(Approximate) Graph Products

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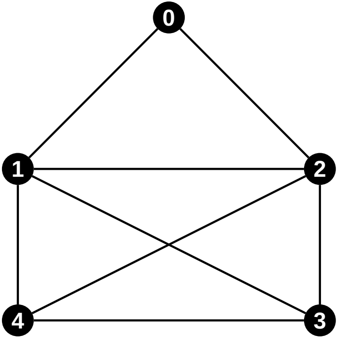
WU WIEN - OCTOBER 14, 2009

Basics

A graph is a pair $G = (V, E)$ with vertex set $V \neq \emptyset$ and edge set E.

here: simple, connected, undirected graphs



Strong and Cartesian Product

The vertex set of the cartesian product ($\Box$) and strong product ($\boxtimes$) is defined as follows:

$$V(G_1 \Box G_2) = V(G_1 \boxtimes G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$$

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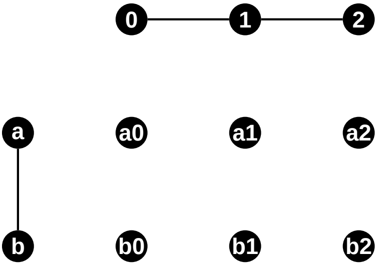
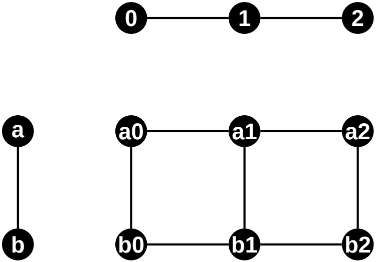


Cartesian Product

Two vertices (x_1,x_2), (y_1,y_2) are adjacent in $G_1 \square G_2$ if

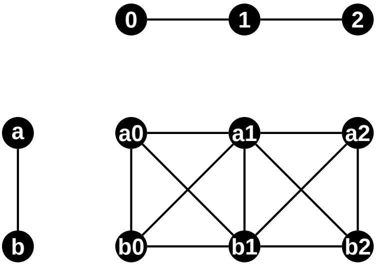
1. $(x_1,y_1) \in E(G_1)$ and $x_2 = y_2$ or if
2. $(x_2,y_2) \in E(G_2)$ and $x_1 = y_1$.



Strong Product

Two vertices (x_1, x_2), (y_1, y_2) are adjacent in $G_1 \boxtimes G_2$ if

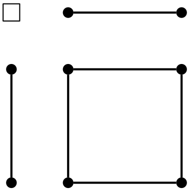
1. $(x_1, y_1) \in E(G_1)$ and $x_2 = y_2$ or if
2. $(x_2, y_2) \in E(G_2)$ and $x_1 = y_1$ or if
3. $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$.

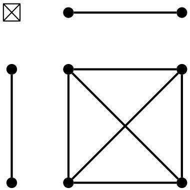


Properties of $\Box$, $\boxtimes$

- commutative
- associative
- unit K_1

Examples





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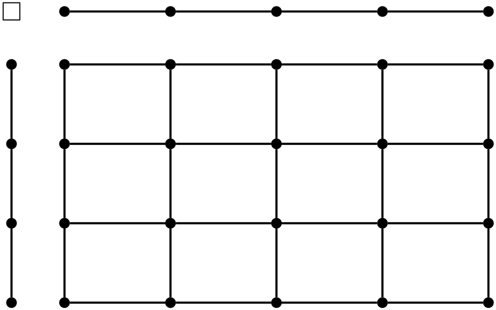

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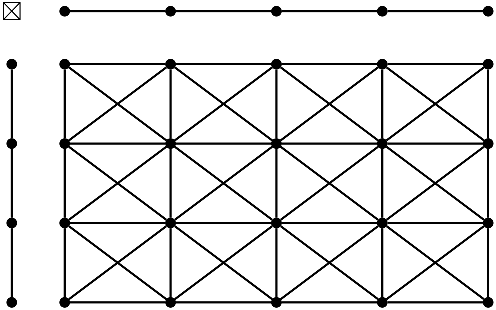
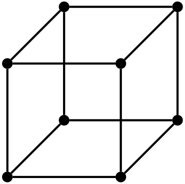


Examples

$\square_{i=1}^{n} K_2 = Q_n$ Hypercube of dimension n



Examples

$$\boxtimes_{i=1}^{n} K_{l_i} = K_m \text{ with } m = \prod_{i=1}^{n} l_i$$

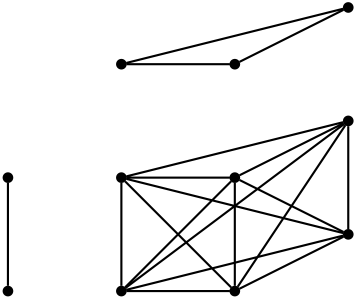


Figure: $K_3 \boxtimes K_2 = K_6$

Motivation

Invariants

Invariants of Factors and invariants of the corresponding Product coincide or give at least an idea or an estimation to each other, e.g.:

- chromatic number:

$$\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$$

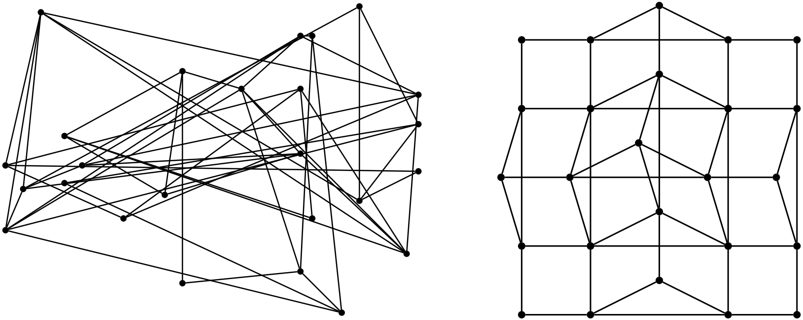
- independent number:

$$\alpha(G) \cdot \alpha(H) \leq \alpha(G \Box H) \leq \min\{\alpha(G) \cdot |V(H)|, \alpha(H) \cdot |V(G)|\}$$

- The Laplacian eigenvalues of $G \Box H$ are equal to all possible sums of eigenvalues of the factors:

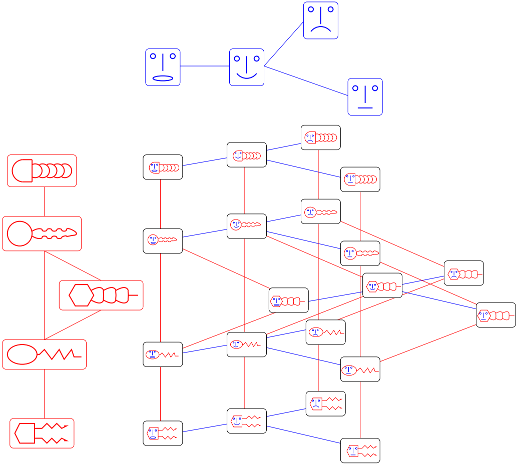
$$\lambda_i(G) + \lambda_j(H),\ i = 1, \dots, |V(G)|,\ \ j = 1, \dots, |V(H)|$$

Visualization : **TopoLayout**



Two isomorphic product graphs.

A Topological Theory of Characters [1]



Characters

Idea:

Characters **can** vary independently

$\Longleftrightarrow$

Factors of pheno-type space

$\Longleftrightarrow$

Factors of graph product

[1] Quasi-Independence, Homology and the Unity of Type: A Topological Theory of Characters, Günter Wagner and Peter F. Stadler, J. theor. Biol., 2003

Problem:

Often real data, that is represented by graphs, is disturbed

and thus the corresponding "product graph" is disturbed.

- How can we recognize original factors of disturbed products?
- How can we recognize at least some parts of a disturbed product as a product?

Decomposition

Definition

G is prime, if $\nexists A * B = G$ with A, B nontrivial, i.e. $|V(A)|, |V(B)| > 1$.
$(* = \Box, \boxtimes)$

Aim: Prime factor decomposition (PFD) of given G.

Prime Factor Decomposition

Theorem (Sabidussi 1959)

PFD of every connected graph w.r.t. the Cartesian product is unique.

Theorem (Dörfler and Imrich (1969))

PFD of every connected graph w.r.t. the strong product is unique.

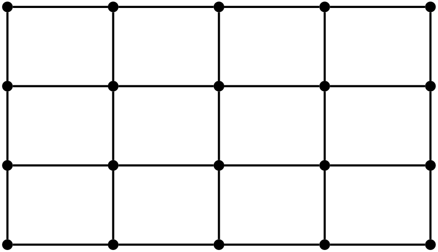
Theorem (Imrich and Peterin (2007))

PFD of every connected graph w.r.t. the Cartesian product can be computed in linear time.

Theorem (Feigenbaum and Schäffer (1992))

PFD of every connected graph w.r.t. the strong product can be computed in $O(n^5)$ time.

Decomposition of Cartesian Product



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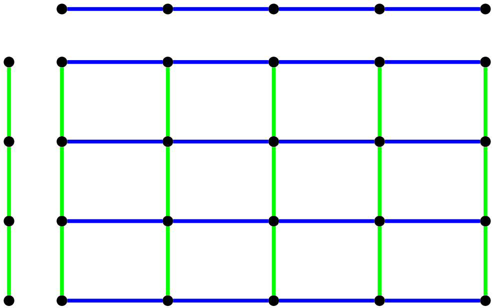


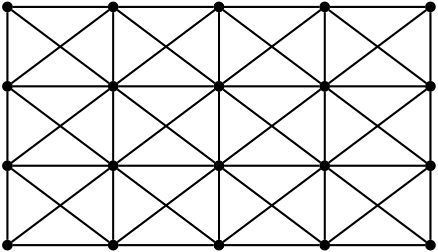
Copies of Factors in Product are called layer or fiber.

MAIN IDEA: Decomposition strong product

Find a spanning subgraph with special properties in G, the so called cartesian skeleton.

The decomposition of the cartesian skeleton w.r.t. cartesian product together with some additional operations leads to the possible factors of the strong product.

MAIN IDEA: Decomposition strong product



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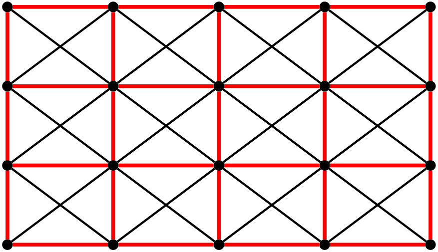

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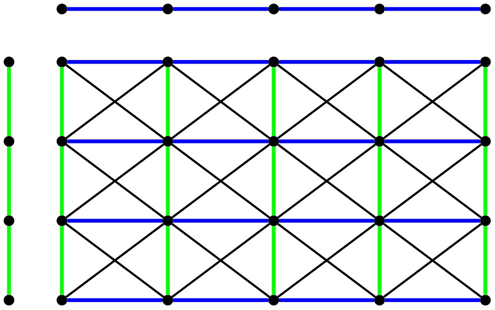
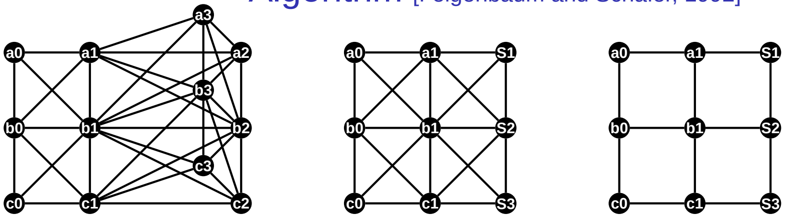
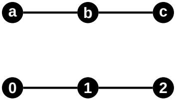
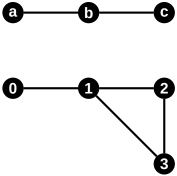


Algorithm [Feigenbaum and Schäfer, 1992]



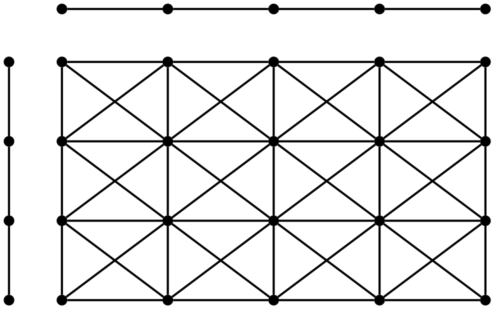
G $\longrightarrow$ G/S $\longrightarrow$ CartSkeleton(G/S)





$\longrightarrow$ Factors of CartSkeleton(G/S) $\longrightarrow$ Factors of G

What, if prime?



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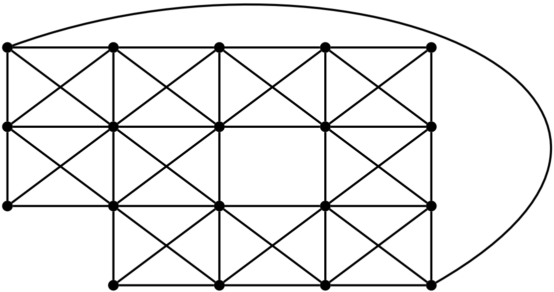
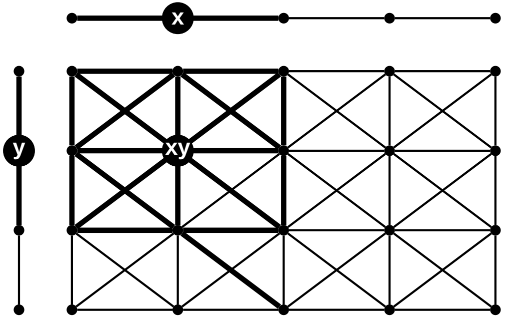


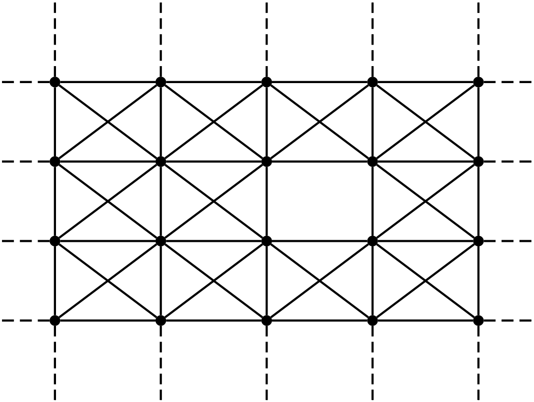
What, if prime?

Aim: Get a product of graphs that is "near" a given prime graph (approximate products).

Remark: Induced neighborhoods in products are products.



IDEA: Approximate Products



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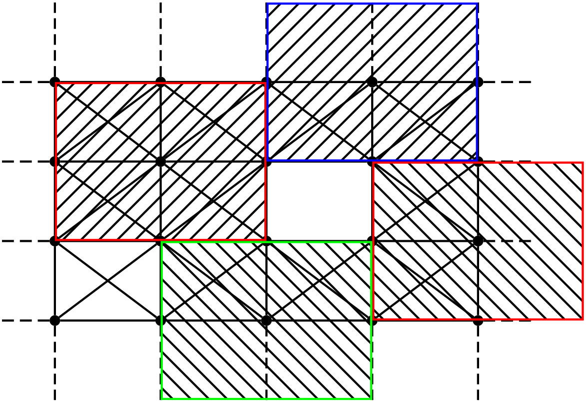

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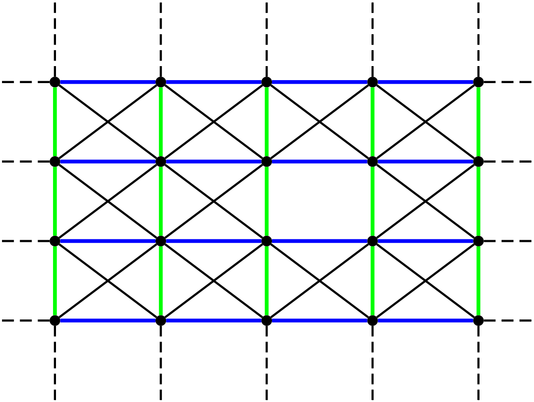


Tools

1. **`S=1-condition`**
2. Backbone $\mathbb{B}(G)$

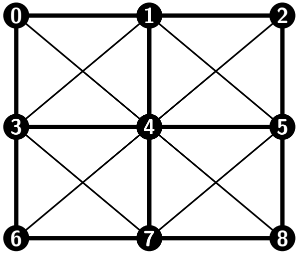
S=1-condition

Let G be a graph and $v, w \in V(G)$.

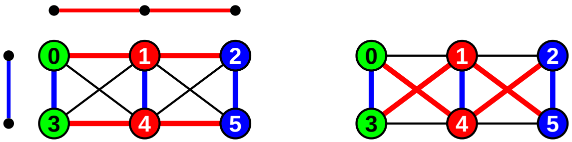
- v, w are in Relation S if N[v] = N[w]
- We call a graph S-thin if no vertices have the same closed neighborhood.

If G is S-thin the Cartesian edges are uniquely determined

S=1-condition



WHAT ARE THE FIBERS ?



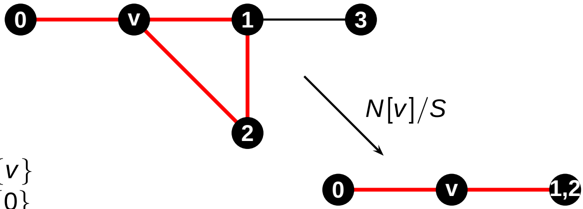
S=1-condition

S class in H that contains x denoted by:

$$S_H(x) = \{y \in V(H) \ : \ N[y] \cap V(H) = N[x] \cap V(H)\}$$

S class in $H = \langle N[v] \rangle$ that contains x denoted by:

$$S_v(x) = \{y \in N[v] \ : \ N[y] \cap N[v] = N[x] \cap N[v]\}$$



$S_v(v) = \{v\}$
$S_v(0) = \{0\}$
$S_v(1) = S_v(2) = \{1,2\}$

S=1-condition

$S_v(x) = \{y \in N[v] : N[y] \cap N[v] = N[x] \cap N[v]\}$ is the S class in $\langle N[v] \rangle$ that contains x.

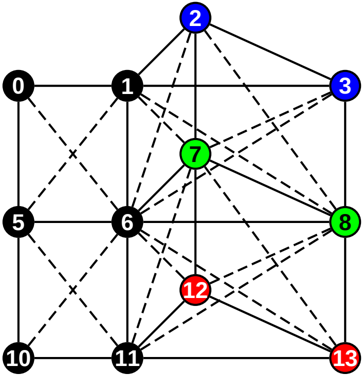
Lemma (S=1-condition)

Let $v \in V(G)$ be a vertex in a thin strong product graph G. Furthermore let $\langle N(v) \rangle$ contain two different S-classes $S_v(x)$ and $S_v(y)$ s.t.

1. $|S_v(x)| = 1$ *or* $|S_v(y)| = 1$ *and*
2. $(S_v(x), S_v(y))$ *is a Cartesian edge* $\langle N(v) \rangle / S$.

Then all edges in $\langle N(v) \rangle$ induced by the vertices of $S_v(x)$ and $S_v(y)$ are Cartesian and all are edges of a copy of the same factor.

S=1-condition



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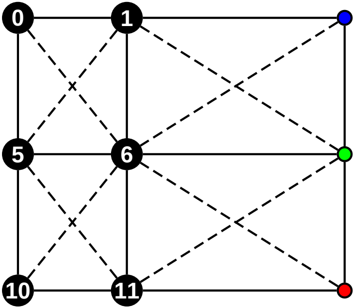


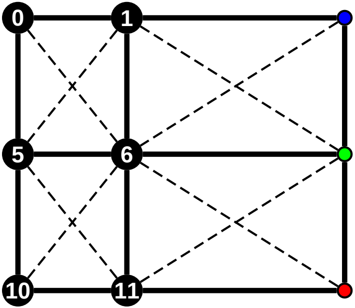
$S_6(3) = \{2, 3\}$

$S_6(8) = \{7, 8\}$

$S_6(13) = \{12, 13\}$

$|S_6(x)| = 1,\ x = 0, 1, 5, 6, 10, 11$

S=1-condition



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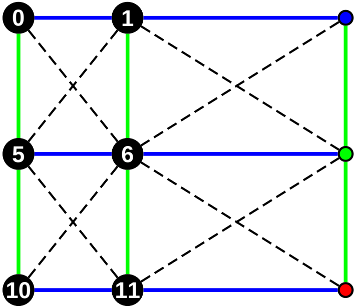

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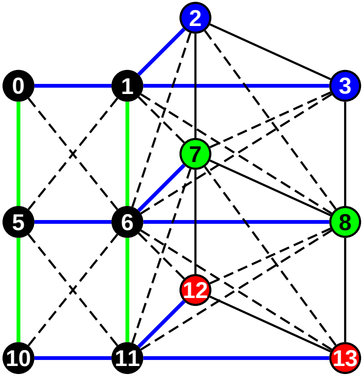
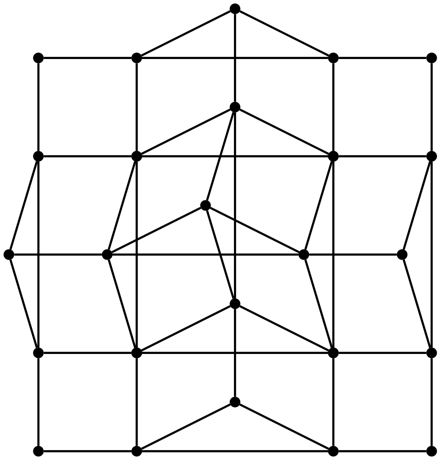


We call those **`S=1-condition`** satisfying Cartesian edges identificable.

Theorem

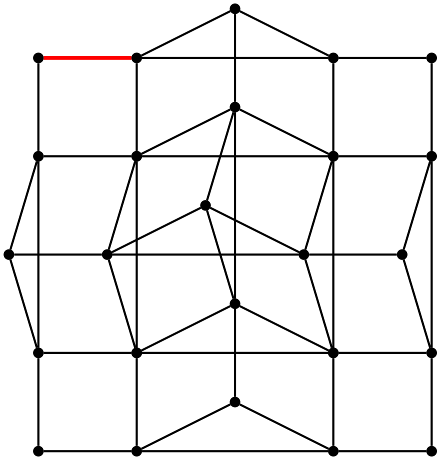
Whenever one edge of some G_i^x-fiber satisfies the ***S=1-condition (identificable)*** *then all edges of this G_i^x-fiber can be determined as Cartesian in an unique way.*

Example



Depicted is the Cartesian Skeleton of 2 times Factor "Triangle with Feets".

Example



one edge of some layer satisfies the **S=1-condition** ⇒
all edges of this layer can be determined as Cartesian.

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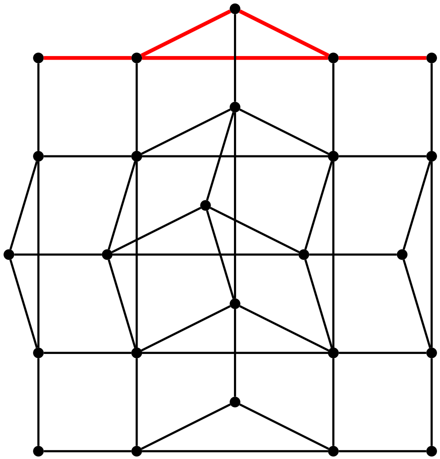

one edge of some layer satisfies the **S=1-condition** $\Rightarrow$
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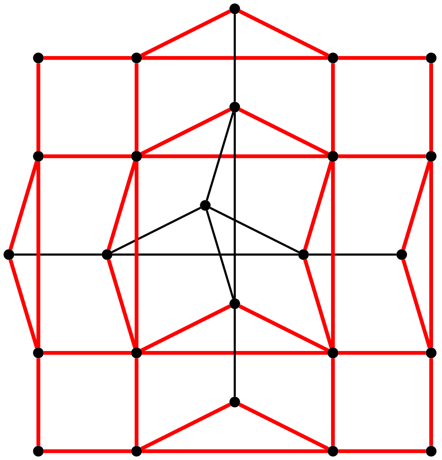
Example

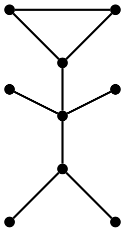


The Backbone $\mathbb{B}(G)$

$$\mathbb{B}(G) := \{v \in V(G) \mid |S_v(v)| = 1\}$$
$$:= \{v \in V(G) \mid N[v] \text{ is strictly maximal in } G\}$$

Theorem

$\mathbb{B}(G)$ is a connected dominating set.



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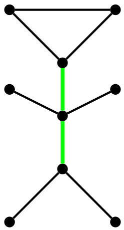

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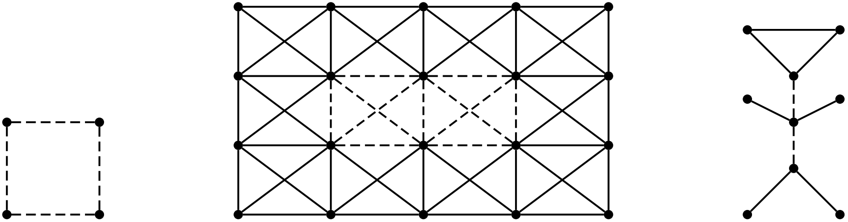
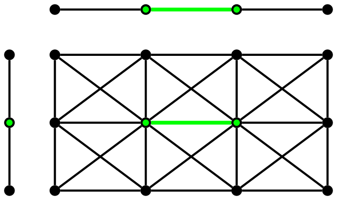


Figure: Examples

The Backbone $\mathbb{B}(G)$

For a local covering we consider neighborhoods of vertices of $\mathbb{B}(G)$ only.

Why?



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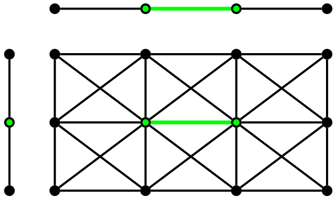
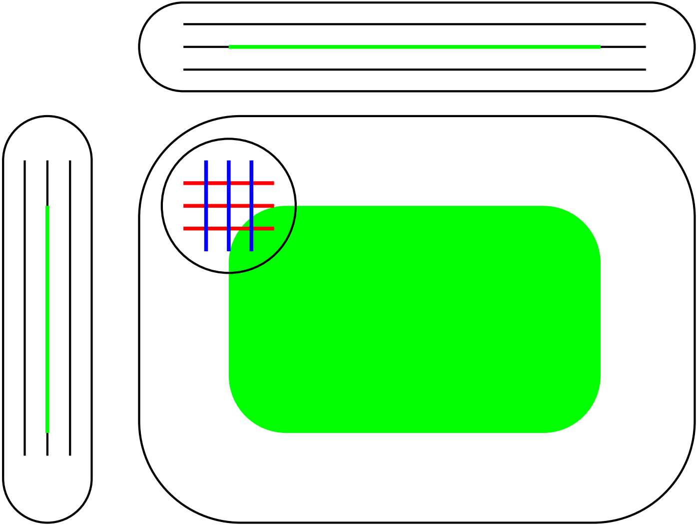


Theorem

All locally identificable edges of fibers are identificable in induced neighborhoods $\langle N[v] \rangle$ with $v \in \mathbb{B}(G)$.

Local Approach

Step 1



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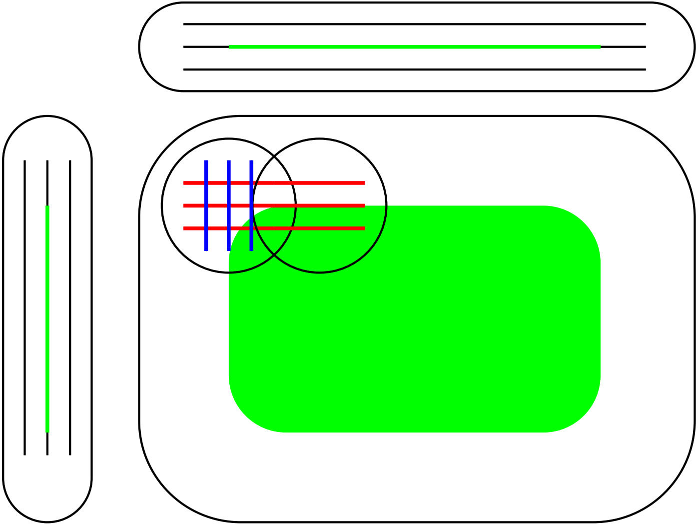

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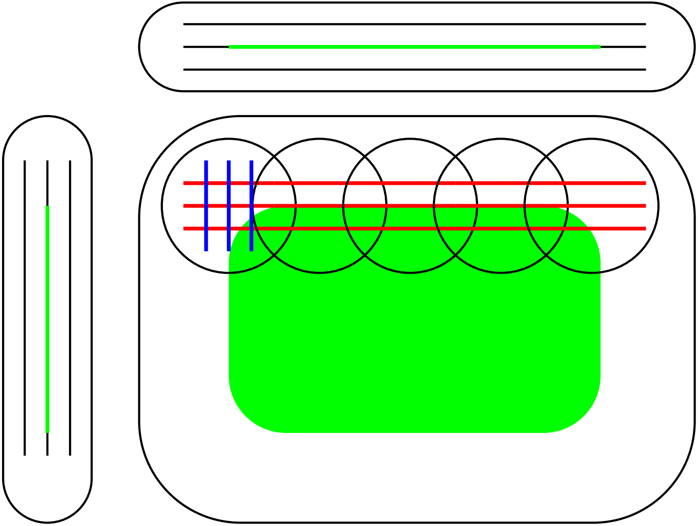

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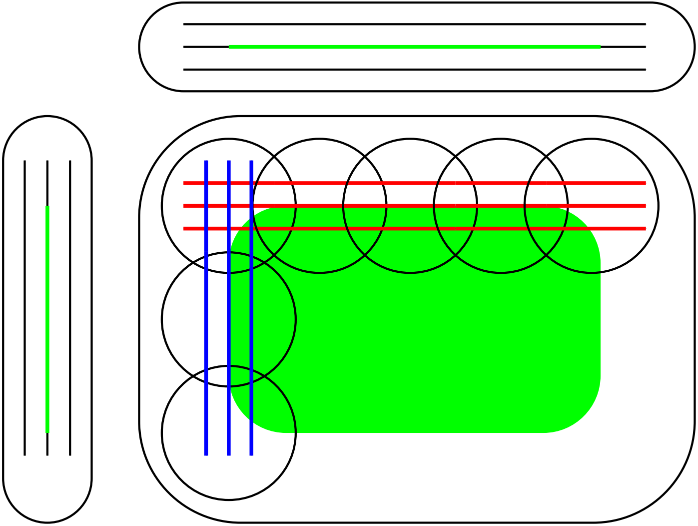

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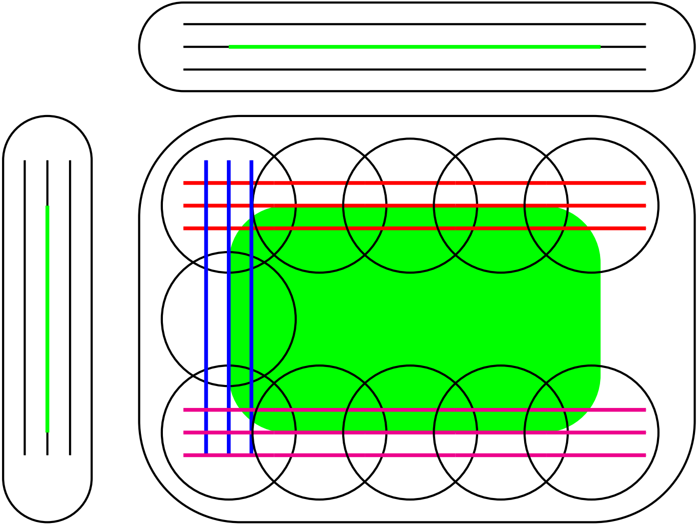

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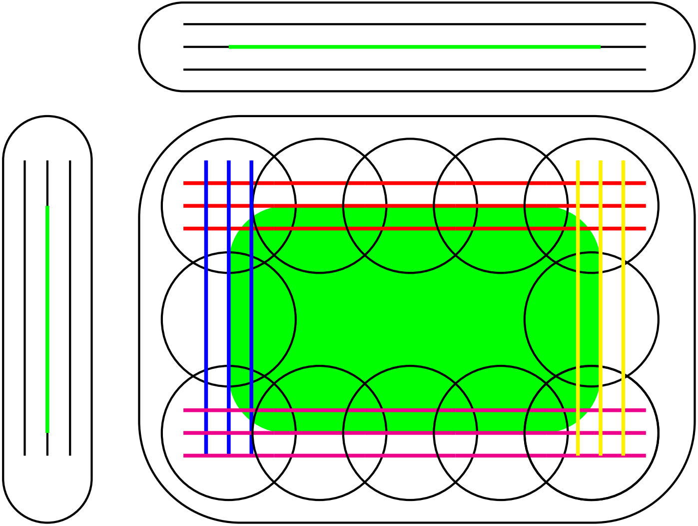
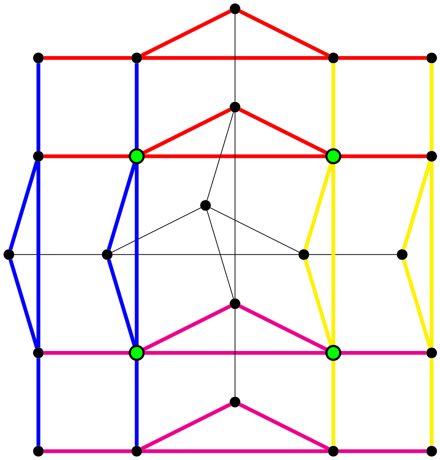


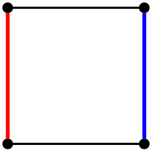
Example



Step 2 : Square Property

Lemma

Any two opposite edges of a chordless square that consists of Cartesian edges only, are edges of copies of one and the same factor.



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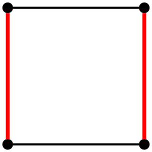
Lemma

Any two opposite edges of a chordless square that consists of Cartesian edges only, are edges of copies of one and the same factor.

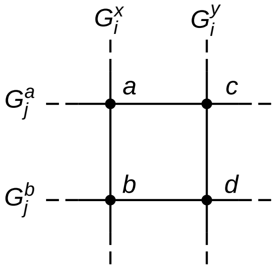


Step 2 : Square Property

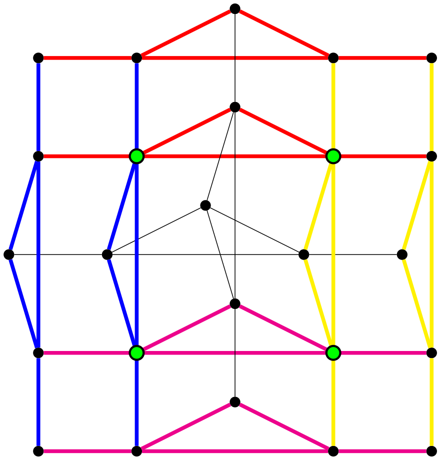
Theorem

All identified G_i-fibers are connected by a path of Cartesian edges.

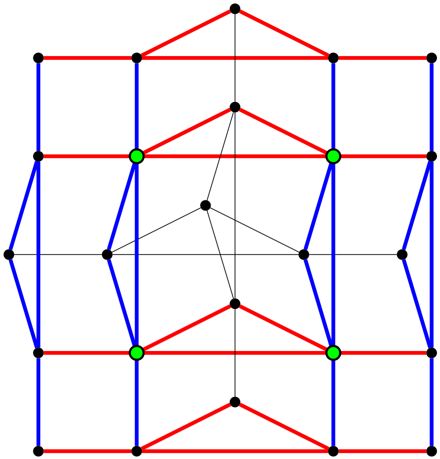
Let G_i^x and G_i^y be different identified G_i-fibers that are connected by a Cartesian edge of some G_j-fiber.
Then there is (chordless) square (a,b,c,d) consisting of Cartesian edges only, such that $(a,b) \in E(G_i^x)$ and $(c,d) \in E(G_i^y)$ and $(a,c) \in G_j^a$ and $(b,d) \in G_j^b$ with G_j^a, G_j^b are identified via the approach above.



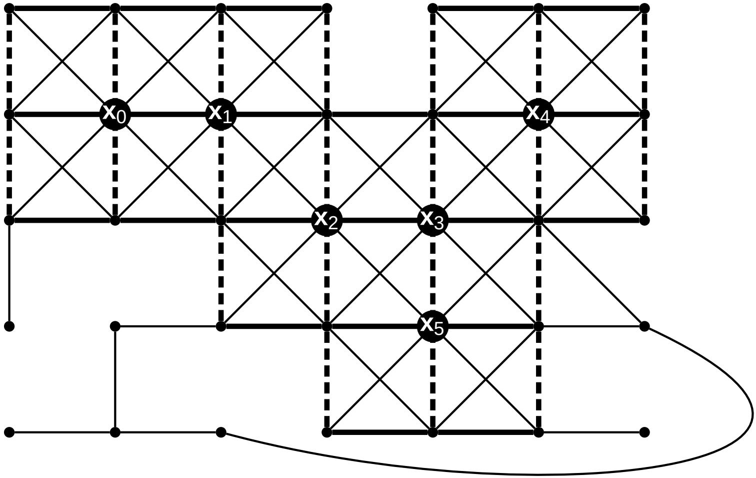
Example



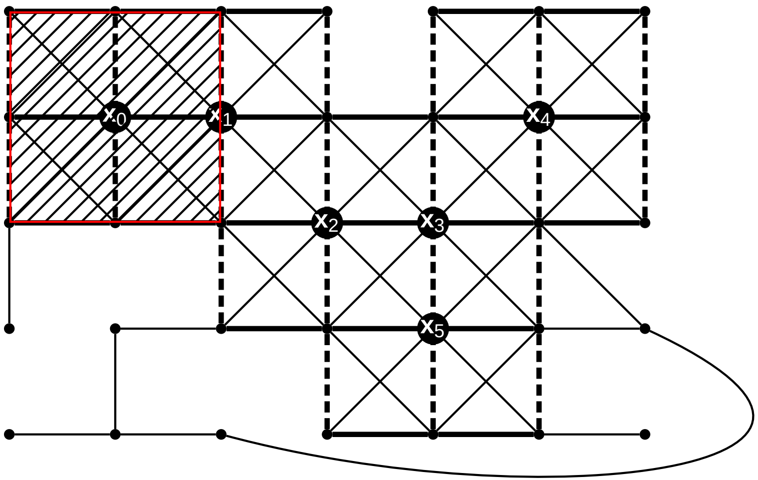
Example



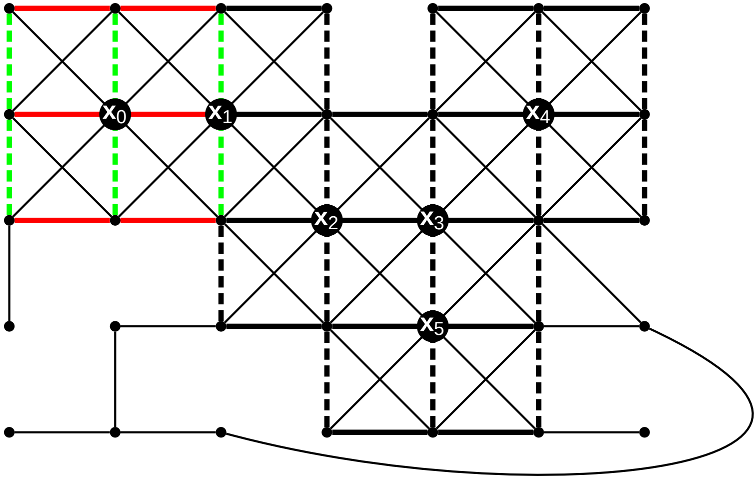
Example



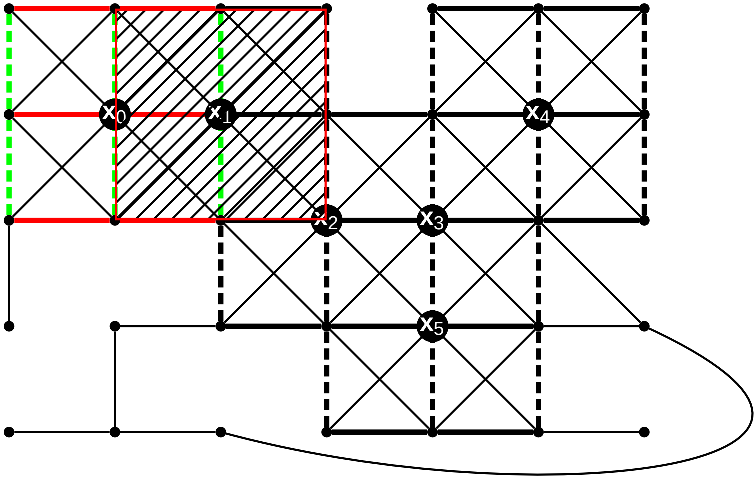
Example



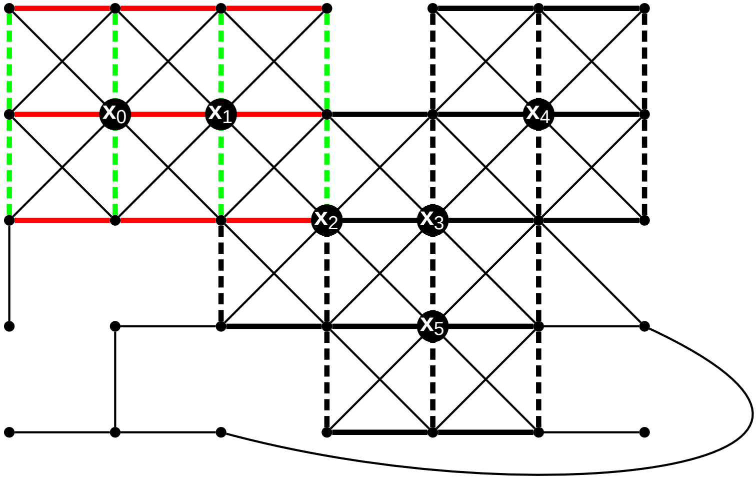
Example



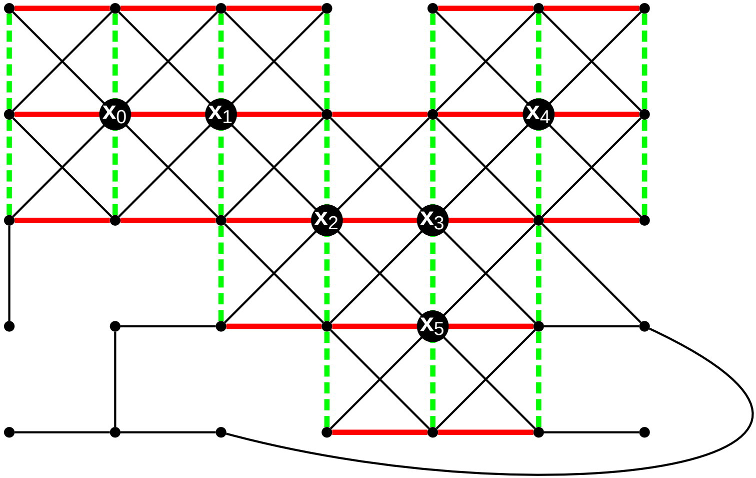
Example



Example



Example



Definition

$$\Gamma := \{G \mid (G \text{ locally unrefined }) \vee (G \text{ can be covered by thin N only})\}$$

Theorem

Let $G = (V, E) = \boxtimes_{j=1}^{n} G_j \in \Gamma$ with bounded maximum degree Δ. Then the prime factors of G can be determined in $O(|V|^2 \cdot \Delta^{10})$ time.

Download

http://www.bioinf.uni-leipzig.de/˜marc/download.html

Thanks to Peter F. Stadler, Wilfried Imrich, Werner Klöckl and Josef Leydold!

Thank you for your attention!