



## Measuring Change: Linear Logistic Models With Relaxed Assumptions (LLRA)

Models and Application

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### What Are Linear Logistic Models With Relaxed Assumptions?

- They allow to analyse categorical **repeated measurement data** (**categorical panel data**)
- They are **Item Response Models**
- They are called “with **Relaxed Assumptions**” because they require neither unidimensionality of items nor distributional assumptions about the latent trait
- They allow to contrast **treatment and/or subject covariate effects**
- They provide a **framework** for testing assumptions on (item) dimensionality, trend and covariate effects
- They measure change on a **metric scale** (actually a ratio scale)
- They are very rarely applied (especially in a business, economics, marketing etc. context)



### About The Nature Of LLRA

- LLRA are Generalised **Rasch Models**
- Each item (or group of items) may - but need not - measure a single latent trait, hence it is a **multi-dimensional** model
- Change is modelled as a function of **trend effects** and **covariate/treatment main and interaction effects**
- Those effects can be specified by **linear decomposition of virtual item parameters**
- Due to **conditional maximum likelihood estimation (CML)**, inference on effect parameters is completely independent of the trait parameters
- Item and person parameters are **not of interest**
- The **same** items are used at each occasion
- Treatment/covariate effects are assumed to be generalisable over subjects of a certain group



### Questions The LLRA Can Answer

- Is there a **treatment/covariate** effect?
- Is there a **trend effect**?
- Are treatment/covariate/trend effects the **same for a group of items**?
- Are there any **interaction effects** between groups (e.g. gender and treatment)?
- Is there a **specific functional form** of trend and covariate/treatment effects **over time** (e.g. linear trend)?
- All **combinations of the above**.



### Conditions And Assumptions

- All changes (positive or negative) must be **independent** of each other, including changes of the same person for other items
- Effect parameters take the **same values for all subjects** in a group
- There are **no latent classes or unobserved heterogeneity** due to subgroups
- **Equal item discrimination** (otherwise effect and discrimination parameters are confounded)

Its benefit is that - given the above holds - **measurement of change is free of systematic influences of nuisance parameters**, like item and person parameters.



### Model Formulation - I

At  $T_1$

$$P(X_{vih1} = 1|T_1) = \frac{\exp(h\theta_{vi1} + \omega_{ih})}{\sum_{l=0}^{m_i} \exp(l\theta_{vi1} + \omega_{il})},$$

At  $T_t$

$$P(X_{vih_t} = 1|T_t) = \frac{\exp(h\theta_{vit} + \omega_{ih})}{\sum_{l=0}^{m_i} \exp(l\theta_{vit} + \omega_{il})} = \frac{\exp(h(\theta_{vi1} + \delta_{vit}) + \omega_{ih})}{\sum_{l=0}^{m_i} \exp(l(\theta_{vi1} + \delta_{vit}) + \omega_{il})},$$

$\theta_{vit}$  ... location of subject  $v$  for item  $i$  at  $T_t$

$\delta_{vit} = \theta_{vit} - \theta_{vi1}$  ... amount of change of person  $v$  for trait  $i$  between time  $T_1$  and  $T_t$

$h$  ...  $h$ -th response category ( $h = 0, \dots, m_i$ )

$\omega_{ih}$  ... parameter for category  $h$  for item  $i$



### Model Formulation - II

The flexibility of LLRA now arises from a **(linear) reparametrisation** of  $\delta_{vit}$  to include different effects:

$$\delta_{vit} = \mathbf{w}_{it}^T \boldsymbol{\eta}$$

$\mathbf{w}_{it}^T$  ... row of design matrix  $W$  (covariate values, e.g. dosages or treatment groups) for trait  $i$  up to  $T_t$ .

$\boldsymbol{\eta}$  is a vector of parameters typically describing treatment or covariate groups and trend and interactions etc.

$$\delta_{vit} = \sum_j q_{vjit} \lambda_{jit} + \tau_{it} + \sum_{j < l} q_{vjit} q_{vlit} \rho_{jlit}$$

$q_{vjit}$  ... dosage of treatments  $j$  for trait  $i$  between  $T_1$  and  $T_t$ .

$\lambda_{jit}$  ... effect of the treatment  $j$  on trait  $i$  at  $T_t$

$\tau_{it}$  ... trend effect on trait  $i$  for  $T_t$ .

$\rho_{jlit}$  ... interaction effects of treatments  $j$  and  $l$  on trait  $i$  at  $T_t$ .



### Estimation Of LLRA - I

Change effects in LLRA can be estimated via CML by using a trick:

Instead of assessing the change on the latent trait, change is estimated from **differences in item difficulties**

- In Rasch models, item and person parameters lie on the same latent trait
- A positive location change on the trait is **equivalent** to a negative shift of the item-(category) threshold
- Hence the same item at  $t$  different time points is viewed as  $t$  different "virtual" items whose locations differ by  $\delta_{vit}$
- CML can be applied to estimate those differences (conditional on difficulties and person parameters)



**Detour: The LLTM For Measuring Change**

If we have unidimensional, Rasch model conforming items, we can assess change by means of the LLTM with **exactly the same trick** of using “virtual items”

- Difficulties for each item are estimated at all time points
- Change is the difference between the difficulty at  $T_1$  and the later time points
- In principle, different items can be used at each timepoint, since difficulties are known from the Rasch model
- The big difference to LLRA is that the **baseline is estimated as well** (the difficulties)



**Estimation Of LLRA - II**

Informal: The likelihood is the **product of all positive and negative change patterns** between two time points, e.g. for (0,1) and (1,0) for dichotomous items.

- Possibly a small part of information considered (all (1,1) and (0,0) patterns are discarded)
- Changes must occur in both directions
- Changes must not be too large
- Assumption of independence of changes



**Data Structure - I**

To fit LLRA with, e.g. **eRm**, the usual wide format has to be changed in to **long format**

Real Persons	$T_1$				$T_2$			
$S_1$	$x_{111}$	$x_{121}$	...	$x_{1k1}$	$x_{112}$	$x_{122}$	...	$x_{1k2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$S_n$	$x_{n11}$	$x_{n21}$	...	$x_{nk1}$	$x_{n12}$	$x_{n22}$	...	$x_{nk2}$

Virtual Persons	$T_1$	$T_2$
$S_{11}^*$	$x_{111}$	$x_{112}$
$\vdots$	$\vdots$	$\vdots$
$S_{n1}^*$	$x_{n11}$	$x_{n12}$
$\vdots$	$\vdots$	$\vdots$
$S_{1k}^*$	$x_{1k1}$	$x_{1k2}$
$\vdots$	$\vdots$	$\vdots$
$S_{nk}^*$	$x_{nk1}$	$x_{nk2}$



**Data Structure - II**

Group membership or covariate values must be specified so we can assign each persons item answers to the according group. This must be multiplied with the according number of items (**assignment group**), e.g. a treatment and a control, 3 items -> six groups

	Virtual Persons	$T_1$	$T_2$	Assignment Group	
Item 1	TG	$S_{(TG)11}^*$	$x_{(TG)111}$	$x_{(TG)112}$	1
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
CG	TG	$S_{(CG)n1}^*$	$x_{(CG)n11}$	$x_{(CG)n12}$	1
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
	CG	$S_{(CG)n1}^*$	$x_{(CG)111}$	$x_{(CG)112}$	2
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$	2
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
Item k	TG	$S_{(TG)1k}^*$	$x_{(TG)1k1}$	$x_{(TG)1k2}$	$2k - 1$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
CG	TG	$S_{(CG)nk}^*$	$x_{(CG)nk1}$	$x_{(CG)nk2}$	$2k - 1$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
	CG	$S_{(CG)1k}^*$	$x_{(CG)1k1}$	$x_{(CG)1k2}$	$2k$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$	$2k$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$S_{(CG)nk}^*$	$x_{(CG)nk1}$	$x_{(CG)nk2}$	$2k$	



**Design Matrix**

Since we are interested in estimation of  $\eta$ , the crucial step is [setting up the design matrix  \$W\$](#) .

In principle,  $W$  can be of any form, but we will use the following [canonical structure](#):

- Each column corresponds to an effect
- The number of rows is the number of time points  $\times$  the number of items  $\times$  the number of covariate groups  $\times$  the number of categories - 1
- The slowest index is the index of time points. Nested within time points are the item indices, and within items the group indices. The fastest index corresponds to response categories in case of polytomous items
- Non existing effects get a zero value
- For items with different numbers of categories, the superfluous rows consist of zeros



**Design Matrix - Generalisation Of Effects**

After setting up a “quasi saturated” LLRA, the model can be [simplified](#) (e.g. effects can be generalised)

- Collapsing columns to equate effects (e.g. two treatment groups get the same effects)
- Specifying certain functional forms (linear effects over time, etc.)
- Collapsing columns to equate effects for certain dimensions (i.e. items)
- Applying any kind of linear contrasts on the effects (e.g. treatment 1 and 2 together are 3 times as effective as treatment 3)
- “Classic” LLRA specifies a general trend over all items and groups and a treatment effect for each group over all items



**Design Matrix - Example I**

2 time points, 2 groups (control and treatment),  $k$  dichotomous items

		$\lambda_1$	$\lambda_2$	$\dots$	$\lambda_k$	$\tau_1$	$\tau_2$	$\dots$	$\tau_k$
$T_1$	Item 1 - TG								
	Item 1 - CG								
	Item 2 - TG								
	Item 2 - CG								
	$\vdots$								
	Item $k$ - TG								
Item $k$ - CG									
$T_2$	Item 1 - TG	1				1			
	Item 1 - CG					1			
	Item 2 - TG		1				1		
	Item 2 - CG						1		
	$\vdots$			$\vdots$				$\vdots$	
	Item $k$ - TG				1				1
Item $k$ - CG								1	



**Design Matrix - Example II**

2 time points, 2 groups (control and treatment), 3 items with 4 categories



## Design Matrix in R - I

Since these design matrices may become huge and are very sparse, setting them up manually might be cumbersome.

Hence we let R do it for us. What can help us is:

- The Kronecker product, `%x%`
- The Matrix package, `library(Matrix)`

For example: Setting up the former two design matrices.

### Dichotomous

```
> des.tmp <- c(0, 1) %x% diag(3)
> design.2 <- cbind(des.tmp %x% c(1, 0), des.tmp %x% c(1, 1))
```

				$\lambda_1$	$\lambda_2$	$\lambda_3$	$\tau_1$	$\tau_2$	$\tau_3$	$\omega_{12}$	$\omega_{13}$	$\omega_{22}$	$\omega_{23}$	$\omega_{32}$	$\omega_{33}$	
$T_1$	Item 1	TG	Cat 1							1						
			Cat 2									1				
			Cat 3											1		
	CG	Cat 1														
		Cat 2														
		Cat 3														
Item 2	TG	Cat 1											1			
		Cat 2												1		
		Cat 3													1	
	CG	Cat 1														
		Cat 2														
		Cat 3														
$T_2$	Item 1	TG	Cat 1	1			1									
			Cat 2	2			2				1					
			Cat 3	3			3					1				
	CG	Cat 1				1										
		Cat 2				2										
		Cat 3				3										
Item 2	TG	Cat 1		1			1									
		Cat 2		2		2						1				
		Cat 3		3		3							1			
	CG	Cat 1				1										
		Cat 2				2										
		Cat 3				3										
Item 3	TG	Cat 1			1			1								
		Cat 2			2			2							1	
		Cat 3			3			3								1
	CG	Cat 1				1										
		Cat 2				2									1	
		Cat 3				3										1



## Design Matrix in R - II

For example: Setting up the former two design matrices.

### Polytomous

```
> pseudodes <- matrix(c(1, 0, 1, 1), 2, 2)
> des0 <- diag(3) %x% pseudodes
> des0 <- des0[, c(1, 3, 5, 2, 4, 6)]
> des0 <- c(0, 1) %x% des0
> des1 <- des0 %x% c(1, 2, 3)
> c0 <- matrix(c(0, 1, 0, 0, 0, 1), 3, 2)
> c1 <- c(1, 1) %x% c0
> c2 <- diag(3) %x% c1
> des2 <- cbind(des1, rbind(c2, c2))
```

Collapsing is simply summing up the values in the according columns

```
> des2 <- cbind(des2[, 1], colSums(des2[, c(2, 3)]), des2[, c(4:12)])
```



## Fitting with eRm - I

Example: Two groups, three items, two time points

```
> data <- matrix(unlist(dat), nc = 2)
> grps6 <- as.numeric(gl(6, 50, 300))
> res.llra <- LPCM(data, W = design.2, mpoints = 2, groupvec = grps6,
+ sum0 = F)
> res.llra
```

Results of LPCM estimation:

Call: LPCM(X = data, W = design.2, mpoints = 2, groupvec = grps6, sum0 = F)

Conditional log-likelihood: -94.49457

Number of iterations: 16

Number of parameters: 6

Basic Parameters eta:

	eta 1	eta 2	eta 3	eta 4	eta 5	eta 6
Estimate	1.8918415	-0.7908925	1.9095409	0.4595322	-0.07410689	-0.9650805
Std.Err	0.8268503	0.5709425	0.6091365	0.3687384	0.38516366	0.4154649



## Fitting with eRm - II

Do it yourself: Fit LLRA as in Hatzinger & Rusch (2009) for

- Example 1
- Example 2
- Example 4
- Example 5
- Example 6
- Example 7

See <http://erm.r-forge.r-project.org/>



## LLRA functions for eRm

As you can see, this is a lot of work...

Functions in **eRm** will automatise this in the near future.

- `LLRA()`: A wrapper that does all of the above; restructure data, build design matrix, build assignment vector, fit LPCM
- `collapse.W()`: A function to conveniently collapse  $W$
- `plotGR()`, `plotTR()`: Plot group and trend effects
- `anova()`: Model comparison with likelihood ratio test



## References

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