Lecture Continuous-time Finance II

Rüdiger Frey

ruediger.frey@wu.ac.at
www.math.uni-leipzig.de˜frey

WU Vienna, Summer 2012

Admin

- Dates of the lecture 6.3., 13.3., 20.3., 27.3., 17.4., 24.4., 8.5.
- Examination: 30 % exercise sheets (3 sets), 70% final oral exam on May 8 (to be discussed), up to 10 % bonus points for participation in lecture

Overview

1. Basics
   - (Further) tools from stochastic calculus
   - Financial Mathematics in Continuous Time
2. Introduction to Term Structure Modelling
3. Time permitting: Introduction to Dynamic Credit Risk Models

Literature. Part 1 and 3 is based on own notes; Part 2 will be based on the survey "Interest Rate Risk" by T. Björk

Martingales

Consider a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$.

Definition

An adapted stochastic process X with $E(|X_t|) < \infty$ for all $t > 0$ is

- a submartingale if $\forall t, s$ with $t > s$ we have $E(X_t|\mathcal{F}_s) \geq X_s$.
- a supermartingale if $\forall t, s$ with $t > s$ we have $E(X_t|\mathcal{F}_s) \leq X_s$.
- a martingale if X is both a sub- and a supermartingale, i.e. if $E(X_t|\mathcal{F}_s) = X_s$ for all t, s.

Important examples for martingales are the Brownian motion and the compensated Poisson process.

Stopping times

Definition

A rv $\tau\colon \Omega \to [0,\infty]$ is called *stopping time wrt.* $\{\mathcal{F}_t\}$ if $\forall t \geq 0$ holds that $\{\tau \leq t\} \in \mathcal{F}_t$.

Most important example: first hitting times.

Definition

Given a stochastic process X and a Borel set A in $\mathbb{R}$. Define $\tau_A := \inf\{t \geq 0\colon X_t \in A\}$. Then the rv τ_A is called *first hitting time* into the set A.

The sigma-field $\mathcal{F}_\tau$.

Definition

Given a stopping time τ. Then we call

$$\mathcal{F}_\tau := \left\{A \in \mathcal{F} \colon A \cap \{\tau \leq t\} \in \mathcal{F}_t \, \forall t \geq 0\right\} \tag{1}$$

the σ-field of the events which are observable up to τ.

Lemma

Given two stopping times S and T. Then

i) $S \wedge T := \min\{S, T\}$ *is a stopping time.*

ii) $S \vee T := \max\{S, T\}$ *is a stopping time.*

iii) $\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

iv) *If* $S \leq T$ *one has* $\mathcal{F}_S \subseteq \mathcal{F}_T$

Stopped processes

Let $(X_t)_{t\geq 0}$ be a right-continuous adapted stochastic process and let T be a stopping time.

- We define the stopped rv X_T by

$$X_T(\omega) := X_{T(\omega)}(\omega) \cdot 1_{\{T<\infty\}}(\omega), \quad \omega \in \Omega. \tag{2}$$

- The stopped process $X^T = (X_t^T)_{t\geq 0}$ is defined by

$$X_t^T(\omega) := X_{t\wedge T(\omega)}(\omega) = \begin{cases} X_{T(\omega)}(\omega), & T(\omega) \leq t. \\ X_t(\omega), & T(\omega) > t. \end{cases} \tag{3}$$

- it holds that $X_T(\omega)$ is $\mathcal{F}_T$-measurable; the stopped process X^T is $\{\mathcal{F}_t\}$-adapted.

The optional sampling theorem

Theorem (Optional sampling theorem)

Consider an adapted stochastic process $X = (X_t)_{t \geq 0}$ with $E(|X_t|) < \infty$, $t \geq 0$. Then the following statements are equivalent.

(1) *X is a martingale.*

(2) *For all bounded stopping times τ ($\tau(\omega) \leq C$ for some $C > 0$, all $\omega \in \Omega$) one has $E(X_\tau) = E(X_0)$.*

(3) *Given two stopping times S and T such that $S \leq T \leq C$ for some $C > 0$. Then $E(X_T \,|\, \mathcal{F}_S) = X_S$.*

Corollary

Let X be a martingale and let τ be a stopping time. Then the stopped process X^τ with $X_t^\tau = X_{t \wedge \tau}$ is also a martingale.

Important Inequalities

Corollary (Martingale inequality)

Let X be a right-continuous martingale such that $X_t > 0$ a.s. Then we have for $C > 0$ $P\left(\sup_{t\geq 0} X_t > C\right) \leq \frac{1}{C} E(X_0)$.

Theorem (Doob inequality)

Let $M_T^* := \sup_{t\leq T} \left|M_t\right|$ *Then*

$$E\left[\left(M_T^*\right)^2\right] \leq 4E\left[M_T^2\right]. \tag{4}$$

Square Integrable Martingales

Throughout we fix some horizon date T.

Definition

The space of all martingales $M = (M_t)_{0 \leq t \leq T}$ with $M_0 = 0$ and $E[M_T^2] < \infty$ is denoted $\mathcal{M}^2$; $\mathcal{M}^{2,c} \subseteq \mathcal{M}^2$ is the subspace of all martingales with continuous trajectories.

Properties.

- $\mathcal{M}^2$ is a Hilbert-space with scalar product

$$\langle M, N \rangle_{\mathcal{M}^2} = E[M_T N_T] = \langle M_T, N_T \rangle_{\mathcal{L}^2}.$$

- $\mathcal{M}^{2,c}$ is a closed subspace of $\mathcal{M}^2$ and hence a Hilbert-space.

Quadratic Variation

$$V_t^2(M;\tau_n) := \sum_{t_i \in \tau_n;\, t_i < t} (M_{t_i} - M_{t_{i-1}})^2 = \sum_{t_i \in \tau_n;\, t_i < t} (\Delta M_{i,n})^2 , \quad (5)$$

is the quadratic variation of M along a partition τ_n of $[0, T]$.

Theorem

Consider a sequence $(\tau_n)_n$ of partitions of $[0, T]$ such that $|\tau_n| \to 0$ and let $M \in \mathcal{M}^{2,c}$. Then

1. $V_t^2(M, \tau_n)$ converges in probability to an increasing and continuous process $\langle M \rangle_t$ with $\langle M \rangle_0 = 0$.

2. $M_t^2 - \langle M \rangle_t$ is a martingale, in particular $E\left[M_t^2\right] = E\left[\langle M \rangle_t\right]$.

3. $\langle M \rangle_t$ is uniquely defined by the requirements

(i) *$\langle M \rangle_t$ is increasing and continuous, $\langle M \rangle_0 = 0$.*

(ii) *$M_t^2 - \langle M \rangle_t$ is a martingale.*

Quadratic Covariation

Corollary

For $M, N \in \mathcal{M}^{2,c}$ the covariation

$$\langle M, N \rangle_t = \lim_{n \to \infty} \sum_{t_i \in \tau_n;\, t_i \leq t} \Delta M_{i,n} \Delta N_{i,n}$$

exists. $\langle M, N \rangle$ is characterized by the following properties

(i) *$\langle M, N \rangle_t$ is a continuous FV-process with $\langle M, N \rangle_0 = 0$.*

(ii) *$M_t N_t - \langle M, N \rangle_t$ is a martingale.*

One has for $M, N \in \mathcal{M}^{2,c}$,

$$E\left[(M_t - M_s)(N_t - N_s)|\mathcal{F}_s\right] = E\left[\langle M, N \rangle_t - \langle M, N \rangle_s | \mathcal{F}_s\right], \quad (6)$$

i.e. $\langle M, N \rangle$ gives the *conditional covariance* of the increments of M and N.

Stochastic Integrals I

Define the class of elementary integrands by

$$\xi := \left\{ H : H_t(\omega) = \sum_{i=0}^{n-1} h_{t_i}(\omega) 1_{(t_i, t_{i+1}]}(t) \right\}, \tag{7}$$

for deterministic $0 \leq t_0 < t_1 < ... < t_n < T = t_{n+1}$, $n \in \mathbb{N}$, and bounded, $\mathcal{F}_{t_i}$-measurable random variables h_{t_i}, $1 \leq i \leq n$.

Definition

For $H \in \xi$ and $M \in \mathcal{M}^{2,c}$ define the stochastic integral $(H.M)$ by

$$(H.M)_t := \int_0^t H_s dM_s := \sum_{0 \leq i \leq n} h_{t_i} \left(M_{t_{i+1} \wedge t} - M_{t_i \wedge t} \right), \ t \leq T,$$

H is called integrand, M integrator.

Stochastic Integrals II

The following result is crucial for extending $(H.M)$ to a larger class of integrands

Lemma (Properties of $(H.M)$ for $H \in \xi$)

We have $(H.M) \in \mathcal{M}^{2,c}$ and the quadratic variation is given by $\langle (H.M) \rangle_t = \int_0^t H_s^2 \, d\langle M \rangle_s$.

Corollary (Itô-isometry)

We have

$$E\left[(H.M)_t^2\right] = E\left[\left(\int_0^t H_s dM_s\right)^2\right] = E\left[\int_0^t H_s^2 d\langle M \rangle_s\right]. \qquad (8)$$

Stochastic Integrals III

Goal. Fix $M \in \mathcal{M}^{2,c}$. We want to extend $(H.M)$ to a larger class of integrands.

- abstract interpretation of $E\left[\int_0^T H_s^2 \, d\langle M\rangle_s\right]$. Put $\bar{\Omega} = \Omega \times [0,T]$, $\bar{\mathcal{F}} = \mathcal{F}_T \otimes \mathcal{B}([0,T])$ and define a measure $\bar{P}_M$ on $\bar{\Omega}$, $\bar{\mathcal{F}}$ by $\bar{P}_M(A) = E\left[\int_0^T 1_A(\omega,t)\, d\langle M\rangle_t\right], \quad A \in \bar{\mathcal{F}}$. We have for $H \in \xi$

$$\|H\|_{\mathcal{L}^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P}_M)} = \bar{E}_M\left[H^2\right] = E\left[\int_0^T H_s^2 \, d\langle M\rangle_s\right]$$

- Itô-isometry $\Rightarrow I : \xi \to \mathcal{M}^{2,c}, \quad H \mapsto (H.M)$ is therefore an isometry from $(\xi, \|\cdot\|_{\mathcal{L}^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P}_M)})$ to $(\mathcal{M}^{2,c}, \|\cdot\|_{\mathcal{M}^2})$

Stochastic Integrals IV

Since $\mathcal{M}^{2,c}$ is a Hilbert space, the following definition makes sense.

Definition

For $H \in \overline{\xi} \subset \mathcal{L}^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}_M)$ the stochastic integral $(H.M)$ is the uniquely defined element of $\mathcal{M}^{2,c}$ such that
$\lim_{n\to\infty} \|(H^n.M) - (H.M)\|_{\mathcal{M}^2} = 0$ for all $H^n \in \xi$ with
$\|H^n - H\|_{\mathcal{L}^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P}_M)} \to 0$.

Properties. Let $M \in \mathcal{M}^{2,c}$, $H_1, H_2 \in \overline{\xi}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then

- Linearity of $(H.M)$:
  $(\alpha_1 H_1 + \alpha_2 H_2 \,.\, M) = \alpha_1 (H_1 \,.\, M) + \alpha_2 (H_2 \,.\, M)$ and
- Itô-isometry: $E\left[(H.M)_t^2\right] = E\left[\int_0^t H_s^2 \, d\langle M\rangle_s\right]$, for $t \leq T$.

Stochastic Integrals V

Theorem

Let $M \in \mathcal{M}^{2,c}$. Then $\overline{\xi}$ contains all adapted, left-continuous processes H such that $E\left[\int_0^T H_s^2 \, d\langle M\rangle_s\right] < \infty$.

Extensions. Definition of $(H.M)$ can be extended in several ways.

- Localisation. (extension to the case where M is only a locally square integrable martingale.)I
- Semimartingales as integrands. Let $X = X_0 + M + A$, where M is a continuous local martingale and A a continuous FV-process. Then one defines for H adapted and left continuous such that $\int_0^T H_s^2 \, d\langle M\rangle_s + \int_0^T |H_s| \, d|A|s < \infty$,

$$\int_0^t H_s \, dX_s = \int_0^t H_s \, dM_s + \int_0^t H_s \, dA_s.$$

Kunita-Watanabe characterization

Theorem (Kunita-Watanabe characterization)

Let $H \in \overline{\xi}$, $M \in \mathcal{M}^{2,c}$. Then we have for all $N \in \mathcal{M}^{2,c}$

$$\langle (H.M), N \rangle_t = \int_0^t H_s \, d\langle M, N \rangle_s.$$

Conversely if $L \in \mathcal{M}^{2,c}$ satisfies $\langle L, N \rangle_t = \int_0^t H_s \, d\langle M, N \rangle_s$, $t \leq T$ for all $N \in \mathcal{M}^{2,c}$, we have $L = (H.M)$.

Further Properties of $(H.M)$.

- *Quadratic covariation.* One has $\langle (H.M) \rangle_t = \int_0^t H_s^2 \, d\langle M \rangle_s$
- *Chain rule.* Consider $H_1, H_2 \in \overline{\xi}$ such that $E\left[\int_0^T (H_{1,s} H_{2,s})^2 \, d\langle M \rangle_s\right] < \infty$. Then

$$\int_0^t H_{1,s} \, d(H_2 \, . \, M)_s = \int_0^t H_{1,s} H_{2,s} \, dM_s. \tag{9}$$

Kunita-Watanabe decomposition

The next result is important in the context of incomplete markets.

Theorem (Kunita Watanabe decomposition)

Consider martingales $N, M_1, \ldots, M_n \in \mathcal{M}^{2,c}$. Then there is a unique decomposition of the form

$$N_t = N_0 + \sum_{i=1}^{n} \int_0^t H_{s,i} dM_{s,i} + L_t, \quad t \leq T,$$

with $H_1, \ldots, H_n \in \overline{\xi}$ and $L \in \mathcal{M}^{2,c}$ strongly orthogonal to $M_1, \ldots, M_n$, that is with $\langle L, M_i \rangle \equiv 0$ for all $1 \leq i \leq n$.

Note that L is also weakly orthogonal to $M_1, \ldots, M_n$, that is $E(L_T M_{T,i}) = L_0 M_{0,i} = 0$ as LM_i is a martingale.

Itô processes.

Itô processes are solutions of SDEs driven by Brownian motion

Definition

Given a d-dimensional Brownian motion $W = (W_{t,1}, ..., W_{t,d})_{t \geq 0}$, a time point $t_0 \geq 0$, some vector $x \in \mathbb{R}^n$ and functions $\mu : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$, the process $X = (X_{t,1}, ..., X_{t,n})_{t \geq 0}$ is called an *Itô process* with initial value t_0, x, drift μ and dispersion matrix σ if X satisfies the SDE

$$X_{t,i} = x_i + \int_{t_0}^{t} \mu_i(s, X_s)\, ds + \sum_{j=1}^{d} \int_{t_0}^{t} \sigma_{ij}(s, X_s)\, dW_{sj}\,, 1 \leq i \leq n. \tag{10}$$

In *short notation* (10) reads $dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$.

Commments

- Euler approximation. For a small time step Δt one has with $t_n = n\Delta t$

$$X_{t_{n+1}} = X_{t_n} + \mu(t_n, X_{t_n})\Delta t + \sum_{j=1}^{d} \sigma_j(t_n, X_{t_n})\epsilon_{n,j}$$

  where $\epsilon_{n,j}$ are iid $\sim N(0, \Delta_t)$. This is the standard approach for simulation of X.
- Define the $n \times n$ matrix $C(t, X) = \sigma(t, X)\sigma'(t, X)$. Then we have $\langle X_i, X_j \rangle_t = \int_0^t C_{ij}(s, X_s)\, ds$. Hence C is often called *instantaneous covariance matrix* of X.

The Generator.

Define for $f : \mathbb{R}^n \to \mathbb{R}$ smooth the differential operator (called *generator* of X)

$$\mathcal{A}f(t,x) = \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x).$$

Lemma. For $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ smooth such that f and its first two derivatives are bounded the process

$$M_t^f = f(t,X_t) - \int_0^t \frac{\partial f}{\partial t}(s,X_s) + \mathcal{A}f(s,X_s)\,ds$$

is a martingale.
The result is the reason for the close connection between SDEs and PDEs and it explains why many pricing problems can be addressed by solving PDEs.

Existence under Lipschitz Conditions

Theorem

Suppose that μ, σ satisfy Lipschitz and growth conditions of the form

$$\begin{aligned} \|\mu(t,X)-\mu(t,Y)\| &\leq K\|X-Y\|, \quad t\geq 0, \\ \|\sigma(t,X)-\sigma(t,Y)\| &\leq K\|X-Y\|, \quad t\geq 0, \\ \|\mu(t,X)+\sigma(t,X)\| &\leq K(1+\|X\|), \quad t\geq 0. \end{aligned}$$

Then for all initial values $(t_0,x)\in[0,\infty)\times\mathbb{R}^n$ a unique solution to (10) exists. Moreover, this solution is $(\mathcal{F}_t^W)$-adapted, where $\mathcal{F}_t^W=\sigma(W_{s,i}:1\leq i\leq d,\, s\leq t)$.

Note that Lipschitz conditions are too strong for many financial applications

One-dimensional Feynman Kac formula

Consider functions $\mu(t,x)$, $\sigma(t,x)$, $r(t,x)$ and some function ϕ on $\mathbb{R}$. Suppose that $F(t,x)$ solves the terminal value problem

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) = r(t,x)F(t,x), \tag{11}$$

$$F(T,x) = \phi(x). \tag{12}$$

Goal: represent F in terms of some Itô process. Consider the SDE

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \quad X_{t_0} = x. \tag{13}$$

Recall that the generator of X equals

$$\mathcal{A}f(t,x) = \mu(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}, \quad f \in C^{1,2},$$

so that (11) can be written as $\frac{\partial F}{\partial t} + \mathcal{A}F = rF$.

One-dimensional Feynman Kac formula II

Then we have, provided that F is sufficiently integrable (eg. bounded)

$$F(t_0, X_{t_0}) = E_{t_0,x}\Big(\exp(-\int_{t_0}^{T} r(s, X_s))\phi(X_T)\Big). \qquad (14)$$

(14) is called *Feynman Kac formula*. It can be used in two ways:

- We can use probabilistic techniques or Monte-Carlo simulation to compute the expectation (14) in order to solve numerically the PDE (11), (12).
- We can solve the PDE (11), (12), perhaps numerically, in order to compute the expectation (14).

For an extension to the d-dimensional case we refer to the literature such as [**?**]

Example: The Black-Scholes PDE.

Recall that fair value $V_t = u(t, S_t)$ of a terminal-value claim with payoff $h(S_T)$ in the Black-Scholes Model solves the terminal value problem

$$u_t + rSu_S + \frac{1}{2}\sigma^2 S^2 u_{SS} = ru, \quad u(T, S) = h(S).$$

In order to give a probabilistic interpretation we consider the SDE $dS_t = rS_t dt + \sigma S_t dW_t$ with generator $\mathcal{A} = rS\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}$, and we obtain from (14)

$$u(t_0, S) = E_{t_0,S}\left(e^{-r(T-t_0)} h(S_T)\right), \quad t_0 \leq T. \tag{15}$$

This is a risk-neutral pricing formula in continuous time.

Measure-Change for Stochastic Processes

Change of measure (eg. from historical measure P to a risk-neutral measure Q or between martingale measures corresponding to different numeraires) is very important in mathematical finance. We discuss

- Densities and density martingales
- Girsanov theorem for Brownian motion
- Applications

Densities and Radon-Nykodym Theorem

Consider some $(\Omega, \mathcal{F}, P)$. A probability measure Q is called absolutely continuous wrt P on $\mathcal{F}$, $(Q \ll P)$, if it holds that $Q(A) = 0$ for all $A \in \mathcal{F}$ with $P(A) = 0$.

Theorem. $Q \ll P$ on $\mathcal{F}$ if and only if there is a $\mathcal{F}$-measurable rv $Z \geq 0$ with $E(Z) = 1$ such that $Q(A) = E^P[Z1_A] = E^P[Z; A]$ $\forall A \in \mathcal{F}$.

Definition. Z is called *(Radon-Nykodym) density* of Q wrt P on $\mathcal{F}$, in short $Z = (dQ/dP)_{|\mathcal{F}}$.

Example. For finite $\Omega = \{\omega_1, ..., \omega_k\}$ $Q \ll P$ iff for $1 \leq i \leq k$: $P(\{\omega_i\}) > 0 \Leftarrow Q(\{\omega_i\}) > 0$; it is immediate that $\frac{dQ}{dP}(\omega) = Q(\{\omega\})/P(\{\omega\})$ in that case.

Density martingales

Setup. Consider a filtered probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{0 \le t \le T}$, and a strictly positive, $\mathcal{F}_T$-measurable random variable Z with $E[Z] = 1$. Define the measure Q by $Z = (dQ/dP)_{|\mathcal{F}_T}$. Note that Q is equivalent to P ($Q \sim P$) that is $Q(A) = 0 \Leftrightarrow P(A) = 0$.
Definition The associated *density martingale* is given by

$$Z_t = E^P[Z|\mathcal{F}_t], \quad 0 \le t \le T. \tag{16}$$

Lemma

$(Z_t)_{0 \le t \le T}$ is a martingale, and for every $\mathcal{F}_t$-measurable random variable Y we have $E^Q[Y] = E^P[YZ_t]$, $t \le T$.

Abstract Bayes Formula

Lemma (abstract Bayes formula)

Consider time points $0 \leq s \leq t \leq T$. *Let* Y *be some* $\mathcal{F}_t$*-measurable integrable random variable. Then*

$$E^Q[Y|\mathcal{F}_s] = \frac{1}{Z_s} E^P[YZ_t|\mathcal{F}_s] \tag{17}$$

Corollary

Let $Q \sim P$ *with* $\frac{dQ}{dP} = Z$. *An adapted process* $(M_t)_{0\leq t\leq T}$ *is a* Q*-martingale if and only if the process* $(M_t Z_t)_{0\leq t\leq T}$ *is a* P*-martingale.*

Girsanov Theorem I

Let $(W_t)_{0\le t\le T}$ be one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{0\le t\le T}$ and consider an adapted process $(\theta_t)_{0\le t\le T}$. Define

$$Z_t = \exp\left(\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right), \quad 0 \le s \le T. \tag{18}$$

Then the Itô-formula shows that $Z_t = Z_0 + \int_0^t \theta_s Z_s \, dW_s$. it follows that

- Z is a positive local martingale and hence a supermartingale
- Z is a true martingale if and only if $E[Z_T] = Z_0 = 1$. In that case Z can be used as a Radon-Nikodym density on $\mathcal{F}_T$ and $Z_t = (dQ/dP)_{|\mathcal{F}_t}$.

Girsanov Theorem II

Theorem (Girsanov for Brownian motion)

Suppose that $(Z_t)_{0\le t\le T}$ defined in (18) is a martingale. Define Q by $(dQ/dP)_{|\mathcal{F}_T} = Z_T$. Then the process $\tilde{W}$ defined by

$$\tilde{W}_t = W_t - \int_0^t \theta_s \, ds, \quad 0 \le t \le T \tag{19}$$

is a Brownian motion under Q.

For the d-dimensional version see [**?**].

The following conditions ensure that $(Z_t)_{0\le t\le T}$ is a martingale

(a) $E\left[\int_0^T \theta_s^2 Z_s^2 \, ds\right] < \infty$ (integrable quadratic variation).

(b) Novikov-criterion: $E\left[\exp\left(\frac{1}{2}\int_0^T |\theta_s|^2 \, ds\right)\right] < \infty$.

Levy-characterization of Brownian motion

The following result is an important tool in the proof of the Girsanov theorem

Theorem (Levy)

Given a filtered probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t\geq 0}$. An adapted process $M = (M_t)_{t\geq 0}$ with continuous trajectories is a one-dimensional Brownian motion if and only if the following conditions hold:

1. *M is a local martingale.*
2. *$\langle M \rangle_t = t$, where $\langle M \rangle$ is the pathwise quadratic variation defined for instance in (5).*

Application: Solving SDEs via a change of measure

Given some bounded function $\theta(s, X) : \mathbb{R}^+ \to \mathbb{R}$, we want to construct a solution X of the SDE

$$dX_t = \theta(t, X_t)\, dt + dW_t, \quad 0 \le t \le T, \tag{20}$$

with some Brownian motion W. Start with some probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{0 \le t \le T}$, supporting a Brownian motion $(X_t)_{0 \le t \le T}$. Define a measure Q via

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = Z_T := \exp\left(\int_0^T \theta(s, X_s)\, dX_s - \frac{1}{2}\int_0^T \theta^2(s, X_s)\, ds\right).$$

By Theorem 25, $W_t := X_t - \int_0^t \theta(s, X_s)\, ds$ is a Q-Brownian motion; hence $X_t = W_t + \int_0^t \theta(s, X_s)\, ds$ solves (20).

Financial Mathematics in Continuous Time

The Model.

- We consider a general model for a frictionless security market with continuous trading on some $(\Omega, \mathcal{F}, P)$ with a given horizon date T and a filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$.
- There are $d+1$ traded assets with price process $S = (S^0, \ldots, S^d)$.
- $S^0, \ldots, S^d$ follow semimartingales
- $S^0(t) > 0$ P a.s. for all t so that S^0 can be used as numeraire; often $S^0(t) = \exp(\int_0^t r_s ds)$ for some adapted r with $r_s \ge 0$. In that case S^0 is the *savings account* and r is the *short rate* of interest.

Selffinancing trading strategies

A *trading strategy* is a $\mathbb{R}^{d+1}$-valued adapted and left-continuous process $\phi(t) = \left(\phi_t^0, \ldots, \phi_t^d\right)$; ϕ_t^i represents the number of units of security i in the portfolio at time t.

Definition

(i) The *value* of the portfolio at time t equals $V_t = V_t^\phi = \sum_{i=0}^d \phi_t^i S_t^i$; $V^\phi = (V_t^\phi)_{0 \le t \le T}$ is called *value process* of the strategy.

(ii) The *gains process* of the strategy is given by

$$G_t^\phi = \int_0^t \phi_u dS_u := \sum_{i=0}^d \int_0^t \phi_u^i dS_u^i$$

(iii) The strategy is called *selffinancing*, if for all $t \in [0, T]$ $V_t^\phi = V_0^\phi + G_t^\phi$. (no frictions)

Numeraire invariance

We choose S^0 as our numeraire and introduce the discounted price process

$$\tilde{S}_t = (1, S_t^1/S_t^0, \ldots, S_t^d/S_t^0), \tag{21}$$

and the discounted value process $\tilde{V}_t^\phi = V_t^\phi/S_t^0 = \sum_{i=0}^d \phi_t^i \tilde{S}_t^i$.

Proposition. Selffinancing strategies remain selffinancing after a change of numeraire, i.e
$d\tilde{V}_t = \sum_{i=1}^d \phi_{t,i} d\tilde{S}_{t,i}$ iff $dV_t = \sum_{i=0}^d \phi_{t,i} dS_{t,i}$

- "A change of numeraire has no substantial economic implications."
- A selffinancing strategy is is completely determined by the initial investment V_0 and the position $\phi^1, \ldots, \phi^d$ in the assets $S^1, \ldots, S^d$.

Martingale Measures and Arbitrage Opportunities

Definition A selffinancing portfolio such that $P(V_t^\phi \geq 0 \quad \text{for all} \quad 0 \leq t \leq T) = 1$, $V_0^\phi = 0$ and $P(V_T^\phi > 0) > 0$ is an arbitrage opportunity.

Arbitrage opportunities represent a chance to create risk free profits and should not exist in a well-functioning market.

Definition.A probability measure Q is called an *equivalent martingale measure* if

(i) $Q \sim P$ on $\mathcal{F}_T$, i.e. for every $A \in \mathcal{F}_T$, $Q(A) = 0 \Leftrightarrow P(A) = 0$.

(ii) The discounted price processes $\tilde{S}^k$, $1 \leq k \leq d$ are Q-martingales.

Spot Martingale Measure

A martingale measure is always related to a given numeraire. If the numeraire is of the form $S_t^0 = \exp(\int_0^t r_s ds)$, r the spot rate of interest, Q is also called *spot martingale measure*.

Lemma

Q is a spot martingale measure, if and only if every price process S^i has Q-dynamics of the form

$$dS_t^i = r_t S_t^i dt + dM_t^i, \quad 0 \leq i \leq d, \tag{22}$$

where $M^0, \ldots, M^d$ are Q-martingales.

The first fundamental theorem of asset pricing

Theorem

If the model $S^0, \ldots, S^d$ admits an equivalent martingale measure Q, there are no arbitrage opportunities.

Remark. Under technical conditions the converse is true as-well, see for instance Chapter 10 of Chapter 10 of [**?**] or the original paper [**?**]

Proof. Two steps:

- Step 1: For any selffinancing strategy ϕ such that $V_t^\phi \geq 0$, the discounted wealth process $\tilde{V}_t^\phi$ is a Q-supermartingale.
- Step 2: Consider ϕ with $V_T^\phi > 0$ a.s. Then $P(V_T^\phi > 0) > 0$ implies $V_0^\phi > 0$.

Hedging and pricing of contingent claims

A *contingent claim* X is an $\mathcal{F}_T$-measurable random variable, to be interpreted as payoff of some financial contract.

Definition

Fix a given martingale measure Q corresponding to the numeraire S^0 and assume that $E^Q(|X|/S^0_T) < \infty$.

(i) A selffinancing trading strategy is called Q-admissible, if the discounted gains process $\tilde{G}^\phi$ is a Q-martingale; $\Phi(Q)$ denotes the linear space of all Q-admissible strategies.

(ii) A contingent claim X such that $\tilde{X} \in \mathcal{L}^1(\Omega, \mathcal{F}_T, Q)$ is called *attainable*, if there is at least one admissible strategy $\phi \in \Phi(Q)$ such that $V^\phi_T = X$, Q a.s.; any such strategy is called replicating strategy for X.

(iii) The financial market model is called *complete*, if any contingent claim X with $\tilde{X} \in \mathcal{L}^1(\Omega, \mathcal{F}_T, Q)$ is attainable.

Attainability and Completeness

Lemma. Consider a contingent claim X such that $\tilde{X} = X/S_T^0$ is Q integrable. Then X is attainable $\Leftrightarrow$ the Q-martingale $M_t = E^Q(\tilde{X}|\mathcal{F}_t)$, $t \leq T$ admits a representation of the form

$$M_t = x + \sum_{i=1}^{d} \int_0^t \phi_u^i d\tilde{S}_u^i, \quad 0 \leq t \leq T, \tag{23}$$

for some some Q-admissible strategy $\phi^1, \ldots, \phi^d$.

Theorem (Second fundamental theorem of asset pricing)

Consider a market that admits (at least) one martingale measure Q. Then the market is complete if and only if Q is unique.

Risk-Neutral Pricing

Denote by $\Pi(t;X)$, $0 \leq t \leq T$ a candidate price process for some claim X.

Theorem (Risk-Neutral Pricing)

1. *If the extended model with price processes* $\left(S^0, S^1, \ldots S^d, \Pi(\cdot, X)\right)$ *is arbitrage free we must have* $X/S^0_T \in \mathcal{L}^1(\Omega, \mathcal{F}_T, Q)$ *and*

$$\Pi(t,X) = S^0_t E^Q\left(\frac{X}{S^0_T}|\mathcal{F}_t\right) \tag{24}$$

   for some martingale measure Q *for the market* $(S^0, \ldots, S^d)$.

2. *If* X *is attainable, we have for any martingale measure* Q *with* $X/S^0_T \in \mathcal{L}^1(\Omega, \mathcal{F}_T, Q)$ *the relation* $S^0_t E^Q\left(\frac{X}{S^0_T}|\mathcal{F}_t\right) = V^\phi_t$, *where* ϕ *is a* Q*-admissible replicating strategy for* X.

Comments.

- Condition (24) is necessary to ensure that there is some martingale measure for the extended market.
- Two different martingale measure Q_1, Q_2 general lead to two different price processes $\Pi_1(t, X)$, $\Pi_2(t, X)$.
- The fair arbitrage-free price of an *attainable* claim is uniquely defined.
- If the market is complete, we must have for any two martingales Q_1 and Q_2 and any bounded random variable X that

$$E^{Q_1}(X) = V^{\phi}(0) = E^{Q_2}(X)$$

(the 'if-part' of the second fundamental theorem)

Change of numeraire

- Change of numeraire is a very useful technique with stochastic interest rates.
- Consider $d+1$ assets $S^0, S^1, \ldots, S^d$ and assume that under Q the discounted asset prices $\tilde{S}_t^i = S_t^i / S_t^0, 1 \leq i \leq d$ are Q-martingales.
- Consider now a new asset $X_t > 0$. Question: which martingale measure do we have to use in order to work with X as new numeraire?

Change of numeraire ctd

Theorem (Change of numeraire)

Consider a non-dividend-paying numeraire X_t such that X_t/S_t^0 is a true Q-martingale. Define a measure Q^X by putting

$$\frac{dQ^X}{dQ}|\mathcal{F}_t = \eta_t := \frac{X_t}{S_t^0} \cdot \frac{S_0^0}{X_0}. \tag{25}$$

Then the processes S_t^i/X_t, $1 \leq i \leq d$ (the basic security process discounted using X) are Q^X-martingales. Moreover, we have for every contingent claim H such that $H/S_T^0 \in \mathcal{L}^1(\Omega, \mathcal{F}_T, Q)$ the change-of-numeraire formula

$$S_t^0 E^Q(H/S_T^0|\mathcal{F}_t) = X_t E^{Q^X}(H/X_T|\mathcal{F}_t). \tag{26}$$

The Black-Scholes Model Revisited

Setup. Two assets (stock and bond) with

$$dS_t^0 = rS_t^0 dt, \quad S_0^0 = 1, \text{ and hence } S_t^0 = \exp(rt). \tag{27}$$

$$dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dW_t\,, \quad S_0^1 = S_0 > 0. \tag{28}$$

Let $D_t = (S_t^0)^{-1} = \exp(-rt)$. We get for the discounted stock price $\tilde{S}_t := D_t S_t^1$ that

$$d\tilde{S}_t = d(D_t S_t^1) \;\; = \;\; (\mu - r)\tilde{S}_t^1 dt + \sigma \tilde{S}_t^1 dW_t$$

Equivalent Martingale Measures

- Recall that $\tilde{S}^1$ is a Q-martingale if and only if S^1 itself has dynamics of the form $S_t^1 = S_0 + \int_0^t rS_s^1 ds + M_t^Q$, where M^Q is a Q-local martingale. Hence we need to define Q such that by going from P to Q the drift is changed from μ to r.
- Let $\tilde{W}_t = W_t + \int_0^t \frac{\mu - r}{\sigma} ds$. Since

$$S_t^1 = S_0 + \int_0^t rS_s^1 ds + \int_0^t \sigma S_s^1 d\tilde{W}_s$$

we may use Girsanov to turn $\tilde{W}$ into a new Brownian motion.

Summary

Proposition. Consider a Black-Scholes model with stock-price dynamics of the form $dS_t^1 = \sigma S_t^1 dW_t + \mu S_t^1 dt$ for constants $\mu,\ \sigma$. Then the (unique) equivalent martingale measure Q is given by the density

$$\frac{dQ}{dP} = G_T := \exp(-\lambda W_T - 1/2\lambda^2 T),$$

where $\lambda = \frac{\mu - r}{\sigma}$ is the market price of risk of the stock. Under Q the stock-price process solves the SDE

$$dS_t^1 = rS_t^1 dt + \sigma S_t^1 d\tilde{W}_t \tag{29}$$

A multidimensional Black-Scholes model

- Traded assets: money-market account S^0 and d risky assets $S^1, \ldots, S^d$.
- Price dynamics: Given $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$, $0 \leq t \leq T$, the asset prices are Itô-processes of the form

$$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sum_{j=1}^{n} \sigma_{ij}(t) dW_t^j, \quad 1 \leq i \leq d, \tag{30}$$

for an n-dimensional Brownian motion $W_t = (W_t^1, \ldots, W_t^n)$ and adapted process α_t^i, $1 \leq i \leq d$ and $\sigma_{ij}(t)$, $1 \leq i \leq d$, $1 \leq j \leq n$. The initial values are given by $S_0 = (S_0^1, \ldots, S_0^d)$.

- The money-market account satisfies

$$S_t^0 = \exp\left(\int_0^t r_s ds\right)$$

for some adapted process $(r_s)_{0 \leq s \leq t}$ modelling the (random) short-rate of interest.

A more intuitive description

Define $\sigma_i(t) = \left(\sum_{j=1}^n \sigma_{ij}(t)^2\right)^{\frac{1}{2}}$ and assume $\sigma_i(t) > 0$ a.s.. Define processes B_t^i, $1 \leq i \leq d$, via

$$B_t^i = \sum_{j=1}^n \int_0^t \frac{\sigma_{ij}(s)}{\sigma_i(s)} dW_s^j. \tag{31}$$

Then $\langle B^i \rangle_t = t$, so that $B^1, \ldots B^d$ are one-dimensional Brownian. Moreover, for $i \neq j$ that

$$\langle B^i, B^j \rangle_t = \int_0^t \rho_{ij}(s) ds \quad \text{with} \quad \rho_{ij}(s) = \frac{\sum_{l=1}^n \sigma_{il}(t)\sigma_{jl}(t)}{\sigma_i(t)\sigma_j(t)}.$$

$\rho_{ij}(t) \in [-1, 1]$ gives instantaneous correlation of B^i and B^j.

A more intuitive description ctd.

Alternative form of asset price dynamics:

$$dS_t^i = \alpha_t^i S_t^i dt + \sigma_i(t) S_t^i dB_t^i, \quad 1 \leq i \leq d, \tag{32}$$

so that $\sigma_i(t)$ is the volatility of S^i at time t. Moreover,

$$\langle \ln S^i, \ln S^j \rangle_t = \int_0^t \sigma_i(s)\sigma_j(s) d\langle B^i, B^j \rangle_s = \int_0^t \sigma_i(s)\sigma_j(s)\rho_{ij}(s) ds.$$

ρ_{ij} therefore models the instantaneous correlation of the logarithmic return processes The discounted asset prices satisfy

$$d\tilde{S}_t^i = (\alpha_t^i - r_t)\tilde{S}_t^i dt + \sigma_i(t)\tilde{S}_t^i dB_t^i$$

Existence of equivalent martingale measures

Proposition. Suppose that for all $0 \leq t \leq T$ there is a solution $\theta_t = (\theta_t^1, \ldots, \theta_t^n)$ of the so-called *market-price-of-risk equations*

$$\alpha_t^i - r_t = \sum_{j=1}^{n} \sigma_{ij}(t)\theta_t^j, \quad 1 \leq i \leq d, \tag{33}$$

and that $E\big(\exp(\frac{1}{2}\int_0^T \|\theta_s\|^2 ds)\big) < \infty$. Then the model (30) admits an equivalent martingale measure.

The converse is true as well. This gives a version of the first fundamental theorem of asset pricing:

Theorem

Modulo integrability conditions, the model (30) *is arbitrage-free if and only if the market-price of risk equation* (33) *admits a solution.*

Characterization of market completeness

Consider (as before) a model with $d+1$ traded assets and dynamics of the form $dS_t^0 = r_t S_t^0 dt$, and

$$dS_t^i = \alpha_t^i S_t^i dt + \sum_{j=1}^{n} \sigma_{ij}(t) S_t^i dW_t^j, \quad ,1 \leq i \leq d, \tag{34}$$

for an n-dimensional Brownian motion $W_t = (W_t^1, \ldots, W_t^n)$.

Theorem

Suppose that the model (34) *is free of arbitrage, and that the underlying filtration is generated by the Brownian motion* W. *Then the market is complete if and only if P.a.s.* $\mathrm{Im}\sigma'(t) = \mathbb{R}^n$, $0 \leq t \leq T$, *where* $\sigma(t)$ *represents the matrix* $\left(\sigma_{ij}(t)\right)_{1\leq i\leq d, 1\leq j\leq n}$.

Comments.

- Since $\sigma(t)$ is a $d \times n$-matrix a necessary condition for completeness is the inequality $d \geq n$ (at least as many assets as Brownian motions)
- The condition $\mathrm{Im}\sigma'(t) = \mathbb{R}^n$ implies that there is at most one solution of the market-price-of-risk equation and hence (in a Brownian model) at most one equivalent martingale measure. (Second fundamental theorem for Brownian models).

Ito representation theorem

The following result is crucial in the proof of the completeness

Theorem

Consider a d-dimensional Brownian motion $W = (W_{t,1}, \ldots, W_{t,d})'$ *on some* $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$, $0 \leq t \leq T$ *and denote by* $\{\mathcal{F}_t^W\}$ *the filtration generated by* W. *Then every* $\{\mathcal{F}_t^W\}$*-adapted square integrable martingale* M *can be written as a stochastic integral with respect to Brownian motion, that is there is a d-dimensional adapted process* $h_t = (h_t^1, \ldots, h_t^d)$ *with* $E\left(\int_0^T (h_s^i)^2 ds\right) < \infty$ *for all* i *such that*

$$M_t = M_0 + \sum_{i=1}^{d} \int_0^t h_s^i dW_{s,i}, \quad 0 \leq t \leq T.$$

Bonds and Interest Rates

We begin with the market for interest-rate products at a given point in time $t \geq 0$.

Zero coupon bond. A zero coupon bond with maturity $T > t$, also called T-bond guarantees the holder €1 to be paid at T. The price in $t < T$ is denoted $p(t, T)$; one has $p(T, T) = 1$ (no default).

The family $p(t, T)$, $T \geq t$ describes the term-structure of interest rates in t. We assume that

- $\exists$ frictionless market for T-bonds for all $T > t$
- For fixed t the mapping $T \mapsto p(t, T)$ is continuously differentiable; we write $p_T(t, T) = \partial_T p(t, T)$.

Note that $t \mapsto p(t, T)$ is usually quite irregular.

Forward price of bonds.

Consider time points $t < S < T$.

- The price, contracted in t, to be paid in S for €1 at time T is the *forward price* of the T-bond with maturity S, denoted by $F(t, S, T)$. It is well-known that

$$F(t, S, T) = \frac{p(t, T)}{p(t, S)}.$$

- Alternative view: a contract at time t to invest €1 over the time period $[S, T]$ yields an amount of $1/F(t, S, T) = p(t, S)/p(t, T)$ at T.

Interest Rates with Continuous Compounding

Let $t < S < T$.

1. $R(t; S, T)$, the forward rate over $(S, T]$ contracted at t, is defined by $\exp((T-S)R(t; S, T)) = -\frac{p(t,S)}{p(t,T)}$ (continuously compounded return on the forward investment). One has

$$R(t; S, T) = \frac{-\ln p(t, T) - \ln p(t, S)}{T - S}.$$

2. The yield with maturity T contracted at t is

$$R(t, T) := R(t; t, T) = -\frac{\ln p(t, T)}{T - t}.$$

3. The instantaneous forward rate with maturity T contracted at t is $f(t, T) := \lim_{S \to T} R(t; S, T) = -\partial_T \ln p(t, T)$.
4. The instantaneous short-rate of interest or spot rate is $r(t) = f(t, t)$, i.e. $r(t) = -\partial_T \ln p(t, T)|_{T=t}$.

Relation between $f(t,\cdot)$ and $p(t,\cdot)$

We get from the fundamental theorem of calculus

$$\ln p(t,T) = \ln p(t,S) + \int_S^T \partial_T \ln p(t,u)du = \ln p(t,S) - \int_S^T f(t,u)du,$$

and therefore $p(t,T) = p(t,S)\exp\left(-\int_S^T f(t,u)du\right)$. In particular

$$p(t,T) = \exp(-\int_t^T f(t,u)du), \tag{35}$$

(a one-to-one relation between bond prices and instantaneous forward rates.)

Dynamic Modelling

Term-structure modelling is the task of constructing of a model for the dynamics of the family $p(\cdot, T),\ T \geq 0$.
Challenges:

- $p(\cdot, T)$ needs to satisfy the final condition $p(T, T) = 1$.
- Absence of arbitrage between the securities $\{p(\cdot, T), T \geq t\}$
- Calibration. At the initial date t_0 the price $p(t_0, T)$ has to be consistent with the bond prices or interest rates observed in the market at time t_0.
- The short rate $r(t)$ should be nonnegative.

Short Rate Models and Martingale Modelling

Consider $(\Omega, \mathcal{F}, Q)$, $(\mathcal{F}_t)$ supporting an adapted process $(r_t)_{t\geq 0}$ modelling the instantaneous short rate and define the savings account by $B_t = \exp(\int_0^t r_s ds)$.
We view Q as risk-neutral spot measure and *define* the price of a T-bond at $t \leq T$ by the risk-neutral pricing rule:

$$p(t,T) = B_t \mathbb{E}\left(\frac{1}{B_T}\middle|\mathcal{F}_t\right) = \mathbb{E}\left(e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right). \qquad (36)$$

The approach of modelling the dynamics of traded securities directly under some martingale measure is termed *martingale modelling*; it is very popular in interest- and credit risk models.

Martingale modelling

Advantages:

- The approach ensures that the model is arbitrage-free.
- Prices are automatically consistent with the terminal condition $p(T,T) = 1$.

Disadvantages/ problems:

- Bond prices defined via (36) are not automatically consistent with the prices observed in the market at time t ($\Rightarrow$ need for model calibration)
- Conceptual difficulties in incomplete markets (less relevant for term structure modelling).

Term Structure Equation

We assume that the short-rate has the following Q-dynamics

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

for a Q-Brownian motion W. Since r is a Markov process, we have for $p(t, T)$ as defined in (36)

$$p(t, T) = E_{t,r_t}\left(\exp\left(-\int_t^T r_s ds\right)\right) =: F^T(t, r_t). \qquad (37)$$

Proposition (Term-structure equation.) The function F^T solves the boundary value problem

$$\partial_t F + \mu\,\partial_r F + \frac{1}{2}\sigma^2\partial_{rr}F = rF,\ t < T, \quad F^T(T, r) = 1. \qquad (38)$$

follows from Feynman-Kac.

Examples for short-rate dynamics

The following short-rate models are frequently used in the literature.

- Vasicek-model: $dr_t = (b - ar_t)dt + \sigma dW_t,\ a, b > 0.$
- CIR-model: $dr_t = (b - ar_t)dt + \sigma\sqrt{r_t}dW_t.$
- Hull-White: $dr_t = (\varphi(t) - ar_t)dt + \sigma dW_t\ .$
- Extended CIR: $dr_t = (\varphi(t) - ar_t)dt + \sqrt{r_t}\sigma dW_t.$
- Black-Derman-Toy: $dr_t = a(t)r_t dt + \sigma r_t dW_t.$

Affine Term Structures

Definition

A short-rate model has an affine term structure (ATS), if the bond-prices are of the form

$$p(t,T) = F^T(t,r_t) = \exp(A(t,T) - B(t,T)r_t) \qquad (39)$$

for deterministic functions $A(t,T)$ and $B(t,T)$, $0 \leq t \leq T$.

Comments.

- In a model with an ATS the yield $R(t,T) = -\frac{1}{T-t}\ln p(t,T)$ is an affine function of r_t.
- Models with an affine term structure are particularly tractable.

Sufficient conditions for an ATS

Theorem (Existence of an ATS)

Assume that μ and σ^2 are of the form

$$\mu(t,r) = \alpha(t)r + \beta(t),\ \sigma^2(t,r) = \gamma(t)r + \delta(t) \tag{40}$$

Then the model has an ATS. A and B satisfy the ODE-system

$$\begin{aligned} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) &= -1 \\ A_t(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) &= 0, \end{aligned} \tag{41}$$

with terminal conditions $B(T,T) = A(T,T) = 0$.

Comments.

- Under additional conditions the converse is true as well.
- The ODE for $B(\cdot, T)$ is a so-called Ricatti equation

Term-structure equation in the Vasicek-model

Here the ODE-system (41) becomes, as $\alpha = -a$, $\beta = b$, $\gamma = 0$, $\delta = \sigma^2$

$$B_t - aB = -1,\ A_t = bB - \frac{1}{2}\sigma^2 B^2.$$

Since the ODE for B is linear, one has, as $B(T,T) = 0$,

$$B(t,T) = \frac{1}{a}(1 - \exp(-a(T-t))),$$

and, by integration,

$$A(t,T) = -\int_t^T bB(s,T)ds + \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds$$

Calibration: two approaches

Statistical estimation from a time series $(r_s)_{s \leq t}$ of the short-rate. Problem: drift of the short-rate may change in transition from P to Q, and bond prices depend on risk-neutral drift.

Calibration to market prices. The approach is as follows:

- Fix a short-rate model with parameter vector θ.
- Solve the term-structure equation for all maturities T, denote the solution by $F^T(t_0, r_{t_0}; \theta)$.
- Go to the market and obtain observed prices $\{p^*(t_0, T_1), \ldots, p^*(t_0, T_m)\}$ and current short rate r_{t_0}.
- Determine θ^* as solution of the minimization problem

$$\min_{\theta} \sum_{i=1}^{m} w_i \left(p^*(t_0, T_i) - F^{T_i}(t_0, r^*_{t_0}; \theta) \right)^2, \qquad (42)$$

where $w_1, \ldots, w_n$ is a vector of weights.

Calibration to market prices ctd.

Calibration to market prices widely adapted in practice.

- Aligning market- and model prices is consistent with fair-value accounting/market consistent valuation
- Calibration is often easier/less subjective than statistical estimation

Problems

- The model may not be flexible enough so that model- and market prices diverge widely even for 'optimal' θ^*.
- The parameter vector θ^* may fluctuate a lot over time.
- Time series properties of the short rate (or of observed bond prices) are ignored completely which might be problematic in risk management.

Heath Jarrow Morton (HJM) Models

In the HJM-approach one models simultaneously the dynamics of all forward-rates $f(t,T)$, $T \geq t$ and computes bond-price dynamics from

$$p(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \tag{43}$$

- The calibration to the observed bond prices is ensured by design,
- (43) ensures that $p(T,T) = \exp(0) = 1$.
- On the other hand, it is a priori not clear if the model generated in this way is actually arbitrage-free (infinitely many securities but typically only finitely many sources of randomness).

Forward-rate dynamics

Assumption. Given $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)$, P the historical measure, let W be a d-dimensional Brownian motion Then for each fixed T the forward rate $f(t, T)$ has a stochastic differential of the form

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t \tag{44}$$

where $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted process with values in $\mathbb{R}$ respective $\mathbb{R}^d$, and where $\int_0^T \sigma(t, T)dW_t$ is shorthand for $\sum_{i=1}^d \int_0^t \sigma_i(t, T)dW_{t,i}$. Moreover α and σ are continuously differentiable in the T-variable.

Bond Price Dynamics

Theorem (Short-rate and bond price dynamics)

If the family of forward rates satisfies (44)*, the short rate* $r(t) = f(t,t)$ *has the differential*

$$\begin{aligned} dr_t &= a(t)dt + b(t)dW_t, \quad \textit{with} \\ a(t) &= \partial_T f(t,t) + \alpha(t,t) \textit{ and } b(t) = \sigma(t,t). \end{aligned} \tag{45}$$

Moreover, $p(t,T) = \exp\big(-\int_t^T f(t,u)du\big)$ *satisfies*

$$\begin{aligned} dp(t,T) &= p(t,T)\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\}dt \\ &\quad + p(t,T)S(t,T)dW_t, \textit{ with} \end{aligned} \tag{46}$$

$$A(t,T) = -\int_t^T \alpha(t,u)du, \quad S(t,T) = -\int_t^T \sigma(t,u)du \tag{47}$$

Proof for short-rate

As $r(t) = f(t,t)$ we get from (44)

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW_s. \qquad (48)$$

Now write $\alpha(s,t) = \alpha(s,s) + \int_s^t \partial_T \alpha(s,u)du$, and similarly for σ. Substitution into (48) gives

$$\begin{aligned} r(t) = &\int_0^t \alpha(s,s)ds + \int_0^t \sigma(s,s)dW_s \\ &+ f(0,t) + \int_0^t \int_s^t \partial_T \alpha(s,u)duds + \int_0^t \int_s^t \partial_T \sigma(s,u)dudW_s. \end{aligned} \qquad (49)$$

Proof for short-rate ctd

Changing the order of integration we get

$$\int_0^t \int_s^t \partial_T \alpha(s,u) du ds = \int_0^t \int_0^t 1_{\{u>s\}} \partial_T \alpha(s,u) du ds$$
$$= \int_0^t \int_0^u \partial_T \alpha(s,u) ds du, \tag{50}$$

and similarly $\int_0^t \int_s^t \partial_T \sigma(s,u) du dW_s = \int_0^t \int_0^u \partial_T \sigma(s,u) dW_s du$. Moreover, $f(0,t) = r(0) + \int_0^t \partial_T f(0,u) du$. On the other hand,

$$\partial_T f(u,u) = \partial_T f(0,u) + \int_0^u \partial_T \alpha(s,u) ds + \int_0^u \partial_T \sigma(s,u) dW_s.$$

Hence the sum in (49) equals $r(0) + \int_0^t \partial_T f(u,u) du$ □

Proof for Bond-Price Dynamics

Define $Y(t,T) := -\int_t^T f(t,s)ds$ so that $p(t,T) = \exp(Y(t,T))$. Recall that **??**,

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)du + \int_0^t \sigma(u,s)dW_u.$$

This gives, after changing the order of integration,

$$Y(t,T) := -\int_t^T f(0,s)ds - \int_0^t \int_t^T \alpha(u,s)dsdu - \int_0^t \int_t^u \sigma(u,s)dsdW_u.$$

Bond price dynamics ctd

Now we can decompose these integrals into

$$
\begin{aligned}
&-\int_0^T f(0,s)ds - \int_0^t \int_u^T \alpha(u,s)dsdu - \int_0^t \int_u^T \sigma(u,s)dsdW_u \\
&+\int_0^t f(0,s)ds + \int_0^t \int_u^t \alpha(u,s)dsdu + \int_0^t \int_u^t \sigma(u,s)dsdW_u \\
&= Y(0,T) + \int_0^t A(u,T)du + \int_0^t S(u,T)dW_u \\
&+\int_0^t f(0,s)ds + \int_0^t \int_0^s \alpha(u,s)duds + \int_0^t \int_0^s \sigma(u,s)dW_u ds\,.
\end{aligned}
$$

The last line equals $\int_0^t f(s,s)ds = \int_0^t r(s)ds$, and we get that $dY(t,T) = \{r(t) + A(t,T)\}dt + S(t,T)dW_t$. (46) now follows by applying the Ito formula to $p(t,T) = \exp(Y(t,T))$.

Absence of arbitrage

We consider a market with infinitely many assets so that we need special conditions to ensure that the model is free of arbitrage.

Theorem

If there is a process $\lambda_t = (\lambda_{t,1}, \ldots, \lambda_{t,d})'$ *such that for all* $0 \leq t \leq T$

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)' ds + \sigma(t,T)\lambda(t) \qquad (51)$$

and such that $Z_t = \exp\left(\int_0^t -\lambda_s dW_s - \frac{1}{2} \int_0^t \|\lambda_s\|^2 ds \right)$ *is a true martingale. Then the model is free of arbitrage.*

Condition (51) is a restriction on the drift of the forward rates, as the equation has to hold simultaneously for all T.

Proof

Recall that bond prices are of the form

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)\left(A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right)dt + p(t,T)S(t,T)dW_t.$$

Define Q by $\frac{dQ}{dP}|\mathcal{F}_T = Z_T$; then $\widetilde{W}_t = W_t + \int_0^t \lambda_s ds$ is a Q-Brownian Motion. Hence under Q the finite-variation part of $p(\cdot, T)$ equals

$$\int_0^t p(s,T)\left\{r(s) + A(s,T) + \frac{1}{2}\|S(s,T)\|^2 - S(s,T)\lambda_s\right\}ds. \tag{52}$$

Proof ctd

We get from (51) and the definition of $S(s,T)$ in (47) that

$$\begin{aligned} S(s,T)\lambda_s &= -\int_s^T \sigma(s,u)\lambda_s du \quad \text{(by (47))} \\ &= -\int_s^T \left\{ \alpha(s,u) - \sigma(s,u) \int_s^u \sigma(s,\tau)' d\tau \right\} du \text{ (by (51))} \\ &= A(s,T) + \frac{1}{2} \|S(s,T)\|^2 \text{ (by (47))}. \end{aligned}$$

Hence the finite variation part of $p(\cdot,T)$ in (52) equals $\int_0^t p(s,T)r(s)ds$, and discounted bond prices are Q-local martingales.

HJM Drift Condition

Under Q the forward-rate dynamics are

$$df(t,T) = \left(\sigma(t,T)\int_0^T \sigma(t,s)'ds\right)dt + \sigma(t,T)d\widetilde{W}_t$$
$$dp(t,T) = r(t)p(t,T)dt + p(t,T)S(t,T)d\widetilde{W}_t,$$

The fact that the risk-neutral drift of $f(t,T)$ is of the form

$$\alpha^Q(t,T) = \sigma(t,T)\int_0^t \sigma(t,s)'ds$$

is known as *HJM drift condition*.

Martingale modelling in HJM

For practical purposes the HJM model is used as follows:

1.) Specify the volatilities $\sigma(t, T)$ (the modelling part)

2.) The forward-rate drift $\alpha^Q(t, T)$ is specified via the HJM-drift condition.

3.) Go to the market and observe current forward rates $f^*(0, T)$.

4.) Forward-rates are given by

$$f(t, T) = f^*(0, T) + \int_0^T \alpha^Q(s, T)ds + \int_0^t \sigma(t, T)d\widetilde{W}_t.$$

5.) Bond prices are given via $p(t, T) = \exp\{-\int_t^T f(t, u)du\}$ or as solutions of

$$dp(t, T) = p(t, T)r(t)dt + p(t, T)S(t, T)d\widetilde{W}_t,$$

for $S(t, T) = -\int_t^T \sigma(t, u)du$.

Example: The Ho-Lee model

For $\sigma(t,T)=\sigma$ the drift of the forward rates equals $\alpha^Q(t,T)=\sigma\int_t^T\sigma ds=\sigma^2(T-t)$ and $f(\cdot,T)$ is given by

$$\begin{aligned} f(t,T) &= f^*(0,T)+\int_0^t \sigma^2(T-s)ds+\sigma W_t \\ &= f^*(0,T)+\sigma^2 t\left(T-\frac{t}{2}\right)+\sigma W_t. \end{aligned}$$

In particular, the short-rate satisfies

$$r(t)=f(t,t)=f^*(0,t)+\sigma^2\frac{t^2}{2}+\sigma W_t,$$

or, in differential form, $dr_t=\left(\partial_T f(0,t)+\sigma^2 t\right)dt+\sigma dW_t$. This is known as *Ho-Lee model* (Vasicek for $a=0$).
Note that the HJM automatically ensures that the model is calibrated to the initial yield curve.

The Vasicek Model

Here we take $\sigma(t,T) = \sigma e^{-a(T-t)}$ for some $a > 0$. Hence we get

$$\alpha^Q(t,T) = \sigma(t,T)\int_0^T \sigma(t,s)ds = \frac{\sigma^2}{a}e^{-a(T-t)}\left(1 - e^{-a(T-t)}\right).$$

The volatility of the short-rate dynamics is given by $\sigma(t,t) = \sigma$. Moreover, we have

$$\mu(t,r_t) = \alpha^Q(t,t) + \partial_T f(t,t) = \partial_T f(t,t),$$

as $\alpha^Q(t,t) = 0$. Using the forward rate dynamics:

$$\partial_T f(t,t) = \partial_T f(0,T) + \int_0^t \partial_T \alpha(s,t)ds + \int_0^t \partial_T \sigma(s,t)dW_s.$$

Vasicek ctd.

As $\sigma(s,t) = \sigma e^{-a(t-s)}$, we get $\partial_T \sigma(s,t) = -a\sigma(s,t)$. Moreover, $\partial_T \alpha^Q(t,s) = -a\alpha^Q(t,s) + \sigma^2 e^{-2a(t-s)}$. This gives

$$\begin{aligned}\partial_T f(t,t) &= \partial_T f(0,t) - a\int_0^t \alpha(s,t)ds - a\int_0^t \sigma(s,t)dW_s \\ &\quad + \int_0^t \sigma^2 e^{-2a(t-s)ds} \\ &= -af(t,t) + g(t)\end{aligned}$$

with $g(t) = \partial_T f(0,t) - af(0,t) + \int_0^t \sigma^2 e^{-2a(t-s)}ds$. Summarizing,

$$dr(t) = (g(t) - ar(t))dt + \sigma dW_t,$$

so that this form of σ leads to the Hull-White or Vasicek-model.

Multi-Currency Derivatives with Interest Rate Risk

- In this section we discuss pricing and hedging of multicurrency derivatives with interest rate risk
- Examples for pricing results under HJM
- A nice application of our mathematical toolbox

The Model

We consider N countries indexed by $n \in \{0, \ldots, N\}$. Country 0 will be the *domestic* country.

- The exchange rate between country 0 and country $n \in \{1, \ldots, N\}$ will be denoted by e^n,
- In all countries zero coupon bonds of all maturities $T \in [0, T_F]$ are traded. The zero coupon bond in country n with maturity date T shall be denoted by $p^n(t, T)$ for $t \in [0, T]$. The short rate in country n, r^n, is given by

$$r_t^n = -\frac{\partial}{\partial T}\Big|_{T=t} \ln p^n(t, T)\,. \tag{53}$$

By $B_{t,T}^n := \exp(\int_t^T r_s^n \, ds)$ we denote the savings-account of country n.

Asset price dynamics

Consider some $(\Omega, \mathcal{F}, Q)$, $(\mathcal{F}_t)_{t\in[0,T_F]}$ supporting a d-dimensional Brownian Motion $W = (W_t)_{0\leq t\leq T_f}$. The domestic assets satisfy

$$dp^0(t,T) = r_t^0 p^0(t,T)dt + \eta^0(t,T)p^0(t,T)dW_t$$

The foreign assets satisfy

$$dp^n(t,T) = (r_t^n - \eta^n(t,T)\cdot\eta^{e^n}(t))p^n(t,T)dt + \eta^n(t,T)p^n(t,T)dW_t$$

Finally the dynamics of the exchange rates are given by

$$de^n(t) = (r_t^0 - r_t^n)e^n(t)dt + \eta^{e^n}(t)e^n(t)dW_t\,. \qquad (54)$$

Here $\eta^n(t,T), \eta^{n,j}(t), \eta^{e^n}(t) : [0,T_F] \to \mathbb{R}^d$ are *deterministic* square integrable functions of time. Moreover we assume that $\eta^n(t,T) = 0\,\forall t \geq T$ and that $\eta^n(t,T)$ is smooth in T

Comments

- The assumption of deterministic dispersion coefficients is essential as we want to obtain explicit pricing formulas.
- An explicit construction of the bond price model is easily done using the HJM approach
- The drift terms of the exchange rate and the foreign assets are determined by absence of arbitrage. Consider e.g. the drift of e^n. By absence of arbitrage the drift of $Y := e^n \beta^n_{(0,\cdot)}$ must equal r^0. This immediately leads to the risk-neutral drift $r^d - r^f$ in the risk-neutral dynamics of e.
- Price process of a discounted foreign asset is not a martingale under the domestic risk-neutral measure, so care must be taken to use the right measure

Risk Neutral Pricing

Assumption.(Market Completeness) There are d traded domestic assets such that for all $t \in [0, T_F]$ the instantaneous covariance matrix of these assets is strictly positive definite.

This assumption guarantees that the market is complete. Hence the domestic risk neutral measure is unique and the price at time t of every domestic contingent claim H with single $\mathcal{F}_T$ measurable and integrable payoff H_T at time T is given by

$$H_t := E^Q \left[\left(\beta^0_{t,T} \right)^{-1} H_T \;\middle|\; \mathcal{F}_t \right] . \tag{55}$$

Lognormal Claims

Definition

A domestic contingent-claim X with a single payoff X_T at a certain date T is called a *lognormal claim* if its price process $(X_t)_{0\leq t\leq T}$ given by

$$X_t := E^Q\left[\left(\beta^d_{t,T}\right)^{-1}\cdot X_T \mid \mathcal{F}_t\right]$$

admits a representation of the form

$$X_t = X_0 \exp\left(\int_0^t \eta^X_s\, dW_s - \frac{1}{2}\int_0^t |\eta^X_s|^2 ds + \int_0^t r^d_s\, ds\right) \qquad (56)$$

with some constant X_0 and with *deterministic* dispersion coefficients $\eta^X : [0,T] \to \mathbb{R}^d$.

Comments

- The name is motivated by the fact that X_T is lognormally distributed
- The main restriction made is the assumption of η^X being deterministic.
- Every domestic asset is a lognormal claim, as the solution of $dX_t = r_t^0 X_t dt + \eta_t^X X_t dW_t$ is given by

$$X_t := X_0 \exp\left(\int_0^t \eta_s^X dW_s - 1/2 \int_0^t |\eta_s^X|^2 ds + \int_0^t r_s^0 ds\right).$$

but the class is larger as will be shown below.

Exchange Options on Lognormal Claims

Theorem (Pricing and hedging of exchange options)

Let X, Y be lognormal claims. Consider a European option with payoff $[X_T - Y_T]^+$. The price process of this option is given by

$$C_t = C(t, X_t, Y_t) \quad := \quad X_t \mathcal{N}(d_t^1) - Y_t \mathcal{N}(d_t^2), \text{ where}$$

$$d_t^1 = \frac{\ln(X_t/Y_t) + \frac{1}{2}\int_t^T |\eta_s^X - \eta_s^Y|^2 ds}{\sqrt{\int_t^T |\eta_s^X - \eta_s^Y|^2 ds}}, \tag{57}$$

$$d_t^2 = d_t^1 - \Big(\int_t^T |\eta_s^X - \eta_s^Y|^2 ds\Big)^{\frac{1}{2}}.$$

The hedge portfolio for this option in terms of X and Y is given by

$$\delta_X^C(t) := \mathcal{N}(d_t^1) \text{ units of } X \quad \text{and } \delta_Y^C(t) := -\mathcal{N}(d_t^2) \text{ units of } Y\,.$$

Example: Currency Option

This has payoff $[e_T - K]^+$. Define $X := e \cdot p^f(\cdot, T)$ and $Y := Kp^d(\cdot, T)$; the corresponding parameters are given by

$$X_0 = e_0 p^f(0,T), \eta^X = \eta^e + \eta^f, Y_0 = Kp^d(0,T), \eta^Y = \eta^d.$$

Since $p^d(T,T) = p^f(T,T) = 1$ the option's payoff equals $[X_T - Y_T]^+$. We get from Theorem 41.

$$C_t = e_t p^f(t,T)\mathcal{N}(d_1) - Kp^d(t,T)\mathcal{N}(d_2) \text{ with}$$

$$d_1 = \frac{\ln(\frac{e_t p^f(t,T)}{p^d(t,T)}) - \ln K + \int_t^T |\eta^e(s) + \eta^f(s,T) - \eta^d(s,T)|^2 ds}{\sqrt{\int_t^T |\eta^e(s) + \eta^f(s,T) - \eta^d(s,T)|^2 ds}}$$

and $d_2 = d_1 - \sqrt{\int_t^T |\eta^e(s) + \eta^f(s,T) - \eta^d(s,T)|^2 ds}$. We can use directly Theorem 41 to compute a feasible hedge portfolio.

Proof

Main tool: change of numeraire technique. Define for a lognormal claim X a new equivalent probability measure Q^X on $\mathcal{F}_T$ by

$$\frac{dQ^X}{dQ} = \frac{X_T \cdot \left(\beta^d_{0,T}\right)^{-1}}{X_0}.$$

Then for every domestic asset Z whose discounted price process is a martingale under Q the process Z/X is a martingale under Q^X. Moreover

$$E^Q\left[\left(\beta^d_{t,T}\right)^{-1} \cdot X_T \cdot Z_T \,|\mathcal{F}_t\right] = X_t \cdot E^{Q^X}[Z_T|\mathcal{F}_t] \qquad (58)$$

Girsanov and form of $dQ^X/dQ \Rightarrow W^X_t := W_t - \int_0^t \eta^X_s ds$ is a Q^X Brownian Motion.

Proof: pricing

According to (55) the price of the option is given by

$$\begin{aligned} C_t &= E^Q\left[\left(\beta^d_{t,T}\right)^{-1}[X_T - Y_T]^+ \,\middle|\, \mathcal{F}_t\right] \\ &= E^Q\left[\left(\beta^d_{t,T}\right)^{-1} X_T \cdot 1_{\left\{\frac{Y_T}{X_T}<1\right\}} \,\middle|\, \mathcal{F}_t\right] - E^Q\left[\left(\beta^d_{t,T}\right)^{-1} Y_T \cdot 1_{\left\{\frac{X_T}{Y_T}>1\right\}}\right. \\ &= X_t \cdot E^{Q^X}\left[1_{\{Y_T/X_T<1\}} \,\middle|\, \mathcal{F}_t\right] - Y_t \cdot E^{Q^Y}\left[1_{\{X_T/Y_T>1\}} \,\middle|\, \mathcal{F}_t\right] \end{aligned}$$

The last line follows from (58) if we take once X and once Y as numeraire. Now we get under Q^X for Y_T/X_T

$$\frac{Y_T}{X_T} = \frac{Y_t}{X_t} \cdot \exp\left(\int_t^T (\eta_s^Y - \eta_s^X) dW_s^X - \frac{1}{2}\int_t^T |\eta_s^Y - \eta_s^X|^2 ds\right)$$

Proof: pricing ctd.

We have

$$Q^X\left(\frac{Y_T}{X_T} < 1 \,\middle|\, \mathcal{F}_t\right) = Q^X\left(\ln Y_T - \ln X_T < 0 \,\middle|\, \mathcal{F}_t\right)$$

$$= Q^X\left(\frac{\int_t^T (\eta_s^Y - \eta_s^X) dW_s^X}{\sqrt{\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}} < \frac{\ln X_T - \ln Y_T + \frac{1}{2}\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}{\sqrt{\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}}\right)$$

Since η^X and η^Y are deterministic, $\int_t^T (\eta_s^Y - \eta_s^X) dW_s^X / \sqrt{\int_t^T |\eta_s^Y - \eta_s^X|^2 ds}$ is a standard normally distributed random variable so that

$$Q^X\left(\frac{Y_T}{X_T} < 1 \,\middle|\, \mathcal{F}_t\right) = \mathcal{N}(d_t^1)\,.$$

Analogously we get $Q^Y\left(X_T/Y_T > 1 \,\middle|\, \mathcal{F}_t\right) = \mathcal{N}(d_t^2)$, and the first part of the theorem follows.

Hedging

We have to verify two conditions need to hold

- We need $d(C)^M = \mathcal{N}(d_t^1)d(X)_t^M - \mathcal{N}(d_t^2)d(Y)_t^M$ where Z^M is the martingale part of a semimartingale Z.
- We need the equality $C_t = V_t$ for all $t \leq T$.

(i) As C_t is a function only of X_t and Y_t we get from Itô's Lemma

$$d(C)_t^M = \frac{\partial C}{\partial x}(t, X_t, Y_t)d(X)_t^M + \frac{\partial C}{\partial y}(t, X_t, Y_t)d(Y)_t^M \ .$$

Hedging ctd.

Next we compute the derivatives of the option price:

$$\begin{aligned}
\frac{\partial C}{\partial x}(t, X_t, Y_t) &= E_t^Q\left[\frac{\partial}{\partial X_t}\left(\left(\beta_{t,T}^d\right)^{-1}[X_T - Y_T]^+\right)\right] \\
&= E_t^Q\left[\left(\beta_{t,T}^d\right)^{-1} 1_{\{X_T \geq Y_T\}} \frac{\partial X_T}{\partial X_t}\right] \\
&= \frac{1}{X_t} E_t^Q\left[\left(\beta_{t,T}^d\right)^{-1} 1_{\{X_T \geq Y_T\}} X_T\right]
\end{aligned}$$

As shown in the first part of the proof this expression equals $\mathcal{N}(d_t^1)$. Similarly we get $\partial C/\partial y\,(t, X_t, Y_t) = -\mathcal{N}(d_t^2)$.

(ii) By Euler's Theorem we get from the linear homogeneity of C in X_t and Y_t

$$C_t = \frac{\partial C}{\partial x}(t, X_t, Y_t) \cdot X_t + \frac{\partial C}{\partial y}(t, X_t, Y_t) \cdot Y_t = \mathcal{N}(d_t^1) X_t - \mathcal{N}(d_t^2) Y_t \,,$$

which shows that also the second condition is satisfied. □

Hedging with other assets

Sometimes X and Y are not traded assets. Suppose that the hedge portfolios for X and Y in terms of traded domestic assets H_i^X and H_i^Y f are given by

$$P_t^X = \sum_{i=1}^{L^X} \delta_i^X(t) H_i^X \quad \text{and} \quad P_t^Y = \sum_{i=1}^{L^Y} \delta_i^Y(t) H_i^Y \, .$$

Then the hedge portfolio for the exchange option on X and Y in terms of H_i^X and H_i^Y is given by

$$P_t = \sum_{i=1}^{L^X} \mathcal{N}(d_t^1) \cdot \delta_i^X(t) H_i^X - \sum_{i=1}^{L^Y} \mathcal{N}(d_t^2) \cdot \delta_i^Y(t) H_i^Y \, .$$

Example: Guaranteed-Exchange-Rate Options:

- *Payoff:* $[\bar{e}S_T^f - \bar{e}K]^+$, where S^f is a foreign asset and $\bar{e}$ some predetermined exchange rate.
- Contract is an option to exchange the lognormal claims X and Y with payoff $X_T = \bar{e}S_T^f$ and $Y_T := \bar{e}K$. Whereas Y_T equals the time T value of $K \cdot \bar{e}$ units of $p^f(\cdot, T)$, there is no traded asset whose value at T is equal to X_T.
- We therefore have to compute the parameters of X. We have

$$\begin{aligned} X_0 &= \bar{e}S_0^f \frac{p^d(0,T)}{p^f(0,T)} \exp\left\{ \int_0^T |\eta^f(s,T)|^2 + \eta^{S^f}(s) \cdot \eta^d(s,T) + \eta^e \right. \\ &\qquad \left. \eta^{S^f}(s) \cdot \eta^f(s,T) - \eta^f(s,T) \cdot \eta^d(s,T) - \eta^{S^f}(s) \cdot \eta^e(s) ds \right\} \\ \eta_s^X &= \eta^{S^f}(s) + \eta^d(s,T) - \eta^f(s,T) \end{aligned}$$

Guaranteed-Exchange-Rate Options ctd

- Price of can be computed by plugging these parameters into the pricing formula of Theorem 41.
- Hedge portfolio. We have to replicate X_T. First note that X_t is given by a function $\tilde{X}$ of the domestic assets $e \cdot S^f$, $p^d(t,T)$ and $e \cdot p^f(t,T)$ with derivatives $\partial\tilde{X}/\partial eS^f = \tilde{X}/(e_t S^f_t)$, $\partial\tilde{X}/\partial ep^f = -\tilde{X}/(e_t p^f(t,T))$ and $\partial\tilde{X}/\partial p^d = \tilde{X}/p^d(t,T)$. As $\tilde{X}$ is linear homogenous in the prices of these assets, we get that the hedge portfolio for X equals

$$\delta^X_{e\cdot S^f}(t) = \frac{X_t}{e_t \cdot S^f_t},\ \delta^X_{e\cdot p^f}(t) = -\frac{X_t}{e_t \cdot p^f(t,T)},\ \delta^X_{p^d}(t) = \frac{X_t}{p^d(t,T)}\,.$$

Options on Interest Rates:

Finally we consider contracts where one of the underlying assets is a foreign or domestic LIBOR rate. For a fixed $\alpha > 0$ the LIBOR rate $L^n(t, \alpha)$ prevailing in country n over the period $[t, t + \alpha]$ is defined by the equation

$$(1 + \alpha \cdot L^n(t, \alpha))p^n(t, t + \alpha) = 1\,,$$

that is $L^n(t, \alpha) = \alpha^{-1}(1/p^n(t, t + \alpha) - 1)$.

Caps

A cap is a portfolio of caplets. The payoff of a caplet with face value V, underlying interest rate process $L^d(t,\alpha)$, level K and maturity date $T+\alpha$ equals

$$V \cdot \alpha \cdot [L^d(T,\alpha) - K]^+ = V \cdot \left[\frac{1}{p^d(T,T+\alpha)} - (\alpha K + 1)\right]^+$$

Payoff of this caplet is known already at $T \Rightarrow$ compute its present value at T which equals $V[1-(\alpha K+1)\cdot p^d(T,T+\alpha)]^+$. Hence price and the hedge portfolio for caplets can be inferred directly from Theorem 41 if we use the lognormal claims $X = p^d(\cdot,T)$ and $Y = (\alpha K+1)\cdot p^d(\cdot,T+\alpha)$.
This choice reflects the fact that caplets can be considered as options on zero coupon bonds.