New Control Variates for Lévy Processes and Asian Options

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Outline

- Control variates for Lévy process models
  - Control variate framework
  - Option pricing examples
- Variance reduction for Asian options
  - A unified framework for non-Gaussian models
  - The proposed method is a combination of
    - Control Variate (CV)
    - Conditional Monte Carlo (CMC)

Monte Carlo (MC) Method: General Principles

- Estimation of an unknown parameter: $\mu = \mathrm{E}[Y]$
- Generation of iid sample $Y_1, Y_2, \ldots, Y_n$
- The estimator: $\hat{\mu} = \frac{\sum_{i=1}^{n} Y_i}{n}$
- To quantify the error $\hat{\mu} - \mu$:
  - Central Limit Theorem: $\frac{\hat{\mu}-\mu}{s_n/\sqrt{n}} \Rightarrow N(0,1)$ as $n \to \infty$.
  - Probabilistic error bound: $\Phi^{-1}(1-\alpha/2)\ s/\sqrt{n}$
  - To get smaller error bound:
    - ★ Increase the sample size n (Larger computational time) $O(1/\sqrt{n})$
    - ★ Decrease the variance s^2 (Variance reduction techniques)

Problem definition

- Lévy process $\{L(t), t \geq 0\}$
  - Stationary and independent increments, and $L(0) = 0$

- Functional of L:
  - $q(L(t_1), \ldots, L(t_d))$
  - Time grid $0 = t_0 < t_1 < t_2 < ... < t_d$ with $t_j = j\Delta t$
  - $t_j = j\Delta t \Rightarrow$ increments $L(t_i) - L(t_{i-1})$ are iid

- Estimation of $\mathrm{E}[q(L(t_1), \ldots, L(t_d))]$ by simulation.

- A new variance reduction method

Problem definition

- In the literature, there exist variance reduction methods suggested for Lévy processes
  - They are often special to the 'process type' or 'problem type'
- A new control variate (CV) method
- It can be applied for any Lévy process for which the probability density function (PDF) of the increments is available in closed form
- Numerical examples: path-dependent options

Control Variate Method

- Estimator: $Y = q(L) - c^T(V - \mathrm{E}[V])$
  - $V = (V_1, \ldots, V_m)^T$ set of CVs with known $\mathrm{E}[V]$
  - $c = (c_1, \ldots, c_m)^T$ the coefficient vector (optimal c^* by linear regression)
- Successful if strong linear dependence: $VRF = 1/(1 - R^2)$.
- Our CV framework:
  - Special CV, tailored to $q()$
  - General CVs, selected from a *basket* of CVs (not tailored to $q()$)

Special CV

- Functional of a Brownian Motion (BM).
- Brownian motion $\{W(t), t \geq 0\}$ with parameters $\{\mu, \sigma\}$:
  - $W(t) = \mu t + \sigma B(t)$
  - $B(t)$ is a standard BM
- Functional $\zeta(W(t_1), \ldots, W(t_d))$
  - Similar to the original function: $\zeta \sim q$.
- **Known expectation**: $\mathrm{E}[\zeta(W)]$ is available in closed form

Special CV

- similarity of paths: $(W(t_1),\ldots,W(t_d)) \sim (L(t_1),\ldots,L(t_d))$ and similarity of functions: $\zeta \sim q$
  $\Rightarrow$ Large correlation between $q(L)$ and $\zeta(W)$
- For similar paths,
  - $\mu = \mathrm{E}[L(1)]$ and $\sigma = \sqrt{\mathrm{Var}(L(1))}$
  - Using CRN (common random numbers) for path simulation
- Comonotonic increments lead to maximal correlation
  - $U \sim U(0,1)$
  - $L(t_i) - L(t_{i-1}) \leftarrow F_L^{-1}(U)$
  - $W(t_i) - W(t_{i-1}) \leftarrow F_{BM}^{-1}(U)$

Special CV

- Inverse CDFs:
  - $F_{BM}^{-1}(U)$, Inverse CDF of normal distribution.
  - $F_{L}^{-1}(U)$, non-tractable.
- Approximation of $F_{L}^{-1}(U)$ by numerical inversion algorithm of Derflinger et al. (2010)
- It requires **only** PDF (probability density function)
- For many Lévy processes, PDF is available in closed form (while CDF and the inverse CDF are not).

General CVs

- Simple path characteristics of L and W (e.g. average, maximum)
- They are not tailored to $q()$
- We call them as 'general CVs' since they are applicable to any $q()$, whereas $\zeta(W)$ is called 'special CV' as it is designed considering the special properties of $q()$.
- Let $\gamma(W, L)$ be a function of the paths of W and L that evaluates the set of path characteristics.

Algorithm

Require: special CV function $\zeta()$, general CV function $\gamma()$

for $i = 1$ to n **do**
  for $j = 1$ to d **do**
    Generate uniform variate $U \sim U(0,1)$.
    Set $X_j \leftarrow F_L^{-1}(U)$ and $Z_j \leftarrow F_{BM}^{-1}(U)$.
    Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
  end for
  Set $Y_i \leftarrow q(L) - c_1\,(\zeta(W) - \mathrm{E}[\zeta(W)]) - c_2^T(\gamma(W,L) - E[\gamma(W,L)])$.
end for

Algorithm

Require: special CV function $\zeta()$, general CV function $\gamma()$

```
for i = 1 to n do
    for j = 1 to d do
        Generate uniform variate U ~ U(0,1).
        Set X_j ← F_L^{-1}(U) and Z_j ← F_{BM}^{-1}(U).
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- **for** $i = 1$ to n **do**
  - **for** $j = 1$ to d **do**
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Algorithm

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end for

CV Selection

- In algorithm, the user has to provide the CV functions $\zeta()$ and $\gamma()$.
- The selection of special CV $\zeta()$ depends on the problem type, as it is tailored to $q()$.
- Our approach for the selection of general CVs:
  - A large *basket* of CV candidates
  - Stepwise backward linear regression.
    - ★ The t-statistics of regression coefficients: $t = \frac{\hat{\beta}}{s.e.(\hat{\beta})}$
    - ★ Check the significancy: $t \in (-5, 5)$?
- pilot simulation run

CV Selection

- Stepwise backward regression
  1. Start with a full regression model
  2. remove the CV with the smallest absolute t statistic from the model, if its value is smaller than 5
  3. recompute the t-statistics of the remaining CVs for the new regression model
  4. Steps 2-3 are repeated until all absolute t values > 5
  5. Use the remaining CVs for the main simulation
- Why not use all CVs in the basket ?
  - Simulation or evaluation of expectation of some CVs can be expensive.
  - Backward regression automatically eliminates the CV if it is not useful.

Complexity

- A single regression with k covariates requires $O(n_p k^2)$ operations
  - k number of CVs
  - n_p sample size of pilot simulation
- The worst case: All CVs are useless $O(n_p k^3)$
- Since $n_p < n$, no substantial increase in the computational time

Basket of CVs

- Path characteristics of which the expectation is available in closed form.
- not exhaustive and depends on our knowledge of the closed form solutions
- No CV that require a numerical method to evaluate the expectation
- Our notation
  - CVL: path characteristics of L (internal CVs)
  - CVW: path characteristics of W (external CVs)

Basket of CVs

Table: Basket of CVs.

Label	CV	Label	CV
CVL1	$L(t_d)$	CVW1	$W(t_d)$
CVL2	$\exp(L(t_d))$	CVW2	$\exp(W(t_d))$
CVL3	$\frac{1}{d}\sum_{i=1}^{d} L(t_i)$	CVW3	$\frac{1}{d}\sum_{i=1}^{d} W(t_i)$
CVL4	$\exp(\frac{1}{d}\sum_{i=1}^{d} L(t_i))$	CVW4	$\exp(\frac{1}{d}\sum_{i=1}^{d} W(t_i))$
CVL5	$\frac{1}{d}\sum_{i=1}^{d} \exp(L(t_i))$	CVW5	$\frac{1}{d}\sum_{i=1}^{d} \exp(W(t_i))$
		CVW6	$\max_{0\le i\le d} W(t_i)$
		CVW7	$\exp(\max_{0\le i\le d} W(t_i))$
		CVW8	$\sup_{0\le u\le t_d} W(u)$
		CVW9	$\exp(\sup_{0\le u\le t_d} W(u))$

Basket of CVs

- All CVs in the basket are easy to simulate
- A bit more difficult CVs: $\sup_{0\le u\le t_d} W(u)$ and $e^{\sup_{0\le u\le t_d} W(u)}$
- Simulation of $\sup_{0\le u\le t_d} W(u)$ conditional on $(W(t_1),\ldots,W(t_d))$
- $$\sup_{0\le u\le t_d} W(u) = \max_{1\le i\le d}\left(\sup_{t_{i-1}\le u\le t_i} W(u)\right).$$
- generate the maxima of d Brownian bridges

Basket of CVs

- CDF of the maximum of a Brownian bridge

$$P\left(\sup_{0\le u\le t} W(u) \le x \,\middle|\, W(t) = y\right) = 1 - \exp\left(-\frac{2x(x-y)}{\sigma^2 t}\right),$$

- Inversion

$$x = 0.5\left(y + \sqrt{y^2 - 2\sigma^2 t \log U}\right),$$

where $U \sim U(0,1)$ is a uniform random number

- $\mathrm{E}\left[\sup_{0\le u\le t_d} W(u) | W(t_1), \ldots, W(t_d)\right]$ as alternative to $\sup_{0\le u\le t_d} W(u)$

- requires numerical integration, not efficient

Basket of CVs

- CVs in the basket are strongly correlated with each other.
- *Multicollinearity*: It inflates the standard errors of the estimates of the regression coefficients
- It can be a problem for the accuracy of the estimates of the t statistics, when the sample size is too small.
- $n_p = 10^4$ is generally sufficient to get relatively stable estimates of the t values.

Basket of CVs: Expectation formulas

Table: Expectation formulas for the CVs depending on the terminal value and the averages.

Label	CV	Expectation
CVL1	$L(t_d)$	$d\,\mathrm{E}[X]$
CVL2	$\exp(L(t_d))$	$M_{\Delta t}(1)^d$
CVL3	$\frac{1}{d}\sum_{i=1}^{d} L(t_i)$	$\mathrm{E}[X](d+1)/2$
CVL4	$\exp(\frac{1}{d}\sum_{i=1}^{d} L(t_i))$	$\prod_{i=1}^{d} M_{\Delta t}(i/d)$
CVL5	$\frac{1}{d}\sum_{i=1}^{d} \exp(L(t_i))$	$\frac{1}{d}\sum_{i=1}^{d} M_{\Delta t}(1)^i$
CVW1	$W(t_d)$	$d\,\mu\,\Delta t$
CVW2	$\exp(W(t_d))$	$e^{(d(\mu\Delta t+\sigma^2\,\Delta t/2))}$
CVW3	$\frac{1}{d}\sum_{i=1}^{d} W(t_i)$	$\mu\,\Delta t(d+1)/2$
CVW4	$\exp(\frac{1}{d}\sum_{i=1}^{d} W(t_i))$	$\exp(\tilde{\mu}+\tilde{\sigma}^2/2)$
CVW5	$\frac{1}{d}\sum_{i=1}^{d} \exp(W(t_i))$	$\frac{1}{d}\sum_{i=1}^{d} e^{(i(\mu\Delta t+\sigma^2\,\Delta t/2))}$

Basket of CVs: Expectation formulas

Table: Expectation formulas for the CVs depending on maximum.

Label	CV	Expectation
CVW6	$\max_{0 \le i \le d} W(t_i)$	by Spitzer's identity
CVW7	$\exp(\max_{0 \le i \le d} W(t_i))$	by Öhgren (2001)
CVW8	$\sup_{0 \le u \le t_d} W(u)$	e.g. Shreve (2004)
CVW9	$\exp(\sup_{0 \le u \le t_d} W(u))$	e.g. Shreve (2004)

A Simple Example

- We use only general CVs in the basket *without* a special CV
- $q(L) = \exp\left(\max_{0 \leq i \leq d} L(t_i)\right)$
- L is a generalized hyperbolic (GH) process
    - $\Delta t = 1/250$
    - $\lambda = 1.5$, $\alpha = 189.3$, $\beta = -5.71$, $\delta = 0.0062$, $\mu = 0.001$
    - the increment distribution is close to normal but has a higher kurtosis
- Variance Reduction Factors (VRFs)
    - For $d = 5$, VRF= 560
    - For $d = 50$, VRF= 395

Examples from Option Pricing

- Underlying stock: $S(t) = S(0)e^{L(t)}$,
  - Non-normal logreturns with high kurtosis.
- Payoff of path dependent options: $\psi(S(t_1), \ldots, S(t_d))$
- Price

$$e^{-r t_d} \mathrm{E}\left[\psi(S(t_1), \ldots, S(t_d))\right],$$

- $q(L) = \psi(S(0)e^L)$

Examples from Option Pricing

- **Special CV:** a similar option with analytically available price under geometric Brownian Motion (GBM)
  - $\zeta()$ corresponds to $\psi_{CV}()$ payoff function of new option.
  - $\{\tilde{S}(t), t \geq 0\}$ stock price under GBM:

$$\tilde{S}(t) = \tilde{S}(0)e^{W(t)} = \tilde{S}(0)\exp((r - \sigma^2/2)t + \sigma B(t)).$$

  - We set $\sigma = \sqrt{\mathrm{Var}(L(1))}$ and $\widetilde{S}(0) = S(0)$

- **General CVs:** Use the basket

Option Examples

- **Asian Option:** $\psi_A(S) = \left(\frac{\sum_{i=1}^{d} S(t_i)}{d} - K\right)^+$

  Special CV: $\psi_G(\tilde{S}) = \left(\left(\prod_{i=1}^{d} \tilde{S}(t_i)\right)^{1/d} - K\right)^+$.

- **Lookback Option**: $\psi_L(S) = \left(\max_{0 \le i \le d} S(t_i) - K\right)^+$,

  Special CV: $\psi_{LC}(\tilde{S}) = \left(\sup_{0 \le u \le t_d} \tilde{S}(u) - K\right)^+$.

Numerical Results

Table: Results for Asian and lookback options under GH process with $T = 1, \Delta t = 1/250, r = 0.05, S(0) = 100, n = 10^4$. Error: 95% error bound.

Option	K	Price	Error	VRF-A	VRF-G	VRF-S
Asian	90	12.239	0.004	1,743	185	78
	100	4.912	0.005	530	51	64
	110	1.240	0.006	121	13	40
Lookback	110	7.534	0.012	294	57	57
	120	3.297	0.012	160	35	44
	130	1.266	0.011	79	17	32

- VRF-A: VRF obtained by using all (significant) CVs,
- VRF-G: VRF obtained by using only general CVs (CVLs and CVWs),
- VRF-S: VRF obtained by using only special CV

Numerical Results

- Efficiency factor: $EF = (\sigma_N^2 t_N)/(\sigma_{CV}^2 t_{CV})$
  - t_N and t_{CV} are the CPU times of naive simulation and CV method.
- In naive simulation, we used the subordination (the standard method in the literature).
- Asian option: $t_N/t_{CV} = 1$
- Lookback option: $t_N/t_{CV} = 0.7$
- time of the pilot simulation run is between 30% and 50% of the main simulation

Success of the method

- Proximity of increment (log-return) distribution to the normal distribution.

- Shape depends on Δt
  - $\Delta t \to \infty$, gets close to normal
  - $\Delta t \to 0$, very high kurtosis

- In option pricing,
  - $\Delta t = 1/4$, quarterly monitoring
  - $\Delta t = 1/12$, monthly monitoring
  - $\Delta t = 1/50$, weekly monitoring
  - $\Delta t = 1/250$, **daily monitoring**
  - $\Delta t \to 0$, continuous monitoring (not possible in practice)

Asian option example for variance gamma (VG) process

K	Δt	Price	Error	**VRF**
70	1/4	31.562	0.004	2,966
	1/12	31.156	0.004	2,582
	1/50	31.002	0.006	894
	1/250	30.975	0.019	87
100	1/4	5.903	0.003	1,949
	1/12	5.229	0.003	1,815
	1/50	4.972	0.005	795
	1/250	4.912	0.015	72
130	1/4	0.082	0.002	114
	1/12	0.034	0.001	101
	1/50	0.025	0.001	43
	1/250	0.020	0.002	15

Table: Using 'special CV' for Asian VG options with $T = 1$ and different Δt's; $n = 10,000$; Error: 95% error bound; VRF: variance reduction factor.

Conclusions

- A general control variate framework for the functionals of Lévy processes.
- The method exploits the strong correlation between the original Lévy process and an auxiliary Brownian motion
  - Numerical inversion of CDFs
- In the CV framework,
  - special control variates tailored to the functionals
  - general control variates selected from a large basket of control variate candidates
- In the application to path dependent options, we observe moderate to large variance reductions

Asian options

- Stock price process $\{S(t), t \geq 0\}$

Arithmetic average call option

$$P_A(S) = \left(\frac{1}{d}\sum_{i=1}^{d} S(t_i) - K\right)^+$$

- Time grid $0 = t_0 < t_1 < t_2 < \dots < t_d = T$ with $t_j = j\Delta t$
- Option price: $e^{-rT}\mathrm{E}\left[P_A(S)\right]$

Geometric Brownian motion (GBM)

$$S(t) = S(0)\exp\left\{(r - \sigma^2/2)\,t + \sigma B(t)\right\}, \quad t \geq 0$$

- No closed form solution for the price

Simulation of Asian options

- Efficient numerical methods under GBM
  - PDE based finite difference methods, e.g. Večeř (2001)
  - Approximations, e.g. Curran (1994); Lord (2006)
- Monte Carlo simulation
  - Advantage : Probabilistic error bound
  - Disadvantage : Slow convergence rate
- Variance reduction method

CVs for Asian options

- Classical CV method of Kemna and Vorst (1990)
  - Arithmetic and geometric averages :

    $A = \frac{1}{d}\sum_{i=1}^{d} S(t_i)$ and $G = \left(\prod_{i=1}^{d} S(t_i)\right)^{1/d}$
  - If $S(t_i)$'s are close to each other, then $A \sim G$
  - $P_A = (A-K)^+ \sim (G-K)^+ = P_G$
- $\mathrm{E}\left[P_G\right]$ is available in closed form under GBM
- Very successful, if σ and T are small

CVs for Asian options

- Lower bound $\mathrm{E}\left[(A-K)\mathbf{1}_{\{G>K\}}\right]$ suggested by Curran (1994) :

$$\begin{aligned}(A-K)^{+} &= (A-K)^{+}\mathbf{1}_{\{G\leq K\}} + (A-K)^{+}\mathbf{1}_{\{G>K\}} \\ &= (A-K)^{+}\mathbf{1}_{\{G\leq K\}} + (A-K)\mathbf{1}_{\{G>K\}},\end{aligned}$$

- New CV by Dingeç and Hörmann (2013)

$$Y_{CV} = P_A - c\left(W - \mathrm{E}\left[W\right]\right),$$

where $W = (A-K)\mathbf{1}_{\{G>K\}}$.

- $\mathrm{E}\left[W\right]$ is available in closed form under GBM

CVs for Asian options

- If we set $c = 1$, then $Y_{CV} = (A-K)^{+}\mathbf{1}_{\{G\leq K\}} + \mathrm{E}[W]$

- Conditional Monte Carlo (CMC) for $Y = (A-K)^{+}\mathbf{1}_{\{G\leq K\}}$
  - New estimator as conditional expectation: $\mathrm{E}[Y|Z] = \int Y\, dF(G)$
  - All variance due to G is removed

- Algorithm in Dingeç and Hörmann (2013)
  - New CV + CMC + additional CVs
  - Larger VRF than the classical CV
  - Special to GBM

Non-Gaussian models

- Under GBM, log-returns are iid normals
- Observed facts
  - Non-normality of log-returns,
    Higher kurtosis, heavier tails than normal
  - Volatility clustering
    - Large *absolute* log-returns are followed by large *absolute* log-returns
    - Non-linear dependency between log-returns
- Alternative models to GBM
  - Lévy process, (i.i.d. log-returns)
  - Stochastic volatility models
  - Regime switching models

A unified framework for non-Gaussian models

- Three models
  - Generalized hyperbolic (GH) Lévy process (Prause, 1999)
  - Heston stochastic volatility (SV) model (Heston, 1993)
  - Regime switching (RS) model (Hardy, 2001)

- A unified framework
  - Stock price process $S(t) = S(0)e^{X(t)}$
  - Log-returns: $\Delta X_i = X(t_i) - X(t_{i-1})$

Unified Framework

- The unified representation

$$\Delta X_i = \Gamma_i + \Lambda_i Z_i, \quad i = 1, \ldots, d,$$

  - Γ_i, Λ_i's are modulated by stochastic process $\{V(t), t \geq 0\}$
  - $\Gamma = f_m(V)$ and $\Lambda = f_v(V)$.
  - Z_i's are i.i.d. standard normal variables independent of $V(t)$.
  - $(\Delta X_1, \ldots, \Delta X_d \,|\, V)$ is multivariate normal

- The variance process $V(t)$
  - GH Lévy: GIG process (subordinator)
  - Heston: CIR process
  - Regime switching: Discrete Time Markov Chain (DTMC)

Typical Control Variate Methods for the Unified Framework

- The standard CV approach mentioned in Glasserman (2004)

$$Y_{CV} = P_A - c(\tilde{P}_G - \mathrm{E}[\tilde{P}_G]).$$

  where $\tilde{P}_G$ denote the payoff of the geometric average option under GBM
    - Using common Z to introduce correlation

- A more elaborate CV approach by Zhang (2011)

$$Y_{CV} = P_A - c(P_G - \mathrm{E}[P_G|V]).$$

- They do not reduce the variance due to V

New Control Variate Method

- CV of Dingeç and Hörmann (2013) : $W = (A - K)\mathbf{1}_{\{G>K\}}$
- It will reduce the variance coming from both random variables (Z and V)
- Evaluation of the expectation $\mu_W = \mathrm{E}[W]$
  - Lemmens et al. (2010) use μ_W as lower bound for the price under Lévy processes
  - Our new observation:
    Formulas of Lemmens et al. (2010) can be used for any model allowing the computation of joint characteristic function (JCF) of the log-return vector $\Delta X = (\Delta X_1, \ldots, \Delta X_d)$.

Expectation of the CV

- Formula of Lemmens et al. (2010) (after simplifications)

$$\mu_W = \frac{g(0)}{2} - \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega L}\mathbf{g}(\omega) - g(0)}{\omega} d\omega,$$

- $$\mathbf{g}(\omega) = \frac{S(0)}{d}\sum_{j=1}^{d} \varphi_{\bar{X},X_j}(\omega,-i) - K\varphi_{\bar{X}}(\omega).$$

  $\varphi_{\bar{X}}(\omega)$: CF of $\bar{X} = \sum_{j=1}^{d} X(t_j)/d$
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- Both CFs can be evaluated, if $\varphi_{\Delta X}(u) = \mathrm{E}\left[e^{iu^T\Delta X}\right]$ is available

Expectation of the CV

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Expectation of the CV

- Formulas for JCF: $\varphi_{\Delta X}(u) = \mathrm{E}\left[e^{iu^T \Delta X}\right]$
- Lévy process: only requires CF of i.i.d. increment
- Heston model: given by Rockinger and Semenova (2005) for affine jump diffusion models
- Regime switching model: it is possible to derive a simple recursion

Improving the CV Method by CMC

- By setting $c = 1$, we get

$$Y_{CV} = Y + \mu_W \qquad \text{with} \qquad Y = (A-K)^+ \mathbf{1}_{\{G \le K\}}.$$

- Conditional Monte Carlo (CMC) for $Y = (A-K)^+ \mathbf{1}_{\{G \le K\}}$
  - Simulation output as a function of two random inputs $Y = q(V, Z)$
  - The idea: Simulation of standard multinormal vector Z in a specific direction $\vartheta \in \Re^d$, $||\vartheta|| = 1$ by the formula

$$Z = \vartheta \Xi + (I_d - \vartheta \vartheta^T) Z', \quad \Xi \sim N(0,1), \quad Z' \sim N(0, I_d), \qquad (1)$$

where I_d is $d \times d$ identity matrix.

  - Select the direction depending on V,

$$\vartheta_i(V) = \frac{(d-i+1)\Lambda_i}{\sqrt{\sum_{j=1}^{d} (d-j+1)^2 \Lambda_j^2}}, \quad i = 1, \dots, d, \qquad (2)$$

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    where $\Lambda = f_v(V)$.

Conditional Monte Carlo

- $Y = q(V, \Xi, Z')$
- Use $\mathrm{E}[Y|Z', V]$ as an estimator.

$$\mathrm{E}[Y|Z' = z, V = v] = \frac{1}{d}\sum_{i=1}^{d} s_i(z,v)\, e^{a_i(v)^2/2}\left[\Phi(k(v) - a_i(v)) - \Phi(b(z,v) - a_i(v))\right]$$
$$- K\left[\Phi(k(v)) - \Phi(b(z,v))\right].$$

  - $\Phi()$: CDF of std. normal dist.
  - $b(z,v)$ is the root of equation, $A(x) - K = 0$, which is found by Newton's method

Algorithm

```
Compute μ_W
for i = 1 to n do
    Simulate a variance path V
    Simulate Z' ~ N(0, I_d)
    Compute E[Y|Z', V]
    Set Y_i ← e^{-rT}(E[Y|Z', V] + μ_W)
end for
return Ȳ and the error bound Φ^{-1}(1 − α/2) s/√n.
```

- Up to 10 times slower than naive simulation

Algorithm

Compute μ_W
for $i = 1$ to n **do**
  Simulate a variance path V
  Simulate $Z' \sim N(0, I_d)$
  Compute $\mathrm{E}[Y|Z', V]$
  Set $Y_i \leftarrow e^{-rT}(\mathrm{E}[Y|Z', V] + \mu_W)$
end for
return $\bar{Y}$ and the error bound $\Phi^{-1}(1-\alpha/2)\ s/\sqrt{n}$.

- Up to 10 times slower than naive simulation

Algorithm

```
Compute $\mu_W$
for $i = 1$ to $n$ do
    Simulate a variance path $V$
    Simulate $Z' \sim N(0, I_d)$
    Compute $\mathrm{E}[Y|Z', V]$
    Set $Y_i \leftarrow e^{-rT}(\mathrm{E}[Y|Z', V] + \mu_W)$
end for
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for $i = 1$ to n **do**
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Algorithm

1: Compute μ_W
2: **for** $i = 1$ to n **do**
3: Simulate a variance path V
4: Simulate $Z' \sim N(0, I_d)$
5: Compute $\mathrm{E}[Y|Z', V]$
6: Set $Y_i \leftarrow e^{-rT}(\mathrm{E}[Y|Z', V] + \mu_W)$
7: **end for**
8: **return** $\bar{Y}$ and the error bound $\Phi^{-1}(1-\alpha/2)\ s/\sqrt{n}$.

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Numerical Results

Model	T	K	Price	Error	VRF
GH	1	90	12.23708	0.00002	8.8 $\times 10^7$
($\Delta t = 1/250$)		100	4.91175	0.00003	2.3 $\times 10^7$
		110	1.24135	0.00004	2.8 $\times 10^6$
	2	90	14.38544	0.00004	3.0 $\times 10^7$
		100	7.56806	0.00005	1.3 $\times 10^7$
		110	3.26088	0.00011	1.4 $\times 10^6$
Heston SV	1	90	12.49622	0.00010	1.8 $\times 10^6$
($\Delta t = 1/12$)		100	4.52669	0.00003	9.9 $\times 10^6$
		110	0.44090	0.00002	2.9 $\times 10^6$
	2	90	14.57217	0.00024	5.5 $\times 10^5$
		100	7.15495	0.00009	2.6 $\times 10^6$
		110	2.19650	0.00005	2.5 $\times 10^6$
RS	1	90	12.46569	0.00004	1.9 $\times 10^7$
($\Delta t = 1/12$)		100	4.93411	0.00003	1.9 $\times 10^7$
		110	1.24898	0.00005	1.8 $\times 10^6$
	2	90	14.53311	0.00007	9.6 $\times 10^6$
		100	7.49728	0.00006	8.1 $\times 10^6$
		110	3.11583	0.00010	1.7 $\times 10^6$

Table: Variance reduction factors (VRF) compared to naive simulation.
$S(0) = 100, r = 0.05, n = 10^4$

Conclusions

- A new efficient simulation method developed for Asian option pricing under a general model framework
  - GH Lévy process
  - Heston stochastic volatility model
  - Discrete-time regime switching model
- Combination of CV and CMC
  - CV is applicable to all models in which the numerical computation of JCF of the log-return vector is possible
  - CMC is applicable to all models having normal mean-variance mixture representation
- Numerical results show significant variance reduction compared to naive simulation

References I

Curran, M., 1994. Valuing Asian and portfolio options by conditioning on the geometric mean price. Management Science 40 (12), 1705–1711.

Derflinger, G., Hörmann, W., Leydold, J., 2010. Random variate generation by numerical inversion when only the density is known. ACM Transactions on Modeling and Computer Simulation 20 (18), 1–25.

Dingeç, K. D., Hörmann, W., 2013. Control variates and conditional Monte Carlo for basket and Asian options. Insurance: Mathematics and Economics 52 (3), 421–434.

Glasserman, P., 2004. Monte Carlo Methods in Financial Engineering. Springer-Verlag, New York.

Hardy, M. R., 2001. A regime-switching model of long-term stock returns. North American Actuarial Journal 5 (2), 41–53.

Heston, S. L., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Fiancial Studies 6 (2), 327–343.

Kemna, A., Vorst, A., 1990. A pricing method for options based on average asset values. Journal of Banking and Finance 14 (1), 113–130.

Lemmens, D., Liang, L., Tempere, J., Schepper, A. D., 2010. Pricing bounds for discrete arithmetic Asian options under Lévy models. Physica A 389 (22), 5193–5207.

References II

Lord, R., 2006. Partially exact and bounded approximations for arithmetic Asian options. Journal of Computational Finance 10 (2), 1–52.

Öhgren, A., 2001. A remark on the pricing of discrete lookback options. Journal of Computational Finance 4 (3), 141–147.

Prause, K., 1999. The generalized hyperbolic model: Estimation, financial derivatives, and risk measures. Ph.D. thesis, University of Freiburg.

Rockinger, M., Semenova, M., 2005. Estimation of jump-diffusion processes via empirical characteristic functions. Tech. rep., Université De Genéve.

Shreve, S., 2004. Stochastic Calculus for Finance II: Continuous Time Models. Springer, New York.

Večeř, J., 2001. A new PDE approach for pricing arithmetic average Asian options. The Journal of Computational Finance 4 (4), 105– 113.

Zhang, K., 2011. Monte carlo methods in derivative modelling. Ph.D. thesis, University of Warwick.

Thank You