Superreplication in Stochastic Volatility Models and Optimal Stopping *

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Abstract

In this paper we discuss the superreplication of derivatives in a stochastic volatility model under the additional assumption that the volatility follows a bounded process. We characterize the value process of our superhedging strategy by an optimal-stopping problem in the context of the Black-Scholes model which is similar to the optimal stopping problem that arises in the pricing of American-type derivatives. Our proof is based on probabilistic arguments. We study the minimality of these superhedging strategies and discuss PDE-characterizations of the value function of our superhedging strategy. We illustrate our approach by examples and simulations.

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1 Introduction

The pricing and hedging of derivative securities is nowadays well-understood in the context of the classical Black-Scholes model of geometric Brownian motion. However, recent empirical research has produced a lot of statistical evidence that is difficult to reconcile with the assumption of independent and normally distributed asset returns. Researchers have therefore attempted to build models for asset price fluctuations that are flexible enough to cope with these empirical deficiencies of the Black-Scholes model. In particular, a lot of work has been devoted to relaxing the assumption of constant volatility in the Black-Scholes model and there is a growing literature on stochastic volatility models (SV-models); see e.g. Ball and Roma (1994) or Frey (1997) for surveys. In this class of models the stochastic differential equation (SDE) that governs the asset price process is driven by a Brownian motion, but the diffusion coefficient of this SDE is modelled as a stochastic process which is only imperfectly correlated to the Brownian motion driving the asset price process.

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SV-models are able to capture the succession of periods with high and low activity we observe in financial markets. However, this increase in realism raises new conceptual problems for the pricing and the hedging of derivative securities: It is well-known that SV-models are *incomplete*, i.e. one cannot replicate the payoff of a typical derivative by dynamic trading in the underlying risky asset ("the stock") and in some riskless money market account. This reflects a real difficulty in the risk management of derivative securities and should therefore not be considered as a disadvantage of this class of models. Today "the uncertain nature of forward volatility is recognized as one of the main factors that drive market-making in options and custom-tailored derivatives;" see Avellaneda and Paras (1996).

Of course, if there is a liquid market for certain standard derivative securities on the stock, the use of dynamic trading strategies in the stock and in these securities might restore market completeness. However, this approach is not always viable. To begin with, there is not always trade in a sufficient number of derivative securities on a particular stock. Even if there are derivative securities available for trading, running a dynamic hedging strategy in these securities might prove impossible because of prohibitive transaction costs. Moreover, this approach requires a precise parametric model for the volatility dynamics of the underlying asset. As volatility is not directly observable, the determination of a good model for the volatility dynamics and the estimation of the corresponding parameters poses difficult problems. Hence there is a considerable risk of model misspecification that might lead to "bad" hedges. This favours approaches to the risk-management of derivative securities which require *dynamic* hedging only in the underlying risky asset and in the money market account; *static* positions in liquidly traded derivatives can then be used in a second step in order to improve the accuracy and reduce the cost of the hedge. Results from the theory of superhedging imply that even in an incomplete market it is possible to "stay on the safe side" by using a particular dynamic trading strategy in the underlying stock and in the money market account; see e.g. Delbaen (1992) or El Karoui and Quenez (1995) for results on continuous processes, and Kramkov (1996) for generalisations to a general semimartingale framework. The cost of implementing such a superhedging strategy is given by the supremum of the expected value of the terminal payoff over all equivalent local martingale measures for the underlying asset.

Unfortunately the concept of superhedging often leads to prices that are too high from a practical viewpoint. For instance Frey and Sin (1999) and Cvitanic, Pham, and Touzi (1997) show that in a typical SV-model where volatility follows an unbounded diffusion process the cost of establishing such a superhedge for a European call option is no smaller than the current price of the underlying stock; hence in this class of models the cheapest superhedging strategy for a European call option is to buy the underlying asset. Additional assumptions are therefore called for, if one wants to obtain superhedging strategies which are at least potentially of some practical interest. In this paper we restrict ourselves to SV-models where the range of the volatility is bounded. Under this additional assumption we are able to obtain "nontrivial" superhedging strategies for a large class of derivatives whose payoff may even be path-dependent. These strategies are universal in the sense that they depend only on the bounds we impose on the volatility and not on a particular parametric model for the volatility dynamics. We characterize the value process of our superhedging strategy by an optimal-stopping problem in the context of the Black-Scholes model. Roughly speaking our result can be phrased as follows: the value of a superhedging strategy for a *European type* derivative under *stochastic volatility* equals value of a corresponding *American type* derivative under *constant volatility.* In particular one can draw on standard numerical methods for the pricing of American type securities to implement our approach. The proof is based on probabilistic arguments. Our main tools are the optional decomposition theorem of El Karoui and Quenez or Kramkov and the results on time-change for continuous martingales.

In practice it may be impossible to determine finite bounds on asset price volatility which hold true with certainty. In those cases we interprete our volatility band as confidence interval for the range of the future volatility. By construction the success-set of our strategy — the set where the terminal value of the hedge portfolio is no smaller than the the payoff of the derivative — contains all asset price trajectories with volatility lying in the volatility band. Moreover, our approach is relatively robust: if the actual volatility exceeds one of the volatility bounds by a small amount the resulting loss will typically be small. Recently Föllmer and Leukert (1998) have developed a general theory of superhedging with a given success probability. In their approach the success set is endogenously determined; it minimizes the superhedging cost over all strategies with a given success probability. This yields a very elegant theory. However, by construction the terminal value of the hedge portfolio is zero on the complement of the success-set. Hence in the approach of Föllmer and Leukert the occurrence of an event belonging to the complement of the success-set may immediately lead to large losses.

It is important to know, if for a given parametric SV-model superhedging strategies can be constructed which are less expensive than our universal superhedging strategy. In Section 3 we study this question for a particular class of SV-models where volatility follows a one-dimensional diffusion. Most parametric models from the financial literature belong to this class. We show that our universal superhedging strategy is in fact a minimal superhedging strategy, provided that the bounds on volatility are sharp and that the lower volatility bound is zero. This generalizes the main result of Frey and Sin (1999); it extends also certain results of Cvitanic, Pham, and Touzi (1997) to path-dependent derivatives.

In most work on superreplication in SV-models with bounded volatility the superhedging cost is characterized by a terminal value problem involving a parabolic PDE, which is in general nonlinear. Important examples of this work are El Karoui, Jeanblanc-Picqué, and Shreve (1998), Avellaneda, Levy, and Paras (1995) and Lyons (1995). In Section 4 we therefore discuss under which conditions the value function of our superhedging strategy can be characterized in terms of some nonlinear parabolic PDE. This gives us also information on the minimality of our universal superhedging strategy in models where the lower volatility bound is strictly positive.

In order to illustrate our approach to superhedging we compute in Section 5 for certain examples the value function of our strategy. We present simulations for the superreplication cost of a call spread and compare our results to those of Avellaneda, Levy, and Paras (1995). We give analytic results on the superhedging cost for a particular barrier option, namely the down-and-out call option. Finally we present an example that shows how static positions in traded derivatives can be used for a reduction of the superhedging cost, an idea which is explored more systematically in Avellaneda and Paras (1996).

2 Superreplication strategies and optimal stopping

2.1 The general stochastic volatility model

We consider a frictionless financial market with continuous security trading where a risky asset (the stock) and a zero coupon bond with maturity T are traded. In our model the short rate of interest is deterministic and given by some constant $r \geq 0$ such that $B(t,T)$, the price of the zero coupon bond at time t, is given by $B(t,T) = \exp(-r(T-t))$. The price of the stock is modelled as a stochastic process $S = (S_t)_{t\geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with $(\mathcal{F}_t)$ satisfying the usual conditions. For the purposes of this paper it is legitimate to assume that P is already a risk-neutral measure for S. More precisely, we assume that the dynamics of S are of the following form:

Assumption 1. (general stochastic volatility model) *Consider an $(\mathcal{F}_t)$-predictable process $(\sigma_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with $P[\sigma_t > 0 \text{ for all } t] = P[\int_0^t \sigma_s^2 ds < \infty \text{ for all } t] = 1$. The stock price process S solves the SDE*

$$dS_t = S_t(\sigma_t dW_t + r dt) \tag{2.1}$$

for a Brownian motion $(W_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$.

This class of SV-models is very general. In fact, it can be shown that in every arbitrage-free asset price model where the price process follows continuous trajectories with absolutely continuous quadratic variation the asset price dynamics are of the form (2.1); see for instance Gallus (1996). Obviously Assumption 1 is satisfied by most SV-models from the financial literature where volatility is assumed to follow a one-dimensional diffusion; see Section 3 for examples.

Fix some maturity date T. By $Z_t := e^{r(T-t)} S_t$ we denote the price of the *forward contract* on S with maturity T. The following set of probability measures Q equivalent to P on $(\Omega, \mathcal{F}_T)$ will be important:

$$\mathbb{M}^e := \{Q \mid Q \sim P \text{ and } (Z_t)_{0\leq t\leq T} \text{ is a } Q\text{-local martingale}\}.$$

For further use we also define the process $M_t := \int_0^t \sigma_s dW_s$. M is a continuous local martingale under all $Q \in \mathbb{M}^e$ with quadratic variation $\langle M \rangle_t = \int_0^t \sigma_s^2 ds$. By Itô's formula S is given by $S_t = S_0 \exp(rt + M_t - \frac{1}{2}\langle M \rangle_t)$.

Remark 2.1. We will use the following notation:

(i) For a process X which is cadlag we put

$$X_{[0,t]}^{\min} := \min_{0\leq s\leq t} X_s \text{ and } X_{[0,t]}^{\max} := \max_{0\leq s\leq t} X_s .$$

(ii) Let $(\mathcal{G}_t)$ be a filtration on some probability space $(\Omega, \mathcal{F}, P)$ and τ_1 and τ_2 be $(\mathcal{G}_t)$-stopping times such that $\tau_1 \leq \tau_2$ a.s. We denote by $\mathcal{G}_{\tau_1,\tau_2}$ the set of all finite $(\mathcal{G}_t)$-stopping times τ with $\tau_1 \leq \tau \leq \tau_2$ a.s.

(iii) By $(\mathcal{B}_t)$ we denote the canonical filtration on $\mathcal{C}_{[0,\infty)}$, the space of all continuos functions from $[0,\infty)$ to $\mathbb{R}$.

2.2 Construction of superreplication strategies via optimal stopping

We consider the following class of contingent claims:

Assumption 2. *The payoff H of the contingent claim is of the form*

$$H = f\left(Z_T, Z^{\min}_{[0,T]}, Z^{\max}_{[0,T]}\right) \tag{2.2}$$

for some function $f : \mathbb{R}^3_+ \to \mathbb{R}$ such that the process $f_t := f\left(Z_t, Z^{\min}_{[0,t]}, Z^{\max}_{[0,t]}\right)_{0\le t\le T}$ is bounded below and cadlag.

This class of payoffs comprises all path-independent options. Most common path-dependent options also satisfy Assumption 2, if we assume that the payoff is defined as a function of the forward price Z of the stock. For instance the payoff of barrier options with barrier condition imposed on Z is of the form (2.2). Note that the payoff of a portfolio of derivatives where each individual contract is of the form (2.2) is again of this form. This facilitates the application of our method to portfolios of derivatives.

We now give a formal definition of superreplication strategies.

Definition 2.2. *Consider a contingent claim with maturity date T whose payoff H is bounded below. A dynamic trading strategy $(\xi, \eta) = (\xi_t, \eta_t)_{0\le t\le T}$ in stock and bond is a superreplicating or superhedging strategy for H if*

(i) *The strategy is admissible, i.e. ξ is predictable, η is adapted, the integral $\int_0^t \xi_s^2 \sigma_s^2 S_s^2 ds$ is a.s. finite and the value process $V_t = \xi_t S_t + \eta_t B(t,T)$ is bounded below.*

(ii) *The terminal value V_T of the strategy equals H. Moreover, the cost process associated with the strategy is nonincreasing, i.e. we have for all $0 \le t \le T$ the representation*

$$V_t = V_0 + \int_0^t \eta_s r B(s,T) ds + \int_0^t \xi_s dS_s + C_t \tag{2.3}$$

for an non-increasing process $C = (C_t)_{0\le t\le T}$ with $C_0 = 0$.

A superreplicating strategy $(\hat{\xi}, \hat{\eta})$ whose value process $\hat{V}$ satisfies for all $0 \le t \le T$ and for any other superhedging strategy (ξ, η) with value process V the inequality $\hat{V}_t \le V_t$ is called minimal. The value process $\hat{V}$ of this strategy, which is uniquely defined, is called the ask-price of the claim H.

Remark 2.3. Superhedging strategies can be characterized using discounted quantities, in our case most conveniently in terms of the forward price $Z_t = S_t/B(t,T)$: A strategy (ξ, η) with value process V is a superreplicating strategy for H if and only if the discounted value process $\tilde{V}_t = V_t/B(t,T) = \xi_t Z_t + \eta_t$ admits a representation of the form

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_s dZ_s + \tilde{C}_t \tag{2.4}$$

for a decreasing process $(\tilde{C}_t)_{0\le t\le T}$ with $\tilde{C}_0 = 0$. The cost processes C and $\tilde{C}$ are related via $d\tilde{C}_t = \exp(r(T-t))dC_t$. The proof, which is an easy application of Itô's product rule, will be omitted.

In this section we make the following assumption on the asset price dynamics.

Assumption 3. (volatility bounds) *There are constants $\sigma_{\min}$ and $\sigma_{\max}$ such that*

$$0 \le \sigma_{\min} \le \sigma_t \le \sigma_{\max} < \infty \text{ for all } t \ge 0 . \tag{2.5}$$

The numbers $\sigma_{\min}$ and $\sigma_{\max}$ reflect expectations about future volatility. For instance one could use econometric techniques in order to obtain an estimate for the distribution of historical volatility and choose $\sigma_{\min}$ and $\sigma_{\max}$ as some lower respectively upper quantile of this distribution; in that case the interval $[\sigma_{\min}, \sigma_{\max}]$ can be interpreted as confidence interval for the future volatility. If there is a liquid market for derivative instruments on S, one could alternatively obtain $\sigma_{\min}$ and $\sigma_{\max}$ from extreme past values of the implied volatilities of these contracts. In either case the volatility band should be wide enough to ensure that current implied volatilities of liquidly traded derivatives are contained in the band. Otherwise the use of static positions in these instruments as additional hedging tool might lead to inconsistencies; see Section 5.2.

Consider a claim H satisfying Assumption 2, and two numbers $0 \le \underline{\sigma} \le \overline{\sigma} \le \infty$ with $\underline{\sigma} < \infty$. Denote by R_z the law of the solution of the SDE $dU_t = U_t dW_t$ with initial value $U_0 = z$ and recall the definition of the set of stopping times $\mathcal{B}_{\underline{\sigma}^2(T-t), \overline{\sigma}^2(T-t)}$ from Remark 2.1. Define the function $V^{\mathrm{Am}} : [0,T] \times \mathbb{R}^3_+ \to \mathbb{R}$ as value function of the following optimal-stopping problem

$$\begin{aligned} &V^{\mathrm{Am}}(t, z, \underline{m}, \overline{m}; \underline{\sigma}, \overline{\sigma}) := \\ &\qquad \sup \left\{ E_z^R \left[f(U_\nu, \underline{m} \wedge U_{[0,\nu]}^{\min}, \overline{m} \vee U_{[0,\nu]}^{\max}) \right], \nu \in \mathcal{B}_{\underline{\sigma}^2(T-t), \overline{\sigma}^2(T-t)} \right\} . \end{aligned} \tag{2.6}$$

V^{Am} has an obvious interpretation as arbitrage price of an American type derivative with partial exercice feature in a standard Black-Scholes model with volatility equal to one and interest rate equal to zero. Wider "volatility bounds" $\underline{\sigma}$ and $\overline{\sigma}$ correspond to a larger time window for the exercice of this American-type security and hence to a larger value of V^{Am}.

The following theorem is the main result of this section.

Theorem 2.4. *Suppose that Assumptions 1 and 3 hold for S, that H satisfies Assumption 2 and that the function $V^{\mathrm{Am}}(t, x, \underline{m}, \overline{m}; \sigma_{\min}, \sigma_{\max})$ is finite for all $(t, x, \underline{m}, \overline{m}) \in [0,T] \times \mathbb{R}^3$. Then the process $V^* = (V_t^*)_{0 \le t \le T}$ defined by*

$$V_t^* := e^{-r(T-t)} V^{\mathrm{Am}} \left(t, Z_t, Z_{[0,t]}^{\min}, Z_{[0,t]}^{\max} ; \sigma_{\min}, \sigma_{\max} \right)$$

is the value process of a superreplicating strategy for H.

COMMENTS:

Consider a payoff of the form $H = f(S_T)$ for some *convex* function f. By Jensens inequality we get for any stopping time $\nu \in \mathcal{B}_{\sigma_{\min}^2(T-t), \sigma_{\max}^2(T-t)}$

$$E_z^R [f(U_\nu)] = E_z^R \left[f \left(E_z^R [U_{\sigma_{\max}^2(T-t)} | \mathcal{B}_\nu] \right) \right] \le E_z^R [f(U_{\sigma_{\max}^2(T-t)})] .$$

It follows that the sup in (2.6) is attained by taking $\nu = \sigma_{\max}^2(T-t)$. Hence V_t^* equals the price of a derivative with payoff $f(S_T)$ in a Black-Scholes model with volatility $\sigma_{\max}$; see El Karoui, Jeanblanc-Picqué, and Shreve (1998).

If H is of the form $H = f(S_T)$ the value function of the optimal-stopping problem (2.6) can be computed using the standard binomial model of Cox, Ross, and Rubinstein (1979).

If one is dealing with path-dependent payoffs some algorithm for the pricing of American path-dependent options such as the forward shooting grid method of Barraquand and Pudet (1996) can be used.

Note that the superreplication cost V^* is subadditive, i.e. the superhedging cost corresponding to a portfolio of two payoffs is no larger than the sum of the superreplication costs of the two individual payoffs. In order to keep the superreplication cost low the method should therefore be applied to large portfolios rather than to individual derivatives.

Our approach can easily be adapted to accommodate portfolios of claims with different maturity dates; see also Avellaneda, Levy, and Paras (1995). Consider the case of two claims H^1 and H^2 with — for notational simplicity path-independent — payoffs $f^1(S_{T_1})$ and $f^2(S_{T_2})$ and maturity dates $T_1 > T_2$. By Theorem 2.4

$$V^{*,1}(S_{T_2}) := \exp(-r(T_1 - T_2))V^{\mathrm{Am}}(T_2, \exp(r(T_1 - T_2))S_{T_2})$$

is the value at time T_2 of the superreplicating strategy for H^1. Define a new claim H with maturity date T_2 and payoff given by $H = f^2(S_{T_2}) + V^{*,1}(S_{T_2})$. Obviously, a superreplicating strategy for H, which can be computed using Theorem 2.4, induces a superreplicating strategy for the portfolio consisting of the claims H^1 and H^2.

2.3 Proof of Theorem 2.4

As a first step we recall some well-known recent results on the existence of minimal superreplicating strategies and in particular the optional decomposition theorems as obtained by El Karoui and Quenez and by Kramkov; the following result is a version of Theorem 2.1 in Kramkov (1996).

Theorem 2.5. (optional decomposition) *Let Z be a locally bounded process on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ such that the set $\mathbb{M}^e$ of equivalent local martingale measures is non-empty. Then a positive process $(\tilde{V}_t)_{t\geq 0}$ is a supermartingale for all $Q \in \mathbb{M}^e$ if and only if there is a predictable, Z–integrable process $(\xi_t)_{t\geq 0}$ and a decreasing, adapted process $(\tilde{C}_t)_{t\geq 0}$ with $\tilde{C}_0 = 0$ such that*

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_s dZ_s + \tilde{C}_t\,. \tag{2.7}$$

This theorem obviously extends to processes which are only bounded below. Together with the following result of El Karoui and Quenez (1995), Theorem 2.5 ensures the existence of minimal superhedging strategies and gives moreover a characterization of the ask-price in terms of the set $\mathbb{M}^e$.

Proposition 2.6. *Let H be a contingent claim with maturity T and payoff which is bounded below. Suppose that $\sup\{E^Q[H]\,,\, Q \in \mathbb{M}^e\} < \infty$. Then there is a RCLL process $\overline{H} = (\overline{H}_t)_{t\geq 0}$ such that $\overline{H}_t = \operatorname{ess\,sup}\{E^Q[H|\mathcal{F}_t]\,,\, Q \in \mathbb{M}^e\}$. Moreover, the process $\overline{H}$ is a Q-supermartingale for all $Q \in \mathbb{M}^e$.*

It is easily seen from (2.4) that whenever (ξ, η) is a superreplicating strategy for H we must have $\tilde{V}_t \geq E^Q[H|\mathcal{F}_t]$ for all $Q \in \mathbb{M}^e$, as the stochastic integral $\int_0^t \xi_s dZ_s$ is a Q-supermartingale. Hence $\tilde{V}_t \geq \overline{H}_t$, and we have the following result, which is well-known in the theory of superreplication:

Corollary 2.7. *Let H be a contingent claim satisfying the hypothesis of Proposition 2.6. Then a minimal superhedging strategy exists; the ask-price is given by $(e^{-r(T-t)}\overline{H}_t)_{0\leq t\leq T}$.*

We next explain the key idea behind the characterization of superhedging strategies given in Theorem 2.4. In view of Corollary 2.7 we want to give an estimate for $\overline{H}$. Consider for notational simplicity some path-independent payoff $H = f(S_T)$. Then we have for every $Q \in \mathbb{M}^e$:

$$E^Q[f(S_T)] = E^Q[f(Z_T)] = E^Q[f(Z_0 \exp(M_T - 1/2\langle M\rangle_T))] .$$

By changing the volatility for $t > T$ if necessary we may assume that $\lim_{t\to\infty}\langle M\rangle_t = \infty$ P-a.s. Define the increasing process A_t via

$$A_t = A(t) := \inf\{s > 0 : \langle M\rangle_s \geq t\} . \tag{2.8}$$

Note that under our assumptions on the volatility the mapping $t \to \langle M\rangle_t$ is P-a.s. a bijection from $[0, \infty)$ onto itself with inverse mapping given by A. Now Levy's characterization of Brownian motion implies that the process $B_t := M_{A_t}$ is a Brownian motion relative to the new filtration $(\mathcal{G}_t) = (\mathcal{F}_{A_t})$ and $M_t = B_{\langle M\rangle_t}$, see e.g. Chapter 3.4 of Karatzas and Shreve (1988). Moreover $\langle M\rangle_T$ is a $(\mathcal{G}_t)$-stopping time which takes its values in the interval $[\sigma^2_{\min}T, \sigma^2_{\max}T]$ by Assumption 3. Hence we get

$$E^Q[f(S_T)] = E^Q[f(Z_0 \exp(B_{\langle M\rangle_T} - 1/2\langle M\rangle_T))] \tag{2.9}$$

$$\leq \sup\left\{E^Q[f(Z_0 \exp(B_\nu - 1/2\nu))],\ \nu \in \mathcal{G}_{\sigma^2_{\min}T, \sigma^2_{\max}T}\right\} . \tag{2.10}$$

We will show in the proof of Proposition 2.10 below that the strong Markov property of the geometric Brownian motion $U_t := Z_0 \exp(B_t - 1/2t)$ implies that the value function of the optimal-stopping problem (2.10) is independent of the particular filtered probability space on which U is defined and equal to $V^{\mathrm{Am}}(0, Z_0)$. Hence the ask price of H is no larger than V^*_0.

Remark 2.8. Obviously, these estimates are valid in the case where $\sigma_{\max} = \infty$ and where the payoff under consideration is path-dependent but satisfies Assumption 2. Hence for every such claim and every $0 \leq t < T$ the ask price is no larger than

$$V^*_t = e^{-r(T-t)} V^{\mathrm{Am}}\left(t, Z_t, Z^{\min}_{[0,t]}, Z^{\max}_{[0,t]}; \sigma_{\min}, \infty\right) .$$

Remark 2.9. The above argument cannot be extended to the case of options on multiple assets $S^1, \ldots, S^d$ with payoff $f(S^1_T, \ldots, S^d_T)$. Assume that the dynamics of S^i are of the form $dS^i_t = S^i_t dM^i_t$ for continuous local martingales M^i, $i = 1, \ldots, d$ with absolutely continuous quadratic variation $\langle M^i\rangle_t$. To illustrate, what "goes wrong" we consider the particular case where

$$\langle M^i, M^j\rangle_t = 0 \text{ for all } 1 \leq i \neq j \leq d \text{ and for all } t \geq 0 .$$

As previously we introduce $A^i_t := \inf\left\{s \geq 0, \langle M^i\rangle_s \geq t\right\}$ and define new processes B^i, $1 \leq i \leq d$ by $B^i_t := M_{A^i_t}$. The F. B. Knight Theorem (see for instance Karatzas and Shreve 1988, Theorem 3.4.13) ensures that $\boldsymbol{B} = (B^1, \ldots, B^d)$ is a d-dimensional standard Brownian motion w.r.t. its own filtration $(\mathcal{G}^{\boldsymbol{B}}_t)$ However, $\boldsymbol{B}$ is in general *not* a d-dimensional Brownian motion w.r.t. the filtration $(\mathcal{G}_t)$ defined by

$$\mathcal{G}_t = \mathcal{F}_{A^1_t} \vee \ldots = \vee \mathcal{F}_{A^d_t} .$$

On the other hand, while $\langle M^i\rangle_t$ is for every $1 \leq i \leq d$ and every t a $(\mathcal{G}_t)$-stopping time, $\langle M^i\rangle_t$ is not necessarily a stopping time for the filtration $(\mathcal{G}^{\boldsymbol{B}}_t)$. Hence we cannot hope to find a representation for $E^Q[f(S^1_T, \ldots, S^d_T)]$ of the form (2.9) in this case.

In view of Corollary 2.7, Theorem 2.4 would follow from the previous estimates if under Assumptions 2 and 3 V_t^* was actually equal to $e^{-r(T-t)}\overline{H}_t$ for all t. As shown in Sections 3 and 4 below, this equality holds true for a large class of general SV-models but is wrong in general. However, Theorem 2.4 follows from Theorem 2.5 and the following proposition.

Proposition 2.10. *The process* $V^{\mathrm{Am}}\left(t, Z_t, Z_{[0,t]}^{\min}, Z_{[0,t]}^{\max}\right)_{0\le t\le T}$ *is a* Q*-supermartingale for all* $Q \in \mathbb{M}^e$*.*

PROOF OF PROPOSITION 2.10: Define for every $0 \le t \le T$ positive random variables $\tau_{\min}(t)$ and $\tau_{\max}(t)$ via

$$\tau_{\min}(t) := A(\sigma_{\min}^2(T-t) + \langle M\rangle_t) \text{ and } \tau_{\max}(t) := A(\sigma_{\max}^2(T-t) + \langle M\rangle_t)\,. \tag{2.11}$$

Lemma 2.11 below shows that $\tau_{\min}(t)$ and $\tau_{\max}(t)$ are $(\mathcal{F}_t)$-stopping times. Moreover, $t \le \tau_{\min}(t) \le \tau_{\max}(t)$ P-a.s. Recall the definition of the process $f_t := f\left(t, Z_t, Z_{[0,t]}^{\min}, Z_{[0,t]}^{\max}\right)$ in Assumption 2 and the definition of the set of $(\mathcal{F}_t)$-stopping times $\mathcal{F}_{\tau_{\min}(t),\tau_{\max}(t)}$ in Remark 2.1(ii). Fix some $Q \in \mathbb{M}^e$ and define a process J_t^Q via the following optimal stopping problem

$$J_t^Q := \operatorname{ess\,sup}\{E^Q[f_\tau \mid \mathcal{F}_t]\,,\, \tau \in \mathcal{F}_{\tau_{\min}(t),\tau_{\max}(t)}\}\,. \tag{2.12}$$

The proof now consists of two steps:

STEP 1: J_t^Q is a Q-supermartingale.
To prove the supermartingale property note first that the set of stopping times $\mathcal{F}_{\tau_{\min}(t),\tau_{\max}(t)}$ is shrinking as t increases. We get that

$$\begin{aligned}
\frac{\partial}{\partial t}\tau_{\min}(t) &= A'(\sigma_{\min}^2(T-t) + \langle M\rangle_t)(-\sigma_{\min}^2 + \sigma_t^2) \ge 0\,,\\
\frac{\partial}{\partial t}\tau_{\max}(t) &= A'(\sigma_{\max}^2(T-t) + \langle M\rangle_t)(-\sigma_{\max}^2 + \sigma_t^2) \le 0.
\end{aligned}$$

The inequalities follow as $A'(x) > 0$ for all $x > 0$ and as $(-\sigma_{\min}^2 + \sigma_t^2) \ge 0$ and $(-\sigma_{\max}^2 + \sigma_t^2) \le 0$ by Assumption 3. Now let $t > s$. We get that

$$\begin{aligned}
E^Q[J_t^Q \mid \mathcal{F}_s] &= E^Q\left[\operatorname{ess\,sup}\{E^Q[f_\tau \mid \mathcal{F}_t]\,,\, \tau \in \mathcal{F}_{\tau_{\min}(t),\tau_{\max}(t)}\} \mid \mathcal{F}_s\right]\\
&\overset{(i)}{=} \operatorname{ess\,sup}\{E^Q[f_\tau \mid \mathcal{F}_s]\,,\, \tau \in \mathcal{F}_{\tau_{\min}(t),\tau_{\max}(t)}\}\\
&\overset{(ii)}{\le} \operatorname{ess\,sup}\{E^Q[f_\tau \mid \mathcal{F}_s]\,,\, \tau \in \mathcal{F}_{\tau_{\min}(s),\tau_{\max}(s)}\}\\
&= J_s^Q.
\end{aligned}$$

Here the equality (i) follows from Proposition A.1 in Appendix A.1; inequality (ii) follows as $\mathcal{F}_{\tau_{\min}(t),\tau_{\max}(t)} \subset \mathcal{F}_{\tau_{\min}(s),\tau_{\max}(s)}$ for $t > s$.

STEP 2: J_t^Q is independent of Q and given by $V^{\mathrm{Am}}\left(t, Z_t, Z_{[0,t]}^{\min}, Z_{[0,t]}^{\max}\right)$.
For notational simplicity we treat only the case $t = 0$. We want to write J_0^Q in a different way using the time change introduced in the beginning of the proof. Define the process U via $U_t = Z_0 \exp(B_t - 1/2t)$, where $B_t = M_{A(t)}$ is Q-Brownian motion. We have

$$Z_t = U_{\langle M\rangle_t}\,,\; Z_{[0,t]}^{\min} = U_{[0,\langle M\rangle_t]}^{\min} \text{ and } Z_{[0,t]}^{\max} = U_{[0,\langle M\rangle_t]}^{\max}\,. \tag{2.13}$$

By (2.13) we get that

$$E^Q[f_\tau] = E^Q\left[f\left(U_{\langle M\rangle_\tau},\, U^{\min}_{[0,\langle M\rangle_\tau]},\, U^{\max}_{[0,\langle M\rangle_\tau]}\right)\right]$$

for every $(\mathcal{F}_t)$-stopping time τ. The following Lemma, whose proof is given in the Appendix A.2, shows that the mapping $\tau \mapsto \langle M\rangle_\tau$ is a bijection from $\mathcal{F}_{\tau_{\min}(0),\tau_{\max}(0)}$ onto $\mathcal{G}_{\sigma^2_{\min}T,\sigma^2_{\max}T}$.

Lemma 2.11. *Let τ be an $(\mathcal{F}_t)$-stopping time. Then $\nu(\tau) := \langle M\rangle_\tau$ is a $(\mathcal{G}_t)$-stopping time. Conversely, if ν is a $(\mathcal{G}_t)$-stopping time, $\tau(\nu) := A_\nu$ is an $(\mathcal{F}_t)$-stopping time.*

Using this Lemma and Proposition A.1 we can write J_0^Q in a different way:

$$\begin{aligned} J_0^Q &= \sup\left\{E^Q\left[f(U_\nu,\, U^{\min}_{[0,\nu]},\, U^{\max}_{[0,\nu]})\right],\ \nu \in \mathcal{G}_{\sigma^2_{\min}T,\sigma^2_{\max}T}\right\} \\ &= E^Q\left[\ \operatorname{ess\,sup}\left\{E^Q\left[f(U_\nu,\, U^{\min}_{[0,\nu]},\, U^{\max}_{[0,\nu]}) \mid \mathcal{F}_{\sigma^2_{\min}T}\right],\ \nu \in \mathcal{G}_{\sigma^2_{\min}T,\sigma^2_{\max}T}\right\}\right]. \end{aligned} \tag{2.14}$$

Now U is a Q-geometric Brownian motion with zero drift, initial value $U_0 = Z_0$ and volatility equal to one; in particular U is Markovian. Hence the process $\boldsymbol{U}_t = \left(U_t, U^{\min}_{[0,t]}, U^{\max}_{[0,t]}\right)$ is a $\mathbb{R}^3$-valued Markov process. Thisd implies that the *Snell-envelope* of the process $f_t = f(\boldsymbol{U}_t)$ with respect to the filtration $(\mathcal{G}_t)$ coincides with the Snell envelope of $f(\boldsymbol{U}_t)$ with respect to the filtration generated by $\boldsymbol{U}$ respectively by U; see Lemma 2.6 of Mulinacci and Pratelli (1998) or Proposition 3.5 of Lamberton and Pagès (1990). Hence

$$\begin{aligned} &\operatorname{ess\,sup}\left\{E^Q\left[f(U_\nu,\, U^{\min}_{[0,\nu]},\, U^{\max}_{[0,\nu]}) \mid \mathcal{F}_{\sigma^2_{\min}T}\right],\ \nu \in \mathcal{G}_{\sigma^2_{\min}T,\sigma^2_{\max}T}\right\} = \\ &= \ \operatorname{ess\,sup}\left\{E^R_{Z_0}\left[f(U_\nu,\, U^{\min}_{[0,\nu]},\, U^{\max}_{[0,\nu]}) \mid \mathcal{B}_{\sigma^2_{\min}T}\right],\ \nu \in \mathcal{B}_{\sigma^2_{\min}T,\sigma^2_{\max}T}\right\} = \\ &= V^{\mathrm{Am}}\left(0, U_{\sigma^2_{\min}T}, U^{\min}_{[0,\sigma^2_{\min}T]}, U^{\max}_{[0,\sigma^2_{\min}T]}; 0, (\sigma^2_{\max} - \sigma^2_{\min})^{1/2}\right). \end{aligned}$$

Plugging this into (2.14) yields

$$\begin{aligned} J_0^Q &= E^Q\left[V^{\mathrm{Am}}\left(0, U_{\sigma^2_{\min}T}, U^{\min}_{[0,\sigma^2_{\min}T]}, U^{\max}_{[0,\sigma^2_{\min}T]}; 0, (\sigma^2_{\max} - \sigma^2_{\min})^{1/2}\right)\right] \\ &= E^R_{Z_0}\left[V^{\mathrm{Am}}\left(0, U_{\sigma^2_{\min}T}, U^{\min}_{[0,\sigma^2_{\min}T]}, U^{\max}_{[0,\sigma^2_{\min}T]}; 0, (\sigma^2_{\max} - \sigma^2_{\min})^{1/2}\right)\right] \\ &= V^{\mathrm{Am}}(0, Z_0; \sigma_{\min}, \sigma_{\max}), \end{aligned}$$

where the last equality follows from the definition of V^{Am}, if we condition on $\mathcal{B}_{\sigma^2_{\min}T}$, apply Proposition A.1 and the Markov-property of $\boldsymbol{U}$. □

3 Minimality of our superhedging strategies

In this section we study under which conditions the superhedging strategy constructed in Theorem 2.4 is actually a minimal superhedging strategy. To analyze this question we have to introduce additional assumptions on the probabilistic structure of the volatility process. We are particularly interested in the case where the volatility follows a one-dimensional diffusion.

Assumption 4. *We assume that S satisfies the equations*

$$dS_t = S_t(|v_t|^{1/2} dW_t^{(1)} + r dt)\,, \tag{3.1}$$

$$dv_t = a(v_t)dt + \eta_1(v_t)dW_t^{(1)} + \eta_2(v_t)dW_t^{(2)}\,, \tag{3.2}$$

for $W_t = (W_t^{(1)}, W_t^{(2)})$ a standard two-dimensional Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. We assume that the coefficients are such that the vector SDE (3.1), (3.2) has a non-exploding and strictly positive solution.

This class of volatility models contains the models considered by Wiggins (1987), Hull and White (1987) or Heston (1993) as special cases. Note that we allow for nonzero η_1 and hence for nonzero instantaneous correlation between volatility innovations and asset returns.

Theorem 3.1. *Suppose that S is given by a SV-model satisfying Assumption 4. Assume moreover that there is some $0 < \sigma_{\max} \le \infty$ such that*

(i) *The real functions a, η_1, η_2 are locally Lipschitz on $(0, \sigma^2_{\max})$; $b(x) := \sqrt{\eta_1^2(x) + \eta_2^2(x)}$ belongs to $\mathcal{C}^1_{(0,\sigma^2_{\max})}$.*

(ii) *$\eta_2(v) > 0$ for all $v \in (0, \sigma^2_{\max})$.*

(iii) *$0 < \sigma_t := \sqrt{v_t} \le \sigma_{\max}$, the last inequality being strict for $\sigma_{\max} = \infty$.*

Then for every claim H satisfying Assumption 2 the ask price equals

$$V_t^* = e^{-r(T-t)} V^{\mathrm{Am}}\left(t, Z_t, Z^{\min}_{[0,t]}, Z^{\max}_{[0,t]}; 0, \sigma_{\max}\right) \text{ for all } 0 \le t \le T\,. \tag{3.3}$$

COMMENTS:

Hypothesis (ii) ensures that volatility innovations and asset returns are not perfectly correlated which in turn implies that the market is incomplete. Moreover, this hypothesis ensures that for all $t > 0$ and all $0 < K_1 \le K_2 < \sigma_{\max}$ we have that $P[\sigma_t < K_1] > 0$ and $P[\sigma_t > K_2] > 0$, i.e. the open interval $(0, \sigma_{\max})$ is contained in the range of σ_t for all $t > 0$. As we assumed existence of a strictly positive solution to (3.2), hypothesis (i) implies pathwise uniqueness for this SDE; see Frey and Sin (1999) for details.

Consider models with unbounded volatility, i.e. $\sigma_{\max} = \infty$. Applying Theorem 3.1 to ordinary call options we get that in a large class of SV-models where the volatility follows a one-dimensional diffusion the ask price of a call option is equal to S_0, the current price of the stock. This is the main result of Frey and Sin (1999).

Cvitanic, Pham, and Touzi (1997) have previously obtained the characterization of the ask price by the optimal-stopping problem (2.6) for a large class of diffusion models with unbounded volatility and path-independent payoffs. Their approach is based on the characterization of the ask price as viscosity supersolution to the Bellman equation corresponding to the infinitesimal generator of the process S. Using that characterization they conclude that the ask price is given by the smallest concave majorant f^* of f. In Lemma 5.4 of their paper it is moreover shown that the solution to the optimal-stopping problem (2.6) is equal to f^*.

As shown below, Theorem 3.1 follows from combining results from Frey and Sin (1999) with the following Proposition which applies to general Markovian SV-models.[1]

[1]The author is grateful for interesting discussions with N. Touzi and H. Pham, which were very helpful in obtaining Proposition 3.2.

Proposition 3.2. *Consider a model where S is given by a stochastic process satisfying Assumption 1. Suppose that there is some $0 < \overline{\sigma} < \infty$ such that the following holds:*
1. The process $X_t := (S_t, \sigma_t)$ is a two-dimensional strong Markov family defined for all initial values $X_0 = (S_0, \sigma_0) \in \mathbb{R}^+ \times (0, \overline{\sigma})$. The corresponding family of measures will be denoted by (P_x), $x \in \mathbb{R}^+ \times (0, \overline{\sigma})$.
2. For every $\delta > 0$ and every $x \in \mathbb{R}^+ \times (0, \overline{\sigma})$ there is a sequence of strictly positive density martingales $G^{1,n} = (G_t^{1,n})_{0 \le t \le T}$ with $G_0^{1,n} = 1$ such that

(i) *$G^{1,n}$ is adapted to the filtration generated by X.*

(ii) *The process $Z_t = e^{r(T-t)} S_t$, $0 \le t \le T$ is a local martingale under the probability measures $Q_x^{1,n}$ defined by $dQ_x^{1,n}/dP_x = G_T^{1,n}$.*

(iii) $\lim_{n\to\infty} Q_x^{1,n}[\langle M \rangle_T > \overline{\sigma}^2 (T - \delta)\,] = 1$.

3. For every compact set $K \subset\subset \mathbb{R}^+ \times (0, \overline{\sigma})$ and every $\delta > 0$ there is a sequence $G^{2,n}$ of strictly positive density martingales $G^{2,n} = (G_t^{2,n})_{0 \le t \le T}$ with $G_0^{2,n} = 1$ such that

(i) *$G^{2,n}$ is adapted to the filtration generated by X.*

(ii) *Z is a local martingale under the measures $Q_x^{2,n}$ defined by $dQ_x^{2,n}/dP_x = G_T^{2,n}$.*

(iii) $\lim_{n\to\infty} \inf_{x \in K} Q_x^{2,n}[\langle M \rangle_T < \delta\,] = 1$.

Then for every claim H satisfying Assumption 2 the ask price is no smaller than

$$e^{-r(T-t)} V^{\mathrm{Am}} \left(t, Z_t, Z^{\min}_{[0,t]}, Z^{\max}_{[0,t]}; 0, \overline{\sigma} \right) .$$

Proof of Proposition 3.2:
While the following proof is rather technical, the underlying idea is simple. Recall the filtration $(\mathcal{G}_t)$ introduced in the proof of Proposition 2.10. We want to show that for every $\nu \in \mathcal{G}_{0,\overline{\sigma}^2}$ and every $\delta > 0$ there is a sequence $Q^n \in \mathbb{M}^e$ such that $Q^n[\langle M \rangle_T \in [\nu,\, \nu + \delta]] \to 1$ as $n \to \infty$. Together with the right-continuity of our payoffs this implies the result. To construct such a sequence of local martingale measures we first choose a sequence of measures $Q^{1,n} \in \mathbb{M}^e$ which put most of the mass on trajectories with "high" volatility. As soon as $\langle M \rangle_t = \nu$ we "drive the volatility down" using another sequence $Q^{2,n} \in \mathbb{M}^e$.

The crucial step in the proof is the following Lemma:

Lemma 3.3. *Suppose that the hypothesises of Proposition 3.2 hold. Then for every $\nu \in \mathcal{G}_{0,\overline{\sigma}^2}$ and for every $\varepsilon, \delta > 0$ there is some $Q \in \mathbb{M}^e$ such that*

$$Q\big[\langle M \rangle_T \in [\nu,\, \nu + \delta]\big] > 1 - \varepsilon\,. \tag{3.4}$$

The proof of Lemma 3.3 is given in Appendix A.3.

We now show that Proposition 3.2 follows from Lemma 3.3. By the Markov property of X it is enough to consider the case $t = 0$. As in the proof of Theorem 2.4 we define the process $U_t := Z_{A(t)}$. We get

$$H = f\left(U_{\langle M \rangle_T}, U^{\min}_{[0,\langle M \rangle_T]}, U^{\max}_{[0,\langle M \rangle_T]} \right) .$$

Let $\tilde{\nu}$ be a stopping time for $(\mathcal{B}_t)$, the canonical filtration on $\mathcal{C}_{[0,\infty)}$. Then $\nu := \tilde{\nu} \circ U$ is a stopping time for $(\mathcal{G}^U_t)$, the sub-filtration of $(\mathcal{G}_t)$ generated by the process U, and hence for $(\mathcal{G}_t)$. Moreover, we have for every $Q \in \mathbb{M}^e$ that $E^Q[f(U_\nu)] = E^R_{Z_0}[U_{\tilde\nu}]$, as U is Q-geometric Brownian motion.

Choose some $\tilde\nu \in \mathcal{B}_{0,\overline{\sigma}^2 T}$ and the corresponding stopping time $\nu := \tilde\nu \circ U$ from $\mathcal{G}_{0,\overline{\sigma}^2 T}$. By Lemma 3.3 there exists a sequence of local martingale measures $Q^n \in \mathbb{M}^e$ such that $Q^n[\langle M\rangle_T \in [\nu,\, \nu+\delta]] > 1 - 1/n$. By Assumption 2 the process $t \to f\left(Z_t, Z^{\min}_{[0,t]}, Z^{\max}_{[0,t]}\right)$ is right continuous such that

$$\lim_{n\to\infty} Q^n\left[\left|H - f\left(U_\nu,\, U^{\min}_{[0,\nu]},\, U^{\max}_{[0,\nu]}\right)\right| > \delta\right] = 0 \tag{3.5}$$

for all $\delta > 0$. Consider first the case of bounded f. It follows from (3.5) that

$$\liminf_{n\to\infty} E^{Q^n}[H] \geq \liminf_{n\to\infty} E^{Q^n}\left[f\left(U_\nu,\, U^{\min}_{[0,\nu]},\, U^{\max}_{[0,\nu]}\right)\right] = E^R_{Z_0}\left[f\left(U_{\tilde\nu},\, U^{\min}_{[0,\tilde\nu]},\, U^{\max}_{[0,\tilde\nu]}\right)\right].$$

As $\tilde\nu \in \mathcal{B}_{0,\overline{\sigma}^2}$ was arbitrary we get that $\liminf_{n\to\infty} E^{Q^n}[H]] \geq V^{\mathrm{Am}}(0, Z_0)$. In the case of unbounded but positive f we consider for $n \in \mathbb{N}$ the claim H_n corresponding to the payoff $f_n := f \wedge n$. Applying the result for bounded f we get that $\overline{H}$ is greater than $V^{\mathrm{Am}}_n(0, Z_0)$, the value function of the optimal-stopping problem (2.6) with f replaced by f_n. Monotone integration now implies that $\overline{H} \geq V^{\mathrm{Am}}(0, Z_0)$. □

Proof of Theorem 3.1:
We have to show that Conditions 2 and 3 of Proposition 3.2 are satisfied under the assumptions of Theorem 3.1. Our argument is based on results obtained by Frey and Sin (1999) for models with unbounded volatility. If $\sigma_{\max} < \infty$ we transform our problem to the case $\sigma_{\max} = \infty$ using some smooth and strictly increasing function ψ that maps the interval $(0, \sigma^2_{\max})$ onto $(0, \infty)$. By Itô's formula $y_t := \psi(v_t)$ solves the SDE

$$dy_t = \tilde a(y_t)dt + \tilde\eta_1(y_t)dW^{(1)}_t + \tilde\eta_2(y_t)dW^{(2)}_t\,,$$

where the coefficients $\tilde a$, $\tilde\eta_1$, $\tilde\eta_2$ and $\tilde b := \sqrt{\tilde\eta_1^2 + \tilde\eta_2^2}$ satisfy hypothesis (i) and (ii) of Theorem 3.1 on $(0,\infty)$. As in Frey and Sin (1999) we consider for $n \in \mathbb{N}$ measures $Q^{1,n}$ and $Q^{2,n} \in \mathbb{M}^e$ with densities given by

$$\frac{dQ^{1,n}}{dP} = \exp\left(nW^{(2)}_t - \frac{1}{2}n^2 T\right) \text{ and } \frac{dQ^{2,n}}{dP} = \exp\left(-nW^{(2)}_t - \frac{1}{2}n^2 T\right).$$

As $\sigma_t = \sqrt{v_t}$ and $\eta_2(v_t)$ are strictly positive, the filtration generated by the two-dimensional$X := (S, v)$ coincides with the filtration generated by $(W^{(1)}, W^{(2)})$; see e.g. Harrison and Kreps (1979). Hence our density martingales are adapted to the filtration generated by X. By Girsanov's theorem $(y_t)_{t\geq 0}$ is under $Q^{i,n}$, $i = 1, 2$, a solution to the SDE

$$dy^n_t = \tilde a(y^n_t) - (-1)^i n\tilde\eta_2(y^n_t)dt + \tilde b(y^n_t)dB^{i,n}_t, \quad y_0 = \psi(v_0). \tag{3.6}$$

for a new $Q^{i,n}$-Brownian motion $B^{i,n}$. The next two results are Lemma 3.5 and Lemma 3.6 of Frey and Sin (1999):

Lemma 3.4. *Assume that the SDE (3.6) has a global and strictly positive solution for any $n \in \mathbb{N}$, any initial value $y \in \mathbb{R}^+$ and any $i \in \{1, 2\}$, and denote by $R^{i,n}_y$ the law of (3.6) with initial value y. Then*

(i) *For every $L > 0$, $T > 0$, $y > 0$ and $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that*

$$R^{1,n}_y[y_t \geq L \text{ for some } 0 \leq t \leq T] > 1 - \varepsilon \text{ for all } n \geq N_1.$$

(ii) *For every $L > 0$, $T > 0$, $y > 0$ and $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that*

$$R_y^{2,n}\left[y_t \le L^{-1} \text{ for some } 0 \le t \le T\right] > 1 - \varepsilon \text{ for all } n \ge N_2.$$

Lemma 3.5. *Assume again that for $n \in \mathbb{N}$, $y \in \mathbb{R}^+$ and for $i = 1, 2$ the SDE (3.6) has a global solution which is strictly positive. Then the following holds:*

(i) *For every $L > 0$, $T > 0$, and $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for $y_0 = 2L$*

$$R_{2L}^{1,n}\left[y_t > L \text{ for all } 0 \le t \le T\right] > 1 - \varepsilon \text{ for all } n \ge N_2.$$

(ii) *For every $L > 0$, $T > 0$ and $\varepsilon > 0$ there exist $N_2 \in \mathbb{N}$ such that for $y_0 = L/2$*

$$R_{L/2}^{2,n}\left[y_t < L \text{ for all } 0 \le t \le T\right] > 1 - \varepsilon \text{ for all } n \ge N_2.$$

To verify that Conditions 2.(iii) and 3.(iii) of Proposition 3.2 are implied by Lemmas 3.4 and 3.5 one now uses exactly the same arguments as in the proof of Frey and Sin (1999), Theorem 3.4. □

4 PDE-Characterisation of the value function V^{Am}

4.1 Previous results

In most of the previous work on superreplication in stochastic volatility models the value process of superhedging strategies is characterized by a terminal value problem involving some — often nonlinear — parabolic PDE. Important examples of this work are the independent papers Avellaneda, Levy, and Paras (1995), Lyons (1995) and El Karoui, Jeanblanc-Picqué, and Shreve (1998). These papers consider mainly path-independent derivatives. Therefore we will concentrate on payoffs of the form $H = f(S_T)$ for some continuous function f. Moreover, we start immediately with discounted quantities.

The main result of Avellaneda, Levy and Paras and Lyons can be stated as follows.

Proposition 4.1. *Suppose that S satisfies Assumptions 1 and 3 and that the terminal value problem*

$$h_t^{\mathrm{AV}} + \frac{1}{2}x^2\left(-\sigma_{\min}^2\left[h_{xx}^{\mathrm{AV}}\right]^- + \sigma_{\max}^2\left[h_{xx}^{\mathrm{AV}}\right]^+\right) = 0\,, \quad h^{\mathrm{AV}}(T,x) = f(x) \tag{4.1}$$

has a solution in $\mathcal{C}^{1,2}([0,t)\times\mathbb{R}^+)$. Then $h^{\mathrm{AV}}(t,S_t)$ is the value at time t of a superhedging strategy for H.

For convenience we recall the simple proof. We get from Itô's formula

$$\begin{aligned} f(S_T) = h^{\mathrm{AV}}(T,S_T) = h^{\mathrm{AV}}(0,S_0) &+ \int_0^T h_x^{\mathrm{AV}}(t,S_t)dS_t \\ &+ \int_0^T \underbrace{(h_t^{\mathrm{AV}} + \frac{1}{2}S_t^2\sigma_t^2 h_{xx}^{\mathrm{AV}})(t,S_t)}_{\le 0 \text{ by } (4.1)}\,dt\,, \end{aligned}$$

such that the strategy with value process $h^{\mathrm{AV}}(t,S_t)$ and stockholdings $h_x^{\mathrm{AV}}(t,S_t)$ has a representation of the form (2.3). □

Remark 4.2. Lyons (1995) has developed an extension of this result to markets with more than one risky asset. Cvitanic, Pham, and Touzi (1997) prove that the terminal value problem (4.1) admits a classical solution if $\sigma_{\min} > 0$ and if the payoff is sufficiently smooth. Moreover, they show that in a large class of SV-models of the form (3.1), (3.2) with $\eta_2(v) > 0$ for all $v \in (\sigma^2_{\min}, \sigma^2_{\max})$ the ask price of a claim with payoff $f(S_T)$ is no smaller than $h^{\mathrm{AV}}(t, S_t)$, provided of course that a solution to (4.1) exists.

Remark 4.3. Note that the above argument also works for functions $h^{\mathrm{AV}} \in \mathcal{C}^{1,1}([0,t) \times \mathbb{R}^+)$, if the space derivative $h^{\mathrm{AV}}_x(t,\cdot)$ is moreover absolutely continuous in x for every t. For an extension of Itô's formula to such situations see e.g. (Krylov 1980, Theorem 2.10.1).

We now discuss the relation between V^{Am} and the nonlinear PDE (4.1). For this we have to distinguish the cases $\sigma_{\min} = 0$ and $\sigma_{\min} > 0$.

4.2 The case $\sigma_{\min} = 0$

In this case the value function V^{Am} of our superhedging strategy will typically not belong to $\mathcal{C}^{1,2}([0,T) \times \mathbb{R}^+)$, as the second derivative V^{Am}_{xx} is usually discontinuous at the optimal stopping boundary of the optimal-stopping problem defining V^{Am}, see also Section 5.1 below. We therefore contend ourselves with a local result.

Proposition 4.4. *Assume that* $\sigma_{\min} = 0$.

(i) *If the value function* $V^{\mathrm{Am}}(t, S_t; 0, \sigma_{\max})$ *defined in (2.6) is of class* $\mathcal{C}^{1,2}$ *in some open set* $B \subset\subset ([0,T) \times \mathbb{R}^+)$, V^{Am} *solves the following version of the PDE (4.1):*

$$V^{\mathrm{Am}}_t(t,x) + \frac{1}{2} x^2 \sigma^2_{\max} [V^{\mathrm{Am}}_{xx}(,x)]^+ = 0 \text{ for all } (t,x) \in B\,. \tag{4.2}$$

(ii) *Suppose that there is a solution* h^{AV} *of the terminal value problem (4.2), which belongs to* $\mathcal{C}^{1,1}([0,t) \times \mathbb{R}^+)$ *and whose space derivative* $h^{\mathrm{AV}}_x(t,\cdot)$ *is moreover absolutely continuous. Then* $V^{\mathrm{Am}} = h^{\mathrm{AV}}$.

Proof: We start with (i). From the characterization of solutions to the optimal-stopping problem (2.6) via variational inequalities we get for all $(t,x) \in B$

$$V^{\mathrm{Am}}_t(t,x) + \frac{1}{2} x^2 \sigma^2_{\max} V^{\mathrm{Am}}_{xx}(t,x) \le 0 \tag{4.3}$$

$$V^{\mathrm{Am}}(t,x) \ge f(x) \text{ and } V^{\mathrm{Am}}(T,x) = f(x)\,, \tag{4.4}$$

where at least one of the two inequalities must holds with equality. For a proof see e.g. Jaillet, Lamberton, and Lapeyre (1990) or Myeni (1992). Moreover, V^{Am} is decreasing in t, i.e. we have $V^{\mathrm{Am}}_t \le 0$ in $B(t_0, x_0)$. Choose some $(t_0, x_0) \in B$. Now we distinguish two cases.

(a) $V^{\mathrm{Am}}_{xx}(t_0,x_0) > 0$: We shall show that this implies $V^{\mathrm{Am}}(t_0,x_0) > f(x_0)$; hence equality must hold in (4.3) which shows that (4.2) holds in this case. Assume to the contrary that $V^{\mathrm{Am}}(t_0,x_0) = f(x_0)$; in that case we must have $V^{\mathrm{Am}}_t(t_0,x_0) = 0$, as $V^{\mathrm{Am}}_t(t_0,x_0) < 0$ would yield a contradiction to (4.4). However, together with $V^{\mathrm{Am}}_{xx}(t_0,x_0) > 0$ this implies that $V^{\mathrm{Am}}_t(t_0,x_0) + V^{\mathrm{Am}}_{xx}(t_0,x_0) > 0$ which contradicts (4.3).

(b) $V^{\mathrm{Am}}_{xx}(t_0,x_0) \le 0$: We show that in that case $V^{\mathrm{Am}}_t(t_0,x_0) = 0$, which implies the result. Assume to the contrary that $V^{\mathrm{Am}}_t(t_0,x_0) < 0$. Hence strict inequality holds in (4.3) such that (4.4) must hold with equality. However, together with $V^{\mathrm{Am}}_t(t_0,x_0) < 0$ this contradicts (4.4) which proves that we must have $V^{\mathrm{Am}}_t(t_0,x_0) = 0$.

Let us now turn to (ii). As shown before h^{AV} induces a superhedging strategy in all SV-models satisfying Assumptions 1 and 3, hence in all models satisfying the hypothesis of Theorem 3.1. As V^{Am} is minimal in these models we have the inequality $h^{\mathrm{AV}} \geq V^{\mathrm{Am}}$. The converse inequality is proved in (Cvitanic, Pham, and Touzi 1997, Remark 6.1). $\square$

4.3 The case $\sigma_{\min} > 0$

To study the relation between V^{Am} and solutions to the nonlinear PDE (4.1) we define the function u: $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$u(t_1, t_2, x) := \sup \left\{ E_x^R[f(U_\nu)] \, , \; \nu \in \mathcal{B}_{t_1, t_1+t_2} \right\} , \tag{4.5}$$

where $(\mathcal{B}_t)$ denotes again the canonical filtration on $\mathcal{C}_{[0,\infty]}$. The function V^{Am} is related to u via

$$V^{\mathrm{Am}}(t, x; \sigma_{\min}, \sigma_{\max}) = u(\sigma_{\min}^2 (T-t), (\sigma_{\max}^2 - \sigma_{\min}^2)(T-t), x) \, . \tag{4.6}$$

Now, using Proposition A.1 we may express u as follows:

$$u(t_1, t_2, x) = E_x^R \left[\operatorname{ess\,sup} \left\{ E_x^R[f(U_\nu) | \mathcal{F}_{t_1}] \, , \; \nu \in \mathcal{B}_{t_1, t_1+t_2} \right\} \right] = E_x^R[h(t_2, U_{t_1})] \, ,$$

where h is defined via the following standard optimal-stopping problem

$$h(t, x) = \sup \left\{ E_x^R[f(U_\nu)] \, , \; \nu \in \mathcal{B}_{0,t} \right\} \, . \tag{4.7}$$

We make the following regularity assumption on h.

Assumption 5. *The function h defined in (4.7) is continuous on $([0,\infty) \times \mathbb{R}^+)$ and is of class $\mathcal{C}^{1,1}((0,\infty) \times \mathbb{R}^+)$. Moreover, for every t there is a finite number of points $x_1, \ldots, x_{n(t)}$, such that for all $x \in \mathbb{R}^+ - \{x_1, \ldots, x_{n(t)}\}$ there is an open environment $B(t,x) \subset ((0,\infty) \times \mathbb{R}^+)$ where h is twice continuously differentiable in x. Moreover the functions h_t, $x h_x$, h_{xx} and $x^2 h_{xx}$ are uniformly bounded on $(0,\infty) \times \mathbb{R}^+$.*

Remark 4.5. These regularity assumptions typically hold for the value function of the optimal-stopping problem (4.7), provided that the terminal value f is sufficiently smooth; see for instance the examples in Section 5.1.

Proposition 4.6. *Suppose that $\sigma_{\min} > 0$. Under Assumption 5 we have the following*

(i) *u belongs to $\mathcal{C}^{1,1,2}((0,\infty) \times (0,\infty) \times \mathbb{R}^+)$.*

(ii) *We have for all $(t_1, t_2, x) \in (0,\infty) \times (0,\infty) \times \mathbb{R}^+$*

$$u_{t_1} = (t_1, t_2, x) = \frac{1}{2} x^2 u_{xx}(t_1, t_2, x) \, , \tag{4.8}$$

$$u_{t_2}(t_1, t_2, x) = E_x^R \left[x^{-2} U_{t_1}^2 h_{xx}(t_2, U_{T_1}) \right] \geq \frac{1}{2} x^2 [u_{xx}(t_1, t_2, x)]^+ \, . \tag{4.9}$$

Equality in (4.9) holds if and only if $h_{xx}(t_2, \cdot)$ is either everywhere nonnegative or everywhere nonpositive.

(iii) *$V^{\mathrm{Am}}(t, x; \sigma_{\min}, \sigma_{\max})$ satisfies the following differential inequality:*

$$V_t^{\mathrm{Am}} + \frac{1}{2} x^2 \big(- \sigma_{\min}^2 [V_{xx}^{\mathrm{Am}}]^- + \sigma_{\max}^2 [V_{xx}^{\mathrm{Am}}]^+ \big) \leq 0 \, . \tag{4.10}$$

Equality holds if and only if equality holds in (4.9), in particular for f convex on $\mathbb{R}^+$ or f concave on $\mathbb{R}^+$. Moreover, we have $V^{\mathrm{Am}}(t,x) \geq h^{\mathrm{AV}}(t,x)$; equality holds if and only if (4.10) holds with equality.

The most important result here is (iii). This result implies that in a model with strictly positive lower volatility bound the ask price of a derivative whose payoff is neither everywhere convex nor everywhere concave may be smaller than V^{Am}. However, as shown in Section 5.1, numerical values of V^{Am} and h^{AV} are typically close to each other.

PROOF: Define for t_2 fixed the function $\tilde{u}(t,x) := u(t,t_2,x)$. It follows immediately from the Feynman-Kac formula that $\tilde{u}$ solves the initial value problem

$$\tilde{u}(t,x) = \frac{1}{2}x^2\tilde{u}_{xx}(t,x), \quad \tilde{u}(0,\cdot) = h(t_2,\cdot)\,.$$

Hence u is $\mathcal{C}^1$ in t_1, $\mathcal{C}^2$ in x and it satisfies (4.8). By Proposition 4.4 and Assumption 5 the function h solves for almost all x the PDE $h_t = \frac{1}{2}x^2[h_{xx}]^+$. Again by Assumption 5 we may exchange differentiation and expectation yielding

$$\begin{aligned}\frac{\partial u}{\partial t_2}(t_1,t_2,x) &= E_x^R\left[\frac{\partial h}{\partial t}(t_2,U_{t_1})\right]\\ &= \frac{1}{2}E_x^R\left[U_{t_1}^2[h_{xx}(t_2,U_{t_1})]^+\right]\\ &\geq \frac{1}{2}\Big[E_x^R\left[U_{t_1}^2 h_{xx}(t_2,U_{t_1})\right]\Big]^+\,,\end{aligned} \tag{4.11}$$

where the last estimate follows from Jensen's inequality. Recall that U is R_x-geometric Brownian motion with initial value $U_0 = x$; hence U_{t_1} is R_x-a.s. equal to $x\exp(W_t - \frac{1}{2}t)$ such that $\frac{\partial}{\partial x}U_{t_1} = U_{t_1}/x$. We use this information to compute the derivatives of $h(t_2,U_{t_1})$ with respect to x and get

$$\frac{\partial^2}{\partial x^2}h(t_2,U_{t_1}) = \frac{\partial}{\partial x}\left(h_x(t_2,U_{t_1})\frac{U_{t_1}}{x}\right) = \frac{U_{t_1}^2}{x^2}h_{xx}(t_2,U_{t_1})\,.$$

As the last expression is bounded by Assumption 5 we may exchange expectation and differentiation and get

$$\frac{\partial^2}{\partial x^2}u(t_1,t_2,x) = x^{-2}E_x^R\left[U_{t_1}^2 h_{xx}(t_2,U_{t_1})\right]\,.$$

Combining this with (4.11) we get $u_{t_2}(t_1,t_2,x) \geq \frac{1}{2}x^2[u_{xx}(t_1,t_2,x)]^+$, i.e. (4.9). Obviously equality holds if and only if equality holds in (4.11), which proves (ii). Let us now turn to (iii). By (4.6) we get that

$$\begin{aligned}V_t^{\mathrm{Am}} &= -\sigma_{\min}^2 u_{t_1} - (\sigma_{\max}^2 - \sigma_{\min}^2)u_{t_2}\\ &\overset{(a)}{\leq} -\sigma_{\min}^2\frac{1}{2}x^2u_{xx} - (\sigma_{\max}^2 - \sigma_{\min}^2)\frac{1}{2}x^2[u_{xx}]^+\\ &\overset{(b)}{=} -\frac{1}{2}x^2(-\sigma_{\min}^2[V_{xx}^{\mathrm{Am}}]^- + \sigma_{\max}^2[V_{xx}^{\mathrm{Am}}]^+)\,,\end{aligned}$$

which is (4.10). Here (a) follows from statement (ii) and (b) from the relation $u_{xx} = V_{xx}^{\mathrm{Am}}$. The inequality $V^{\mathrm{Am}}(t,x) \geq h^{\mathrm{AV}}(t,x)$ follows now from the maximum principle for viscosity solutions of nonlinear parabolic PDE's; see Remark 6.1 of Cvitanic, Pham, and Touzi (1997). $\square$

5 Examples and simulations

5.1 Path-independent derivatives

In this section we consider path-independent derivatives whith payoff given by some function $f(S_T)$. As in Section 4 we distinguish between the cases $\sigma_{\min} = 0$ and $\sigma_{\min} > 0$.

i) THE CASE $\sigma_{\min} = 0$: In the case of unbounded volatility, i.e. $\sigma_{\max} = \infty$, V^{Am} is given by the smallest concave majorant f^* of f, see the comments following Theorem 3.1. Cvitanic, Pham, and Touzi (1997) give the following description of f^* as affine envelope of f:

$$f^*(x) = \inf \{c > 0\,,\ \exists \Delta \in \mathbb{R} \text{ such that } c + \Delta(z - x) \geq f(z) \text{ for all } z > 0\}\,. \tag{5.1}$$

Let us now consider a call-spread with strike prices $K_1 < K_2$ as more specific example; the payoff of this derivative is given by $f(x) := [x - K_1]^+ - [x - K_2]^+$. This payoff is interesting in our context as it is neither everywhere convex nore everywhere concave. Hence the superreplication price is not simply the Black-Scholes price corresponding to one of the volatility bounds. Using the description (5.1) it is immediately seen that for $\sigma_{\max} = \infty$ the superreplicating cost is given by

$$f^*(x) := \begin{cases} \frac{K_2 - K_1}{K_2}\, x & , \quad 0 < x \leq K_2 \\ K_2 - K_1 & , \quad x > K_2 \end{cases}\,. \tag{5.2}$$

For $\sigma_{\max} < \infty$ we have to use numerical techniques to obtain values for V^{Am}. Figure 1 shows V^{Am}, the superreplication cost for a standard call-spread with $K_1 = 90$, $K_2 = 100$, time to maturity equal to 6 month and volatility bounds given by $\sigma_{\min} = 0$ and $\sigma_{\max} = 0.4$. Observe that $V^{\mathrm{Am}}(t,x) = K_2 - K_1$ whenever $x \geq K_2$, as for $x \geq K_2$ immediate exercice is the optimal strategy in the stopping problem defining V^{Am}.

Remark 5.1. Note that the left limit $\lim_{x \to K_2^-} V_x^{\mathrm{Am}}(t,x)$ must be larger than $(K_2 - K_1)/K_2$, as $V^{\mathrm{Am}} \leq f^*$, the superreplicating cost for $\sigma_{\max} = \infty$. V_x^{Am} is therefore discontinuous in $x = K_2$, hence in particular not absolutely continuous with respect to x; compare also Figure 1. By Proposition 4.4 (ii) the terminal value problem (4.1) does therefore not admit a classical solution.This shows that at least for $\sigma_{\min} = 0$ the PDE-approach to superhedging is not always as straightforward as it seems at first sight.

In Figure 2 we have graphed the superhedging cost for a "call-spread" with smooth terminal payoff f,[2] time to maturity equal to 6 month and volatility bounds given by $\sigma_{\min} = 0$ and $\sigma_{\max} = 0.4$, together with the terminal payoff f. Recall the definition of the stopping boundary B^* for the stopping problem defining V^{Am}. Here B^* is given by

$$B^* = \{(t, b^*(t))\,,\ b^*(t) = \inf\{x > 0,\, V^{\mathrm{Am}}(t,x) = f(x)\}\,.$$

We see that in the example with smooth terminal payoff we have "smooth fit", i.e. the space derivative V_x^{Am} is continuous at b^*. However, the second derivative V_{xx}^{Am} is discontinuous at b^*: On the one hand we have

$$\lim_{x \to b^*(t)^+} V^{\mathrm{Am}}(t,x) = f''(b^*(t)) < 0\,.$$

[2] f is given by the Black-Scholes price of a standard call-spread with $K_1 = 90$, $K_2 = 100$, time to maturity one week and volatility 0.2.

On the other hand it follows from the characterization of V^{Am} via variational inequalities that $V^{\mathrm{Am}}_{xx}(t,x) \geq 0$ whenever $V(t,x) > f(x)$; see also the proof of Proposition 4.4. The regularity properties of this example, which are typical for optimal-stopping problems with sufficiently smooth payoff, motivate some of the hypothesises in Assumption 5.

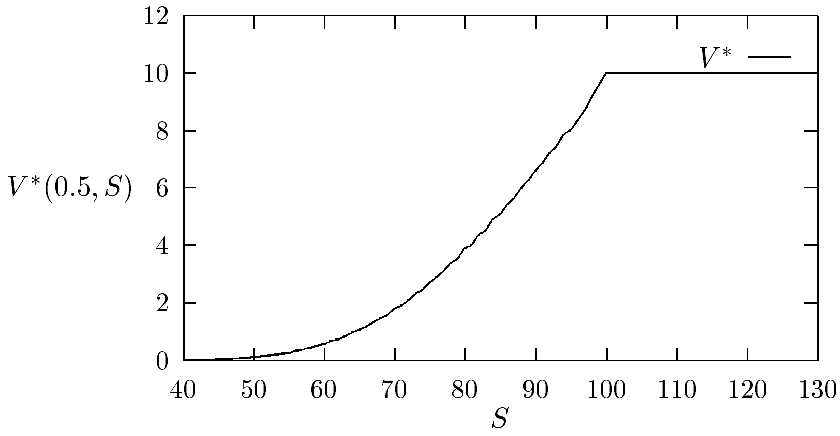

Figure 1: Superreplication cost for a standard call-spread with $K_1 = 90$, $K_2 = 100$, time to maturity 6 month and volatility bounds $\sigma_{\min} = 0$ and $\sigma_{\max} = 0.4$.

ii) The case $\sigma_{\min} > 0$: We know from Proposition 4.6 (iii) that V^{Am} is typically not equal to the ask-price of a path-independent derivative whenever the terminal payoff is of mixed convexity. To get a feeling for the numerical size of the difference between V^{Am} and the discounted ask-price, which is given by the solution h^{AV} to the terminal value problem (4.1), we computed V^{Am} for the standard call-spread considered above. In Table 1 we present for different values of S_0 our solution V^{Am} together with values for h^{AV} taken from Avellaneda, Levy, and Paras (1995), assuming that the volatility is bounded by $\sigma_{\min} = 0.1$ and $\sigma_{\max} = 0.4$. We see that the difference between the two functions is relatively small.

S_0	75	80	85	90	95
V^{Am}	2.71	3.92	5.09	6.53	7.78
h^{AV}	2.69	3.73	4.90	6.15	7.44
$V^{\mathrm{Am}} - h^{\mathrm{AV}}$	0.02	0.19	0.19	0.38	0.34

Table 1: Superreplication price for a standard call-spread with $K_1 = 90$, $K_2 = 100$, time to maturity 6 month, and volatility bounds $\sigma_{\min} = 0.1$ and $\sigma_{\max} = 0.4$. V^{Am} gives the superreplication cost according to our approach, h^{AV} is the solution to the terminal value problem (4.1).

5.2 Barrier options

We now consider a particular barrier option namely a *down-and-out call* on the forward price with strike price K and barrier H as example of a derivative with path-dependent payoff. In the notation introduced in Section 2.2 its payoff is given by

$$f\left(Z_T, Z^{\min}_{[0,T]}\right) := [Z_T - K]^+ 1_{\left\{Z^{\min}_{[0,T]} \geq H\right\}}.$$

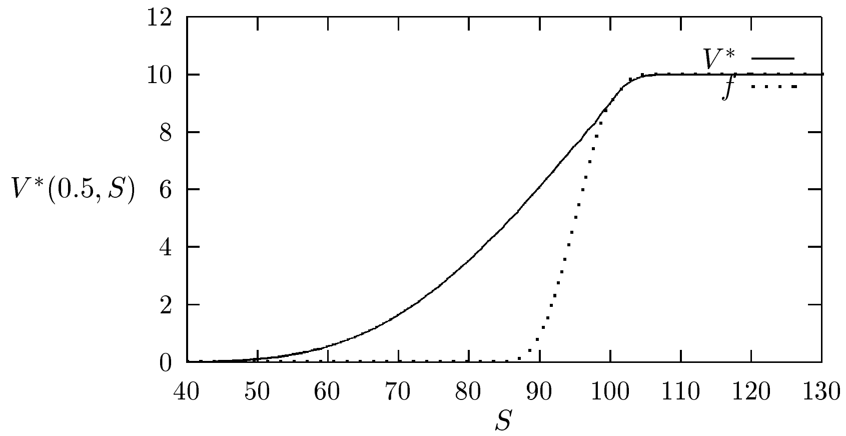

Figure 2: Superreplication cost for a "smooth call-spread" with terminal payoff $f(x)$, time to maturity 6 month and volatility bounds $\sigma_{\min} = 0$ and $\sigma_{\max} = 0.4$.

For this particular payoff we may give an analytic expression for the superhedging strategy V^{Am}. For our analysis we have to distinguish if $H > K$ or if $H \le K$.

i) THE CASE $H \le K$: It is well-known that in this case a rational investor will never exercise an American down-and-out call before maturity; see e.g. Reimer and Sandmann (1995) for the corresponding portfolio argument. Hence our superhedging cost V^{Am} equals the price of the down-and-out call in a Black-Scholes model with constant volatility equal to the *upper* volatility bound $\sigma_{\max}$ and zero interest rate, independently of $\sigma_{\min}$. This price is well known, see e.g. Reimer and Sandmann (1995) or Chapter 9 of Musiela and Rutkowski (1997). By Theorem 3.1 this is the ask-price for the down-and out call in a large class of SV-models with volatility range $[0, \sigma_{\max}]$. As the optimum in the stopping problem for V^{Am} is attained at a deterministic stopping time, the proof of Theorem 3.1 shows that V^{Am} is the ask price for the barrier option even if $\sigma_{\min} > 0$.

Le us now consider the case $\sigma_{\max} = \infty$. Inspection of the formula for the Black-Scholes price of our barrier call shows that in that case V^{Am} is given by

$$V^{\mathrm{Am}}(t, Z_t, Z^{\min}_{[0,t]}) = 1_{\left\{Z^{\min}_{[0,t]} \ge H\right\}}(Z_t - H)\,.$$

By Theorem 3.1 this is the ask-price of the down-and-out call in most of the standard SV-models with unbounded volatility . The corresponding hedging strategy is a buy and hold strategy. At $t = 0$ we buy one share of the stock and sell H zero-coupon bonds with maturity T. If the barrier is hit the value of our position is zero and we dissolve the portfolio immediately, otherwise we hold our position until maturity.

ii) THE CASE $H > K$: An elementary argument shows that for $\sigma_{\min} = 0$ the function V^{Am} equals

$$V^{\mathrm{Am}}(t, Z_t, Z^{\min}_{[0,t]}) = 1_{\left\{Z^{\min}_{[0,t]} \ge H\right\}}(Z_t - K)\,, \tag{5.3}$$

independently of $\sigma_{\max}$. By Theorem 3.1 this is the ask-price for the down-and-out call in a large class of SV-models with $\sigma_{\min} = 0$. Let us now look what happens if $\sigma_{\min} > 0$.

Here we have

$$
\begin{aligned}
V^{\mathrm{Am}}(t, Z_t, Z^{min}_{[0,t]}) &= 1_{\left\{Z^{\min}_{[0,t]} \geq H\right\}} E^R_{Z_t}\Big[1_{\left\{U^{\min}_{[0,\sigma^2_{\min}(T-t)]} \geq H\right\}} \\
&\qquad \sup\left\{E^R_{U(\sigma^2_{\min}(T-t))}\left[f(U_\nu, U^{\min}_{[0,\nu]})\right] \; ; \; \nu \in \mathcal{B}_{0,(\sigma^2_{\max}-\sigma^2_{\min})(T-t)}\right\}\Big] \\
&= E^R_{Z_t}\left[1_{\left\{U^{\min}_{[0,\sigma^2_{\min}(T-t)]} \geq H\right\}}\left(U(\sigma^2_{\min}(T-t)) - K\right)\right], \qquad (5.4)
\end{aligned}
$$

where the last equality follows from (5.3). Obviously (5.4) and hence V^{Am} is equal to the price of the option in a Black-Scholes model with constant volatility equal to the *lower* volatility bound $\sigma_{\min}$. For an explicit formula see again Reimer and Sandmann (1995) or Chapter 9 of Musiela and Rutkowski (1997). It is not difficult to see that V^{Am} is the ask-price of the option in a large class of SV-models with volatility range $[\sigma_{\min}, \infty)$.
iii) USING "VANILLA OPTIONS" TO REDUCE THE HEDGE COST: We now present a numerical example that explains how traded "vanilla options" can be used to reduce the superhedging cost. We want to hedge a down-and-out call with strike price $K = 80$, barrier $H = 100$ and time to maturity 3 month, assuming that the volatility range is given by $\sigma_{\min} = 0.15$ and $\sigma_{\max} = 0.4$. We moreover assume that we can take arbitrary positions in a standard call option with $K = 100$ and time to maturity 3 month, trading at an implied volatility of $\sigma_{\mathrm{impl}} = 0.3$. If we do not take any position in the vanilla call the superhedging cost for the barrier option is given by the price of the option in a Black-Scholes model with volatility $\sigma = 0.15$. If we add a position of λ standard calls to our portfolio, the hedge cost is given by the sum $V^{\mathrm{Am}}_\lambda + \lambda C(S_0)$. Here V^{Am}_λ represents the superhedging cost of the payoff

$$
f_\lambda\left(Z_T, Z^{\min}_{[0,T]}\right) := [Z_T - 80]^+ 1_{\left\{Z^{\min}_{[0,T]} \geq 100\right\}} - \lambda[Z_T - 100]^+ ,
$$

and $C(S_0)$ denotes the current markety price of the vanilla call. The following table gives the superreplication cost for $\lambda = 0$ and for $\lambda = -2.5$.

Superhedging cost for $\lambda = 0$:	25.7
Superhedging cost for $\lambda = -2.5$:	23.6
Black-Scholes price for $\sigma = 0.225$:	21.8

We see that by using a static position in the vanilla call we can achieve a drastic reduction of our hedge cost. Our superhedging price is now much closer to the Black-Scholes price for a "reasonable" input volatility of $\sigma = 0.225$. Of course in our situation one should choose λ so that the superhedging cost of the portfolio is minimized. This idea is developed systematically in Avellaneda and Paras (1996).

A Appendix

A.1 Exchanging conditional expectation and essential supremum

The following proposition, which is adapted from El Karoui (1981), justifies the exchange of essential supremum and conditional expectation carried out at various places throughout the paper.

Proposition A.1. *Consider two stopping times $\underline{\tau} \leq \overline{\tau}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Let $(f_t)_{t\geq 0}$ denote some adapted and RCLL-stochastic process, which is bounded below. Then we have for two points in time $0 \leq s < t \leq \overline{\tau}$*

$$\operatorname{ess\,sup}\left\{E[f_\tau \mid \mathcal{F}_s]\,,\, \tau \in \mathcal{F}_{\underline{\tau},\overline{\tau}}\right\} = E\left[\operatorname{ess\,sup}\left\{E[f_\tau \mid \mathcal{F}_t]\,,\, \tau \in \mathcal{F}_{\underline{\tau},\overline{\tau}}\right\} \mid \mathcal{F}_s\right]. \tag{A.1}$$

PROOF: We may write the lhs of (A.1) as $\operatorname{ess\,sup}\left\{E[E[f_\tau|\mathcal{F}_t] \mid \mathcal{F}_s]\,,\, \tau \in \mathcal{F}_{\underline{\tau},\overline{\tau}}\right\}$. Hence the lhs of (A.1) is a priori smaller than the rhs. Define for $\tau \in \mathcal{F}_{\underline{\tau},\overline{\tau}}$ the $\mathcal{F}_t$-measurable random variable Y^τ by $Y^\tau := E[f_\tau|\mathcal{F}_t]$. To show equality in (A.1) we have to show that there is a sequence $(\tau_n^*)_{n\in\mathbb{N}}$ such that

$$Y^{\tau_n^*} \text{ converges monotonically to } \operatorname{ess\,sup}\left\{Y^\tau\,,\, \tau \in \mathcal{F}_{\underline{\tau},\overline{\tau}}\right\}\,; \tag{A.2}$$

in that case the claim follows by monotone integration. It is well-known from general properties of the essential supremum that there exists a countable sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $\operatorname{ess\,sup}\left\{Y^\tau\,,\, \tau \in \mathcal{F}_{\underline{\tau},\overline{\tau}}\right\} = \sup\left\{Y^{\tau_n}\,,\, n \in \mathbb{N}\right\}$. Hence a sequence $(\tau_n^*)_{n\in\mathbb{N}}$ with (A.2) exists, if $\mathbb{M}_{\underline{\tau}\overline{\tau}} := \{Y^\tau\,,\, \tau \in \mathcal{F}_{\underline{\tau},\overline{\tau}}\}$ is *sup-stable.*

Consider τ_1 and τ_2 from $\mathcal{F}_{\underline{\tau},\overline{\tau}}$ and define a set $A \in \mathcal{F}_t$ by $A := \{Y^{\tau_1} > Y^{\tau_2}\}$. Define the random variable $\tau_{\max}(\tau_1, \tau_2)$ by $\tau_{\max}(\tau_1, \tau_2) = \tau_1 1_A + \tau_2 1_{A^C}$. It is easily seen that $\tau_{\max}(\tau_1, \tau_2)$ is a stopping time, so that $\tau_{\max}(\tau_1, \tau_2) \in \mathcal{F}_{\underline{\tau},\overline{\tau}}$. Moreover, as $A \in \mathcal{F}_t$,

$$Y^{\tau_{\max}} = E[f_{\tau_1}|\mathcal{F}_t]1_A + E[f_{\tau_2}|\mathcal{F}_t]1_{A^C} = Y^{\tau_1} \vee Y^{\tau_2}.$$

Hence $Y^{\tau_1} \vee Y^{\tau_2}$ belongs to $\mathbb{M}_{\underline{\tau}\overline{\tau}}$, showing that this set of random variables is sup-stable. $\square$

A.2 Proof of Lemma 2.11

Let τ be an $(\mathcal{F}_t)$-stopping time. Then we have for any $t_0 \geq 0$

$$\{\langle M\rangle_\tau \leq t_0\} = \{\tau \leq A_{t_0}\} \overset{(i)}{\in} \mathcal{F}_{A_{t_0}} = \mathcal{G}_{t_0}\,,$$

where (i) follows as τ and A_{t_0} are $(\mathcal{F}_t)$-stopping times. Conversely, as $\langle M\rangle_{t_0}$ is a $(\mathcal{G}_t)$-stopping time we have for any $(\mathcal{G}_t)$-stopping time ν

$$\{A_\nu \leq t_0\} = \{\nu \leq \langle M\rangle_{t_0}\} \in \mathcal{G}_{\langle M\rangle_{t_0}} = \mathcal{F}_{t_0}\,,$$

which proves that A_ν is an $(\mathcal{F}_t)$-stopping time. $\square$

A.3 Proof of Lemma 3.3

Define for a given $(\mathcal{G}_t)$-stopping time ν a random time τ by $\tau = A(\nu)$. By Lemma 2.11 τ is an $(\mathcal{F}_t)$-stopping time. Denote by $\mathbb{D}^2_{[0,T]}$ the two-dimensional Skorohod space. Hypothesis 2.(i) and 3.(i) imply that the densities $G^{i,n}$ can be written as functions of the trajectories of X:

$$G_t^{i,n} = G^{i,n}\left(t; (X_s)_{0\leq s\leq t}\right) \text{ for a function } G^{i,n} : [0,T] \times \mathbb{D}^2_{[0,T]} \to \mathbb{R}^+ \text{ with}$$

$$y^1, y^2 \in \mathbb{D}^2_{[0,T]},\ y^1 = y^2 \text{ on } [0,t] \Rightarrow G^{i,n}(t; y^1) = G^{i,n}(t; y^2)\,.$$

Now define for given n_1, n_2 an equivalent martingale measure $Q \in \mathbb{M}^e$ by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_T} := G^{1,n_1}(\tau \wedge T;\, X)\, G^{2,n_2}\left(T - \tau \wedge T;\, \theta_{\tau\wedge T}(X)\right)\,, \tag{A.3}$$

where θ denotes the shift operator on $\mathbb{D}^2$. By definition $\tau = \inf\{t > 0,\, \langle M\rangle_t \geq \nu\}$. Hence

$$\begin{aligned} Q[\langle M\rangle_T \in [\nu,\, \nu+\delta]] &= Q[\langle M\rangle_T \geq \nu;\, \langle M\rangle_T - \langle M\rangle_\tau \leq \delta] && \text{(A.4)} \\ &= Q[\tau \leq T;\, (\langle M\rangle_{T-\tau} \circ \theta_\tau \,\leq \delta)]\,. && \text{(A.5)} \end{aligned}$$

Now by definition of Q it follows that (A.5) equals

$$E^P\left[1_{\{\tau\leq T\}} G^{1,n_1}(\tau\wedge T;\, X) 1_{\{(\langle M\rangle_{T-\tau\wedge T})\,\circ\,\theta_{\tau\wedge T}\,\leq\,\delta\}} G^{2,n_2}\left(T-\tau\wedge T;\, \theta_{\tau\wedge T}(X)\right)\right]\,.$$

Conditioning on $\mathcal{F}_{\tau\wedge T}$ we get from the strong Markov property that this is equal to

$$E^P\left[1_{\{\tau\leq T\}} G^{1,n_1}(\tau\wedge T;\, X)\, E^P_{X_\tau}\left[G^{2,n_2}(T-\tau\wedge T;\, X);\, \langle M\rangle_{T-\tau\wedge T} \leq \delta\right]\right]. \tag{A.6}$$

Now, using that G^{2,n_2} is a martingale and that $\langle M\rangle_t$ is increasing, we get

$$\begin{aligned} E^P_{X_\tau}\left[G^{2,n_2}(T-\tau\wedge T;\, X);\, \langle M\rangle_{T-\tau\wedge T}\leq\delta\right] &= E^P_{X_\tau}\left[G^{2,n_2}(T;\, X);\, \langle M\rangle_{T-\tau\wedge T}\leq\delta\right] \\ &\geq E^P_{X_\tau}\left[G^{2,n_2}(T;\, X);\, \langle M\rangle_T\leq\delta\right]\,. \end{aligned}$$

Moreover we may obviously replace $G^{1,n_1}(\tau\wedge T;\, X)$ by $G^{1,n_1}(T;\, X)$ in (A.6). Hence

$$Q\left[\langle M\rangle_T \in [\nu,\, \nu+\delta]\right] \geq E^P\left[1_{\{\tau\leq T\}} G^{1,n_1}(T;\, X) Q^{2,n}_{X_\tau}\left[\langle M\rangle_T\leq\delta\right]\right] \tag{A.7}$$

Fix some $\varepsilon > 0$. Choose n_1 large enough so that $Q^{1,n_1}[\tau \geq T] < \varepsilon/3$, and choose $K \subset\subset \mathbb{R}^+ \times [0,\, \overline{\sigma}]$ with $Q^{1,n_1}[X_t \notin K$ for some $t \in [0,T]] < \varepsilon/3$. Now choose finally for this set K some n_2 such that

$$Q^{2,n_2}_x\left[\langle M\rangle_T \leq \delta\right] > 1 - \varepsilon/3 \text{ for all } x \in K\,.$$

This is possible by hypothesises 2(iii) and 3(iii). Hence the rhs of (A.7) is minorized by

$$\begin{aligned} &E^P\left[1_{\{\tau\leq T\}} 1_{\{X_\tau\in K\}}\, G^{1,n_1}(T;\, X) Q^{2,n_2}_{X_\tau}\left[\langle M\rangle_T\leq\delta\right]\right] \\ &\quad\geq (1-\varepsilon/3) Q^{1,n_1}\left[(\tau\leq T)\cap(X_\tau\in K)\right] \\ &\quad\geq (1-\varepsilon/3)\left(1 - Q^{1,n_1}[\tau\geq T] - Q^{1,n_1}\left[X_t\notin K \text{ for some } t\in[0,T]\right]\right) \\ &\quad\geq (1-\varepsilon/3)(1-2\varepsilon/3) \\ &\quad> (1-\varepsilon)\,. \end{aligned}$$

□

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