Cross-correlation

Y_{jt} is a (second-order) stationary process with $\mu_j = \mathrm{E}(Y_{jt})$ and $\sigma_j^2 = \mathrm{Var}(Y_{jt})$. Process of the standard scores of Y_{jt}:

$$SC_{jt} = \frac{Y_{jt} - \mu_j}{\sigma_j}$$

- $SC_{jt} > 0$: at time t, process Y_{jt} is above the long-run mean μ_j
- $SC_{jt} < 0$: at time t, process Y_{jt} is below the long-run mean μ_j

Cross-correlation

Autocorrelation (lag h):

$$\rho_{jj}(h) = \mathrm{E}\left(\frac{(Y_{jt} - \mu_j)(Y_{j,t-h} - \mu_j)}{\sigma_j^2}\right) = \mathrm{E}(SC_{jt} \cdot SC_{j,t-h})$$

Cross correlation:

$$\rho_{jk}(h) = \mathrm{E}\left(\frac{(Y_{jt} - \mu_j)(Y_{k,t-h} - \mu_k)}{\sigma_j \sigma_k}\right) = \mathrm{E}(SC_{jt} \cdot SC_{k,t-h})$$

Like the autocorrelation, the cross correlation lies between -1 and 1.

Cross-correlation

$h = 0$ - simultaneous cross correlation between process Y_{jt} and Y_{kt}:

- $\rho_{jk}(0) > 0$: both processes Y_{jt} and Y_{kt} are on average either above or below their respective long-run means.

- $\rho_{jk}(0) < 0$: deviations of Y_{jt} and Y_{kt} from their respective long-run means are negatively correlated.

- $\rho_{jk}(0) = 0$: deviations of Y_{jt} and Y_{kt} from their respective long-run means are not correlated

$h \neq 0$ - $\rho_{jk}(h)$ measures delayed cross correlation between process $Y_{j,\cdot}$ and $Y_{k,\cdot}$.

Cross-correlation

$h > 0$: lag h - backward cross correlation: measures dependence of Y_{jt} on the past of process $Y_{k,t-h}$

- If the cross correlation $\rho_{jk}(h)$ between Y_{jt} and $Y_{k,t-h}$ at lag h is equal to 0, then the expected level of Y_{jt} is not influenced by the value of $Y_{k,t-h}$.
- If $\rho_{jk}(h) > 0$ for lag h and the process $Y_{k,t-h}$ is above (below) the long-run mean μ_k at time $t - h$, then the expected value of $Y_{j,t}$ will be above (below) the long-run mean μ_j at time t.

Cross-correlation

$h < 0$: lead $|h|$ - forward cross correlation: measures impact of Y_{jt} on the future of process $Y_{k,t+|h|}$

- If the cross correlation $\rho_{jk}(h)$ between Y_{jt} and $Y_{k,t+|h|}$ at lead h is equal to 0, then the actual value of Y_{jt} does not influence the expected future level of $Y_{k,t+h}$.

- If $\rho_{jk}(h) > 0$ for lead $|h|$ and the process Y_{jt} is above (below) the long-run mean μ_j at time t, then the expected value of $Y_{k,t+|h|}$ will be above (below) the long-run mean μ_k at time $t + |h| \Rightarrow$ $Y_{j,\cdot}$ is a leading indicator for $Y_{k,\cdot}$.

Autocorrelation matrix

Note that $\rho_{jk}(-h)$ is usually different from $\rho_{jk}(h)$, however, the following result holds:

$$\rho_{jk}(-h) = \rho_{kj}(h), \qquad h > 0.$$

Entire dependence structure is described by the cross correlations matrix at lag h:

$$\boldsymbol{\rho}(h) = \begin{pmatrix} \rho_{11}(h) & \cdots & \rho_{1m}(h) \\ \vdots & \ddots & \vdots \\ \rho_{m1}(h) & \cdots & \rho_{mm}(h) \end{pmatrix}$$

2.2 VAR(1)-Prozesse

- Define a model for multivariate time series that captures autocorrelation, simultaneous cross-correlation, and delayed cross-correlation

- VAR stands for **V**ector **A**uto**R**egressive

- VAR(1) stands for a model with lag 1, i.e. a model where only observations of lag 1 enter the model definition, like for a univariate AR(1)-model.

- Use of matrix notation simplifies notation.

VAR(1)-Prozesse

Introduce matrix notation for individual modelling of two time series Y_{1t} and Y_{2t} using an AR(1)-model:

$$
\begin{aligned}
Y_{1t} &= \varphi_1 Y_{1,t-1} + c_1 + u_{1t}, \quad u_{1t} \sim \text{Normal}\left(0, \sigma^2_{u,1}\right), \\
Y_{2t} &= \varphi_2 Y_{2,t-1} + c_2 + u_{2t}, \quad u_{2t} \sim \text{Normal}\left(0, \sigma^2_{u,2}\right).
\end{aligned}
$$

Or, equivalently:

$$
\begin{aligned}
Y_{1t} &= \varphi_1 Y_{1,t-1} + 0 \cdot Y_{2,t-1} + c_1 + u_{1t}, \\
Y_{2t} &= 0 \cdot Y_{1,t-1} + \varphi_2 Y_{2,t-1} + c_2 + u_{2t}.
\end{aligned}
$$

VAR(1)-Prozesse

Rewrite these two equations as:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}.$$

Define

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \boldsymbol{u}_t = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2_{u,1} & 0 \\ 0 & \sigma^2_{u,2} \end{pmatrix}$$

VAR(1)-Prozesse

Matrix notation for individual modelling of two AR(1)-processes Y_{1t} and Y_{2t}:

$$\mathbf{Y}_t = \mathbf{\Phi}\mathbf{Y}_{t-1} + \mathbf{c} + \boldsymbol{u}_t, \quad \boldsymbol{u}_t \sim \text{Normal}\left(\mathbf{0}, \mathbf{\Sigma}\right). \tag{3}$$

- Special case of a bivariate VAR(1)-model, where $\mathbf{\Phi}$ and $\mathbf{\Sigma}$ are diagonal matrices
- In a more general bivariate VAR(1)-model, $\mathbf{\Phi}$ and $\mathbf{\Sigma}$ are not reduced to be diagonal matrices.
- It is evident from (3) why this model is called VAR(1).

VAR(1)-Prozesse

General bivariate VAR(1)-model:

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} \Rightarrow \mathbf{\Phi} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix},$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2_{u,1} & 0 \\ 0 & \sigma^2_{u,2} \end{pmatrix} \Rightarrow \mathbf{\Sigma} = \begin{pmatrix} \sigma^2_{u,1} & \sigma^2_{u,12} \\ \sigma^2_{u,21} & \sigma^2_{u,2} \end{pmatrix}.$$

Rewrite the bivariate VAR(1)-model as a system of two equations:

$$Y_{1t} = \Phi_{11} Y_{1,t-1} + \Phi_{12} Y_{2,t-1} + c_1 + u_{1t},$$

$$Y_{2t} = \Phi_{21} Y_{1,t-1} + \Phi_{22} Y_{2,t-1} + c_2 + u_{2t}.$$

Capturing cross correlation

- The conditional expectation $\mathrm{E}(Y_{1t}|\mathbf{Y}_{t-1})$ depends not only on $Y_{1,t-1}$, but also on $Y_{2,t-1}$, as long as $\Phi_{12} \neq 0$; similarly for $\mathrm{E}(Y_{2t}|\mathbf{Y}_{t-1})$.

- Conditional variance: $\mathrm{Var}\left(Y_{jt}|\mathbf{Y}_{t-1}\right) = \sigma^2_{u,j}$, $j = 1, 2$.

- Conditional covariance: $\mathrm{Cov}(Y_{1t}, Y_{2t}|\mathbf{Y}_{t-1}) = \sigma^2_{u,12} \Rightarrow$ simultaneous correlation of Y_{1t} and Y_{2t}, if $\sigma^2_{u,12} \neq 0$.

- Regression model for Y_{1t} and Y_{2t} with identical predictors $Y_{1,t-1}$ and $Y_{2,t-1}$, however different coefficients.

Stationary VAR(1)-Models

- $\mathbf{c}$ is a vector with parameters c_1 and c_2 which are unconstrained.

- $\mathbf{\Sigma}$ is a covariance matrix, hence a symmetric matrix with the 3 parameters $\sigma^2_{u,1}$, $\sigma^2_{u,21}$, and $\sigma^2_{u,2}$, such that the simultaneous correlation $\rho_{12}(0) = \sigma^2_{u,21}/(\sigma_{u,1}\sigma_{u,2})$ lies between -1 and 1.

- $\mathbf{\Phi}$ is a square matrix with the 4 parameters Φ_{11}, Φ_{12}, Φ_{21}, and Φ_{22}. For a stationary bivariate VAR(1)-model, the parameters in $\mathbf{\Phi}$ are selected in such a way, that the resulting process is second order stationary.

Not all matrices $\mathbf{\Phi}$ lead to stationary processes.

Stationarity condition

All eigenvalues of the matrix $\mathbf{\Phi}$ lie within the unit circle.

Under this condition the following holds:

- If the errors are normally distributed, then the VAR(1)-process is strictly stationary

- If the errors are not normally distributed, but are second order stationary, i.e. $\mathrm{E}(\boldsymbol{u}_t) = \mathbf{0}$ and $\mathrm{Var}\left(\boldsymbol{u}_t\right) = \mathbf{\Sigma}$, then the VAR(1)-process is second order stationary.

Long-run mean $\mu_j = \mathrm{E}(Y_{jt})$ and variance $\mathrm{Var}\left(Y_{jt}\right)$ and all cross correlations $\rho_{12}(h)$ available from $\mathbf{\Phi}$, $\mathbf{c}$, and $\mathbf{\Sigma}$.

Stationarity condition

For instance, the long-run mean $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, where $\mathrm{E}(Y_{jt}) = \mu_j$, satisfies the equation:

$$\mathrm{E}(\mathbf{Y}_t) = \boldsymbol{\Phi}\mathrm{E}(\mathbf{Y}_{t-1}) + \mathbf{c} + \mathrm{E}(\boldsymbol{u}_t),$$
$$\boldsymbol{\mu} = \boldsymbol{\Phi}\boldsymbol{\mu} + \mathbf{c},$$

which may be solved for $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} = (\mathbf{I} - \boldsymbol{\Phi})^{-1}\mathbf{c},$$

where $\mathbf{I}$ is the identity matrix.

Computation of $\mathrm{Var}\,(Y_{jt})$ and $\rho_{12}(h)$ is more involved, but possible.

Stationarity condition

Use $\mathbf{c} = (\mathbf{I} - \mathbf{\Phi})\boldsymbol{\mu} = \boldsymbol{\mu} - \mathbf{\Phi}\boldsymbol{\mu}$ to represent a stationary VAR(1)-process as deviations from the long-run mean:

$$\mathbf{Y}_t - \boldsymbol{\mu} = \mathbf{\Phi}(\mathbf{Y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{u}_t.$$

Rewrite this representation as a system of two equations:

$$Y_{1t} - \mu_1 = \Phi_{11}(Y_{1,t-1} - \mu_1) + \Phi_{12}(Y_{2,t-1} - \mu_2) + u_{1t},$$
$$Y_{2t} - \mu_2 = \Phi_{21}(Y_{1,t-1} - \mu_1) + \Phi_{22}(Y_{2,t-1} - \mu_2) + u_{2t}.$$

Example: individual AR(1)-Modeling

$$\mathbf{\Phi} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

$$\mathrm{Var}\left(\mathbf{Y}_t\right) = \begin{pmatrix} 5.5556 & 0 \\ 0 & 4 \end{pmatrix}, \quad \boldsymbol{\rho}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example: individual AR(1)-Modeling

Cross correlation between Y_{jt} and $Y_{k,t-h}$

h	1	2	3	4	5
$j=k=1$	0.800	0.640	0.512	0.410	0.328
$j=1,k=2$	0	0	0	0	0
$j=2,k=1$	0	0	0	0	0
$j=k=2$	0.500	0.250	0.125	0.063	0.031

Example: bivariate VAR(1)-Model

$$\mathbf{\Phi} = \begin{pmatrix} 0.5 & 0.2 \\ -0.3 & 0.7 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix},$$

$$\mathrm{Var}\left(\mathbf{Y}_t\right) = \begin{pmatrix} 1.6341 & 0.8116 \\ 0.8116 & 1.5808 \end{pmatrix}, \quad \boldsymbol{\rho}(0) = \begin{pmatrix} 1 & 0.5049 \\ 0.5049 & 1 \end{pmatrix}$$

Example: bivariate VAR(1)-Model

Cross correlation between Y_{jt} and $Y_{k,t-h}$

h	1	2	3	4	5
$j=k=1$	0.599	0.309	0.125	0.024	-0.023
$j=1,k=2$	0.449	0.332	0.214	0.121	0.057
$j=2,k=1$	0.049	-0.149	-0.199	-0.177	-0.131
$j=k=2$	0.546	0.245	0.070	-0.016	-0.048

VAR(1)-Model

VAR(1)-Modeling of m time series:

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1m} \\ \vdots & \ddots & \vdots \\ \Phi_{m1} & \cdots & \Phi_{mm} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix},$$

$$\boldsymbol{u}_t = \begin{pmatrix} u_{1t} \\ \vdots \\ u_{mt} \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \sigma^2_{u,11} & \cdots & \sigma^2_{u,1m} \\ \vdots & \ddots & \vdots \\ \sigma^2_{u,m1} & \cdots & \sigma^2_{u,mm} \end{pmatrix}.$$

VAR(1)-Model

$\mathbf{\Phi}$ has m^2 parameters, $\mathbf{\Sigma}$ has $m(m+1)/2$ parameters, $\mathbf{c}$ has m free parameters. Total number of parameters is equal to $m^2 + m(m+1)/2 + m$ and increases rapidly:

- $m = 2$: 9 parameters
- $m = 3$: 18 parameters
- $m = 4$: 30 parameters

Trivariate VAR(1)-Model

Rewrite this representation as a system of three equations:

$$
\begin{aligned}
Y_{1t} &= \Phi_{11}Y_{1,t-1} + \Phi_{12}Y_{2,t-1} + \Phi_{13}Y_{3,t-1} + c_1 + u_{1t}, \\
Y_{2t} &= \Phi_{21}Y_{1,t-1} + \Phi_{22}Y_{2,t-1} + \Phi_{23}Y_{3,t-1} + c_2 + u_{2t}, \\
Y_{3t} &= \Phi_{31}Y_{1,t-1} + \Phi_{32}Y_{2,t-1} + \Phi_{33}Y_{3,t-1} + c_3 + u_{3t}.
\end{aligned}
$$

Regression model for Y_{1t}, Y_{2t}, and Y_{3t} with identical predictors $Y_{1,t-1}$, $Y_{2,t-1}$, and $Y_{3,t-1}$, however different coefficients.

VAR(p)-Process

Definition:

$$\mathbf{Y}_t = \mathbf{\Phi}_1 \mathbf{Y}_{t-1} + \ldots + \mathbf{\Phi}_p \mathbf{Y}_{t-p} + \mathbf{c} + \boldsymbol{u}_t, \quad \boldsymbol{u}_t \sim \mathrm{Normal}\left(\mathbf{0}, \mathbf{\Sigma}\right)$$

- $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_p$ are $m \times m$-Matrizen $\Rightarrow pm^2$ parameters
- If $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_p$ and $\mathbf{\Sigma}$ are diagonal matrices, then individual AR(p)-modeling of each time series results.

Trivariate VAR(2)-Model

Rewrite this representation as a system of three equations:

$$
\begin{aligned}
Y_{1t} &= \Phi_{1,11}Y_{1,t-1} + \Phi_{1,12}Y_{2,t-1} + \Phi_{1,13}Y_{3,t-1} \\
&+ \Phi_{2,11}Y_{1,t-2} + \Phi_{2,12}Y_{2,t-2} + \Phi_{2,13}Y_{3,t-2} + c_1 + u_{1t}, \\
Y_{2t} &= \Phi_{1,21}Y_{1,t-1} + \Phi_{1,22}Y_{2,t-1} + \Phi_{1,23}Y_{3,t-1} \\
&+ \Phi_{2,21}Y_{1,t-2} + \Phi_{2,22}Y_{2,t-2} + \Phi_{2,23}Y_{3,t-2} + c_2 + u_{2t}, \\
Y_{3t} &= \Phi_{1,31}Y_{1,t-1} + \Phi_{1,32}Y_{2,t-1} + \Phi_{1,33}Y_{3,t-1} \\
&+ \Phi_{2,31}Y_{1,t-2} + \Phi_{2,32}Y_{2,t-2} + \Phi_{2,33}Y_{3,t-2} + c_3 + u_{3t}.
\end{aligned}
$$

Conditional expectation $\mathrm{E}(Y_{jt}|\mathbf{Y}_{t-1}, \mathbf{Y}_{t-2})$ depends not only on $Y_{j,t-1}$ and $Y_{j,t-2}$, but also on $Y_{k,t-1}$ and $Y_{k,t-2}$.