Worst-Case Value-at-Risk of Non-Linear Portfolios

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Portfolio Optimization

- Consider a market consisting of m assets.

Optimal Asset Allocation Problem

Choose the weights vector $\boldsymbol{w} \in \mathbb{R}^m$ to make the portfolio return high, whilst keeping the associated risk $\rho(\boldsymbol{w})$ low.

- Portfolio optimization problem:

$$\begin{array}{ll} \underset{\boldsymbol{w} \in \mathbb{R}^m}{\text{minimize}} & \rho(\boldsymbol{w}) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}. \end{array}$$

- Popular risk measures ρ:
  - Variance $\rightarrow$ Markowitz model
  - Value-at-Risk $\rightarrow$ Focus of this talk

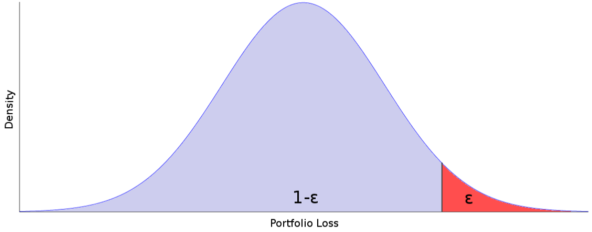
Value-at-Risk: Definition

- Let $\tilde{\boldsymbol{r}}$ denote the random returns of the m assets.
- The portfolio return is therefore $\boldsymbol{w}^T\tilde{\boldsymbol{r}}$.

Value-at-Risk (VaR)

The minimal level $\gamma \in \mathbb{R}$ such that the probability of $-\boldsymbol{w}^T\tilde{\boldsymbol{r}}$ exceeding γ is smaller than ϵ.

$$\text{VaR}_\epsilon(\boldsymbol{w}) = \min\left\{\gamma \ : \ \mathbb{P}\left\{\gamma \leq -\boldsymbol{w}^T\tilde{\boldsymbol{r}}\right\} \leq \epsilon\right\}$$



Theoretical and Practical Problems of VaR

- VaR lacks some desirable theoretical properties:
  - **Not** a **coherent** risk measure.
  - Needs **precise knowledge** of the distribution of $\tilde{\boldsymbol{r}}$.
  - **Non-convex** function of $\boldsymbol{w}$
    $\rightarrow$ VaR minimization **intractable**.
- To optimize VaR: resort to VaR approximations.
- Example: assume $\tilde{\boldsymbol{r}} \sim \mathcal{N}(\boldsymbol{\mu_r}, \boldsymbol{\Sigma_r})$, then

$$\text{VaR}_\epsilon(\boldsymbol{w}) = -\boldsymbol{\mu_r}^T \boldsymbol{w} - \Phi^{-1}(\epsilon)\sqrt{\boldsymbol{w}^T \boldsymbol{\Sigma_r} \boldsymbol{w}},$$

- Normality assumption **unrealistic**
  $\rightarrow$ may **underestimate** the actual VaR.

Worst-Case Value-at-Risk

- Only know means $\boldsymbol{\mu_r}$ and covariance matrix $\boldsymbol{\Sigma_r} \succ \mathbf{0}$ of $\tilde{\boldsymbol{r}}$.
- Let $\mathcal{P}_{\boldsymbol{r}}$ be the set of all distributions of $\tilde{\boldsymbol{r}}$ with mean $\boldsymbol{\mu_r}$ and covariance matrix $\boldsymbol{\Sigma_r}$.

Worst-Case Value-at-Risk (WCVaR)

$$\mathrm{WCVaR}_\epsilon(\boldsymbol{w}) = \min\left\{\gamma \;:\; \sup_{\mathbb{P}\in\mathcal{P}_{\boldsymbol{r}}} \mathbb{P}\left\{\gamma \leq -\boldsymbol{w}^T\tilde{\boldsymbol{r}}\right\} \leq \epsilon\right\}$$

- WCVaR is immunized against uncertainty in $\mathbb{P}$: distributionally robust.
- Unless the most pessimistic distribution in $\mathcal{P}_{\boldsymbol{r}}$ is the true distribution, actual VaR will be lower than WCVaR.

Robust Optimization Perspective on WCVaR

- El Ghaoui *et al.* have shown that

$$\text{WCVaR}_\epsilon(\boldsymbol{w}) = -\boldsymbol{\mu}^T\boldsymbol{w} + \kappa(\epsilon)\sqrt{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}},$$

where $\kappa(\epsilon) = \sqrt{(1-\epsilon)/\epsilon}$.

- Connection to robust optimization:

$$\text{WCVaR}_\epsilon(\boldsymbol{w}) = \max_{\boldsymbol{r}\in\mathcal{U}_\epsilon} -\boldsymbol{w}^T\boldsymbol{r},$$

where the ellipsoidal uncertainty set $\mathcal{U}_\epsilon$ is defined as

$$\mathcal{U}_\epsilon = \left\{\boldsymbol{r} : (\boldsymbol{r}-\boldsymbol{\mu}_{\boldsymbol{r}})^T\boldsymbol{\Sigma}_{\boldsymbol{r}}^{-1}(\boldsymbol{r}-\boldsymbol{\mu}_{\boldsymbol{r}}) \leq \kappa(\epsilon)^2\right\}.$$

- Therefore,

$$\min_{\boldsymbol{w}\in\mathcal{W}} \text{WCVaR}_\epsilon(\boldsymbol{w}) \equiv \min_{\boldsymbol{w}\in\mathcal{W}} \max_{\boldsymbol{r}\in\mathcal{U}_\epsilon} -\boldsymbol{w}^T\boldsymbol{r}.$$

Worst-Case VaR for Derivative Portfolios

- Assume that the market consists of:
  - $n \leq m$ basic assets with returns $\tilde{\xi}$, and
  - $m - n$ derivatives with returns $\tilde{\eta}$.
  - $\tilde{\xi}$ are only risk factors.

  We partition asset returns as $\tilde{\boldsymbol{r}} = (\tilde{\xi}, \tilde{\eta})$.

- Derivative returns $\tilde{\eta}$ are uniquely determined by basic asset returns $\tilde{\xi}$. There exists $f : \mathbb{R}^n \to \mathbb{R}^m$ with $\tilde{\boldsymbol{r}} = f(\tilde{\xi})$.
- f is highly non-linear and can be inferred from:
  - Contractual specifications (option payoffs)
  - Derivative pricing models

Worst-Case VaR for Derivative Portfolios

- WCVaR is applicable but **not** suitable for portfolios containing derivatives:
  - Moments of $\tilde{\eta}$ are **difficult** to estimate accurately.
  - **Disregards** perfect dependencies between $\tilde{\eta}$ and $\tilde{\xi}$.
- WCVaR severly **overestimates** the actual VaR, because:
  - $\Sigma_{\boldsymbol{r}}$ only accounts for **linear** dependencies
  - $\mathcal{U}_\epsilon$ is **symmetric** but derivative returns are **skewed**

Generalized Worst-Case VaR Framework

- We develop two new Worst-Case VaR models that:
  - Use first- and second-order moments of $\tilde{\xi}$ but not $\tilde{\eta}$.
  - Incorporate the non-linear dependencies f

Generalized Worst-Case VaR

Let $\mathcal{P}$ denote set of all distributions of $\tilde{\xi}$ with mean μ and covariance matrix Σ.

$$\min \left\{ \gamma \; : \; \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P} \left\{ \gamma \leq -\boldsymbol{w}^T f(\tilde{\xi}) \right\} \leq \epsilon \right\}$$

- When $f(\tilde{\xi})$ is:
  - convex polyhedral $\rightarrow$ Worst-Case Polyhedral VaR (SOCP)
  - nonconvex quadratic $\rightarrow$ Worst-Case Quadratic VaR (SDP)

Piecewise Linear Portfolio Model

- Assume that the $m - n$ derivatives are European put/call options maturing at the end of the investment horizon T.
- Basic asset returns: $\tilde{r}_j = f_j(\tilde{\xi}) = \tilde{\xi}_j$ for $j = 1, \ldots, n$.
- Assume option j is a call with strike k_j and premium c_j on basic asset i with initial price s_i, then $\tilde{r}_j$ is

$$
\begin{aligned}
f_j(\tilde{\xi}) &= \frac{1}{c_j} \max\left\{0, s_i(1+\tilde{\xi}_i) - k_j\right\} - 1 \\
&= \max\left\{-1, a_j + b_j\tilde{\xi}_i - 1\right\}, \text{ where } a_j = \frac{s_i - k_j}{c_j},\ b_j = \frac{s_i}{c_j}.
\end{aligned}
$$

- Likewise, if option j is a put with premium p_j, then $\tilde{r}_j$ is

$$
f_j(\tilde{\xi}) = \max\left\{-1, a_j + b_j\tilde{\xi}_i - 1\right\}, \text{ where } a_j = \frac{k_j - s_i}{p_j},\ b_j = -\frac{s_i}{p_j}.
$$

Piecewise Linear Portfolio Model

- In compact notation, we can write $\tilde{\boldsymbol{r}}$ as

$$\tilde{\boldsymbol{r}} = f(\tilde{\xi}) = \begin{pmatrix} \tilde{\xi} \\ \max\left\{-\boldsymbol{e}, \boldsymbol{a} + \mathbf{B}\tilde{\xi} - \boldsymbol{e}\right\} \end{pmatrix}.$$

- Partition weights vector as $\boldsymbol{w} = (\boldsymbol{w}^{\xi}, \boldsymbol{w}^{\eta})$.
- No derivative short-sales: $\boldsymbol{w} \in \mathcal{W} \implies \boldsymbol{w}^{\eta} \geq \mathbf{0}$.
- Portfolio return of $\boldsymbol{w} \in \mathcal{W}$ can be expressed as

$$\begin{aligned} \boldsymbol{w}^T \tilde{\boldsymbol{r}} &= \boldsymbol{w}^T f(\tilde{\xi}) \\ &= (\boldsymbol{w}^{\xi})^T \tilde{\xi} + (\boldsymbol{w}^{\eta})^T \max\left\{-\boldsymbol{e}, \boldsymbol{a} + \mathbf{B}\tilde{\xi} - \boldsymbol{e}\right\}. \end{aligned}$$

Worst-Case Polyhedral VaR

- Use the piecewise linear portfolio model:

$$\boldsymbol{w}^T f(\tilde{\xi}) = (\boldsymbol{w}^{\xi})^T \tilde{\xi} + (\boldsymbol{w}^{\eta})^T \max\left\{-\boldsymbol{e}, \boldsymbol{a} + \mathbf{B}\tilde{\xi} - \boldsymbol{e}\right\}.$$

Worst-Case Polyhedral VaR (WCPVaR)

For any $\boldsymbol{w} \in \mathcal{W}$, we define $\text{WCPVaR}_{\epsilon}(\boldsymbol{w})$ as

$$\text{WCPVaR}_{\epsilon}(\boldsymbol{w}) = \min\left\{\gamma \;:\; \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\left\{\gamma \leq -\boldsymbol{w}^T f(\tilde{\xi})\right\} \leq \epsilon\right\}.$$

Theorem: SDP Reformulation of WCPVaR

WCPVaR of $\boldsymbol{w}$ can be computed as an SDP:

$$\begin{aligned}
\mathrm{WCPVaR}_\epsilon(\boldsymbol{w}) = \min \quad & \gamma \\
\text{s. t.} \quad & \mathbf{M} \in \mathbb{S}^{n+1}, \quad \boldsymbol{y} \in \mathbb{R}^{m-n}, \quad \tau \in \mathbb{R}, \quad \gamma \in \mathbb{R} \\
& \langle \boldsymbol{\Omega}, \mathbf{M} \rangle \leq \tau\epsilon, \quad \mathbf{M} \succcurlyeq \mathbf{0}, \quad \tau \geq 0, \quad \mathbf{0} \leq \boldsymbol{y} \leq \boldsymbol{w}^\eta \\
& \mathbf{M} + \begin{bmatrix} \mathbf{0} & \boldsymbol{w}^\xi + \mathbf{B}^T \boldsymbol{y} \\ (\boldsymbol{w}^\xi + \mathbf{B}^T \boldsymbol{y})^T & -\tau + 2(\gamma + \boldsymbol{y}^T \boldsymbol{a} - \boldsymbol{e}^T \boldsymbol{w}^\eta) \end{bmatrix} \succcurlyeq \mathbf{0}
\end{aligned}$$

Where we use the second-order moment matrix $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T & \boldsymbol{\mu} \\ \boldsymbol{\mu}^T & 1 \end{bmatrix}$$

Worst-Case Polyhedral VaR: Convex Reformulations

Theorem: SOCP Reformulation of WCPVaR

WCPVaR of $\boldsymbol{w}$ can be computed as an SOCP:

$$\text{WCPVaR}_{\epsilon}(\boldsymbol{w}) = \min_{\boldsymbol{0} \leq \boldsymbol{g} \leq \boldsymbol{w}^{\eta}} -\boldsymbol{\mu}^T(\boldsymbol{w}^{\xi} + \mathbf{B}^T\boldsymbol{g}) + \kappa(\epsilon) \left\| \boldsymbol{\Sigma}^{1/2}(\boldsymbol{w}^{\xi} + \mathbf{B}^T\boldsymbol{g}) \right\|_2 \dots$$
$$\dots - \boldsymbol{a}^T\boldsymbol{g} + \boldsymbol{e}^T\boldsymbol{w}^{\eta}$$

- SOCP has better scalability properties than SDP.

Robust Optimization Perspective on WCPVaR

- WCPVaR minimization is equivalent to:

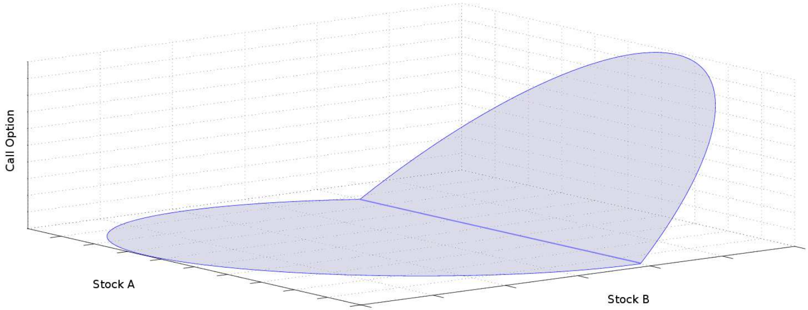
$$\min_{\boldsymbol{w} \in \mathcal{W}} \max_{\boldsymbol{r} \in \mathcal{U}^p_\epsilon} -\boldsymbol{w}^T \boldsymbol{r}.$$

where the uncertainty set $\mathcal{U}^p_\epsilon \subseteq \mathbb{R}^m$ is defined as

$$\mathcal{U}^p_\epsilon = \left\{ \boldsymbol{r} \in \mathbb{R}^m : \begin{array}{ll} \exists \boldsymbol{\xi} \in \mathbb{R}^n & \text{such that} \\ (\boldsymbol{\xi} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\xi} - \boldsymbol{\mu}) \leq \kappa(\epsilon)^2 & \text{and} \\ \boldsymbol{r} = f(\boldsymbol{\xi}) & \end{array} \right\}$$

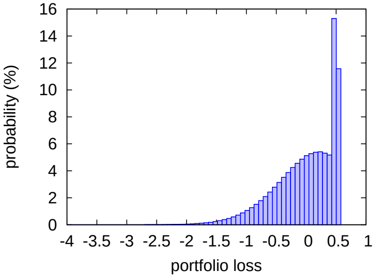
- Unlike $\mathcal{U}_\epsilon$, the set $\mathcal{U}^p_\epsilon$ is not symmetric!

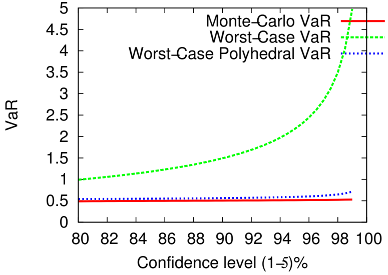
Robust Optimization Perspective on WCPVaR



Example: WCPVaR vs WCVaR

- Consider Black-Scholes Economy containing:
  - Stocks A and B, a call on stock A, and a put on stock B.
  - Stocks have drifts of 12% and 8%, and volatilities of 30% and 20%, with instantaneous correlation of 20%.
  - Stocks are both $100.
  - Options mature in 21 days and have strike prices $100.
- Assume we hold equally weighted portfolio.
- Goal: calculate VaR of portfolio in 21 days.
  - Generate 5,000,000 end-of-period stock and option prices.
  - Calculate first- and second-order moments from returns.
  - Estimate VaR using: Monte-Carlo VaR, WCVaR, and WCPVaR.





- At confidence level $\epsilon = 1\%$:
  - WCVaR unrealistically high: 497%.
  - WCVaR is 7 times larger than WCPVaR.
  - WCPVaR is much closer to actual VaR.

Delta-Gamma Portfolio Model

- $m - n$ derivatives can be exotic with arbitrary maturity time. Value of asset $i = 1 \dots m$ is representable as $v_i(\tilde{\xi}, t)$.
- For short horizon time T, second-order Taylor expansion is accurate approximation of $\tilde{r}_i$:

$$\tilde{r}_i = f_i(\tilde{\xi}) \approx \theta_i + \boldsymbol{\Delta}_i^T \tilde{\xi} + \frac{1}{2} \tilde{\xi}^T \boldsymbol{\Gamma}_i \tilde{\xi} \quad \forall i = 1, \dots, m.$$

- Portfolio return approximated by (possibly non-convex):

$$\boldsymbol{w}^T \tilde{\boldsymbol{r}} = \boldsymbol{w}^T f(\xi) \approx \theta(\boldsymbol{w}) + \boldsymbol{\Delta}(\boldsymbol{w})^T \tilde{\xi} + \frac{1}{2} \tilde{\xi}^T \boldsymbol{\Gamma}(\boldsymbol{w}) \tilde{\xi},$$

  where we use the auxiliary functions

$$\theta(\boldsymbol{w}) = \sum_{i=1}^{m} w_i \theta_i, \quad \boldsymbol{\Delta}(\boldsymbol{w}) = \sum_{i=1}^{m} w_i \boldsymbol{\Delta}_i, \quad \boldsymbol{\Gamma}(\boldsymbol{w}) = \sum_{i=1}^{m} w_i \boldsymbol{\Gamma}_i.$$

- We now allow short-sales of options in $\boldsymbol{w}$

Worst-Case Quadratic VaR

Worst-Case Quadratic VaR (WCQVaR)

For any $\boldsymbol{w} \in \mathcal{W}$, we define WCQVaR as

$$\min \left\{ \gamma \ : \ \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P} \left\{ \gamma \leq -\theta(\boldsymbol{w}) - \boldsymbol{\Delta}(\boldsymbol{w})^T \tilde{\xi} - \frac{1}{2} \tilde{\xi}^T \boldsymbol{\Gamma}(\boldsymbol{w}) \tilde{\xi} \right\} \leq \epsilon \right\}$$

Theorem: SDP Reformulation of WCQVaR

WCQVaR can be found by solving an SDP:

$$\begin{aligned} \mathrm{WCQVaR}_\epsilon(\boldsymbol{w}) = \min \quad & \gamma \\ \text{s. t.} \quad & \mathbf{M} \in \mathbb{S}^{n+1}, \quad \tau \in \mathbb{R}, \quad \gamma \in \mathbb{R} \\ & \langle \boldsymbol{\Omega}, \mathbf{M} \rangle \leq \tau\epsilon, \quad \mathbf{M} \succcurlyeq \mathbf{0}, \quad \tau \geq 0, \\ & \mathbf{M} + \begin{bmatrix} \boldsymbol{\Gamma}(\boldsymbol{w}) & \boldsymbol{\Delta}(\boldsymbol{w}) \\ \boldsymbol{\Delta}(\boldsymbol{w})^T & -\tau + 2(\gamma + \theta(\boldsymbol{w})) \end{bmatrix} \succcurlyeq \mathbf{0} \end{aligned}$$

- There seems to be no SOCP reformulation of WCQVaR.

Robust Optimization Perspect on WCQVaR

- WCQVaR minimization is equivalent to:

$$\min_{\boldsymbol{w}\in\mathcal{W}} \max_{\mathbf{Z}\in\mathcal{U}_\epsilon^q} -\langle \mathbf{Q}(\boldsymbol{w}), \mathbf{Z}\rangle$$

  where

$$\mathbf{Q}(\boldsymbol{w}) = \begin{bmatrix} \frac{1}{2}\boldsymbol{\Gamma}(\boldsymbol{w}) & \frac{1}{2}\boldsymbol{\Delta}(\boldsymbol{w}) \\ \frac{1}{2}\boldsymbol{\Delta}(\boldsymbol{w})^T & \theta(\boldsymbol{w}) \end{bmatrix},$$

  and the uncertainty set $\mathcal{U}_\epsilon^q \subseteq \mathbb{S}^{n+1}$ is defined as

$$\mathcal{U}_\epsilon^q = \left\{ \mathbf{Z} = \begin{bmatrix} \mathbf{X} & \boldsymbol{\xi} \\ \boldsymbol{\xi}^T & 1 \end{bmatrix} \in \mathbb{S}^{n+1} \ : \ \boldsymbol{\Omega} - \epsilon\mathbf{Z} \succcurlyeq \mathbf{0}, \ \mathbf{Z} \succcurlyeq \mathbf{0} \right\}$$

- $\mathcal{U}_\epsilon^q$ is lifted into $\mathbb{S}^{n+1}$ to compensate for non-convexity.

Robust Optimization Perspect on WCQVaR

- There is a connection between $\mathcal{U}_\epsilon \subseteq \mathbb{R}^m$ and $\mathcal{U}_\epsilon^q \subseteq \mathbb{S}^{n+1}$.
- If we impose: $\boldsymbol{w} \in \mathcal{W} \implies \boldsymbol{\Gamma}(\boldsymbol{w}) \succcurlyeq \boldsymbol{0}$ then robust optimization problem reduces to:

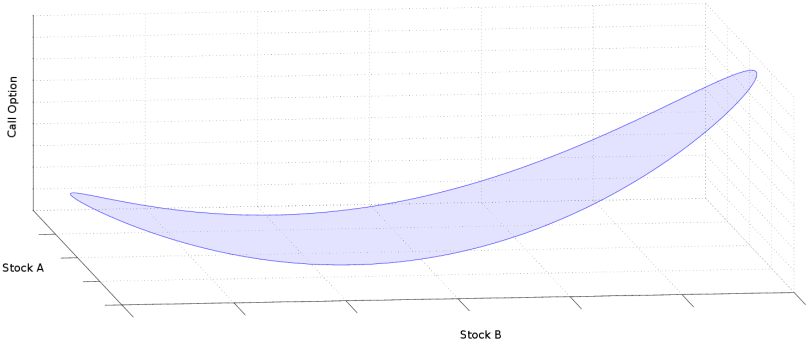
$$\min_{\boldsymbol{w} \in \mathcal{W}} \max_{\boldsymbol{r} \in \mathcal{U}_\epsilon^{q'}} -\boldsymbol{w}^T \boldsymbol{r}$$

where the uncertainty set $\mathcal{U}_\epsilon^{q'} \subseteq \mathbb{R}^m$ is defined as

$$\mathcal{U}_\epsilon^{q'} = \left\{ \boldsymbol{r} \in \mathbb{R}^m : \begin{array}{l} \exists \boldsymbol{\xi} \in \mathbb{R}^n \quad \text{such that} \\ (\boldsymbol{\xi} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\xi} - \boldsymbol{\mu}) \leq \kappa(\epsilon)^2 \quad \text{and} \\ r_i = \theta_i + \boldsymbol{\xi}^T \boldsymbol{\Delta}_i + \frac{1}{2} \boldsymbol{\xi}^T \boldsymbol{\Gamma}_i \boldsymbol{\xi} \ \ \forall i = 1, \ldots, m \end{array} \right\}$$

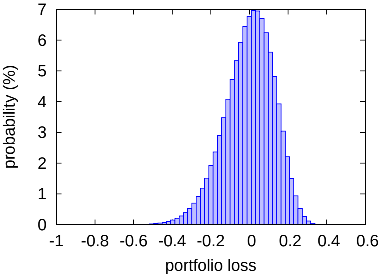
- Unlike $\mathcal{U}_\epsilon$, the set $\mathcal{U}_\epsilon^{q'}$ is not symmetric!

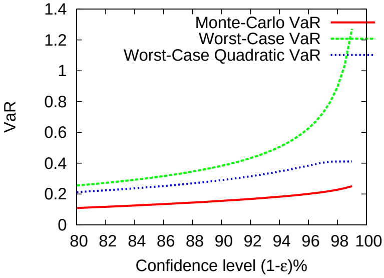
Robust Optimization Perspective on WCQVaR



Example: WCQVaR vs WCVaR

- Now we want to estimate VaR after 2 days (not 21 days).
- VaR not evaluated at option maturity times
  $\longrightarrow$ use WCQVaR (not WCPVaR).
- Use Black-Scholes to calculate prices and greeks.

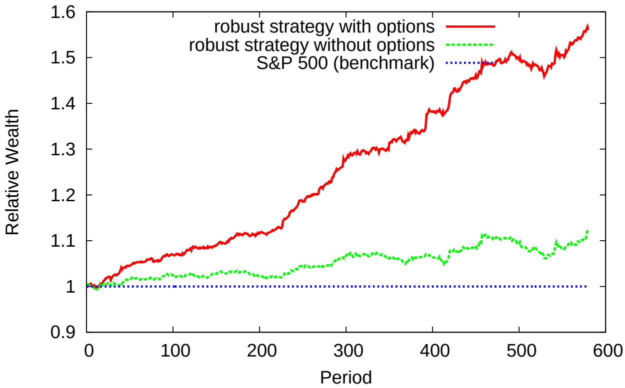




- At $\epsilon = 1\%$: WCVaR still 3 times larger than WCQVaR.

Index Tracking using Worst-Case Quadratic VaR

- Total test period: Jan. 2nd, 2004 – Oct. 10th, 2008.
- Estimation Window: 600 days. Out-of-sample returns: 581.



- Outperformance: option strat 56%, stock-only strat 12%.
- Sharpe Ratio: option strat 0.97, stock-only strat 0.13.
- Allocation option strategy: 89% stocks, 11% options.

Questions?

- Paper available on optimization-online.