Scenario-Free Stochastic Programming

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Outline

1. Deterministic Optimisation
2. Stochastic Programming
3. Scenario-Based Stochastic Programming
4. Scenario-Free Stochastic Programming
   - Linear Decision Rules
   - Bounding the Optimality Gap
   - Piecewise Linear Decision Rules
5. Case Study: Capacity Expansion in Power Systems

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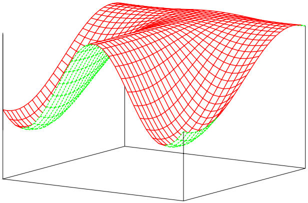
Deterministic Optimisation

Deterministic optimisation model:

$$\begin{array}{ll} \underset{x}{\text{minimise}} & f(x,\xi) \\ \text{subject to} & g(x,\xi) \le 0, \end{array}$$



where

- x are *decision variables*
- ξ are (precisely known) *parameters*

Real world is *uncertain*. Why not use $\xi = \mathbb{E}[\,\tilde{\xi}\,]$?

Deterministic Optimisation

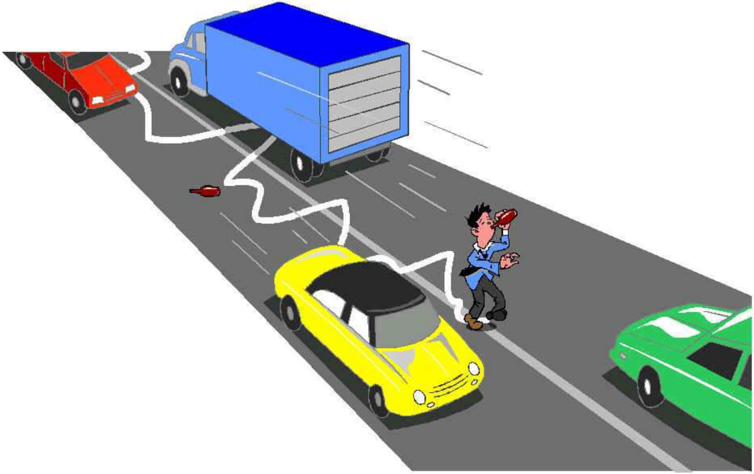
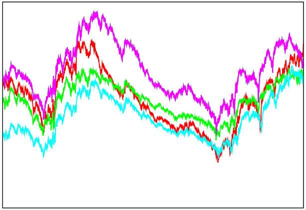


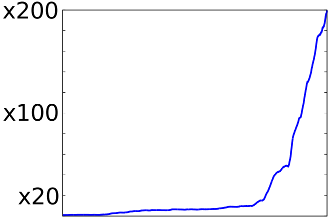
Image Source: Sam L. Savage, Stanford University

$\text{alive}(\mathbb{E}[\text{position}]) = \text{true}$, but $\mathbb{E}[\text{alive}(\text{position})] = \text{false}$!

Deterministic Optimisation

Portfolio optimisation:





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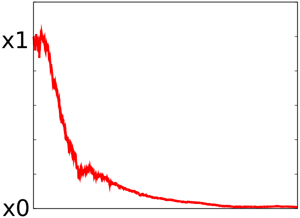

$\text{wealth}\left(\mathbb{E}\left[\text{stock returns}\right]\right)$ vs. $\mathbb{E}\left[\text{wealth}\left(\text{stock returns}\right)\right]$

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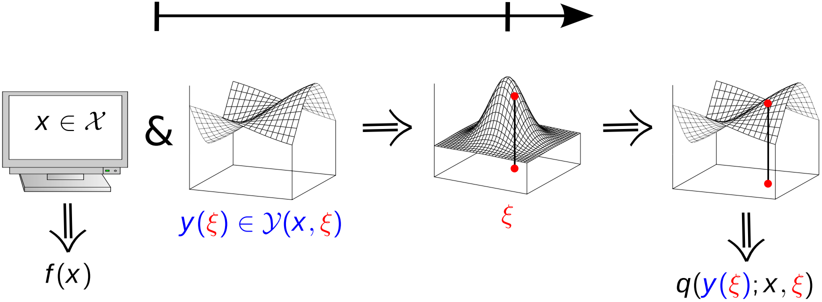
Stochastic Programming

Two-stage stochastic program:

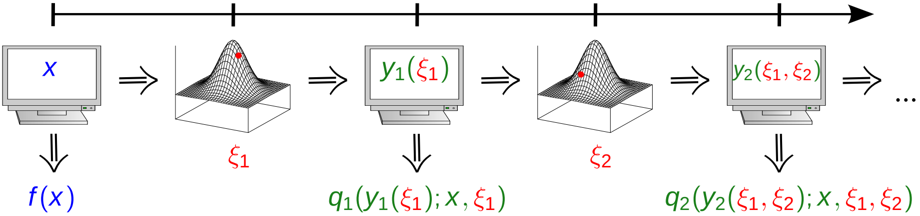
$$
\begin{aligned}
\underset{x,y}{\text{minimise}} \quad & f(x) + \mathbb{E}\left[q(y(\xi); x, \xi)\right] \\
\text{subject to} \quad & x \in \mathcal{X}, \\
& y(\xi) \in \mathcal{Y}(x, \xi) \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$



Stochastic Programming

Multi-stage stochastic program: *several* recourse decisions

- capacity expansion: several investment stages
- production planning: annual production plan (seasonalities)
- portfolio optimisation: rebalancing, asset & liability mgmt.
- ...



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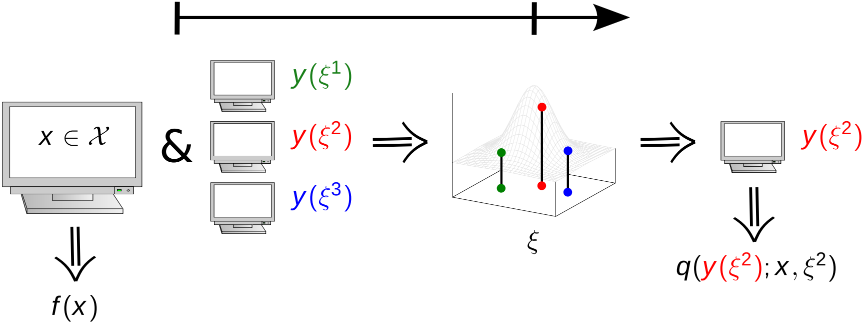
Scenario-Based S.P.

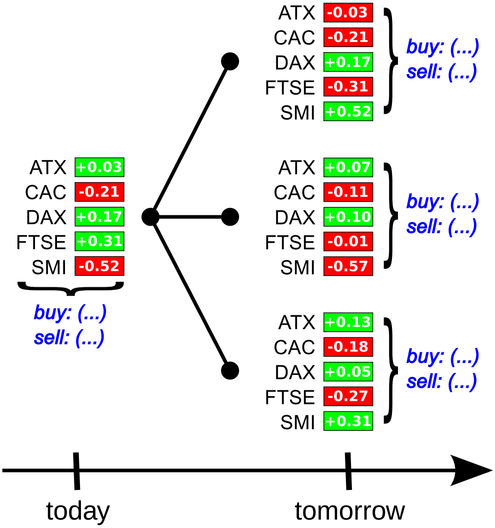
Discretise distribution: into *scenarios*

$$\begin{aligned} \underset{x,y}{\text{minimise}} \quad & f(x) + \sum_{s \in \mathcal{S}} p_s \, q(y^s; x, \xi^s) \\ \text{subject to} \quad & x \in \mathcal{X}, \\ & y^s \in \mathcal{Y}(x, \xi^s) \quad \forall s \in \mathcal{S}. \end{aligned}$$



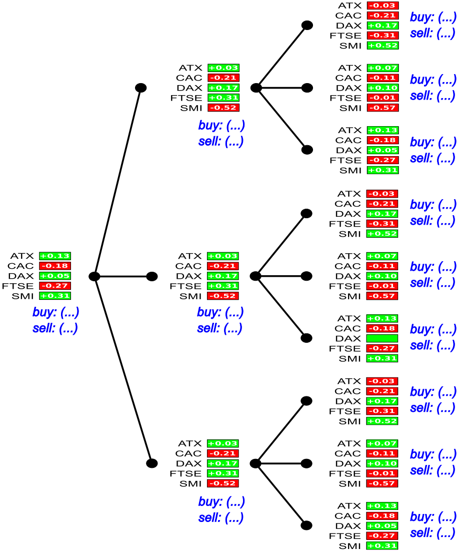
Scenario-Based S.P.

Portfolio optimisation:

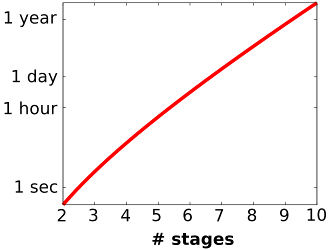


Scenario-Based S.P.

Multi-stage stochastic program:



Solution time:



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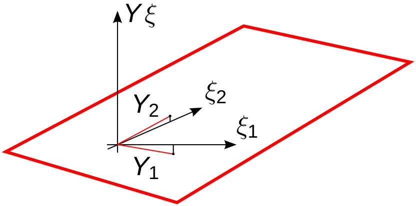
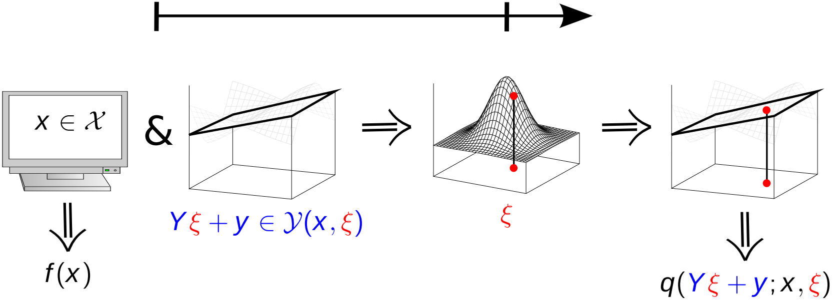
Linear Decision Rules[1]

Step 1: linearise decision rule

$$\begin{array}{ll} \underset{x, Y, y}{\text{minimise}} & f(x) + \mathbb{E}\left[q(Y\xi + y; x, \xi)\right] \\ \text{subject to} & x \in \mathcal{X}, \\ & Y\xi + y \in \mathcal{Y}(x, \xi) \quad \mathbb{P}\text{-a.s.} \end{array}$$



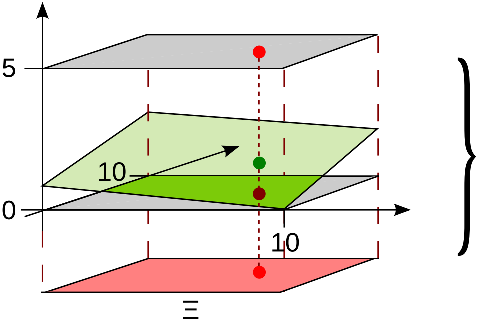


[1] Ben-Tal *et al.*, Math. Programming, 2004.

Linear Decision Rules

Step 2: reformulate semi-infinite $\mathbb{P}$-a.s. constraint

for $\mathcal{Y}(x, \xi) = \{y(\xi) \,:\, y(\xi) \in [0, 5]\}$:



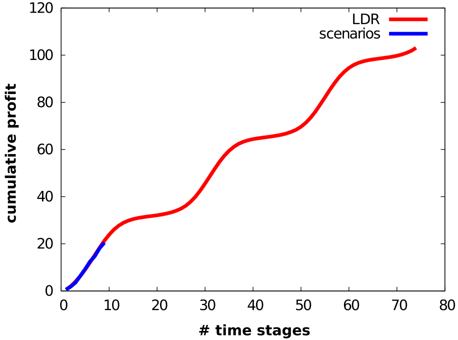
$$\left.\begin{array}{l} y_0 \geq 10\,[-Y_1]^+ + 10\,[-Y_2]^+ \\ y_0 \leq 5 - 10\,[Y_1]^+ - 10\,[Y_2]^+ \end{array}\right.$$

If $\mathcal{Y}$ (conic) convex: can be achieved by duality theory!

Linear Decision Rules

Example problem:

- three factories produce single good, one warehouse
- limited per-period production and storage capacities
- demand uniformly distributed among known nominal demand
- nominal demand seasonal: $d_t = 1{,}000 \times \left(1 + \frac{1}{2}\sin\left[\frac{\pi(t-1)}{12}\right]\right)$



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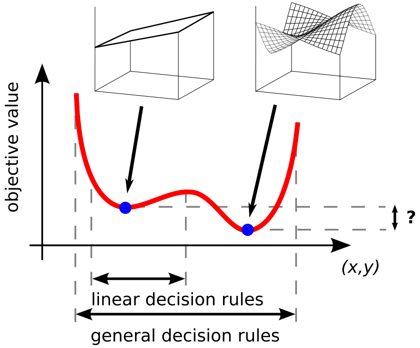
Bounding the Optimality Gap[2]

So far: obtained upper bound on optimal value via restriction from general to linear decision rules

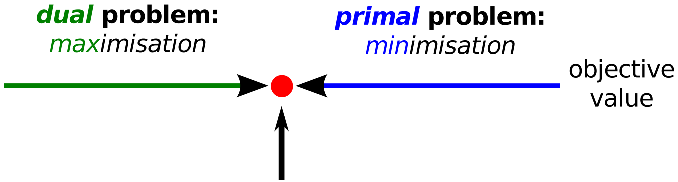


Idea: obtain lower bound on optimal value to bound suboptimality

[2]Kuhn *et al.*, Math. Programming, 2010.

Bounding the Optimality Gap

Step 1: dualise stochastic program with general decision rules

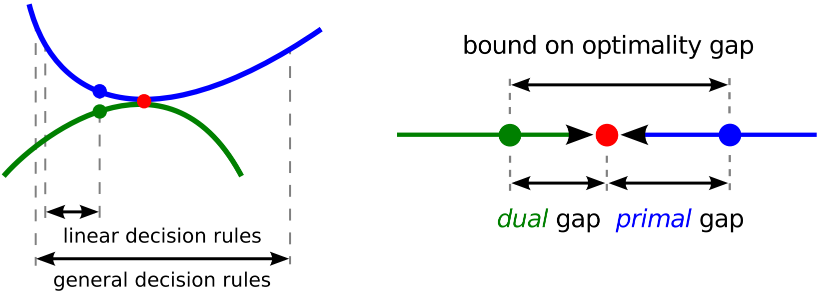


duality theory: under certain regularity conditions,
optimal values of primal and dual problems coincide!

Result: dual stochastic program with general decision rules

Bounding the Optimality Gap

Step 2: restrict dual stochastic program to linear decision rules

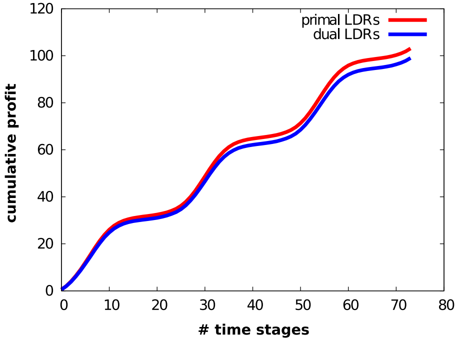


Result: dual stochastic program with linear decision rules allows us to bound incurred suboptimality

Example Problem Revisited

Example problem:

- three factories produce single good, one warehouse
- limited per-period production and storage capacities
- demand uniformly distributed among known nominal demand
- nominal demand seasonal: $d_t = 1{,}000 \times \left(1 + \frac{1}{2}\sin\left[\frac{\pi(t-1)}{12}\right]\right)$



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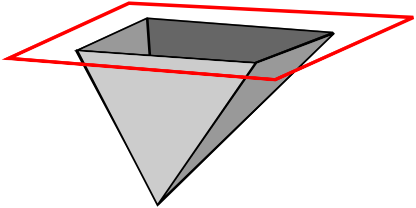
Piecewise Linear Decision Rules[3]

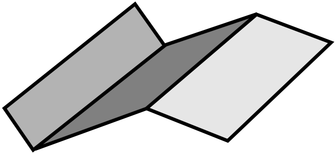
Linear decision rules can fail:

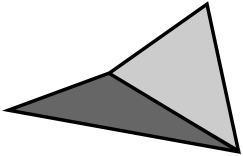
$$\begin{array}{lll} \underset{y}{\text{minimise}} & \mathbb{E}\left[y(\xi)\right] & \\ \text{subject to} & y(\xi) \geq \|\xi\|_1 & \mathbb{P}\text{-a.s.} \end{array}$$

where $\xi \sim \mathcal{U}[-1, 1]^k$.



Remedy: use *piecewise linear* decision rules instead

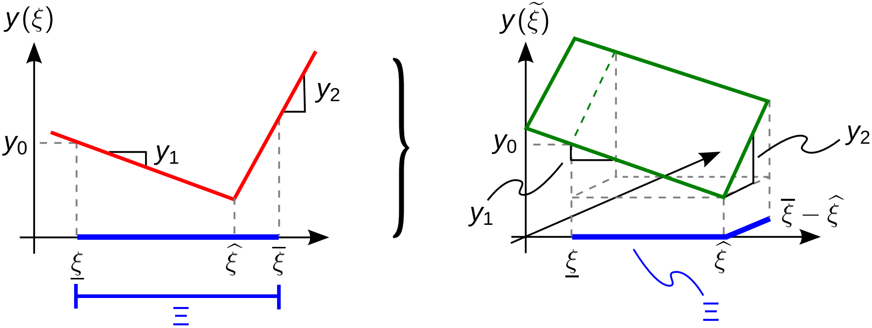




[3]Goh & Sim, Oper. Research, 2010; Georghiou *et al.*, Optimization Online, 2010.

Piecewise Linear Decision Rules

Piecewise linear decision rule $\equiv$ linear decision rule in *lifted* space:



$$y(\xi) = y_0 + y_1 \left(\min\left\{\xi, \widehat{\xi}\right\} - \underline{\xi} \right) + y_2 \left(\max\left\{\xi, \widehat{\xi}\right\} - \widehat{\xi} \right)$$

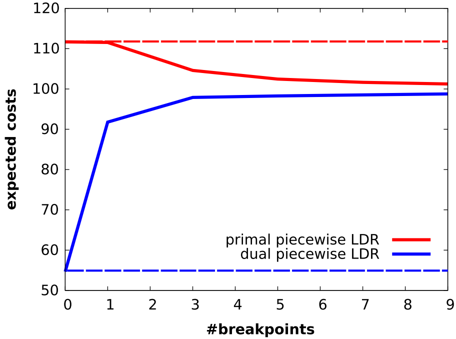
$$y(\widetilde{\xi}) = y_0 + y_1 \widetilde{\xi}_1 + y_2 \widetilde{\xi}_2$$

where $\widetilde{\xi} = \begin{pmatrix} \min\left\{\xi, \widehat{\xi}\right\} - \underline{\xi} \\ \max\left\{\xi, \widehat{\xi}\right\} - \widehat{\xi} \end{pmatrix}$

Piecewise Linear Decision Rules

Example problem: capacity expansion of a power grid

- 10 regions with uncertain demand
- 5 power plants with known capacity, uncertain operating costs
- 24 transmission lines with known capacity
- **goal:** meet demand at lowest expected costs, via
  - capacity expansion plan (here-and-now)
  - plant operating policies (wait-and-see)



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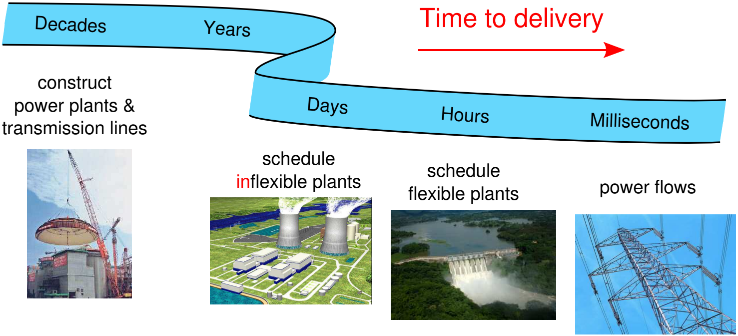
Capacity Expansion in Power Systems

Multiple time scales:

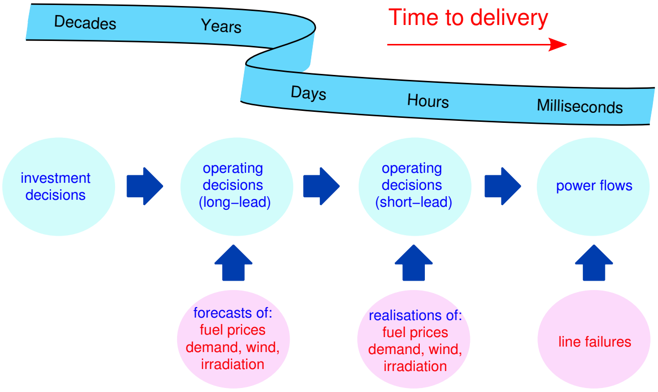


Nonrenewable energy sources: Natural gas, coal-fired
Renewable energy sources: Solar and wind power

Capacity Expansion in Power Systems

Four-stage stochastic program:

Objective: minimise investment costs + expected operating costs
over next 30 years



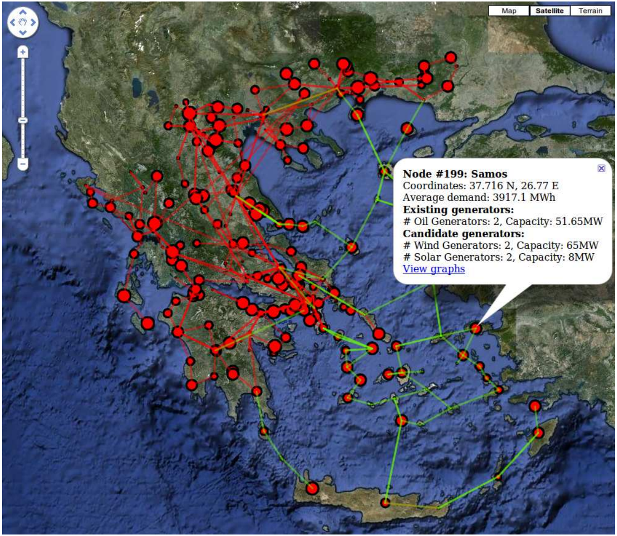
Optimisation Model

$$
\begin{aligned}
&\text{minimise} && \sum_{n\in N_c} c_n u_n + \sum_{m\in M_c} d_m v_m + \mathbb{E}\left(\sum_{n\in N} \gamma_n g_n\right) \\
&\text{subject to}
\end{aligned}
$$

$$
\left.
\begin{array}{ll}
g_n \quad \mathcal{F}_l\text{-measurable} & \forall n \in N_l \\
g_n \quad \mathcal{F}_s\text{-measurable} & \forall n \in N_s \\
f_m \quad \mathcal{F}\text{-measurable} & \forall m \in M \\
u_n \in \{0,1\},\ v_m \in \{0,1\} & \forall n \in N,\ \forall m \in M \\
u_n = 1 & \forall n \in N_e \\
v_m = 1 & \forall m \in M_e \\
0 \le g_n \le \overline{g}_n u_n & \forall n \in N \\
g_n \le \zeta_n & \forall n \in N_r \\
|f_m| \le \varphi_m \overline{f}_m v_m & \forall m \in M \\
\sum_{n\in N(k)} g_n - \sum_{m\in M_-(k)} f_m + \sum_{m\in M_+(k)} f_m \ge \delta_k & \forall k \in K
\end{array}
\right\} \mathbb{P}\text{-a.s.}
$$

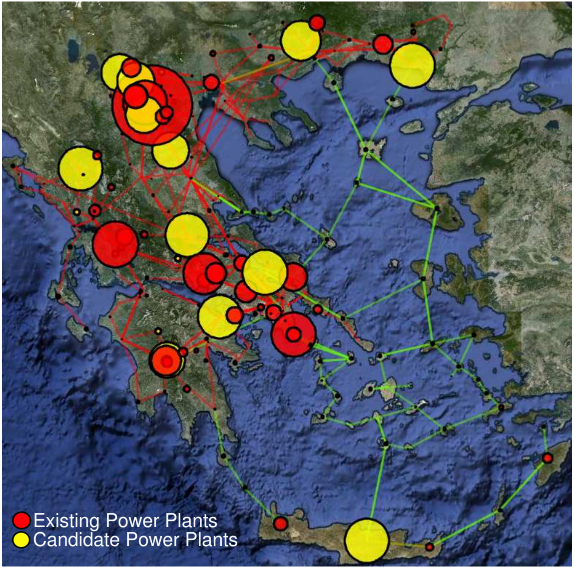
Existing Infrastructure

Power system: 265 buses, 429 transmission lines



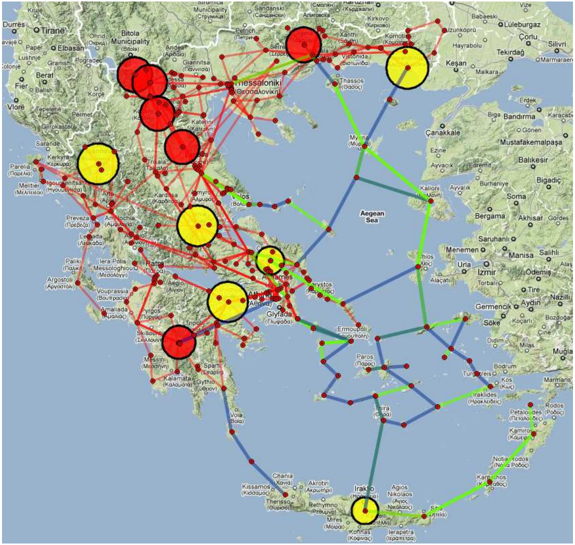
Extension Options

Possible additions: 217 power plants, 81 transmission lines



Results

Solution time: 2.5 hours



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3. J. Goh, M. Sim. *Distributionally Robust Optimization and its Tractable Approximations*. Operations Research 58(4):902–917, 2010.

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