Econometrics III

Sylvia Frühwirth-Schnatter

Department of Finance, Accounting and Statistics
Vienna University of Economics and Business
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Advanced Econometrics

- Modelling Volatility: ARCH and GARCH Models
- Multivariate time series (spurios regression, cointegration, VAR)
- Endogeneity and IV estimation
- Panel data analysis

Wooldridge, Introductory Econometrics, Thompson 2009; Hackl, Einführung in die Ökonometrie, Pearson Verlag, 2005.

Mile stone I

Modelling Volatility: ARCH and GARCH Models

- Modeling (conditional) volatility
- ARCH models
- GARCH models

Modeling (conditional) volatility

- In finance, the volatility is the standard deviation of logarithmic returns derived for some financial instrument (stock prices, exchange rates).
- In many econometric time series such as returns the variance of the disturbance term is not constant, i.e. the assumption of homoscedasticity is violated.
- ARIMA models are models for the conditional mean of a time series. ARCH and GARCH models are models for the conditional variance.

EVIEWS Exercise : Case Study `nasdaq`

- Monthly equity prices p_t of a US firm listed at the NASDAQ (January 1992 to December 2011)

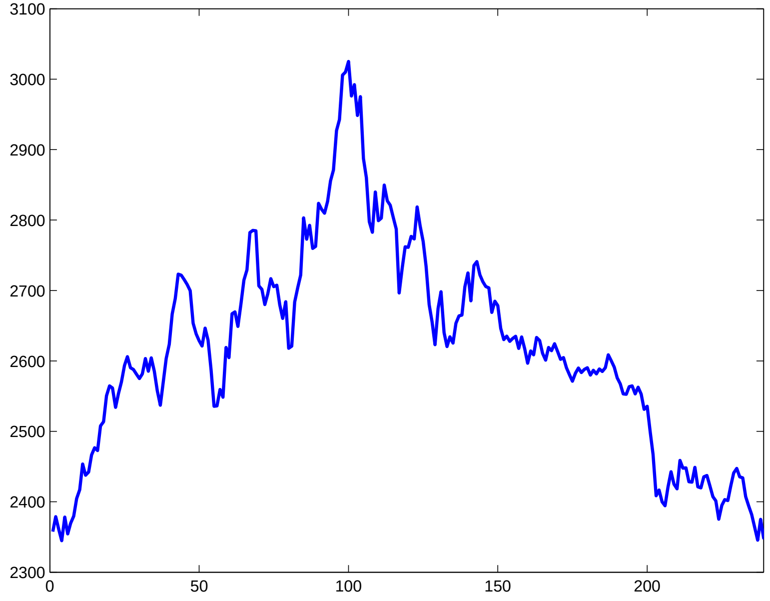
- Is the Random Walk $Y_t = Y_{t-1} + u_t$ a good model for the time series $Y_t = \log p_t$?

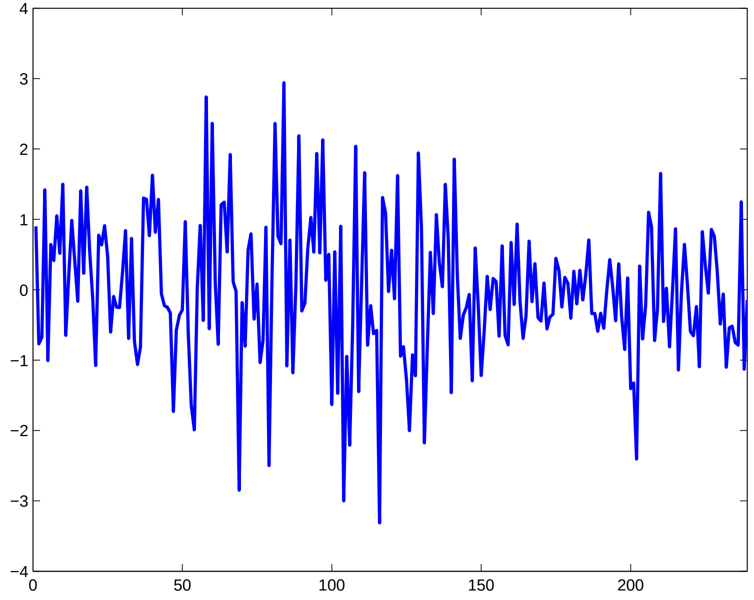
- $\hat{\sigma}^2 = 1.0015$

- Are the returns u_t autocorrelated?

- Is the variance $\mathrm{Var}(u_t)$ of the returns constant?

EVIEWS Exercise : Case Study `nasdaq`





The returns of `nasdaq` has periods with smaller variance than $\hat{\sigma}^2 = 1.0015$ and periods with larger variance than $\hat{\sigma}^2$.

EVIEWS Exercise : Case Study `nasdaq`

- The series shows volatility clusters, i.e. small and high volatility persists for many months.

- The uncertainty of short-term forecasts derived from a constant volatility model is overestimated in periods of small volatility and underestimated in periods of high volatility

- Periods of high volatility are periods of great potential and high risk. Capturing dependence in the volatility would be important.

EVIEWS Exercise : Case Study `nasdaq`

- How to measure dependence (autocorrelation) in volatility (which is unobserved)?

- The volatility $(u_t - \mu)^2$ (deviation from the mean), seems to be correlated

- Consider correlogram of the squared residuals u_t^2 and test for autocorrelation.

ARCH(1) processes

- In an Autoregressive Conditional Heteroscedasticity, or ARCH Model the variance of Y_t is modelled as a function of past values of the time series.

- Y_t is an ARCH(1) process with mean μ if u_t is a white noise process with $\mathrm{Var}(u_t) = 1$ and

$$Y_t - \mu = u_t \cdot \sqrt{\alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2}, \qquad \alpha_0, \alpha_1 > 0$$

Hence,

$$\sigma_t^2 = \mathrm{Var}(Y_t | Y_{t-1}) = \alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2.$$

ARCH(1) processes

- An ARCH(1) process is stationary, if $\alpha_1 < 1$.

- Moments of a stationary ARCH(1) Process

$$E(Y_t) = \mu, \qquad \mathrm{Var}(Y_t) = \frac{\alpha_0}{1 - \alpha_1}$$

- The conditional mean equals the long-run mean

$$E(Y_t|Y_{t-1}, \ldots,) = E(Y_t) = \mu$$

- The conditional variance $\mathrm{Var}(Y_t|Y_{t-1})$ differs from the marginal variance:

ARCH(1) processes

- The conditional variance $\mathrm{Var}(Y_t|Y_{t-1}) = \alpha_0 + \alpha_1(Y_{t-1} - \mu)^2$ grows with the deviation of the past value Y_{t-1} from the long-run mean μ. In particular:

$$\mathrm{Var}(Y_t|Y_{t-1}) > \mathrm{Var}(Y_t) \Leftrightarrow (Y_{t-1} - \mu)^2 > \mathrm{Var}(Y_t)$$

Proof:

$$\mathrm{Var}(Y_t|Y_{t-1}) = \alpha_0 + \alpha_1(Y_{t-1} - \mu)^2 > \frac{\alpha_0}{1-\alpha_1} = \mathrm{Var}(Y_t)$$

$$\Leftrightarrow \alpha_0 - \alpha_0\alpha_1 + \alpha_1(1-\alpha_1)(Y_{t-1} - \mu)^2 > \alpha_0$$

$$\Leftrightarrow (1-\alpha_1)(Y_{t-1} - \mu)^2 > \alpha_0$$

ARCH(1) processes

- Y_t is not correlated with Y_{t-s} for $s \neq 0$

$$E(Y_t - \mu)(Y_{t-s} - \mu) = 0$$

- Adjacent values however are not independent, as the conditional variance of Y_t depends on Y_{t-1}

- Squared residuals follow an AR(1) process.

- The actual volatility persists for a long time if α_1 is close to 1.

ARCH(1) processes

The process of squared residuals $(Y_t - \mu)^2$ is correlated as

$$\begin{aligned}(Y_t - \mu)^2 = u_t^2 \cdot \big(\alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2\big) = \\ = \alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2 + u_t^*\end{aligned}$$

where $u_t^* = (u_t^2 - 1)\big(\alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2\big)$

- $E(u_t^*) = 0$
- u_t^* is uncorrelated

Normal ARCH(1)Processes

Assumes that $u_t \sim$ i.i.d. $\mathrm{Normal}\,(0,1)$ is normal.

- The conditional distribution of $Y_t|Y_{t-1}$ is a normal distribution with mean μ and variance $\sigma_t^2 = \alpha_0 + \alpha_1(Y_{t-1} - \mu)^2$

- The marginal distribution of Y_t however is not normal
  - its 3. moment is 0 $\Longrightarrow$ symmetric
  - its kurtosis (4. central moment) is $K = \frac{3(1-\alpha_1^2)}{1-3\alpha_1^2}$

    K is finite if $3\alpha_1^2 < 1$, otherwise infinite

  The marginal distribution has fatter tails than the normal distribution

ARCH(r) Processes

- In an ARCH(r) process the conditional variance does not only depend on Y_{t-1}, but also on $Y_{t-1}, \ldots, Y_{t-r}$

- $Y_t = \mu + u_t \sigma_t$ is an ARCH(r) process with mean μ, if u_t is a white noise process with $\mathrm{Var}(u_t) = 1$ and

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{r} \alpha_i (Y_{t-i} - \mu)^2, \qquad \alpha_0, \alpha_1, \ldots, \alpha_r > 0$$

  - r is the order of the ARCH process
  - $\alpha_0, \alpha_1, \ldots, \alpha_r$ are the ARCH parameters

ARCH(r) Processes

- An ARCH(r) process is stationary, if $\sum_{i=1}^{r} \alpha_i < 1$
- The moments of a stationary ARCH(r) process are

$$\mathrm{E}(Y_t) = \mu$$

$$\mathrm{Var}(Y_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{r} \alpha_i}$$

- $(Y_t - \mu)^2$ is an autoregressive process of order r

$$(Y_t - \mu)^2 = \alpha_0 + \sum_{i=1}^{r} \alpha_i (Y_{t-i} - \mu)^2 + u_t^*$$

EVIEWS Exercise : Case Study `nasdaq`

Discuss fitting of ARCH(r) processes in EViews.

- Inspect the marginal distribution of `dlog(nasdaq)`
- Fit ARCH(r) processes of different order to `dlog(nasdaq)`
- Determine whether the process is stationary.

Estimation of ARCH(r) Processes

An ARCH(r) model $Y_t - \mu \sim \text{Normal}\left(0, \sigma_t^2(\alpha_0, \alpha_1, \ldots, \alpha_r)\right)$, where

$$\sigma_t^2(\alpha_0, \alpha_1, \ldots, \alpha_r) = \alpha_0 + \sum_{i=1}^{r} \alpha_i (Y_{t-i} - \mu)^2$$

is to be fitted to the time series $y_1, \ldots, y_T$

Estimation Problem:

- How can the parameters $\theta = (\mu, \alpha_0, \alpha_1, \ldots, \alpha_r)$ be estimated from the data?

- What are the statistical properties of the estimator?

Estimation of ARCH(r) Processes

- Maximize the log of the likelihood-function

$$p(y_1, \ldots, y_T|\theta) = p(y_T|y_{T-1}, \cdot)p(y_{T-1}|y_{T-2}, \cdot) \cdots p(y_{r+1}|y_r, \ldots, y_1),$$

which is given by:

$$l(\theta) = \text{constant} - \frac{1}{2}\Big(\sum_{t=r+1}^{T} \frac{(y_t - \mu)^2}{\sigma_t^2(\alpha_0, \alpha_1, \ldots, \alpha_r)} + \ln \sigma_t^2(\alpha_0, \alpha_1, \ldots, \alpha_r)\Big)$$

- The log-likelihood-function is maximized numerically using an iterative algorithm

Statistical properties of the ML estimator

- The MLE is consistent and for normal ARCH models also efficient
- Standard errors of the MLE can be determined from the Hesse Matrix of the log-likelihood function $J(\theta) = \left(\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta'}\right)^{-1}$
- ML-estimators are asymptotically normally distributed (under mild regularity assumptions)
- To test hypotheses about the parameters $t-$ and $F-$statistics can be used.

Selection of model order

- Overfitting : Choose r large and test $H_0 : \alpha_r = 0$

- Model selection criteria: choose the model where r minimizes the AIC or the Schwarz criterion

$$AIC = -\frac{2}{n} \log l(\hat{\theta}) + \frac{2k}{n}$$
$$SC = -\frac{2}{n} \log l(\hat{\theta}) + \frac{\ln n \cdot k}{n},$$

  where $\hat{\theta}$ is the ML estimator and k is the number of parameters, e.g. $k = r + 2$, if $\mu \neq 0$.

Model Checking

Model checking is based on the standardized prediction error

$$r_t = \frac{(y_t - \hat{\mu})^2}{\sigma_t^2(\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_r, \hat{\mu})}$$

- no autocorrelation of r_t: correlogram of r_t
- no autocorrelation of r_t^2: correlogram of r_t^2
- normal distribution of r_t (validity of standard errors): histogramm, Jarque-Bera
- homoscedasticity of r_t: time series plot of r_t (no more volatility clusters)

Violations of model assumptions

Autocorrelations in r_t:

- the model for the conditional mean $E(Y_t|Y_{t-1},\dots) = \mu$ is not appropriate $\Longrightarrow$ Combine the ARCH Model for the conditional variance with an ARMA Model for the conditional mean

Autocorrelations in r_t^2:

- Model order is to small $\Longrightarrow$ Increase the model order
- Use a different model specification $\Longrightarrow$ Use a GARCH Model

Violations of model assumptions

r_t is not normally distributed:

- MLEs are consistent, but not efficient
- Standard errors are biased (usually downwards, i.e. underestimated)
- Correct estimators of standard errors are obtained by multiplication of the matrix J with a correction matrix C as $J^* = J \cdot C$
- Using the option `Heteroskedasticity Consistent Covariance` in EVIEWS produces corrected standard errors

GARCH Models

- In a generalized ARCH model, or GARCH model the variance of Y_t is modelled as a function of past deviations of Y_t from the mean and past values of the variance.

- Y_t is a GARCH(1,1) process with mean μ if u_t is a white noise process with $\text{Var}(u_t) = 1$ and

$$Y_t - \mu = u_t \cdot \sigma_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2 + \gamma_1 \sigma_{t-1}^2, \qquad \alpha_0, \alpha_1, \gamma_1 \geq 0$$

- The ARCH(1) process is a special case of the GARCH(1,1) process for $\gamma_1 = 0$

The GARCH(1,1) Process

- A GARCH(1,1) process is stationary, if

$$\alpha_1 + \gamma_1 < 1$$

- Moments of a stationary GARCH(1,1) Process

$$E(Y_t) = \mu, \quad \text{Var}(Y_t) = \frac{\alpha_0}{1 - \alpha_1 - \gamma_1}$$

- The conditional mean equals the long-run mean

$$E(Y_t|Y_{t-1}, \ldots,) = E(Y_t) = \mu$$

Conditional variance of the GARCH(1,1) process

The conditional variance of the GARCH(1,1) process is

$$\mathrm{Var}(Y_t|Y_{t-1},\ldots,) = \sigma_t^2 = \alpha_0 + \alpha_1(Y_{t-1}-\mu)^2 + \gamma_1\sigma_{t-1}^2$$

Substituting σ_{t-1}^2 gives

$$\sigma_t^2 = \alpha_0 + \alpha_1(Y_{t-1}-\mu)^2 + \gamma_1\Big(\alpha_0 + \alpha_1(Y_{t-2}-\mu)^2 + \gamma_1\sigma_{t-2}^2\Big)$$

Substituting $\sigma_{t-2}^2, \sigma_{t-3}^2, \ldots$ successively shows that a GARCH(1,1) model corresponds to an ARCH model of order ∞

$$\sigma_t^2 = \frac{\alpha_0}{1-\gamma_1} + \alpha_1 \sum_{j=1}^{\infty} \gamma_1^{j-1}(Y_{t-j}-\mu)^2 = \alpha_0^* + \sum_{j=1}^{\infty} \alpha_j^*(Y_{t-j}-\mu)^2$$

Conditional variance of the GARCH(1,1) process

The coefficients α_j^* of $(Y_{t-j} - \mu)^2$ decay exponentially:

Lag	squared deviation	α_j^*
1	$(Y_{t-1} - \mu)^2$	α_1
2	$(Y_{t-2} - \mu)^2$	$\alpha_1 \gamma_1 < \alpha_1$
3	$(Y_{t-3} - \mu)^2$	$\alpha_1 \gamma_1^2 < \alpha_1 \gamma_1$

The actual information ($=$ squared deviation from the mean) has the largest weight, information into the past is down weighted with $\gamma_1 (< 1)$ for each step.

In classical variance estimation, all squared deviation have equal weight

$$\hat{\sigma}^2 = \frac{1}{t-2} \sum_{i=1}^{t-1} (Y_{t-i} - \mu)^2$$

Autocorrelation

- The process $(Y_t - \mu)$ is not autocorrelated

- The process of squared residuals $(Y_t - \mu)^2$ however is correlated and has a representation as an ARMA(1,1) process:

$$(Y_t - \mu)^2 = \alpha_0 + (\alpha_1 + \gamma_1)(Y_{t-1} - \mu)^2 + u_t^* - \gamma_1 u_{t-1}^*$$

  where u_t^* is white noise process.

- The actual volatility persists for a long time if $\alpha_1 + \gamma_1$ is close to 1.

GARCH(r,s) Processes

- $Y_t = \mu + u_t \cdot \sigma_t$ is a GARCH(r,s) process with mean μ, if u_t is a white noise process and

$$\sigma_t^2 = \alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2 + \cdots + \alpha_r (Y_{t-r} - \mu)^2 + \\ + \gamma_1 \sigma_{t-1}^2 + \cdots + \gamma_s \sigma_{t-s}^2$$

- The GARCH(r,s) process is stationary if

$$\sum_{i=1}^{r} \alpha_i + \sum_{j=1}^{s} \gamma_j < 1$$

GARCH(r,s) Processes

- Moments of a stationary GARCH(r,s) Process

$$E(Y_t) = \mu, \qquad \mathrm{Var}(Y_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{r} \alpha_i - \sum_{j=1}^{s} \gamma_j}$$

- The process of squared residuals $(Y_t - \mu)^2$ is an ARMA(r,s) process

- Parameters of a GARCH(r,s) process are estimated by the ML-method.

- Model selection using overfitting and AIC/Schwarz criterion

EVIEWS Exercise : Case Study `exchange rates`

Daily exchange rate of the Swedish krona against the Euro (January 3, 2000 to April 4, 2012)

- Discuss fitting an ARCH(1) model and a GARCH(1,1) model in EViews
- Discuss forecasting with a GARCH(1,1) model
- Compare volatility forecasts with ARCH(1) and GARCH(1,1) models
- Discuss fitting of GARCH(r,s) models and forecasting with GARCH(r,s) models in EViews

ARMA and regression models with ARCH errors

- ARMA models and regression models are models for the conditional mean - the variance is assumed to be constant (homoscedasticity)

- GARCH Models are models for the conditional variance, the conditional mean is assumed to be constant

- Combining both types of models allows to model the conditional mean and the conditional variance