$ be left-continuous.

Because $x^*(t)$ is continuous at t^*, $x^*(t^{*+}) = t^*$, so $C = 2t^*$. Hence, $x^*(t) = -t +$
Then $x^*(1) = 0$ implies that $t^* = 1/2$. We conclude that the optimal solution is

$$u^*(t) = \begin{cases} 1 & \text{in } [0, 1/2] \\ -1 & \text{in } (1/2, 1] \end{cases} \qquad x^*(t) = \begin{cases} t & \text{in } [0, 1/2] \\ 1 - t & \text{in } (1/2, 1] \end{cases}$$

To find $p(t)$, note that $\dot{p}(t) = 2x^*(t) - 2 = 2t - 2$ in $[0, 1/2]$. Because $p(1/2) = 0$, on
$p(t) = t^2 - 2t + 3/4$. In the interval $(1/2, 1]$, $(**)$ implies that $\dot{p}(t) = -2t$, and bec
$p(t)$ is continuous with $p(1/2) = 0$, the adjoint function is $p(t) = -t^2 + 1/4$. For
function $p(t)$ the maximum condition $(*)$ is satisfied.

The last example shows a typical kind of argument needed to prove that $p_0 \ne 0$.

EXAMPLE 4 Consider Example 3 again. Including the multiplier p_0, the Hamiltonian function (4) is
$p_0(2x - x^2) + pu$, and the differential equation (6) for p is $\dot{p} = -H'_x = -p_0(2 - 2x^*(t))$. Sup
$p_0 = 0$. Then $\dot{p} = 0$ and so p is a constant, $\bar{p}$. Because $(p_0, p(t)) = (p_0, \bar{p}) \ne (0, 0)$, that con
$\bar{p}$ is not 0. Now, an optimal control must maximize $pu = \bar{p}u$ subject to $u \in [-1, 1]$. If $\bar{p} > 0$,
obviously $u^*(t) = 1$ for all t in $[0, 1]$. This means that $\dot{x}^*(t) \equiv 1$, with $x^*(0) = 0$, so $x^*(t) \equiv t$.
violates the terminal condition $x^*(1) = 0$. If $\bar{p} < 0$, then obviously $u^*(t) \equiv -1$, and $\dot{x}^*(t) \equiv$
for all t in $[0, 1]$, with $x^*(0) = 0$, so $x^*(t) \equiv -t$. This again violates the terminal condition
conclude that $p_0 = 0$ is impossible, so $p_0 = 1$.

PROBLEMS FOR SECTION 9.4

1. What is the obvious solution to the problem

$$\max \int_0^T x(t)\,dt, \quad \dot{x}(t) = u(t), \quad x(0) = 0, \quad x(T) \text{ free}, \quad u(t) \in [0, 1]$$

where T is a fixed positive constant? Compute the associated value, $V(T)$, of
objective function. Find the solution also by using Theorem 9.4.2.

2. Solve the problem: $\max \int_0^1 (1 - x^2 - u^2)\,dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(1) \ge 1, \quad u \in$

3. Consider the problem in Example 9.2.1.

   (a) Replace $u \in \mathbb{R}$ by $u \in [0, 1]$ and find the optimal solution.

   (b) Replace $u \in \mathbb{R}$ by $u \in [-1, 1]$ and find the optimal solution, provided $T > 2$.

4. Solve the following problems. Also compute the corresponding value of the object
function.

   (a) $\max_{u \in [0,1]} \int_0^{10} x\,dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(10) = 2$

   (b) $\max_{u \in [0,1]} \int_0^T x\,dt, \quad \dot{x} = u, \quad x(0) = x_0, \quad x(T) = x_1$ (with $x_0 < x_1 < x_0 + T$)

5. (a) Given the fixed positive number T, write down the conditions in Theorem 9.4.1 for the problem

$$\max \int_0^T -(u^2 + x^2)\, dt, \quad \dot{x} = au, \quad x(0) = 1, \quad x(T) \text{ free}, \quad u(t) \in [0, 1]$$

and find the solution when $a \geq 0$.

(b) Find the solution if $a < 0$. (*Hint:* Try $u^*(t) \in (0, 1)$ for all t.)

6. Solve the following special case of Problem I in Example 9.1.2:

$$\max \int_0^5 [10u - (u^2 + 2)]e^{-0.1t}\, dt, \quad \dot{x} = -u, \; x(0) = 10, \; x(5) \geq 0, \; u \geq 0$$

7. (From Kamien and Schwartz (1991).) A firm has an order of B units of a commodity to be delivered at time T. Let $x(t)$ be the stock at time t. We assume that the cost per unit of time of storing $x(t)$ units is $ax(t)$. The increase in $x(t)$, which equals production per unit of time, is $u(t) = \dot{x}(t)$. Assume that the total cost of production per unit of time is equal to $b(u(t))^2$. Here a and b are positive constants. So the firm's natural problem is

$$\min \int_0^T \left[ax(t) + bu(t)^2\right] dt, \quad \dot{x}(t) = u(t), \; x(0) = 0, \; x(T) = B, \; u(t) \geq 0$$

(a) Write down the necessary conditions implied by Theorem 9.4.1.

(b) Find the only possible solution to the problem and explain why it really is a solution. (*Hint:* Distinguish between the cases $B \geq aT^2/4b$ and $B < aT^2/4b$.)

8. Find the only possible solution to the problem

$$\max \int_0^2 (x^2 - 2u)\, dt, \quad \dot{x} = u, \quad x(0) = 1, \quad x(2) \text{ free}, \quad u \in [0, 1]$$

(*Hint:* Show that $p(t)$ is strictly decreasing.)

9. Consider the problem $\max \int_0^2 u\, dt$, $\dot{x} = u$, $x(0) = 0$, $x(2) \leq 1$, $u \in [-1, 1]$.

(a) Prove that the associated adjoint variable $p(t)$ is a constant, and show that this constant has to be -1. But then $H \equiv 0$, and $u^*(t)$ is not determined by the maximum condition (5).

(b) Show that *any* control which implies that $x^*(2) = 1$ solves the problem.

10. Consider the problem $\max \int_0^1 -u\, dt$, $\dot{x} = u^2$, $x(0) = x(1) = 0$, $u \in \mathbb{R}$.

(a) Explain why $u^*(t) = x^*(t) = 0$ solves the problem.

(b) Show that the conditions in the maximum principle are satisfied only for $p_0 = 0$.

9.5 The Maximum Principle and the Calculus of Variations

The introduction to this chapter claimed that optimal control theory extends the cla calculus of variations. Consider what the maximum principle has to say about the sta variational problem

$$\max \int_{t_0}^{t_1} F(t, x, \dot{x})\, dt, \qquad x(t_0) = x_0, \qquad \begin{cases} \text{(a)} \; x(t_1) = x_1 \\ \text{(b)} \; x(t_1) \geq x_1 \\ \text{(c)} \; x(t_1) \text{ free} \end{cases}$$

To transform this to a control problem, simply let $\dot{x}(t)$ be a control variable with $\dot{x}(t) =$ Because there are no restrictions on $\dot{x}(t)$ in the variational problem, there are no restric on the control function $u(t)$. Hence, $U = \mathbb{R}$.

The control problem has the particularly simple differential equation $\dot{x}(t) = u(t)$. Hamiltonian is $H(t, x, u, p) = p_0 F(t, x, u) + pu$. The maximum principle states tl $u^*(t)$ solves the problem, then H as a function of u must be maximized at $u^*(t)$. Bec $U = \mathbb{R}$, a necessary condition for this maximum is

$$H'_u(t, x, u, p(t)) = p_0 F'_u(t, x, u) + p(t) = 0$$

Since $(p_0, p(t)) \neq (0, 0)$, equation $(*)$ implies that $p_0 \neq 0$, so $p_0 = 1$. The differe equation for $p(t)$ is

$$\dot{p}(t) = -H'_x(t, x, u, p) = -F'_x(t, x, u)$$

Differentiating $(*)$ with respect to t yields

$$\frac{d}{dt}\Big(F'_u(t, x, u)\Big) + \dot{p}(t) = 0 \qquad (*$$

Since $u = \dot{x}$, it follows from $(**)$ and $(***)$ that

$$F'_x(t, x, \dot{x}) - \frac{d}{dt}\Big(F'_{\dot{x}}(t, x, \dot{x})\Big) = 0$$

which is the Euler equation. Moreover, $(*)$ implies that

$$p(t) = -F'_{\dot{x}}(t, x, \dot{x})$$

Using (3) it is easy to check that the transversality conditions in (9.4.7) are precisely th set out in Section 8.5. Note also that concavity of the Hamiltonian with respect to $(x, u$ equivalent to concavity of $F(t, x, \dot{x})$ with respect to $(x, \dot{x})$.

Thus the maximum principle confirms all the main results found in Chapter 8. Actua it contains more information about the solution of the optimization problem. For instan according to the maximum principle, for every t in $[t_0, t_1]$ the Hamiltonian attains its m imum at $u^*(t)$. Assuming that F is a C^2 function, not only is $H'_u = 0$, but also $H''_{uu} \leq$ implying that $F''_{\dot{x}\dot{x}} \leq 0$. This is the so-called **Legendre condition** in the calculus of va ations. (Also, continuity of $p(t)$ and (3) together give the **Weierstrass–Erdmann corr condition**, requiring $F'_{\dot{x}}$ to be continuous. This is a well known result in the classical theor

PROBLEMS FOR SECTION 9.5

1. Find the only possible solution to the following problem by using both the calculus of variations and control theory:

$$\max \int_0^1 (2xe^{-t} - 2x\dot{x} - \dot{x}^2)\,dt, \quad x(0) = 0, \ x(1) = 1$$

2. Solve the following problem by using both the calculus of variations and control theory:

$$\max \int_0^2 (3 - x^2 - 2\dot{x}^2)\,dt, \quad x(0) = 1, \ x(2) \geq 4$$

3. Solve the following problem by using both the calculus of variations and control theory:

$$\max \int_0^1 (-2\dot{x} - \dot{x}^2)e^{-t/10}\,dt, \quad x(0) = 1, \ x(1) = 0$$

4. At time $t = 0$ an oil field contains $\bar{x}$ barrels of oil. It is desired to extract all of the oil during a given time interval $[0, T]$. If $x(t)$ is the amount of oil left at time t, then $-\dot{x}$ is the extraction rate (which is ≥ 0 when $x(t)$ is decreasing). Assume that the world market price per barrel of oil is given and equal to $ae^{\alpha t}$. The extraction costs per unit of time are assumed to be $\dot{x}(t)^2 e^{\beta t}$. The profit per unit of time is then $\pi = -\dot{x}(t)ae^{\alpha t} - \dot{x}(t)^2 e^{\beta t}$. Here a, α, and β are constants, $a > 0$. This leads to the variational problem

$$\max \int_0^T \left[-\dot{x}(t)ae^{\alpha t} - \dot{x}(t)^2 e^{\beta t}\right]e^{-rt}\,dt, \qquad x(0) = \bar{x}, \ x(T) = 0, \qquad (**)$$

where r is a positive constant. Find the Euler equation for problem $(**)$, and show that at the optimum $\partial\pi/\partial\dot{x} = ce^{rt}$ for some constant c. Derive the same result by using control theory.

5. S. Strøm considers the problem

$$\max_x \int_0^T \{U(x(t)) - b(x(t)) - gz(t)\}\,dt, \quad \dot{z}(t) = ax(t), \ z(0) = z_0, \ z(T) \text{ free}$$

Here $U(x)$ is the utility enjoyed by society consuming x, whereas $b(x)$ is total cost and $z(t)$ is the stock of pollution at time t. Assume that U and b satisfy $U' > 0$, $U'' < 0$, $b' > 0$, and $b'' > 0$. The control variable is $x(t)$, whereas $z(t)$ is the state variable. The constants a and g are positive.

(a) Write down the conditions implied by the maximum principle. Show that the adjoint function is given by $p(t) = g(t - T)$, $t \in [0, T]$, and prove that if $x^*(t) > 0$ solves the problem, then

$$U'(x^*(t)) = b'(x^*(t)) + ag(T - t) \qquad (*)$$

(b) Prove that a solution of $(*)$ with $x^*(t) > 0$ must solve the problem. Show that $x^*(t)$ is strictly increasing. (*Hint:* Differentiate $(*)$ with respect to t.)

9.6 Adjoint Variables as Shadow Prices

Like the Lagrange multipliers used to solve static constrained optimization proble Chapter 3, the adjoint function $p(t)$ in the maximum principle can be given an inter price interpretation.

Consider the standard endconstrained problem (9.4.1)–(9.4.3). Suppose that it unique optimal solution $(x^*(t), u^*(t))$ with unique corresponding adjoint function The corresponding value of the objective function will depend on x_0, x_1, t_0, and t_1. S denoted by

$$V(x_0, x_1, t_0, t_1) = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))\,dt$$

We call V the **(optimal) value function**. (When $x(t_1)$ is free, x_1 is not an argument o

Suppose x_0 is changed slightly. In general, both $u^*(t)$ and $x^*(t)$ will change ove whole interval $[t_0, t_1]$. For typical problems in control theory, there is no guarantee tl is differentiable at a particular point. But at any point where it is differentiable,

$$\frac{\partial V(x_0, x_1, t_0, t_1)}{\partial x_0} = p(t_0)$$

The number $p(t_0)$ therefore measures the marginal change in the optimal value functi x_0 increases.

EXAMPLE 1 In Example 9.2.1 the objective function was $\int_0^T [1 - tx(t) - u(t)^2]\,dt$, and the solu was $u^*(t) = -\frac{1}{4}(T^2 - t^2)$, $x^*(t) = x_0 - \frac{1}{4}T^2 t + \frac{1}{12}t^3$, with $p(t) = -\frac{1}{2}(T^2 - t^2)$. So value function is

$$V(x_0, T) = \int_0^T \left[1 - tx^*(t) - (u^*(t))^2\right]dt = \int_0^T \left[1 - x_0 t + \tfrac{1}{4}T^2t^2 - \tfrac{1}{12}t^4 - \tfrac{1}{16}(T^2 - t^2)^2\right.$$

This last integral could be evaluated exactly, but fortunately we do not need to. Inst simply differentiating V w.r.t. x_0 under the integral sign using formula (4.2.1) gives

$$\frac{\partial V(x_0, T)}{\partial x_0} = \int_0^T (-t)\,dt = -\tfrac{1}{2}T^2$$

On the other hand, $p(0) = -\frac{1}{2}T^2$, so (2) is confirmed.

Formula (2) interprets $p(t)$ at time $t = t_0$. What about $p(t)$ at an arbitrary $t \in (t_0, t_1)$? want an interpretation that relates to the value function for the problem defined over the wh interval $[t_0, t_1]$, not only the subinterval $[t, t_1]$. Consider again problem (9.4.1)–(9.4.3), assume that all admissible $x(t)$ are forced to have a jump equal to v at $t \in (t_0, t_1)$, so t $x(t^+) - x(t^-) = v$. Suppose all admissible $x(t)$ are continuous elsewhere. The optir value function V for this problem will depend on v. Suppose that $(x^*(t), u^*(t))$ is optimal solution of the problem for $v = 0$. Then, under certain conditions, it can be sho that V as a function of v is defined in a neighbourhood of $v = 0$, that V is differentiable w.r.t. v at $v = 0$, and that

$$\left(\frac{\partial V}{\partial v}\right)_{v=0} = p(t) \tag{3}$$

The adjoint variable $p(t)$ is approximately the change in the value function (1) due to a unit increase in $x(t)$.[5]

A General Economic Interpretation

Consider a firm that seeks to maximize its profit over a planning period $[t_0, t_1]$. The state of the firm at time t is described by its capital stock $x(t)$. At each time t the firm can partly influence its immediate profit, as well as the change in its future capital stock. Let the firm's decision or control variable at time t be $u(t)$. Let the rate of profit at time t be $f(t, x(t), u(t))$, so that the total profit in the time period $[t_0, t_1]$ is

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

The firm can choose $u(t)$ within certain limits, so that $u(t) \in U = [u_0, u_1]$, but it cannot directly influence $x(t)$. The rate of change in the capital stock depends on the present capital stock as well as on the value chosen for $u(t)$ at time t. Thus,

$$\dot{x}(t) = g(t, x(t), u(t)), \qquad x(t_0) = x_0$$

where x_0 is the given capital stock at time $t = t_0$. The control variable $u(t)$ not only influences the immediate profit but also, via the differential equation, influences the rate of change of the capital stock and thereby the future capital stock, which again changes the total profit.

Suppose we have found the solution to this problem, with corresponding adjoint function $p(t)$. According to (3), $p(t)$ is a "**shadow price**" of the capital stock, since $p(t)$ measures the marginal profit of capital. The Hamiltonian is $H = f(t, x, u) + p(t)g(t, x, u)$. Consider a small time interval $[t, t + \Delta t]$. Over this time interval, $\Delta x \approx g(t, x, u)\, \Delta t$ and so

$$H\, \Delta t = f(t, x, u)\, \Delta t + p(t)g(t, x, u)\, \Delta t \approx f(t, x, u)\, \Delta t + p(t)\, \Delta x$$

Hence $H\, \Delta t$ is the sum of the instantaneous profit $f(t, x, u)\, \Delta t$ earned in the time interval $[t, t + \Delta t]$ and the contribution $p(t)\, \Delta x$ to the total profit produced by the extra capital Δx at the end of this time period. The maximum principle requires choosing at each time the value of u that maximizes H, and hence $H\, \Delta t$.

---

[5] Economists have realized for a long time that the adjoint can be interpreted as a shadow price. Dorfman (1969) has an illuminating discussion on the economic interpretations, extending the material in the next subsection.

Other Sensitivity Results

Consider once again the standard end constrained problem (9.4.1)–(9.4.3) and its op value function (1). It turns out that, provided V is differentiable, the effects on V of s changes in x_1, t_0, and t_1 can also be expressed very simply. Define

$$H^*(t) = H(t, x^*(t), u^*(t), p(t))$$

Then

$$\frac{\partial V}{\partial x_0} = p(t_0), \quad \frac{\partial V}{\partial x_1} = -p(t_1), \quad \frac{\partial V}{\partial t_0} = -H^*(t_0), \quad \frac{\partial V}{\partial t_1} = H^*(t_1)$$

The first of these equations was discussed above. As for the second, it is like the first, ex that requiring the state x_1 to be larger at time t_1, has an effect that is opposite of allowin to be larger at time t_0. For example, in the capital accumulation interpretation in the prev subsection, increasing the initial capital stock x_0 by one unit increases the total profi approximately $p(t_0)$. On the other hand, increasing the capital which must be left at the of the planning period t_1 decreases the total profit earned by approximately $p(t_1)$. The t equality is similar to the fourth except for the change of sign. In the capital accumula interpretation, increasing t_1 makes the planning period longer and the total profit incre (if it is positive). On the other hand, increasing t_0 makes the planning period shorter, so total profit decreases. The last equality is illustrated in the next example.

NOTE 1 Consider the standard end constrained problem with $x(t_1)$ free. If $(x^*(t), u^*$ is an optimal pair with corresponding adjoint function $p(t)$, then according to condi (9.4.7)(c′), $p(t_1) = 0$. This makes sense in light of the second formula in (5): The $(x^*(t), u^*(t))$ will solve the problem with terminal condition $x(t_1) = x^*(t_1) = x_1$, the optimal value function V is given in (1). Since the optimal path in the problem w $x(t_1)$ free ends at $x^*(t_1)$, small changes in $x_1 = x^*(t_1)$ will not change V, and theref $p(t_1) = -\partial V/\partial x_1 = 0$. With the economic interpretation given above the result is a natural: If there is no reason to care about the capital stock at the end of the planning peri its shadow price should be equal to 0.

EXAMPLE 2 Verify the last equality in (5) for the problem in Example 1.

Solution: Differentiating the value function $V(x_0, T)$ from Example 1 w.r.t. T, usi the Leibniz rule (4.2.3) yields

$$\frac{\partial V}{\partial T} = 1 - x_0 T + \tfrac{1}{4}T^4 - \tfrac{1}{12}T^4 + \int_0^T \left[\tfrac{1}{2}t^2 T - \tfrac{1}{8}(T^2 - t^2)2T\right] dt$$

Integrating and simplifying gives

$$\frac{\partial V}{\partial T} = 1 - x_0 T + \tfrac{1}{6}T^4$$

Now, $H^*(T) = 1 - Tx^*(T) - (u^*(T))^2 + p(T)u^*(T) = 1 - x_0 T + \frac{1}{6}T^4$, because $u^*(T) =$ and $x^*(T) = x_0 - \frac{1}{6}T^3$. Thus the last result in (5) is confirmed.

EXAMPLE 3 **(Economic Growth)** Consider the following problem in economic growth theory due to Shell (1967):

$$\max \int_0^T (1 - s(t))e^{\rho t} f(k(t))e^{-\delta t}\, dt$$

$$\dot{k}(t) = s(t)e^{\rho t} f(k(t)) - \lambda k(t), \quad k(0) = k_0, \quad k(T) \geq k_T > k_0, \quad 0 \leq s(t) \leq 1$$

Here $k(t)$ is the capital stock (a state variable), $s(t)$ is the savings rate (a control variable) and $f(k)$ is a production function. Suppose that $f(k) > 0$ whenever $k \geq k_0 e^{-\lambda T}$, that $f'(k) > 0$, and that ρ, δ, λ, T, k_0, and k_T are all positive constants.

(a) Suppose $(k^*(t), s^*(t))$ solves the problem. Write down the conditions in the maximum principle in this case. What are the possible values of $s^*(t)$?

(b) Put $\rho = 0$, $f(k) = ak$, $a > 0$, $\delta = 0$ and $\lambda = 0$. Suppose that $T > 1/a$ and that $k_0 e^{aT} > k_T$. Try to find the only possible solution to the problem.

(c) Compute the value function for the problem in (b) and then verify the relevant equalities in (5).

Solution: (a) The Hamiltonian is $H = (1 - s)e^{\rho t} f(k)e^{-\delta t} + p(se^{\rho t} f(k) - \lambda k)$. If $(k^*(t), s^*(t))$ solves the problem, then in particular, $s^*(t)$ must solve

$$\max\ (1 - s)e^{\rho t} f(k^*(t))e^{-\delta t} + p(t)\big[se^{\rho t} f(k^*(t)) - \lambda k^*(t)\big] \text{ subject to } s \in [0, 1]$$

Disregarding the terms that do not depend on s, $s^*(t)$ must maximize the expression $e^{\rho t} f(k^*(t))\big(-e^{-\delta t} + p(t)\big)s$ for $s \in [0, 1]$. Hence we must choose

$$s^*(t) = \begin{cases} 1 & \text{if } p(t) > e^{-\delta t} \\ 0 & \text{if } p(t) < e^{-\delta t} \end{cases} \tag{i}$$

A possible optimal control can therefore only take the values 1 and 0 (except if $p(t) = e^{-\delta t}$).

Except where $s^*(t)$ is discontinuous,

$$\dot{p}(t) = -(1 - s^*(t))e^{\rho t} f'(k^*(t))e^{-\delta t} - p(t)s^*(t)e^{\rho t} f'(k^*(t)) + \lambda p(t) \tag{ii}$$

The transversality condition (9.4.7)(b′) gives

$$p(T) \geq 0 \quad (p(T) = 0 \text{ if } k^*(T) > k_T) \tag{iii}$$

For more extensive discussion of the model, see Shell (1967).

(b) Briefly formulated, the problem reduces to

$$\max \int_0^T (1 - s)ak\, dt, \qquad \dot{k} = ask, \quad k(0) = k_0, \quad k(T) \geq k_T > k_0$$

with $s \in [0, 1]$, $a > 0$, $T > 1/a$, and $k_0 e^{aT} > k_T$.

The Hamiltonian is $H = (1 - s)ak + pask$. The differential equation (ii) is now

$$\dot{p}(t) = -a + s^*(t)a(1 - p(t)) \tag{iv}$$

whereas (i) implies that

$$s^*(t) = \begin{cases} 1 & \text{if } p(t) > 1 \\ 0 & \text{if } p(t) < 1 \end{cases}$$

From (iv) and (v) it follows that

$$\dot{p}(t) = -a < 0 \quad \text{if} \quad p(t) < 1, \text{ while } \dot{p}(t) = -ap(t) \quad \text{if} \quad p(t) > 1$$

In all cases $\dot{p}(t) < 0$, so $p(t)$ is strictly decreasing.

Suppose $p(0) < 1$, which implies that $p(t) < 1$ throughout $(0, T]$. Then by (v), $s^*($ 0, and so $k^*(t) \equiv k_0$, which contradicts $k^*(T) \geq k_T > k_0$. Hence $p(0) > 1$. Then ther two possible paths for $p(t)$, which are shown in Fig. 1. In the first case $p(T) = 0$; i second case, $p(T) > 0$.

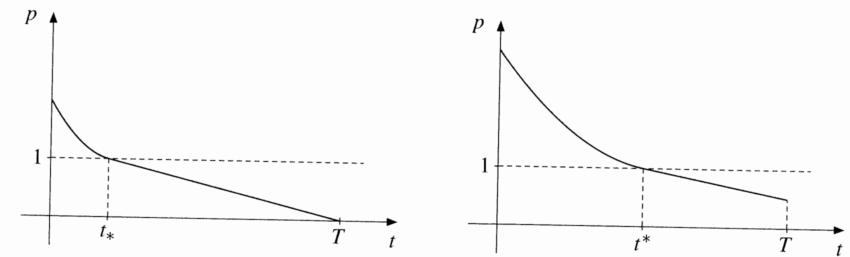

Figure 1: Two possible paths for $p(t)$.

Case I: $p(T) = 0$. Since $p(t)$ is continuous and strictly decreasing with $p(0) > 1$ $p(T) = 0$, there is a unique $t_* \in (0, T)$ such that $p(t_*) = 1$, with $p(t) > 1$ in [0 and $p(t) < 0$ in $(t_*, T]$. Then $s^*(t) = 1$ in $[0, t_*)$ and $s^*(t) = 0$ in $(t_*, T]$. By ($\dot{p}(t) = -ap(t)$ in $[0, t_*)$ and $\dot{p}(t) = -a$ in $(t_*, T]$. On $(t_*, T]$, we have $p(t) = -a(t -$ because we have assumed $p(T) = 0$. But $p(t_*) = 1$, so $1 = -a(t_* - T)$, implying $t_* = T - 1/a$. Furthermore, $p(t) = e^{a(T-t)-1}$ on $[0, T - 1/a]$. This gives the follow solution candidate:

For $t \in [0, T - 1/a]$, $s^*(t) = 1$, $k^*(t) = k_0 e^{at}$, and $p(t) = e^{a(T-t)-1}$ (

For $t \in (T - 1/a, T]$, $s^*(t) = 0$, $k^*(t) = k_0 e^{aT-1}$, and $p(t) = -a(t - T)$ (v

To verify all the conditions in the maximum principle, it remains to check that $k(T) \geq$ Here this reduces to $k_0 e^{aT-1} \geq k_T$, or $e^{aT-1} \geq k_T/k_0$, or

$$t_* = T - \frac{1}{a} \geq \frac{1}{a} \ln\left(\frac{k_T}{k_0}\right) \qquad ($$

If this inequality holds, then (vii)–(ix) give a possible solution to the problem.

Case II: $p(T) > 0$. In this case, by (iii), $k^*(T) = k_T$. If it were true that $p(T) \geq 1$, then c would have $p(t) > 1$ and so $s^*(t) = 1$ for all t in $[0, T)$, implying that $k^*(T) = k_0 e^{aT} >$ a contradiction. So there exists a unique t^* in $[0, T)$ such that $p(t^*) = 1$. Similar argume to those for case I suggest the following as an optimal solution:

For $t \in [0, t^*]$, $s^*(t) = 1$, $k^*(t) = k_0 e^{at}$, and $p(t) = e^{a(t^*-t)}$ (

For $t \in (t^*, T]$, $s^*(t) = 0$, $k^*(t) = k_0 e^{at^*}$, and $p(t) = 1 - a(t - t^*)$ (

From $k^*(T) = k_T$ it follows that $e^{at^*} = k_T/k_0$, so

$$t^* = \frac{1}{a}\ln\Big(\frac{k_T}{k_0}\Big) \tag{xii}$$

We note that $t^* < T$ is equivalent to $k_0e^{aT} > k_T$, as assumed. All of this was derived under the assumption that $p(T) > 0$, i.e. $1 - a(T - t^*) > 0$, which gives

$$T - \frac{1}{a} < \frac{1}{a}\ln\Big(\frac{k_T}{k_0}\Big) = t^* \tag{xiii}$$

Putting the two cases together, there is only one solution candidate, with

$$s^*(t) = 1 \text{ in } [0, \bar{t}\,], \qquad s^*(t) = 0 \text{ in } (\bar{t}, T\,] \tag{xiv}$$

where $\bar{t} = \max\big\{T - 1/a,\ (1/a)\ln(k_T/k_0)\big\}$.

Example 9.7.3 proves that we have found the optimal solution.

(c) For case I in (b) we have

$$V(k_0, k_T, T) = \int_{T-1/a}^{T} ak_0e^{aT-1}\,dt = ak_0e^{aT-1}[T - (T - 1/a)] = k_0e^{aT-1}$$

so $\partial V/\partial k_0 = e^{aT-1} = p(0)$, using (vii). Also $\partial V/\partial k_T = 0 = -p(T)$. Finally, $H^*(T) = (1 - s^*(T))ak^*(T) + p(T)as^*(T)k^*(T) = ak^*(T) = ak_0e^{aT-1} = \partial V/\partial T$.

For case II,

$$V(k_0, k_T, T) = \int_{t^*}^{T} ak_0e^{at^*}\,dt = ak_0e^{at^*}(T - t^*) = ak_T\big(T - \frac{1}{a}\ln k_T + \frac{1}{a}\ln k_0\big)$$

Hence $\partial V/\partial k_0 = k_T/k_0$, and we see that $p(0) = e^{at^*} = k_T/k_0$ also. Moreover, $\partial V/\partial k_T = a\big(T - \frac{1}{a}\ln k_T + \frac{1}{a}\ln k_0\big) - 1 = a(T - t^*) - 1$, and $-p(T) = a(T - t^*) - 1$ also. Finally, $\partial V/\partial T = ak_T$ and $H^*(T) = ak^*(T) = ak_0e^{at^*} = ak_0(k_T/k_0) = ak_T$.

PROBLEMS FOR SECTION 9.6

1. (a) Solve the control problem

$$\max \int_0^T (x - \tfrac{1}{2}u^2)\,dt, \quad \dot{x} = u, \quad x(0) = x_0, \quad x(T) \text{ free}, \quad u(t) \in \mathbb{R}$$

(b) Compute the optimal value function $V(x_0, T)$, and verify the first and the last equalities in (5).

2. Verify that $V'(T) = H^*(T)$ for Problem 9.4.1.

3. Verify (5) for Problem 9.4.4(b).

HARDER PROBLEMS

4. (a) Given the positive constant T, find the only possible solution to the problem:

$$\max \int_0^T (2x^2e^{-2t} - ue^t)\,dt, \ \dot{x} = ue^t, \ x(0) = 1, \ x(T) \text{ free}, \ u \in [0, 1]$$

(b) Compute the value function $V(T)$ and verify that $V'(T) = H^*(T)$.

5. Consider the problem $\max \int_0^1 ux\,dt$, $\dot{x} = 0$, $x(0) = x_0$, $x(1)$ free, $u \in [0, 1]$.

(a) Prove that if $x_0 < 0$, then the optimal control is $u^* = 0$, and if $x_0 > 0$, then the optimal control is $u^* = 1$.

(b) Show that the value function $V(x_0)$ is not differentiable at $x_0 = 0$.

9.7 Sufficient Conditions

The maximum principle provides necessary conditions for optimality. Only solution candidates fulfilling these necessary conditions can possibly solve the problem. However, the maximum principle by itself cannot tell us whether a given candidate is optimal or not, nor does it tell us whether or not an optimal solution exists.

The following result, originally due to Mangasarian, has been referred to before. In fact it is quite easy to prove.

THEOREM 9.7.1 (MANGASARIAN)

Consider the standard end constrained problem (9.4.1)–(9.4.3) with U an interval of the real line. Suppose the admissible pair $(x^*(t), u^*(t))$ satisfies all the conditions (9.4.5)–(9.4.7) of the maximum principle, with the associated adjoint function $p(t)$, and with $p_0 = 1$. Then, if

$$H(t, x, u, p(t)) \quad \text{is concave w.r.t. } (x, u) \text{ for all } t \text{ in } [t_0, t_1] \tag{1}$$

the pair $(x^*(t), u^*(t))$ solves the problem.

If $H(t, x, u, p(t))$ is strictly concave w.r.t. (x, u), then the pair $(x^*(t), u^*(t))$ is the unique solution to the problem.

NOTE 1 Suppose that U is an open interval (u_0, u_1). Then the concavity of $H(t, x, u, p(t))$ in u implies that the maximization condition (A) in Theorem 9.4.1 is equivalent to the first-order condition $\partial H^*/\partial u = \partial H(t, x^*(t), u^*(t), p(t))/\partial u = 0$. (See Theorem 3.1.2.) The concavity of $H(t, x, u, p(t))$ in (x, u) is satisfied, for example, if f and g are concave in (x, u) and $p(t) \geq 0$, or if f is concave whereas g is linear in (x, u).

Suppose that $U = [u_0, u_1]$. If $u^*(t) \in (u_0, u_1)$, then $\partial H^*/\partial u = 0$. If the lower limit $u^*(t) = u_0$ maximizes the Hamiltonian, then $\partial H^*/\partial u \leq 0$, because otherwise if $\partial H^*/\partial u > 0$, then the Hamiltonian would be increasing to the right of u_0. If the upper limit $u^*(t) = u_1$ maximizes the Hamiltonian, then we see in a similar way that $\partial H^*/\partial u \geq 0$. Because the Hamiltonian is concave in u, it follows that if $U = [u_0, u_1]$, then the maximum condition (9.4.5) is *equivalent* to the conditions:

$$\frac{\partial H^*}{\partial u} \begin{cases} \leq 0 & \text{if } u^*(t) = u_0 \\ = 0 & \text{if } u^*(t) \in (u_0, u_1) \\ \geq 0 & \text{if } u^*(t) = u_1 \end{cases} \tag{2}$$

These conditions are illustrated in Fig. 1.

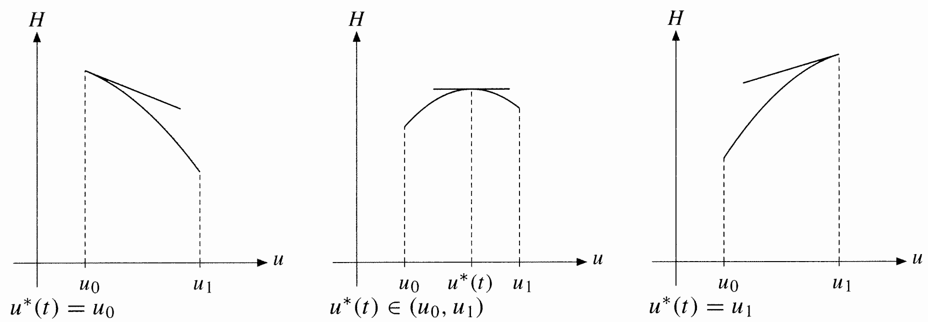

Figure 1

If the Hamiltonian is concave in u, the maximization condition in (9.4.5) can be *replaced* by the inequality

$$\frac{\partial H^*}{\partial u}(u^*(t) - u) \geq 0 \quad \text{for all} \quad u \in [u_0, u_1] \tag{3}$$

If $u^*(t) \in (u_0, u_1)$, condition (3) reduces to $\partial H^*/\partial u = 0$. If $u^*(t) = u_0$, then $u^*(t) - u = u_0 - u \leq 0$ for all $u^*(t) \in [u_0, u_1]$, so (3) is equivalent to $\partial H^*/\partial u \leq 0$. On the other hand, if $u^*(t) = u_1$, then $u^*(t) - u = u_1 - u \geq 0$ for all $u^*(t) \in [u_0, u_1]$, so (3) is equivalent to $\partial H^*/\partial u \geq 0$.

Proof of Theorem 9.7.1: Suppose that $(x, u) = (x(t), u(t))$ is an arbitrary alternative admissible pair. We must show that

$$D_u = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))\,dt - \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt \geq 0$$

First, simplify notation by writing H^* instead of $H(t, x^*(t), u^*(t), p(t))$ and H instead of $H(t, x(t), u(t), p(t))$, etc. Then, using the definition of the Hamiltonian and the fact that $\dot{x}^*(t) = g(t, x^*(t), u^*(t))$ and $\dot{x}(t) = g(t, x(t), u(t))$, we have $f^* = H^* - p\dot{x}^*$ and $f = H - p\dot{x}$. Therefore

$$D_u = \int_{t_0}^{t_1} (H^* - H)\,dt + \int_{t_0}^{t_1} p(\dot{x} - \dot{x}^*)\,dt \tag{$*$}$$

Because H is concave in (x, u), Theorem 2.4.1 implies that

$$H - H^* \leq \frac{\partial H^*}{\partial x}(x - x^*) + \frac{\partial H^*}{\partial u}(u - u^*)$$

Now, $\dot{p} = -\partial H^*/\partial x$, so $(*)$ and $(**)$ together imply that

$$D_u \geq \int_{t_0}^{t_1} \left[\dot{p}(x - x^*) + p(\dot{x} - \dot{x}^*)\right] dt + \int_{t_0}^{t_1} \frac{\partial H^*}{\partial u}(u^* - u)\,dt$$

Because of (3), the second integral is ≥ 0. Moreover, according to the rule for differentia a product, $\dot{p}(x - x^*) + p(\dot{x} - \dot{x}^*) = (d/dt)(p(x - x^*))$. Hence,

$$D_u \geq \int_{t_0}^{t_1} \frac{d}{dt}[p(x - x^*)]\,dt = \Big|_{t_0}^{t_1} p(t)[x(t) - x^*(t)] = p(t_1)(x(t_1) - x^*(t_1)) \quad (*$$

where the last equality holds because the contribution from the lower limit of integratic 0 because $x(t_0) - x^*(t_0) = x_0 - x_0 = 0$.

Now one can use the terminal condition (9.4.3) and the transversality condition (9. to show that the last term in $(***)$ is always ≥ 0. Indeed, if (9.4.3)(a) holds, then $x(t_1$ $x^*(t_1) = x_1 - x_1 = 0$. But if (9.4.3)(b) holds, then $p(t_1) \geq 0$ and so if $x^*(t_1) =$ then $p(t_1)(x(t_1) - x^*(t_1)) = p(t_1)[x(t_1) - x_1] \geq 0$ because $x(t_1) \geq x_1$. Alternativel $x^*(t_1) > x_1$, then $p(t_1) = 0$, and the term is 0. Finally, if (9.4.3)(c) holds, then $p(t_1) =$ and the term is 0. In all cases, therefore, one has $D_u \geq 0$.

If H is strictly concave in (x, u), then the inequality $(**)$ is strict for $(x, u) \neq (x^*,$ and so $D_u > 0$ unless $x(t) = x^*(t)$ and $u(t) = u^*(t)$ for all t. Hence (x^*, u^*) is the uni solution to the problem.

Most of the control problems presented so far can be solved by using Mangasarian's suffic conditions. However, in many important economic models the Hamiltonian is not conca Arrow has suggested a weakening of this concavity condition. Define

$$\widehat{H}(t, x, p) = \max_{u \in U} H(t, x, u, p)$$

assuming that the maximum value is attained. The function $\widehat{H}(t, x, p)$ is called the **ma mized Hamiltonian**. Then one can show:

THEOREM 9.7.2 (ARROW'S SUFFICIENT CONDITIONS)

Suppose that $(x^*(t), u^*(t))$ is an admissible pair in the standard end constrained problem (9.4.1)–(9.4.3) that satisfies all the requirements in the maximum principle, with $p(t)$ as the adjoint function, and with $p_0 = 1$. Suppose further that

$$\widehat{H}(t, x, p(t)) \text{ is concave in } x \text{ for every } t \in [t_0, t_1]$$

Then $(x^*(t), u^*(t))$ solves the problem.

A proof and further discussion of this result is postponed to Section 10.1.

NOTE 2 Here is an important generalization of the theorem: Suppose the problem imposes the constraint that $x(t)$ belongs to a convex set $A(t)$ for all t. Suppose also that $x^*(t)$ is an interior point of $A(t)$ for every t. Then Theorem 9.7.2 is still valid, and $x \mapsto \widehat{H}(t, x, p(t))$ need only be concave for x in $A(t)$.

EXAMPLE 1 Consider the problem

$$\max \int_0^2 (u^2 - x)\,dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(2) \text{ free}, \quad 0 \le u \le 1$$

(a) Find the only possible solution candidate by using the maximum principle.

(b) Use Theorem 9.7.2 to prove that the pair found in (a) is optimal.

Solution: (a) The Hamiltonian with $p_0 = 1$ is $H(t, x, u, p) = u^2 - x + pu$. Because $H'_x = -1$, the differential equation for $p = p(t)$ becomes $\dot{p} = -H'_x = 1$. The solution of this equation with $p(2) = 0$ is $p(t) = t - 2$. According to the maximum condition (9.4.5), for each t in $[0, 2]$, an optimal control $u^*(t)$ must maximize $u^2 - x^*(t) + (t-2)u = -x^*(t) + u^2 + tu - 2u$ subject to $u \in [0, 1]$. The term $-x^*(t)$ is independent of u, so $u^*(t)$ must maximize $g(u) = u^2 + tu - 2u$ subject to $u \in [0, 1]$. Note that $g(u)$ is a strictly convex function, so its maximum cannot occur at an interior point of $[0, 1]$. At the end points, $g(0) = 0$ and $g(1) = t - 1$. Thus the maximum of g depends on the value of t. Clearly, if $t < 1$ the maximum of g occurs at $u = 0$, and if $t > 1$, the maximum occurs at $u = 1$. Thus the only possible optimal control which is continuous on the left at $t = 1$ is the bang-bang control

$$u^*(t) = \begin{cases} 0 & \text{if } t \in [0, 1] \\ 1 & \text{if } t \in (1, 2] \end{cases}$$

In the interval $[0, 1]$ one has $\dot{x}^*(t) = u^*(t) = 0$ and $x^*(0) = 0$, so $x^*(t) = 0$. In the interval $(1, 2]$ one has $\dot{x}^*(t) = u^*(t) = 1$ and $x^*(1) = 0$, so $x^*(t) = t - 1$. We have found the only possible pair that can solve the problem.

(b) The Hamiltonian with $p(t) = t - 2$ is $H(t, x, u, p) = u^2 - x + (t - 2)u$, which is convex in u. The maximized Hamiltonian is seen to be

$$\widehat{H}(t, x, p(t)) = \max_{u \in [0,1]} u^2 - x + (t-2)u = \begin{cases} -x & \text{if } t \in [0, 1] \\ -x + t - 1 & \text{if } t \in (1, 2] \end{cases}$$

For each t in $[0, 2]$, the maximized Hamiltonian is linear in x, hence concave. The conclusion follows from Theorem 9.7.2.

The following example illustrates an important aspect of Theorem 9.7.2: It suffices to show that the maximized Hamiltonian is concave as a function of x with $p(t)$ as the adjoint function derived from the maximum principle.

EXAMPLE 2 Use Theorem 9.7.2 to prove that for the problem

$$\max \int_0^1 3u\,dt, \quad \dot{x} = u^3, \quad x(0) = 0, \quad x(1) \le 0, \quad u \in [-2, \infty)$$

$u^*(t) = 1$ in $[0, 8/9]$ and $u^*(t) = -2$ in $(8/9, 1]$ is an optimal control with $p(t) \equiv -1$.

Solution: The Hamiltonian with $p(t) \equiv -1$ is $H(t, x, u, p) = 3u - u^3$, which concave in (x, u). (See Fig. 2.) However, the maximized Hamiltonian is $\widehat{H}(t, x, p($ $\max_{u \in [-2,\infty)}(3u - u^3) \equiv 2$, which *is* concave. Note that $p(t) \equiv -1$ satisfies $-\partial H^*/\partial x = 0$. Moreover, both $u^*(t) = 1$ and $u^*(t) = -2$ maximize $3u - u$ $u \in [-2, \infty)$ (see Fig. 2). Because $p(1) = -1$, the result in Note 9.4.2 implie $x^*(1) = 0$. The function $x^*(t)$ must satisfy the equation $\dot{x}^*(t) = (u^*(t))^3$ for each also have $x^*(0) = 0$ and $x^*(1) = 0$. One possibility is $x^*(t) = t$ in $[0, 8/9]$, with $u^*(t)$ and $x^*(t) = 8 - 8t$ in $(8/9, 1]$, with $u^*(t) = -2$. Because all the conditions in The 9.7.2 are satisfied, this is a solution. (But the solution is not unique. One could also for example, $x^*(t) = -8t$ in $[0, 1/9]$ with $u^*(t) = -2$, and $x^*(t) = t - 1$ in $(1/9, 1]$ $u^*(t) = 1$.)

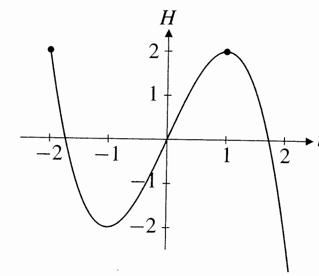

Figure 2

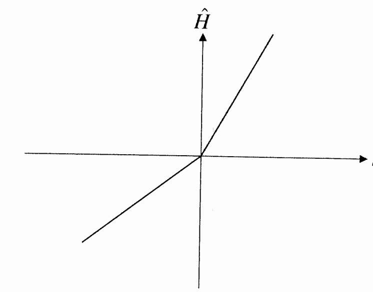

Figure 3

Our final example makes use of Note 2.

EXAMPLE 3 Consider the capital accumulation model of Example 9.6.3(b). Prove that the prop solution candidate is optimal.

Solution: The Hamiltonian is $H(t, k, s, p) = (1 - s)ak + pask = ak[1 + (p -$ This function is not concave in (k, s) for $p \ne 1$, because $H''_{kk}H''_{ss} - (H''_{ks})^2 = -a^2(p -$ The function $\widehat{H}$ defined in (4) is

$$\widehat{H}(t, k, p(t)) = ak \max_{s \in [0,1]} [1 + (p(t) - 1)s]$$

Given $\bar{t}$ defined as in the solution to Example 9.6.3(b), we found that for $t \in [0, \bar{t})$, adjoint variable is $p(t) > 1$. It follows that $\widehat{H}(t, k, p(t)) = ap(t)k$ for $k \ge 0$, w $\widehat{H}(t, k, p(t)) = ak$ for $k \le 0$. For $t \in (\bar{t}, T)$, however, the adjoint variable is $p(t) < 1$, it follows that $\widehat{H}(t, k, p(t)) = ak$ for $k \ge 0$, while $\widehat{H}(t, k, p(t)) = ap(t)k$ for $k \le 0$. tempting to suggest that because $\widehat{H}$ is linear in each case, $\widehat{H}$ concave in k. But the grap Fig. 3 shows that $\widehat{H}$ is convex, and not concave.

Define $A(t) = \{k : k \ge 0\}$. Certainly, the optimal $k^*(t)$ is positive for all t, so $k^*(t$ an interior point of $A(t)$ for every t. For $k \ge 0$,

$$\widehat{H}(t, k, p(t)) = \begin{cases} akp(t) & \text{if } p(t) > 1 \\ ak & \text{if } p(t) \le 1 \end{cases}$$

Thus for each t, $\widehat{H}(t, k, p(t))$ is concave as a function of $k \in A = [0, \infty)$. Accordin Theorem 9.7.2 and Note 2, the suggested candidate in Example 9.6.3(b) is optimal.

NOTE 3 To give a complete solution of an optimal control problem using the Mangasarian (or Arrow) sufficiency results, it is necessary to prove that there is a pair $(x^*(t), u^*(t))$ satisfying all the requirements. In problems where it is impossible to find explicit solutions for $x^*(t)$ and $u^*(t)$, this means that we must prove that there exist admissible solutions of the differential equations which are valid for the whole interval $[t_0, t_1]$. (This is almost never done in economics literature.)

NOTE 4 **(What to do if the Arrow condition fails)** If the maximized Hamiltonian is not concave, then the Mangasarian condition also fails. For the corresponding case in static optimization problems we turned next to the extreme value theorem, which promises that under certain conditions, there exists an optimal solution. In Section 10.4 we discuss analogous existence theorems for control problems.

PROBLEMS FOR SECTION 9.7

1. (a) Solve the control problem

$$\max \int_0^1 (100 - x - \tfrac{1}{2}u^2)\, dt, \quad \dot{x} = u, \; x(0) = x_0, \; x(1) = x_1, \; u \in (-\infty, \infty)$$

(b) Verify that $\partial V/\partial x_0 = p(0)$ and $\partial V/\partial x_1 = -p(1)$, where V is the optimal value function.

2. (a) Find the only possible solution to

$$\max \int_0^{10} (1-s)\sqrt{k}\, dt, \quad \dot{k} = s\sqrt{k}, \quad k(0) = 1, \quad k(10) \text{ free}, \quad s \in [0, 1]$$

(b) Use Theorem 9.7.2 to prove that the solution candidate in (a) is optimal.

3. Solve the problem (where T, α, and β are positive constants, $\alpha \neq 2\beta$)

$$\max \int_0^T e^{-\beta t}\sqrt{u}\, dt \text{ when } \dot{x}(t) = \alpha x(t) - u(t), \; x(0) = 1, \; x(T) = 0, \; u(t) \geq 0$$

What happens if the terminal condition $x(T) = 0$ is changed to $x(T) \geq 0$?

4. Let f be a C^1-function defined on a set A in $\mathbb{R}^n$, and let S be a convex set in the interior of A. Show that if $\mathbf{x}^0$ maximizes $f(\mathbf{x})$ in S, then $\nabla f(\mathbf{x}^0) \cdot (\mathbf{x}^0 - \mathbf{x}) \geq 0$ for all $\mathbf{x}$ in S. (*Hint:* Define the function $g(t) = f(t\mathbf{x} + (1-t)\mathbf{x}^0)$ for t in $[0, 1]$. Then $g(0) \geq g(t)$ for all t in $[0, 1]$.)

9.8 Variable Final Time

In the optimal control problems studied so far the time interval has been fixed. Yet for s
control problems in economics, the final time is a variable to be chosen optimally, along
the path $u(t)$, $t \in [t_0, t_1]$. One instance is the optimal extraction problem of Example 9.
in which it is natural to have the length of the extraction period as a variable (in addi
to the rate of extraction). Another important case is the minimal time problem in which
objective is to steer a system from its initial state to a desired state as quickly as possib

The **variable final time problem** considered here can be briefly formulated as foll
(note that the choice variables u and t_1 are indicated below the max instruction):

$$\max_{u, t_1} \int_{t_0}^{t_1} f(t, x, u)\, dt, \quad \dot{x} = g(t, x, u), \quad x(t_0) = x_0, \quad \begin{cases} \text{(a)} \; x(t_1) = x_1 \\ \text{(b)} \; x(t_1) \geq x_1 \\ \text{(c)} \; x(t_1) \text{ free} \end{cases}$$

The only difference from the standard end constrained problem is that t_1 can now be cho
Thus, the problem is to maximize the integral in (1) over all admissible control functi
$u(t)$ that, over the time interval $[t_0, t_1]$, bring the system from x_0 to a point satisfying
terminal conditions. In contrast to the previous problems, the admissible control func
can be defined on different time intervals.

Suppose $(x^*(t), u^*(t))$ is an optimal solution defined on $[t_0, t_1^*]$. Then the conditi
(9.4.5)–(9.4.7) in the maximum principle are still valid on the interval $[t_0, t_1^*]$, because
pair $(x^*(t), u^*(t))$ must be optimal for the corresponding fixed time problem with $t_1 =$
In fact, the result is this[6]:

THEOREM 9.8.1 (THE MAXIMUM PRINCIPLE WITH VARIABLE FINAL TIME)

Let $(x^*(t), u^*(t))$ be an admissible pair defined on $[t_0, t_1^*]$ which solves problem (1) with t_1 free ($t_1 \in (t_0, \infty)$). Then all the conditions in the maximum principle (Theorem 9.4.1) are satisfied on $[t_0, t_1^*]$, and, in addition,

$$H(t_1^*, x^*(t_1^*), u^*(t_1^*), p(t_1^*)) = 0$$

Compared with a fixed final time problem there is one additional unknown t_1^*. Fortunat
(2) is one extra condition.

One method for solving variable final time problems is first to solve the problem w
t_1 fixed for every $t_1 > t_0$. The optimal final time t_1^* must then maximize the opti
value function V as a function of t_1. According to (9.6.5), if V is differentiable, t
$\partial V/\partial t_1 = H^*(t_1) = H^*(t_1^*, x^*(t_1^*), u^*(t_1^*), p(t_1^*))$. Thus, condition (2) is precisely
expected.

[6] For a proof see Hestenes (1966).

NOTE 1 **(A common misunderstanding)** Concavity of the Hamiltonian in (x, u) is *not* sufficient for optimality when t_1 is free. For sufficiency results when the final time is variable, see Seierstad and Sydsæter (1987), Sections 2.9 and 6.7.

EXAMPLE 1 Consider Problem II in Example 9.1.2 for the special case when the cost function $C = C(t, u)$ is independent of x and convex in u, with $C''_{uu} > 0$. Thus, the problem is

$$\max_{u,T} \int_0^T [q(t)u(t) - C(t, u(t))]e^{-rt}\, dt, \quad \dot{x}(t) = -u(t), \quad x(0) = K, \quad x(T) \geq 0, \quad u(t) \geq 0$$

What does the maximum principle imply for this problem?

Solution: Suppose $(x^*(t), u^*(t))$, defined on $[0, T^*]$, solves this problem. The Hamiltonian with $p_0 = 1$ is $H(t, x, u, p) = [q(t)u - C(t, u)]e^{-rt} + p(-u)$, and the maximum principle states that there exists a continuous function $p(t)$ such that

$$u^*(t) \text{ maximizes } \big[q(t)u - C(t, u)\big]e^{-rt} - p(t)u \text{ subject to } u \geq 0 \tag{i}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = 0, \quad p(T^*) \geq 0 \; (p(T^*) = 0 \text{ if } x^*(T^*) > 0) \tag{ii}$$

$$\big[q(T^*)u^*(T^*) - C(T^*, u^*(T^*))\big]e^{-rT^*} = p(T^*)u^*(T^*) \tag{iii}$$

Because $p(t)$ is continuous, (ii) implies that $p(t) = \bar{p} \geq 0$, where $\bar{p}$ is a constant.

Put $g(u) = \big[q(t)u - C(t, u)\big]e^{-rt} - \bar{p}u$. Because $C(t, u)$ is convex in u and the other terms are linear in u, the function $g(u)$ is concave. According to (i), $u^*(t)$ maximizes $g(u)$ subject to $u \geq 0$. If $u^*(t) = 0$, then $g'(u^*(t)) = g'(0) \leq 0$. If $u^*(t) > 0$, then $g'(u^*(t)) = 0$. Therefore (i) implies that

$$\big[q(t) - C'_u(t, u^*(t))\big]e^{-rt} - \bar{p} \leq 0 \quad (= 0 \text{ if } u^*(t) > 0) \tag{iv}$$

Because g is concave, this condition is also sufficient for (i) to hold.

At any time t where $u^*(t) > 0$, equation (iv) implies that

$$q(t) - C'_u(t, u^*(t)) = \bar{p}e^{rt} \tag{v}$$

The left hand side is the marginal profit from extraction, $\partial\pi/\partial u$. Therefore, whenever it is optimal to have positive extraction, we have the following rule that was discovered by Hotelling (1931).

(HOTELLING'S RULE)

Positive optimal extraction requires the marginal profit to increase exponentially at a rate equal to the discount factor r. (3)

Putting $t = T^*$ in (v), and using (iii), we deduce that if $u^*(T^*) > 0$, then

$$C'_u(T^*, u^*(T^*)) = \frac{C(T^*, u^*(T^*))}{u^*(T^*)} \tag{vi}$$

Terminate extraction at a time when the marginal cost of extraction is equal to average cost!

If the problem has a solution with $u^*(t) > 0$, then (v) and (vi) both hold. If $C(T^*, 0) > 0$, then $u^*(T^*) > 0$, because $u^*(T^*) = 0$ contradicts (iii).

We have not proved that there exists an optimal solution. (For a more thorough discussion of this problem, see Seierstad and Sydsæter (1987), Section 2.9, Example 11.)

PROBLEMS FOR SECTION 9.8

1. Find the only possible solution to the following variable final time problems:

(a) $\max_{u,T} \int_0^T (x - t^3 - \frac{1}{2}u^2)\, dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(T) \text{ free}, \quad u \in \mathbb{R}$

(b) $\max_{u,T} \int_0^T (-9 - \frac{1}{4}u^2)\, dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(T) = 16, \quad u \in \mathbb{R}$

2. Solve problem 9.4.7 with T free.

3. Consider the optimal extraction problem over a fixed extraction period,

$$\max_{u(t)\geq 0} \int_0^T \big[ae^{\alpha t}u(t) - (u(t))^2 e^{\beta t} - c\big]e^{-rt}\, dt, \quad \dot{x}(t) = -u(t), \quad x(0) = K, \quad x(T) = 0$$

Here $x(t)$ and $u(t)$ have the same interpretation as in Example 1, with $q(t) = ae^{\alpha t}$ as the world market price, and $(u(t))^2 e^{\beta t} - c$ as the cost of extraction, with $c > 0$.

(a) One can prove that if $u^*(t)$ is optimal, then $u^*(t) > 0$ for all t. (You are not required to show this.) The adjoint function is a constant $\bar{p}$. Find $u^*(t)$ expressed in terms of $\bar{p}$. Then find $x^*(t)$ and $\bar{p}$ for the case $\alpha = \beta = 0$, $r \neq 0$.

(b) Let $T > 0$ be subject to choice (keeping the assumptions $\alpha = \beta = 0$, $r \neq 0$). Prove that the necessary conditions lead to an equation for determining the optimal T^* which has a unique positive solution. Assume that $\max_u(au - u^2 - c) > 0$, i.e. $a^2 > 4c$.

9.9 Current Value Formulations

Many control problems in economics literature have the following structure:

$$\max_{u\in U\subseteq\mathbb{R}} \int_{t_0}^{t_1} f(t, x, u)e^{-rt}\, dt, \quad \dot{x} = g(t, x, u), \quad x(t_0) = x_0, \quad \begin{cases} \text{(a)}\ x(t_1) = x_1 \\ \text{(b)}\ x(t_1) \geq x_1 \\ \text{(c)}\ x(t_1) \text{ free} \end{cases} \tag{1}$$

The new feature is the explicit appearance of the discount factor e^{-rt}. For such problems it is often convenient to formulate the maximum principle in a slightly different form. The ordinary Hamiltonian is $H = p_0 f(t, x, u)e^{-rt} + pg(t, x, u)$. Multiply it by e^{rt} to obtain the **current value Hamiltonian** $H^c = He^{rt} = p_0 f(t, x, u) + e^{rt} pg(t, x, u)$. Introducing $\lambda = e^{rt} p$ as the **current value shadow price** for the problem, one can write H^c in the form (where we put $p_0 = \lambda_0$)

$$H^c(t, x, u, \lambda) = \lambda_0 f(t, x, u) + \lambda g(t, x, u) \tag{2}$$

Note that if $\lambda = e^{rt} p$, then $\dot\lambda = re^{rt} p + e^{rt}\dot p = r\lambda + e^{rt}\dot p$ and so $\dot p = e^{-rt}(\dot\lambda - r\lambda)$. Also, $H^c = He^{rt}$ implies that $\partial H^c/\partial x = e^{rt}(\partial H/\partial x)$. So $\dot p = -\partial H/\partial x$ takes the form $\dot\lambda - r\lambda = -\partial H^c/\partial x$. In fact, one can prove the following:

THEOREM 9.9.1 (THE MAXIMUM PRINCIPLE. CURRENT VALUE FORMULATION)

Suppose that the admissible pair $(x^*(t), u^*(t))$ solves problem (1) and let H^c be the current value Hamiltonian (2). Then there exists a continuous function $\lambda(t)$ and a number λ_0, either 0 or 1, such that for all $t \in [t_0, t_1]$ we have $(\lambda_0, \lambda(t)) \neq (0, 0)$, and:

(A) $u = u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t))$ for $u \in U$ (3)

(B) $\dot\lambda(t) - r\lambda(t) = -\dfrac{\partial H^c(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$ (4)

(C) Finally, the transversality conditions are:

(a′) $\lambda(t_1)$ no condition

(b′) $\lambda(t_1) \geq 0$ ($\lambda(t_1) = 0$ if $x^*(t_1) > x_1$) (5)

(c′) $\lambda(t_1) = 0$

The Mangasarian and Arrow sufficiency results from Section 9.7 have immediate extensions to problem (1). The conditions in Theorem 9.9.1 are sufficient for optimality if $\lambda_0 = 1$ and

$$H^c(t, x, u, \lambda(t)) \text{ is concave in } (x, u) \qquad \text{(Mangasarian)} \tag{6}$$

or (more generally)

$$\widehat{H}^c(t, x, \lambda(t)) = \max_{u \in U} H^c(t, x, u, \lambda(t)) \text{ is concave in } x \qquad \text{(Arrow)} \tag{7}$$

EXAMPLE 1 Solve the following problem using the current value formulation.

$$\max_{u \geq 0} \int_0^{20} (4K - u^2)e^{-0.25t}\, dt, \quad \dot K = -0.25K + u, \quad K(0) = K_0, \quad K(20) \text{ is free}$$

Economic interpretation: $K(t)$ is the value of a machine, $u(t)$ is the repair effort, $4K - u^2$ is the instantaneous net profit at time t, and $e^{-0.25t}$ is the discount factor.

Solution: The current value Hamiltonian is $H^c = 4K - u^2 + \lambda(-0.25K + u)$ (w $\lambda_0 = 1$), and so $\partial H^c/\partial u = -2u + \lambda$ and $\partial H^c/\partial K = 4 - 0.25\lambda$. Assuming that $u^*(t)$ > (we try this assumption in the following), $\partial(H^c)^*/\partial u = 0$, so $u^*(t) = 0.5\lambda(t)$. The adjo function λ satisfies

$$\dot\lambda - 0.25\lambda = -\partial(H^c)^*/\partial K = -4 + 0.25\lambda, \qquad \lambda(20) = 0$$

It follows that

$$\lambda(t) = 8(1 - e^{0.5t-10}) \quad \text{and} \quad u^*(t) = 0.5\lambda = 4(1 - e^{0.5t-10})$$

Note that $u^*(t) > 0$ in $[0, 20)$. The time path of $K^*(t)$ is found from $\dot K^* = -0.25K^* + u^*$ $-0.25K^* + 4(1 - e^{0.5t-10})$. Solving this linear differential equation with $K^*(0) = K_0$, get

$$K^*(t) = (K_0 - 16 + \tfrac{16}{3}e^{-10})e^{-0.25t} + 16 - \tfrac{16}{3}e^{0.5t-10}$$

Here $H^c = (4K - u^2) + \lambda(-0.25K + u)$ is concave in (K, u), so we have found t complete solution.

Note that the pair $(K^*(t), \lambda(t))$ must satisfy the system

$$\dot\lambda = 0.5\lambda - 4, \quad \lambda(20) = 0$$
$$\dot K = -0.25K + 0.5\lambda, \quad K(0) = K_0$$

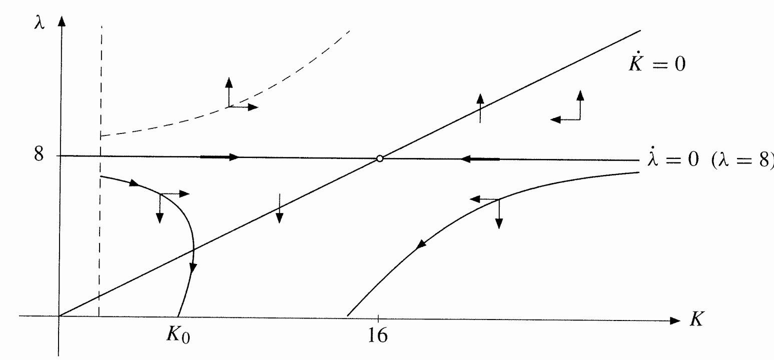

Figure 1 Phase diagram for Example 1.

Figure 1 shows a phase diagram for this system. When $K_0 < 16$ as in the figure, t curve drawn with a solid line is consistent with the indicated arrows. Initially the valu of the machine increases, and the repair effort is reduced. Then, after the curve hits t line $\dot K = 0$, the value decreases and the repair effort is reduced till it eventually is 0. T dotted curve is also consistent with the arrows, but there is no way the curve can satis $\lambda(20) = 0$—the required repair effort is too high to lead to an optimal solution. (When λ large, so is $u = 0.5\lambda$, and the integrand $4K - u^2$ becomes large negative.)

The diagrammatic analysis related to Fig. 1 in the last example is in a way superfluous sinc the solution has already been completely specified. But it is very useful in some problen where explicit solutions are unobtainable. See Section 9.12.

PROBLEMS FOR SECTION 9.9

1. Find the solution to the following problem using the current value formulation:

$$\max_{u(t)\in\mathbb{R}} \int_0^T (-x^2 - \tfrac{1}{2}u^2)e^{-2t}\,dt, \quad \dot{x} = x + u, \quad x(0) = 1, \quad x(T) \text{ free}$$

2. Find the solution of Problem 9.4.6 using the current value formulation.

3. Find the solution of Problem 9.5.3 using the current value formulation.

9.10 Scrap Values

In some economic optimization problems it is natural to include within the optimality criterion an additional function representing the value or utility associated with the terminal state. This gives the typical problem

$$\max_{u(t)\in U} \left\{ \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt + S(x(t_1)) \right\}, \quad \dot{x}(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0 \quad (1)$$

The function $S(x)$ is called a **scrap value function**, and we shall assume that it is C^1.

Suppose that $(x^*(t), u^*(t))$ solves this problem (with no additional condition on $x(t_1)$). Then, in particular, that pair is a solution to the corresponding problem with fixed terminal point $(t_1, x^*(t_1))$. For all admissible pairs in this new problem, the scrap value function $S(x^*(t_1))$ is constant. But then $(x^*(t), u^*(t))$ must satisfy all the conditions in the maximum principle, except the transversality conditions. Then the correct transversality condition for "normal" problems is

$$p(t_1) = S'(x^*(t_1)) \quad (2)$$

This is quite natural if we use the general economic interpretation explained in Section 9.6. In fact, if $x(t)$ denotes the capital stock of a firm, then according to (2), the shadow price of capital at the end of the planning period is equal to the marginal scrap value of the terminal stock.

NOTE 1 If $S(x) \equiv 0$, then (2) reduces to $p(t_1) = 0$, which is precisely as expected in a problem with no restrictions on $x(t_1)$.

One way to show that (2) is the correct transversality condition involves transforming problem (1) into one studied before. Indeed, suppose that $(x(t), u(t))$ is an admissible pair for the problem (1). Then $\frac{d}{dt}S(x(t)) = S'(x(t))\dot{x}(t) = S'(x(t))g(t, x(t), u(t))$. So by integration,

$$S(x(t_1)) - S(x(t_0)) = \int_{t_0}^{t_1} S'(x(t))g(t, x(t), u(t))\,dt$$

Here $S(x(t_0)) = S(x_0)$ is a constant, so if the objective function in (1) is replaced by

$$\int_{t_0}^{t_1} [f(t, x(t), u(t)) + S'(x(t))g(t, x(t), u(t))]\,dt \quad (3)$$

then the new problem is of a type studied previously with no scrap value, still with $x(t_1)$ free. Let the Hamiltonian for this new problem be $H_1 = f + S'(x)g + qg = f + (q + S'(x))g$, with adjoint variable q. An optimal pair $(x^*(t), u^*(t))$ for this problem must have the properties:

(a) $u^*(t)$ maximizes $H_1(t, x^*(t), u, q(t))$ for $u \in U$

(b) $\dot{q}(t) = -\partial H_1^*/\partial x, \quad q(t_1) = 0$

Define $p(t) = q(t) + S'(x^*(t))$. Problem 7 asks you to prove that, if $H = f + pg$ is the ordinary Hamiltonian associated with problem (1), then $u^*(t)$ maximizes $H(x^*(t), u, p(t))$ for $u \in U$ and $\dot{p}(t) = -\partial H^*/\partial x$, with $p(t_1) = 0$.

Appropriate concavity conditions again ensure optimality as shown in the next theorem.

THEOREM 9.10.1 (SUFFICIENT CONDITIONS WITH SCRAP VALUE)

Suppose $(x^*(t), u^*(t))$ is an admissible pair for the scrap-value problem (1) and suppose there exists a continuous function $p(t)$ such that for all t in $[t_0, t_1]$,

(A) $u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$ w.r.t. $u \in U$

(B) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)), \quad p(t_1) = S'(x^*(t_1))$

(C) $H(t, x, u, p(t))$ is concave in (x, u) and $S(x)$ is concave

Then $(x^*(t), u^*(t))$ solves the problem.

Proof: Suppose that $(x, u) = (x(t), u(t))$ is an arbitrary admissible pair. We must show that

$$D_u = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))\,dt + S(x^*(t_1)) - \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt - S(x(t_1)) \ge 0$$

Because $S(x)$ is C^1 and concave, $S(x^*(t_1)) - S(x(t_1)) \ge S'(x(t_1))[x^*(t_1) - x(t_1)]$. Combining this with the inequality $\int_{t_0}^{t_1}(f^* - f)\,dt \ge p(t_1)(x(t_1 - x^*(t_1))$ that was derived in the proof of the Theorem 9.7.1, we get

$$D_u \ge [p(t_1) - S'(x^*(t_1))](x(t_1) - x^*(t_1)) = 0$$

where the last equality follows from (B). So $D_u \ge 0$. ∎

NOTE 2 The theorem still holds if the concavity of H in (x, u) is replaced by the Arrow condition requiring $\widehat{H}(t, x, p(t))$ to exist and be concave in x.

EXAMPLE 1 Solve the problem

$$\max_{u \in (-\infty,\infty)} \left\{ \int_0^1 -\tfrac{1}{2}u^2\,dt + \sqrt{x(1)} \right\}, \quad \dot{x} = x + u, \quad x(0) = 0, \quad x(1) \text{ free}$$

Solution: We have $H = -\frac{1}{2}u^2 + p(x+u)$ and $S(x) = \sqrt{x} = x^{1/2}$. Hence $H'_u = -u + p$ and $H'_x = p$. Since $u \in (-\infty, \infty)$, $H'_u = 0$, which gives $u = p$, and we have the following differential equations, $\dot{x} = x + u = x + p$, $\dot{p} = -H'_x = -p$. The latter has the solution $p(t) = Ae^{-t}$. Then $\dot{x} = x + p = x + Ae^{-t}$, and this linear differential equation has the solution $x = Be^t - \frac{1}{2}Ae^{-t}$, where the constant B is determined by $x(0) = B - \frac{1}{2}A = 0$. Hence, $B = \frac{1}{2}A$, so that $x(t) = \frac{1}{2}A(e^t - e^{-t})$. The constant A is determined by the transversality condition $p(1) = Ae^{-1} = S'(x(1)) = \frac{1}{2}(x(1))^{-1/2} = \frac{1}{2}[\frac{1}{2}A(e^1 - e^{-1})]^{-1/2}$. Solving for A we find $A = e[2(e^2 - 1)]^{-1/3}$. Thus we have the following candidate for an optimal solution:

$$u(t) = p(t) = Ae^{-t}, \quad x(t) = \tfrac{1}{2}A(e^t - e^{-t}), \quad A = e[2(e^2 - 1)]^{-1/3}$$

Because the Hamiltonian is concave in (x, u) and the scrap value function is concave in x, this is the solution.

Current Value Formulation

Many control problems in economics literature have the following structure:

$$\max_{u \in U \subseteq \mathbb{R}} \left\{ \int_{t_0}^{t_1} f(t, x, u)e^{-rt}\,dt + S(x(t_1))e^{-rt_1} \right\}, \quad \dot{x} = g(t, x, u), \quad x(t_0) = x_0 \tag{4}$$

$$\text{(a) } x(t_1) = x_1 \quad \text{(b) } x(t_1) \geq x_1 \quad \text{or} \quad \text{(c) } x(t_1) \text{ free} \tag{5}$$

The new features compared to problem (1) are the discount factor (or interest rate) r, and the reintroduction of the alternative terminal conditions in the standard problem. (If $x(t_1)$ is fixed as in 5(a), the scrap value function is a constant.)

The current value Hamiltonian for the problem is

$$H^c(t, x, u, \lambda) = \lambda_0 f(t, x, u) + \lambda g(t, x, u) \tag{6}$$

and the correct necessary conditions are as follows:

THEOREM 9.10.2 (CURRENT VALUE MAXIMUM PRINCIPLE WITH SCRAP VALUE)

Suppose that the admissible pair $(x^*(t), u^*(t))$ solves problem (4)–(5). Then there exists a continuous function $\lambda(t)$ and a number λ_0, either 0 or 1, such that for all $t \in [t_0, t_1]$ we have $(\lambda_0, \lambda(t)) \neq (0, 0)$, and:

(A) $u = u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t))$ for $u \in U$

(B) $\dot{\lambda}(t) - r\lambda(t) = -\dfrac{\partial H^c(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$

(C) Finally, the transversality conditions are:

(a′) $\lambda(t_1)$ no condition

(b′) $\lambda(t_1) \geq \lambda_0 \dfrac{\partial S(x^*(t_1))}{\partial x}$ (with = if $x^*(t_1) > x_1$)

(c′) $\lambda(t_1) = \lambda_0 \dfrac{\partial S(x^*(t_1))}{\partial x}$

The following sufficiency result is a straightforward extension of Theorem 9.10.1:

THEOREM 9.10.3 (SUFFICIENT CONDITIONS)

The conditions in Theorem 9.10.2 with $\lambda_0 = 1$ are sufficient if U is convex, $H^c(t, x, u, \lambda(t))$ is concave in (x, u), and $S(x)$ is concave in x.

EXAMPLE 2 Consider the following problem:

$$\max\left\{ \int_0^T (x - u^2)e^{-0.1t}\,dt + ax(T)e^{-0.1T} \right\}$$

$$\dot{x} = -0.4x + u, \quad x(0) = 1, \quad x(T) \text{ is free}, \quad u \in \mathbb{R}$$

where a is a positive constant. Solve the problem.

Solution: The current value Hamiltonian, with $\lambda_0 = 1$, is $H^c(t, x, u, \lambda) = x - u^2$ $\lambda(-0.4x + u)$, which is concave in (x, u). Moreover, $S(x) = ax$ is linear, and hen concave in x. The conditions in the maximum principle are therefore sufficient. Becau H^c is concave in u and $u \in \mathbb{R}$, the maximum of the Hamiltonian occurs when

$$\frac{\partial H^c(t, x^*(t), u^*(t), \lambda(t))}{\partial u} = -2u^*(t) + \lambda(t) = 0 \quad ($$

Next, the differential equation for λ is

$$\dot{\lambda}(t) - 0.1\lambda(t) = -\frac{\partial H^c}{\partial x} = -1 + 0.4\lambda(t) \quad ($$

Because $x(T)$ is free and $S(x) = ax$, condition (C)(c′) yields

$$\lambda(T) = a \tag{iii}$$

By integrating the linear differential equation (ii), using (iii), we obtain

$$\lambda(t) = (a-2)e^{-0.5(T-t)} + 2$$

From (i), $u^*(t) = \frac{1}{2}\lambda(t)$. Because $x^*(t)$ satisfies the linear differential equation $\dot{x}^* = -0.4x^* + u^* = -0.4x^* + \frac{1}{2}(a-2)e^{-0.5(T-t)} + 1$, with $x^*(0) = 1$, one has

$$x^*(t) = \tfrac{5}{2} + \tfrac{5}{9}(a-2)e^{-0.5(T-t)} - \left(\tfrac{3}{2} + \tfrac{5}{9}(a-2)e^{-0.5T}\right)e^{-0.4t}$$

All the sufficient conditions are satisfied, so this is the solution.

EXAMPLE 3 **(Optimal Feeding of Fish)** Let $x(t)$ be the weight of a fish at time t and let $P(t,x)$ be the price per kilogram of a fish whose weight is x at time t. Furthermore, let $u(t)$ denote the amount of fish food per unit of time measured as a proportion of the weight of a fish, and let c be a cost of a kilogram of fish food. If the interest rate is r, then the present value of the profit from feeding the fish and then catching it at the fixed time T is

$$x(T)P(T, x(T))e^{-rT} - \int_0^T cx(t)u(t)e^{-rt}\,dt \tag{i}$$

Suppose that

$$\dot{x}(t) = x(t)g(t, u(t)), \quad x(0) = x_0 > 0 \tag{ii}$$

so that the proportional rate of growth in the weight of the fish is a known function $g(t, u(t))$. Assuming that $u(t) \geq 0$, the natural problem is to find the feeding function $u^*(t)$ and the corresponding weight function $x^*(t)$ that maximize (i) subject to the constraint (ii) and $u(t) \geq 0$.

(a) Write down necessary conditions for $(x^*(t), u^*(t))$, with corresponding adjoint function $\lambda(t)$, to solve the problem. Deduce an equation that $u^*(t)$ must satisfy if $u^*(t) > 0$.

(b) Suppose $c(t) = c$, $P(t,x) = a_0 + a_1 x$, and $g(t,u) = a - be^{st}/u$, where all the constants are positive, with $s > r$. Characterize the only possible solution.

Solution: (a) The current value Hamiltonian is $H^c(t,x,u,\lambda) = -cxu + \lambda x g(t,u)$, and the scrap value function is $S(x) = xP(t,x)$. Thus $\partial H^c/\partial x = -cu + \lambda g(t,u)$, $\partial H^c/\partial u = x(-c + \lambda g'_u(t,u))$, and $S'_x(t,x) = P(t,x) + xP'_x(t,x)$.

According to the maximum principle, there exists a continuous function $\lambda(t)$ such that

$$u^*(t) \text{ maximizes } x^*(t)(-cu + \lambda(t)g(t,u)) \text{ for } u \geq 0 \tag{iii}$$

and

$$\dot{\lambda}(t) - r\lambda(t) = -\frac{\partial (H^c)^*}{\partial x} = cu^*(t) - \lambda(t)g(t, u^*(t)) \tag{iv}$$

Furthermore, condition (C)(c′) takes the form

$$\lambda(T) = P(T, x^*(T)) + x^*(T)P'_x(T, x^*(T))$$

From (iii) it follows that if $u^*(t) > 0$, then $\partial(H^c)^*/\partial u = 0$. If $x^*(t)$ is not 0 then

$$\lambda(t)g'_u(t, u^*(t)) = c$$

(b) We have $g'_u(t,u) = be^{st}/u^2$, so (vi) yields $\lambda(t)be^{st}/(u^*(t))^2 = c$. Then $\lambda(t) > 0$, with $u^*(t) > 0$, we obtain

$$u^*(t) = \sqrt{b/c}\, e^{\frac{1}{2}st}(\lambda(t))^{1/2}$$

Equation (iv) is now $\dot{\lambda}(t) - r\lambda(t) = cu^*(t) - \lambda(t)[a - be^{st}/u^*(t)]$, which reduces to

$$\dot{\lambda}(t) = (r-a)\lambda(t) + 2\sqrt{bc}\, e^{\frac{1}{2}st}(\lambda(t))^{1/2}$$

Finally, (v) reduces to

$$\lambda(T) = a_0 + a_1 x^*(T) + a_1 x^*(T) = a_0 + 2a_1 x^*(T)$$

The standard trick for solving the Bernoulli equation (vii) is to introduce a new variab defined by $z = \lambda^{1/2}$. (See (5.6.2).) Then $\lambda = z^2$, so $\dot{\lambda} = 2z\dot{z}$, and (vii) yields

$$2z\dot{z} = (r-a)z^2 + 2\sqrt{bc}\,e^{\frac{1}{2}st}z, \quad \text{or} \quad \dot{z} = \tfrac{1}{2}(r-a)z + \sqrt{bc}\,e^{\frac{1}{2}st}$$

According to (5.4.4) this has the solution

$$z = Ae^{\frac{1}{2}(r-a)t} + \sqrt{bc}\,e^{\frac{1}{2}(r-a)t}\int e^{\frac{1}{2}(s-r+a)t}\,dt = Ae^{\frac{1}{2}(r-a)t} + \frac{2\sqrt{bc}}{s-r+a}e^{\frac{1}{2}st}$$

where A is a constant. Since $u^*(t) = \sqrt{b/c}e^{\frac{1}{2}st}z$, we get

$$u^*(t) = A\sqrt{b/c}\,e^{\frac{1}{2}(s+r-a)t} + \frac{2b}{s-r+a}e^{st}$$

Inserting $u^*(t)$ into (ii) yields a separable differential equation for $x^*(t)$, with a uni solution satisfying $x^*(0) = x_0$. The constant A is finally determined by equation (viii).

PROBLEMS FOR SECTION 9.10

1. Find the solution to the control problem

$$\max\left\{\int_0^1 (1 - tu - u^2)\,dt + 2x(1) + 3\right\}, \quad \dot{x} = u, \quad x(0) = 1, \quad u \in (-\infty, \infty$$

2. In a study of savings and inheritance, Atkinson (1971) considers the problem

$$\max\left\{\int_0^T U(rA(t) + w - u(t))e^{-\rho t}\,dt + e^{-\rho T}\varphi(A(T))\right\}, \quad \dot{A} = u, \quad A(0) =$$

An economic interpretation is given in Example 8.5.3, except that the objective functi now includes an extra term which measures the individual's discounted benefit fr bequeathing $A(T)$. Suppose that $\varphi' > 0$, $\varphi'' < 0$. Give a set of sufficient conditi for the solution of this problem.

3. Solve the following control problem from economic growth theory:

$$\max\left\{\int_0^{10}(1-s)\sqrt{k}\,dt + 10\sqrt{k(10)}\right\}, \qquad \dot{k} = s\sqrt{k}, \quad k(0) = 1, \quad s \in [0, 1]$$

where $k = k(t)$ is the capital stock, and $s = s(t)$ is the savings ratio. (See Problem 9.7.2.)

4. (a) Solve the problem

$$\max\left\{\int_0^{1}(x-u)\,dt + \tfrac{1}{2}x(1)\right\}, \quad \dot{x} = u, \quad x(0) = \tfrac{1}{2}, \quad x(1) \text{ free}, \quad u \in [0, 1]$$

(b) Solve the problem with the objective function $\int_0^1 (x-u)\,dt - \frac{1}{4}(x(1)-2)^2$.

5. Consider the problem:

$$\max\left\{\int_0^{T} -u^2\,dt - x(T)^2\right\}, \qquad \dot{x} = -x + u, \quad x(0) = x_0, \quad u \in \mathbb{R}$$

(a) Solve the problem using Theorem 9.10.3.

(b) Compute the optimal value function, $V(x_0, T)$, and show that $\partial V/\partial x_0 = p(0)$ and $\partial V/\partial T = H^*(T)$.

6. Solve the following problem using the current value formulation

$$\max_{u\in\mathbb{R}}\left\{\int_0^{T} -e^{-rt}(x-u)^2\,dt - e^{-rT}x(T)^2\right\} \text{ s.t. } \dot{x} = u - x + a, \quad x(0) = 0, \quad x(T) \text{ free}$$

The constants r, a, and T are all positive.

7. Consider the control problem

$$\max \int_{t_0}^{t_1} [f(t,x,u) + S'(x)g(t,x,u)]\,dt, \quad \dot{x} = g(t,x,u), \quad x(t_0) = x_0, \quad u \in U$$

(See (3).) Let the Hamiltonian be $H_1 = f + S'(x)g + qg = f + (q + S'(x))g$, with q as the adjoint variable. Then an optimal pair (x^*, u^*) for this problem must satisfy conditions (a) and (b) above Theorem 9.10.1. Define $p = q + S'(x^*)$ and let $H = f + pg$. Prove that properties (a) and (b) imply that u^* maximizes $H(t, x^*, u, p)$ for $u \in U$, while $\dot{p} = -\partial H^*/\partial x$, with $p(t_1) = S'(x^*(t_1))$. Thus conditions (A)–(C) in Theorem 9.10.1 are satisfied.

9.11 Infinite Horizon

Most of the optimal growth models appearing in literature have an infinite time hor
The Nobel laureate Ragnar Frisch (1970) has the following to say about infinite ho
growth models:

> Questions of convergence under an infinite time horizon will depend so much on epsi
> refinements in the system of assumptions—and on the infinite constancy of these refinements-
> we are humanly speaking absolutely certain of getting infinite time horizon results which ha
> relevance to concrete reality. And in particular we are absolutely certain of getting irrelevant r
> if such epsilontic exercises are made under the assumption of a constant technology. "In the lor
> we are all dead". These words by Keynes ought to be engraved in marble and put on the desk
> epsilontologists in growth theory under an infinite horizon.

Clearly, choosing an infinite horizon makes sense in economic models only if the di
future has no significant influence on the optimal path for the near future in which w
most interested. Nevertheless, the infinite horizon assumption often does simplify forr
and conclusions, though at the expense of some new mathematical problems that ne
be sorted out.

A typical infinite horizon optimal control problem in economics literature take
following form:

$$\max \int_{t_0}^{\infty} f(t, x(t), u(t))e^{-rt}\,dt\,, \quad \dot{x}(t) = g(t, x(t), u(t)), \; x(t_0) = x_0, \; u(t) \in U$$

Often no condition is placed on $x(t)$ as $t \to \infty$, but many problems do impose the const

$$\lim_{t\to\infty} x(t) \geq x_1 \qquad (x_1 \text{ is a fixed number})$$

The pair $(x(t), u(t))$ is *admissible* if it satisfies $\dot{x}(t) = g(t, x(t), u(t))$, $x(t_0) =$
$u(t) \in U$, along with (2) when this is imposed. Suppose the integral (1) converges wher
the pair $(x(t), u(t))$ is admissible. For example, the integral will converge for al
missible $(x(t), u(t))$ if r is a positive constant, and if there exists a number M such
$|f(t, x, u)| \leq M$ for all (x, u).

One can then show (Halkin (1974)) that all the necessary conditions in the maxi
principle hold, except the transversality conditions. With no transversality condition w
too many solution candidates.

NOTE 1 It is tempting to assume that all results for finite horizon problems can be ca
over in a simple way to the infinite horizon case. This is wrong. For example, in a f
horizon problem with $x(t_1)$ free, the transversality condition is $p(t_1) = 0$. However,
no terminal condition, the "natural" transversality condition, $p(t) \to 0$ as $t \to \infty$, i
correct. A well known counterexample is due to Halkin (1974). This example also sl
that the condition $p(t)x(t) \to 0$ is *not* a necessary condition for optimality, contrary
widespread belief in economic literature, including some popular textbooks.

However, in economic models with $x(\infty)$ free, it is in most cases a sensible wor
hypothesis that $p(t)$ does tend to 0 as t tends to ∞. But ultimately, this must be confir

Because of the presence of the discount factor e^{-rt} in the problem above, it is convenient to use the current value formulation with the current value Hamiltonian

$$H^c(t, x, u, \lambda) = \lambda_0 f(t, x, u) + \lambda g(t, x, u)$$

and with λ as the current value shadow price.

THEOREM 9.11.1 (SUFFICIENT CONDITIONS WITH AN INFINITE HORIZON)

Suppose that an admissible pair $(x^*(t), u^*(t))$ for problem (1), with or without terminal condition (2), satisfies the following conditions for some $\lambda(t)$ for all $t \geq t_0$, with $\lambda_0 = 1$:

(a) $u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t))$ w.r.t. $u \in U$

(b) $\dot{\lambda}(t) - r\lambda = -\partial H^c(t, x^*(t), u^*(t), \lambda(t))/\partial x$

(c) $H^c(t, x, u, \lambda(t))$ is concave w.r.t. (x, u)

(d) $\lim\limits_{t\to\infty} \lambda(t)e^{-rt}[x(t) - x^*(t)] \geq 0$ for all admissible $x(t)$

Then $(x^*(t), u^*(t))$ is optimal.

Proof: For any admissible pair $(x(t), u(t))$ and for all $t \geq t_0$, define

$$D_u(t) = \int_{t_0}^{t} f(\tau, x^*(\tau), u^*(\tau))e^{-r\tau}\,d\tau - \int_{t_0}^{t} f(\tau, x(\tau), u(\tau))e^{-r\tau}\,d\tau = \int_{t_0}^{t} (f^* - f)e^{-r\tau}\,d\tau$$

in simplified notation. Now, $f^* = (H^c)^* - \lambda g^* = (H^c)^* - \lambda\dot{x}^*$ and $f = H^c - \lambda\dot{x}$, so

$$D_u(t) = \int_{t_0}^{t} [(H^c)^* - H^c]e^{-r\tau}\,d\tau + \int_{t_0}^{t} \lambda e^{-r\tau}(\dot{x} - \dot{x}^*)\,d\tau$$

By concavity of H^c, one has

$$(H^c)^* - H^c \geq \frac{\partial (H^c)^*}{\partial x}(x - x^*) + \frac{\partial (H^c)^*}{\partial u}(u^* - u)$$
$$= (\dot{\lambda} - r\lambda)(x - x^*) + \frac{\partial (H^c)^*}{\partial u}(u^* - u)$$

so

$$D_u(t) \geq \int_{t_0}^{t} e^{-r\tau}[(\dot{\lambda} - r\lambda)(x - x^*) + \lambda(\dot{x} - \dot{x}^*)]\,d\tau + \int_{t_0}^{t} \frac{\partial (H^c)^*}{\partial u}(u^* - u)\,d\tau$$

As in the proof of Theorem 9.7.1, we see that the second integral is ≥ 0 and so

$$D_u(t) \geq \int_{t_0}^{t} \frac{d}{d\tau}[e^{-r\tau}\lambda(\tau)(x(\tau) - x^*(\tau))]\,d\tau = \Big|_{t_0}^{t} e^{-r\tau}\lambda(\tau)(x(\tau) - x^*(\tau))$$

The contribution from the lower limit of integration is 0 because $x^*(t_0) - x(t_0) = x_0 - x_0 = 0$, so $D_u(t) \geq e^{-rt}\lambda(t)(x(t) - x^*(t))$. Passing to the limit as $t \to \infty$ in this inequality and using (d), one concludes that $(x^*(t), u^*(t))$ is optimal. ∎

NOTE 2 Condition (d) is well known in economics literature, but is often not prop checked. Note that the inequality (d) must be shown for *all* admissible $x(t)$, which is o problematic. Suppose for example that $\lim_{t\to\infty} \lambda(t)e^{-rt} \geq 0$, $\lim_{t\to\infty} \lambda(t)e^{-rt}x^*(t) =$ and $x(t) \geq 0$ for all t. Do these conditions ensure that (d) is satisfied? The answer is For a counterexample consider what happens when $\lambda(t) = -1$, $r = 1$, $x(t) = e^t$, $x^*(t) = 1$. Then $\lambda(t)e^{-t}[x(t) - x^*(t)] = -e^{-t}(e^t - 1) = e^{-t} - 1 \to -1$ as $t \to \infty$.

NOTE 3 Suppose the terminal condition is $\lim_{t\to\infty} x(t) \geq x_1$. Rewrite the brack expression in (d) as

$$\lambda(t)e^{-rt}(x(t) - x_1) + \lambda(t)e^{-rt}(x_1 - x^*(t))$$

We claim that, provided the following three conditions are all satisfied, then condition *is* satisfied.

(A) $\lim_{t\to\infty} \lambda(t)e^{-rt}(x_1 - x^*(t)) \geq 0$

(B) There exists a number M such that $|\lambda(t)e^{-rt}| \leq M$ for all $t \geq t_0$

(C) There exists a number t' such that $\lambda(t) \geq 0$ for all $t \geq t'$

Because of (A), in order to prove (d), it suffices to show that the first term in (∗) tends number ≥ 0. If $\lim_{t\to\infty} x(t) = x_1$, then $x(t) - x_1$ tends to 0 as t tends to ∞, so becaus (B), the first term in (∗) tends to 0. If $\lim_{t\to\infty} x(t) > x_1$, then $x(t) - x_1 > 0$ for t sufficie large. Then, because of (C), $\lambda(t)e^{-rt}(x(t) - x_1)$ tends to a number ≥ 0. We conclude t if (A)–(C) are all satisfied for all admissible pairs, then (d) holds.

NOTE 4 Suppose that we introduce additional conditions for admissibility in Theo 9.11.1. Then the inequality in Theorem 9.11.1(d) needs to hold only for pairs satisfying additional conditions.

In particular, if it is required that $x(t) \geq x_1$ for all t, then it suffices to check condit (A) and (C) in Note 3. This result is referred to as the **Malinvaud transversality conditi**

EXAMPLE 1 Consider the problem

$$\max \int_0^\infty -u^2 e^{-rt}\,dt, \quad \dot{x} = ue^{-at}, \quad x(0) = 0, \quad \lim_{t\to\infty} x(t) \geq K, \quad u \in \mathbb{R}$$

The constants r, a, and K are positive, with $a > r/2$. Find the optimal solution.

Solution: The current value Hamiltonian is $H^c = -u^2 + \lambda ue^{-at}$, which is obvio concave in x and u. We find $\partial H^c/\partial x = 0$ and $\partial H^c/\partial u = -2u + \lambda e^{-at}$. It follows $u = \frac{1}{2}\lambda e^{-at}$. The differential equation for λ is $\dot{\lambda} - r\lambda = -\partial H^c/\partial x = 0$, with the solu $\lambda = Ae^{rt}$, where A is a constant. Thus $u = \frac{1}{2}Ae^{(r-a)t}$. The differential equation for x becomes

$$\dot{x} = ue^{-at} = \tfrac{1}{2}Ae^{(r-2a)t}, \quad x(0) = 0, \quad \text{with solution} \quad x(t) = \frac{A}{2(2a - r)}(1 - e^{(r-2a}$$

Thus $x(t)$ converges to $A/2(2a-r)$ as t approaches ∞. Hence we must have $A/2(2a-r) \geq K$, or $A \geq 2K(2a-r)$. In particular, $A \geq 0$, and (B) and (C) in Note 3 are satisfied. To check condition (A) requires considering

$$\lambda(t)e^{-rt}\big(K - x(t)\big) = Ae^{rt}e^{-rt}\left[K - \frac{A}{2(2a-r)}(1 - e^{(r-2a)t})\right]$$

which tends to $A[K - A/2(2a-r)]$ as t tends to ∞. We conclude that if we choose $A = 2K(2a-r)$, all the conditions in Theorem 9.11.1 are satisfied and we have found the optimal solution. Note that $p(t) = \lambda e^{-rt} = 2K(2a-r)$, which does not tend to 0 as t tends to ∞. Nor does $p(t)x^*(t)$.

EXAMPLE 2 Consider the following version of Example 8.5.3:

$$\max \int_0^\infty \frac{1}{1-\delta}[rA(t) + w - u(t)]^{1-\delta}e^{-\rho t}\,dt$$

$$\dot{A}(t) = u(t), \quad A(0) = A_0 > 0, \quad \lim_{t\to\infty} A(t) \geq -w/r, \quad u \in \mathbb{R}$$

Assume that $0 < \delta < 1$ and $0 < r < \rho$, and then solve the problem.

Solution: The current value Hamiltonian is $H^c = \frac{1}{1-\delta}(rA + w - u)^{1-\delta} + \lambda u$, and the differential equation for $\lambda(t)$ is

$$\dot{\lambda}(t) - \rho\lambda(t) = -\frac{\partial (H^c)^*}{\partial A} = -r\big[rA^*(t) + w - u^*(t)\big]^{-\delta} \tag{i}$$

The control function $u^*(t)$ maximizes

$$\varphi(u) = \frac{1}{1-\delta}[rA^*(t) + w - u]^{1-\delta} + \lambda u \quad \text{for} \quad u \in \mathbb{R} \tag{ii}$$

Now the function H^c is concave in (A, u), as a concave function of a linear function. (Alternatively, look at the Hessian.) In particular, $\varphi(u)$ is concave in u, so $u^*(t)$ maximizes $\varphi(u)$ provided $\varphi'(u^*(t)) = 0$, or if

$$-[rA^*(t) + w - u^*(t)]^{-\delta} + \lambda(t) = 0, \quad \text{or} \quad u^*(t) = rA^*(t) + w - \lambda(t)^{-1/\delta} \tag{iii}$$

Combining (i) and (iii), it follows that $\dot{\lambda}(t) - \rho\lambda(t) = -r\lambda(t)$, so $\dot{\lambda} = (\rho - r)\lambda(t)$, with solution

$$\lambda(t) = C_1 e^{(\rho-r)t} \tag{iv}$$

for some constant C_1. Because $\dot{A}^* = u^*$, it follows that

$$\dot{A}^*(t) - rA^*(t) = w - C_1^{-1/\delta}e^{-at} \quad \text{where} \quad a = (\rho - r)/\delta$$

The general solution of this linear differential equation is

$$A^*(t) = C_2e^{rt} - w/r + C_1^{-1/\delta}e^{-at}/(a+r)$$

We must now find suitable values of the constants C_1 and C_2. It seems reasonable to ass that $\lim_{t\to\infty} A^*(t) = -w/r$. This is only possible if $C_2 = 0$. Then C_1 is determine the initial condition $A^*(0) = A_0$, which gives $A_0 = -w/r + C_1^{-1/\delta}/(a+r)$. Hence find that $C_1^{-1/\delta}/(a+r) = A_0 + w/r$. We therefore have the following candidate fo optimum:

$$A^*(t) = (A_0 + w/r)e^{-at} - w/r, \quad u^*(t) = -a(A_0 + w/r)e^{-at}, \quad \lambda(t) = \bar{\lambda}e^{(\rho-r)t}$$

where $\bar{\lambda} = \big((a+r)(A_0 + w/r)\big)^{-\delta}$.

It remains to verify (d) in Theorem 9.11.1. According to Note 3 it suffices to show conditions (A), (B), and (C) are satisfied. In our case (A) holds because

$$\lim_{t\to\infty} \lambda(t)(w/r + A^*(t)) = \bar{\lambda}(A_0 + w/r)\lim_{t\to\infty} e^{-(r+a)t} = 0$$

and (B) and (C) are evidently satisfied. Hence we have shown that $(A^*(t), u^*(t))$ solves problem.

Many economists seem to believe that for problems with an infinite horizon, no neces transversality conditions are generally valid. This is wrong. But certain growth conditi are needed for such conditions to hold. A special result of this type is given in the theorem. (See Seierstad and Sydsæter (1987), Section 3.9, Theorem 16 for a more gen result. Please correct a misprint in that theorem: Replace $b > k$ by $b > (n-m)k$.)

THEOREM 9.11.2 (NECESSARY CONDITION FOR AN INFINITE HORIZON)

Assume that $(x^*(t), u^*(t))$ is optimal in problem (1), with no condition on the limiting behaviour of $x(t)$ as $t \to \infty$. Assume that $\int_{t_0}^\infty |f(t, x(t), u(t))|dt < \infty$ for all admissible $(x(t), u(t))$. Suppose too that there exist positive constants A and k with $r > k$ such that

$$|\partial f(t, x, u^*(t))/\partial x| \leq A \quad \text{for all } x$$

and

$$\partial g(t, x, u^*(t))/\partial x \leq k \quad \text{for all } x$$

Then there exists a continuous function $\lambda(t)$ such that, with $\lambda_0 = 1$,

$$H^c(t, x^*(t), u, \lambda(t)) \leq H^c(t, x^*(t), u^*(t), \lambda(t)) \text{ for all } u \in U$$

The function $\lambda(t)$ equals $\lim_{T\to\infty} \lambda(t, T)$, where $\lambda(t, T)$ is the solution of

$$\dot{\lambda} - r\lambda = -\partial H(t, x^*(t), u^*(t), \lambda)/\partial x, \quad \lambda(T, T) = 0$$

NOTE 5 In fact, if $G(t)$ is a set containing $x(t)$ for all admissible $x(t)$, and N is some posi number, then, for any t, (3) need only hold for $x \in B(x^*(t), 2Ne^{kt}) \cap G(t)$ and (4) need only l for $|x| \geq N$, $x \in G(t)$.

PROBLEMS FOR SECTION 9.11

1. Solve the problem

$$\max \int_0^\infty (\ln u)\, e^{-0.2t}\, dt, \quad \dot{x} = 0.1x - u, \quad x(0) = 10, \quad \lim_{t\to\infty} x(t) \geq 0, \quad u > 0$$

using Theorem 9.11.1 and Note 3.

2. Find the only possible solution to the problem

$$\max \int_0^\infty x(2-u)e^{-t}\, dt, \quad \dot{x} = uxe^{-t}, \quad x(0) = 1, \quad x(\infty) \text{ is free}, \quad u \in [0, 1]$$

3. Compute the optimal value V of the objective function in Example 2. How does V change when ρ increases and when w increases? Show that $\partial V / \partial A_0 = \lambda(0)$.

4. Solve the problem

$$\max \int_{-1}^\infty (x-u)e^{-t}\, dt, \quad \dot{x} = ue^{-t}, \quad x(-1) = 0, \quad x(\infty) \text{ is free}, \quad u \in [0, 1]$$

9.12 Phase Diagrams

Consider the following problem

$$\max \int_{t_0}^{t_1} f(x, u)e^{-rt}\, dt, \quad \dot{x} = g(x, u), \quad x(t_0) = x_0, \quad u \in U \subseteq \mathbb{R} \tag{1}$$

with the standard end constraints, and with t_1 finite or ∞. In this case the functions f and g do not depend explicitly on t. Nor, therefore, does the current value Hamiltonian H^c.

Suppose that $u = u(x, \lambda)$ maximizes $H^c = f(x, u) + \lambda g(x, u)$ w.r.t. u for $u \in U$. Replacing u by $u = u(x, \lambda)$ in the differential equations for x and λ gives

$$\begin{aligned} \dot{x} &= F(x, \lambda) \\ \dot{\lambda} &= G(x, \lambda) \end{aligned} \tag{2}$$

This is an *autonomous* system that is simpler to handle than one in which $\dot{x}$ and $\dot{\lambda}$ depend explicitly on t as well as on x and λ. In particular, *phase plane analysis* (see Section 6.7) can be used to shed light on the evolution of an autonomous system even when explicit solutions are not obtainable. Example 9.9.1 showed a simple case.

We study two examples.

EXAMPLE 1 Write down the system of equations (2), and draw a phase diagram for the problem

$$\max \int_0^\infty (x - u^2)e^{-0.1t}\, dt, \quad \dot{x} = -0.4x + u, \quad x(0) = 1, \quad x(\infty) \text{ is free}, \quad u \in (0, \infty)$$

Try to find the solution of the problem. (See Example 9.10.2.)

Solution: In this case the Hamiltonian $H^c(t, x, u, \lambda) = (x - u^2) + \lambda(-0.4x +$
concave in (x, u). The maximization of H^c w.r.t. u gives $u = 0.5\lambda$, assuming that λ is
(We try this assumption in the following.) Hence, $\dot{x} = -0.4x + 0.5\lambda$. System (2) is h

$$\begin{aligned} \dot{x} &= -0.4x + 0.5\lambda, \quad x(0) = 1 \\ \dot{\lambda} &= 0.5\lambda - 1 \end{aligned}$$

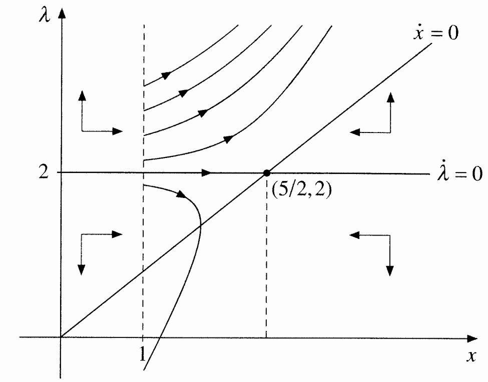

Figure 1: Phase diagram for system (∗) in Example 1.

Figure 1 shows a phase diagram for (∗). Any path $(x(t), \lambda(t))$ that solves the problem
start at some point on the vertical line $x = 1$, but no restrictions are imposed on $x($
$t \to \infty$. If we start above or below the line $\lambda = 2$, it appears that $(x(t), \lambda(t))$ will "wa
off to infinity", which makes it difficult to satisfy requirement (d) in Theorem 9.11.1.

In fact, the general solution of (∗) is $x^*(t) = \frac{5}{9}Ae^{0.5t} + \frac{5}{2} - (\frac{5}{9}A + \frac{3}{2})e^{-0.4t}$
$\lambda(t) = Ae^{0.5t} + 2$. The expression we need to consider in Theorem 9.11.1(d) is the differ
of the two terms, $\lambda(t)e^{-0.1t}x(t)$ and $\lambda(t)e^{-0.1t}x^*(t)$. For large values of t, the latter pro
is dominated by the term $\frac{5}{9}A^2e^{0.9t}$, which tends to infinity as t tends to infinity w
$A \neq 0$; and that does not seem promising. It approaches 0 as t approaches infinity if A
(then $\lambda \equiv 2$), and then the product is equal to $5e^{-0.1t} - 3e^{-0.5t}$, which does approach 0
approaches infinity. It is easy to see that $x(t) > 0$ for all $t \geq 0$ so $\lambda(t)e^{-0.1t}x(t)$ is > 0 fo
$t \geq 0$. It follows that condition (d) in Theorem 9.11.1 is satisfied, and $x^*(t) = -\frac{3}{2}e^{-0.4t}$
is therefore optimal.

Coming back to the phase diagram, if we start at the point $(x, \lambda) = (1, 2)$, then $\lambda(t)$
while $x(t)$ converges to the value $\frac{5}{2}$, which is x-coordinate of the point of intersec
between the curves $\dot{\lambda} = 0$ and $\dot{x} = 0$. The phase diagram therefore suggests the opt
solution to the problem.

The point $(\frac{5}{2}, 2)$ is an equilibrium point for system (∗). Let us see what Theorem 6
says about this equilibrium point. Defining $f(x, \lambda) = -0.4x + 0.5\lambda$ and $g(x, \lambda) = 0.5\lambda$
we find that the determinant of the Jacobian matrix in Theorem 6.9.1 is

$$\begin{vmatrix} -0.4 & 0.5 \\ 0 & 0.5 \end{vmatrix} = -0.2 < 0$$

so $(\frac{5}{2}, 2)$ really is a saddle point.

EXAMPLE 2 Consider an economy with capital stock $K = K(t)$ and production per unit of time $Y = Y(t)$, where $Y = aK - bK^2$, with a and b as positive constants. Consumption is $C > 0$, whereas $Y - C = aK^2 - bK - C$ is investment. Over the period $[0, \infty)$, the objective is to maximize total discounted utility. Specially, the problem is

$$\int_0^{\infty} \frac{1}{1-v} C^{1-v} e^{-rt} dt, \quad \dot{K} = aK - bK^2 - C, \qquad K(0) = K_0 > 0$$

where $a > r > 0$ and $v > 0$, and C is the control variable. We require that

$$\lim_{t \to \infty} K(t) \geq 0$$

The current value Hamiltonian is $H^c = \frac{1}{1-v} C^{1-v} + \lambda(aK - bK^2 - C)$. An interior maximum of H^c requires $\partial H^c/\partial c = 0$, i.e.

$$C^{-v} = \lambda \tag{i}$$

The differential equation for $\lambda = \lambda(t)$ is $\dot{\lambda} = -\lambda(a - 2bK) + r\lambda$, or

$$\dot{\lambda} = \lambda(r - a + 2bK) = 2b\lambda\Big(K - \frac{a-r}{2b}\Big) \tag{ii}$$

Now (i) implies that $C = \lambda^{-1/v}$, which inserted into the differential equation for K yields

$$\dot{K} = aK - bK^2 - \lambda^{-1/v} \tag{iii}$$

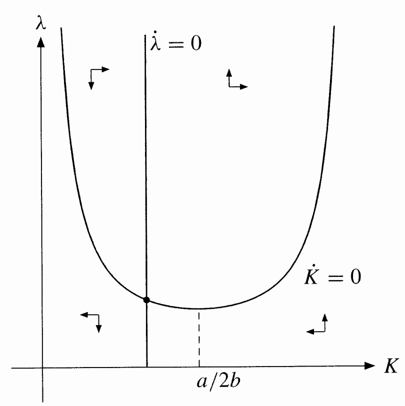

Figure 2

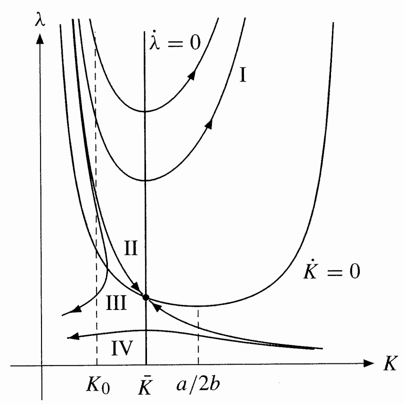

Figure 3

Figure 2 presents a phase diagram for the system given by (ii) and (iii). We see that $\dot{K} = 0$ for $\lambda = (aK - bK^2)^{-v}$, with $v > 0$. Here $z = aK - bK^2$ represents a concave parabola with $z = 0$ for $K = 0$ and for $K = a/b$. For $z = 0$, one has $\lambda = \infty$. The graph of $\dot{K} = 0$ is symmetrical about $K = a/2b$. Note that $\dot{\lambda} = 0$ when $K = (a - r)/2b$, which gives a straight line parallel to the λ-axis. Because $0 < (a - r)/2b < a/2b$, the graph of $\dot{\lambda} = 0$ will be as suggested in the figure. The equilibrium point $(\bar{K}, \bar{\lambda})$ is given by $\bar{K} = (a - r)/2b$, $\bar{\lambda} = [(a^2 - r^2)/4b]^{-v}$.

Figure 2 shows the $K\lambda$-plane divided into four parts. The arrows indicate the directi of the integral curves in each of these four parts. From (ii) we see that $K > (a - r)/$ implies $\dot{\lambda} > 0$, whereas $K < (a - r)/2b$ implies $\dot{\lambda} < 0$. Also, the right-hand side of (i $aK - bK^2 - \lambda^{-1/v}$, increases as λ increases for each fixed K, so that $\dot{K} > 0$ above curve $\dot{K} = 0$, and $\dot{K} < 0$ below this curve.

Figure 3 shows some integral curves that $(K(t), \lambda(t))$ could follow as t increases. In t figure we have assumed that $K_0 < \bar{K}$. Of particular interest are paths that start at $K =$ but other curves, which start with larger values of K, are also drawn. Note that, although is known, the quantity $\lambda(0)$ must be regarded as an unknown parameter. In this particu problem $\lambda(0)$ can be determined as follows: If $\lambda(0)$ is large, the point $(K(t), \lambda(t))$ st high up on the line $K = K_0$ and moves along a curve like that marked I in Figure 3. $\lambda(0)$ is small, then $(K(t), \lambda(t))$ starts low down on the line $K = K_0$ and moves alon the curve like III in the figure. If $\lambda(0)$ is even smaller, and $(K_0, \lambda(0))$ lies below the cu $\dot{K} = 0$, then $(K(t), \lambda(t))$ moves steadily "southwest", like curve IV. At some point on line $K = K_0$, continuity suggests that there should be some particular value $\lambda^*(0)$ of λ such that the resultant curve is of type II, which converges to the stationary point $(\bar{K}, \bar{\lambda}$

Here is a more precise argument: Curve I was obtained using a high initial value for $\lambda($ Along curve I the point $(K(t), \lambda(t))$ moves down to the right until it reaches a minim point where it crosses the line $\dot{\lambda} = 0$. Let $\lambda(0)$ decrease. Then curve I shifts downwar Its minimum point on the line $\dot{\lambda} = 0$ will then shift downwards to the equilibrium p $(\bar{K}, \bar{\lambda})$. Actually, $\lambda^*(0)$ is precisely that value of $\lambda(0)$ which makes this minimum occu the point $(\bar{K}, \bar{\lambda})$. This initial value $\lambda^*(0)$ leads to a special path $(K^*(t), \lambda^*(t))$. Both $\dot{K}^*$ and $\dot{\lambda}^*(t)$ approach zero as $t \to \infty$. For all finite t, the path $(K^*(t), \lambda^*(t))$ never reac the point $(\bar{K}, \bar{\lambda})$, but $(K^*(t), \lambda^*(t)) \to (\bar{K}, \bar{\lambda})$ as $t \to \infty$.

So far we have argued that the conditions of the maximum principle are satisfied al a curve $(K^*(t), \lambda^*(t))$ of type II in Figure 3, where $K^*(t) \to \bar{K}$ and $\lambda^*(t) \to \bar{\lambda}$ as $t \to$ Let us prove that this candidate solution is optimal.

The present value Hamiltonian H^c is concave as a function of (K, C). With $\lambda(t)$ given $C^*(t) = (\lambda(t))^{-1/v}$, the first-order condition for a maximum of H^c is satisfied, and beca H^c is concave in C, it reaches a maximum at $C^*(t)$. Moreover, (A) in Note 9.11.3 is a satisfied: $\lambda(t)e^{-rt}K^*(t) \to 0$ as $t \to \infty$ and $\lim_{t\to\infty} \lambda(t)e^{-rt}K(t) \geq 0$ as $\lim_{t\to\infty} K(t$ 0, because $\lambda(t)e^{-rt}$ is positive and bounded (it even approaches 0). These properties im that

$$\lim_{t \to \infty} \lambda(t)e^{-rt}[K(t) - K^*(t)] \geq 0$$

for all admissible $K(t)$. This verifies all the sufficient conditions, so $(K^*(t), C^*(t)$ optimal.[7]

Any solution of the system (ii) and (iii) will depend on K_0 and on $\lambda(0) = \lambda^0$, s can be denoted by $K(t) = K(t; K_0, \lambda^0)$ and $\lambda(t) = \lambda(t; K_0, \lambda^0)$. In this problem, K given, whereas λ^0 is determined by the requirement that $\lim_{t\to\infty} \lambda(t; K_0, \lambda^0) = \bar{\lambda}$. Figu actually shows *two* curves of type II that converge to $(\bar{K}, \bar{\lambda})$. The alternative solution of differential equations converges to $(\bar{K}, \bar{\lambda})$ from the "southeast". This path does not so

[7] We did not formally require $K(t) \geq 0$ for all t, but it is indeed a natural requirement. Then need to check only conditions (A) and (C) in Note 9.11.3.

the optimization problem, however, because it must start from the wrong value of K at time $t = 0$. (It *does* solve the problem when $K_0 > \bar{K}$, however.)

The equilibrium point $(\bar{K}, \bar{\lambda}) = \left((a-r)/2b, \left[(a^2 - r^2)/4b\right]^{-v}\right)$ is an example of a *saddle point* (see Section 6.9). We show this by applying Theorem 6.9.1. To do so, define the functions $f(K, \lambda) = aK - bK^2 - \lambda^{-1/v}$ and $g(K, \lambda) = 2b\lambda(K - (a-r)/2b)$ corresponding to the right-hand sides of (iii) and (ii) respectively. Then at the point $(\bar{K}, \bar{\lambda})$ one has $\partial f/\partial K = a - 2b\bar{K} = r$, $\partial f/\partial \lambda = (1/v)\bar{\lambda}^{-1/v-1}$, $\partial g/\partial K = 2b\bar{\lambda}$ and $\partial g/\partial \lambda = 2b(\bar{K} - (a-r)/2b) = 0$. The determinant of the matrix $\mathbf{A}$ in Theorem 6.9.1 is therefore

$$\begin{vmatrix} r & (1/v)\bar{\lambda}^{-1/v-1} \\ 2b\bar{\lambda} & 0 \end{vmatrix} = -\frac{2b}{v}\bar{\lambda}^{-1/v} < 0$$

This confirms that $(\bar{K}, \bar{\lambda})$ really *is* a saddle point.

PROBLEMS FOR SECTION 9.12

1. (a) Consider the problem

$$\max \int_0^\infty (ax - \tfrac{1}{2}u^2)e^{-rt}\,dt, \qquad \dot{x} = -bx + u, \;\; x(0) = x_0, \;\; x(\infty) \text{ free}, \;\; u \in \mathbb{R}$$

where a, r, and b are all positive. Write down the current value Hamiltonian H^c for this problem, and determine the system (2). What is the equilibrium point?

(b) Draw a phase diagram for $(x(t), \lambda(t))$ and show that for the two solutions which converge to the equilibrium point, $\lambda(t)$ is a constant.

(c) Use sufficient conditions to solve the problem.

(d) Show that $\partial V/\partial x_0 = \lambda(0)$, where V is the optimal value function.

2. In Problem 9.9.1 we studied a problem closely related to

$$\max \int_0^T \left(-x^2 - \tfrac{1}{2}u^2\right)e^{-2t}\,dt, \quad \dot{x} = x + u, \;\; x(0) = 1, \;\; x(T) \geq 0, \;\; u \in \mathbb{R}$$

Solve this problem in the case $T = \infty$. (*Hint:* $\lim_{t\to\infty} p(t) = 0$.)

3. (a) Consider the problem

$$\max \int_0^T e^{-rt} \ln c(t)\,dt$$

$$\dot{K}(t) = A(K(t))^\alpha - c(t), \quad K(0) = K_0, \;\; K(T) = K_T$$

where the constants A and r are positive, and $\alpha \in (0, 1)$. Here $K(t)$ denotes the capital stock of an economy and the control variable $c(t)$ denotes consumption

at time t. The horizon T is fixed and finite. Prove that if $K = K^*(t) >$ $c = c^*(t) > 0$ solve the problem, then

$$\dot{K} = AK^\alpha - c$$
$$\dot{c} = c(\alpha A K^{\alpha - 1} - r)$$

(b) Suppose $A = 2$, $\alpha = 1/2$, and $r = 0.05$. Prove that the equilibrium is a point. In Problem 6.7.3 you were asked to draw a phase diagram of the syst

(c) Indicate in the diagram for Problem 6.7.3 a possible integral curve for th $K_0 = 100$ and $K_T = 600$? What is the solution when $K_0 = 100$ and $T = \infty$ $K(T) > 0$ for all t?

4. Consider the problem

$$\max_{u \in \mathbb{R}} \int_0^\infty [-(x-1)^2 - \tfrac{1}{2}u^2]e^{-t}\,dt, \quad \dot{x} = x - u, \;\; x(0) = \tfrac{1}{2}, \;\; x(\infty) \text{ free}$$

(a) Solve the problem qualitatively by a saddle point argument.

(b) Find an explicit solution.