Expert Opinions and Dynamic Portfolio Optimization Under Partial Information

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Agenda

Introduction

Initial capital	$x_0 > 0$
Horizon	$[0, T]$
Aim	Maximize expected utility of terminal wealth
Problem	Find an optimal investment strategy
	How many shares
	of which asset
	have to be held at which time by the portfolio manager ?
Market model	continuously tradable assets
	drift depends on unobservable factor process
	investor only observes stock prices and
	expert opinions

Financial Market with Partial Information

$(\Omega, \mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}, P)$ filtered probability space

Money market with interest rate 0

Stock market prices $S_t = (S_t^1, \dots, S_t^n)^\top$, returns $dR_t^i = dS_t^i / S_t^i$

$$dR_t = \mu_t \, dt + \sigma \, dW_t^R$$

$W^R = (W_t^R)_{t\in[0,T]}$ n-dimensional Brownian motion

$\mu = (\mu_t)_{t\in[0,T]}$ stochastic drift, independent on W^R

σ volatility, non-singular

Information investor filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]} \subset \mathbb{G}$

classical situation $\mathcal{F}_t = \mathcal{F}_t^R = \mathcal{F}_t^S \subset \mathcal{F}^{W^R,\mu} \subset \mathcal{G}$

filtering with observation R and signal μ

we may also consider additional information leading to $\mathcal{F}_t^H \subset \mathcal{G}$

optimal strategies depend on filter $\widehat{\mu}_t^H = E[\mu_t | \mathcal{F}_t^H]$ and its dynamics

Trading and Portfolio Optimization

- X_t wealth (portfolio value) at time t
- $\pi = (\pi_t)_{t\in[0,T]}$ trading strategy

  π_t^i is fraction of wealth X_t invested in stock i

  π has to be $\mathbb{F}^H$-adapted
- Wealth $X_t = X_t^\pi$ is controlled by π and satisfies

$$dX_t = X_t \pi_t^\top dR_t = X_t \pi_t^\top (\mu_t dt + \sigma dW_t^R), \quad X_0 = x_0$$

- Evaluation of terminal wealth with utility function U, e.g.

$$U_\theta(x) = \frac{x^\theta}{\theta}, \quad \theta < 1, \ \theta \neq 0, \qquad \text{or} \qquad U_0(x) = \log(x)$$

- Stochastic control problem: maximize expected utility

$$E[U(X_T^\pi)] \quad \text{over admissible strategies } \pi \quad \text{for } x_0 > 0$$

Optimal Strategies in Special Cases

- For constant μ: $\pi_t^* = \frac{1}{1-\theta}(\sigma\sigma^\top)^{-1}\mu = \text{const}$ Merton strategy
- For stochastic μ, information $\mathbb{F}^H$ and $U = U_0 = \log$
  optimal strategy is obtained by substituting filter $\widehat{\mu}_t^H$ for μ
  (Certainty equivalence principle)
  - Proof: (for $n = 1$ and $x_0 = 1$):
    $$\log X_T^\pi = \int_0^T \Big(\pi_t\,\mu_t - \frac{1}{2}(\sigma\pi_t)^2\Big)dt + \int_0^T \pi_t\sigma dW_t^R$$
  - For $\mathbb{F}^H$-adapted π we obtain
    $$\begin{aligned} E[\log X_T^\pi] &= \int_0^T E\left[\pi_t\, E\left[\mu_t \mid \mathcal{F}_t^H\right] - \frac{1}{2}(\sigma\pi_t)^2\right] dt \; + \; 0 \\ &= \int_0^T E\left[\pi_t\,\widehat{\mu}_t^H - \frac{1}{2}(\sigma\pi_t)^2\right] dt \end{aligned}$$
  - Pointwise maximization yields $\pi_t^* = \sigma^{-2}\widehat{\mu}_t^H$
- In general we expect a dependency of π_t^* on filter $\widehat{\mu}_t^H$ and its dynamics

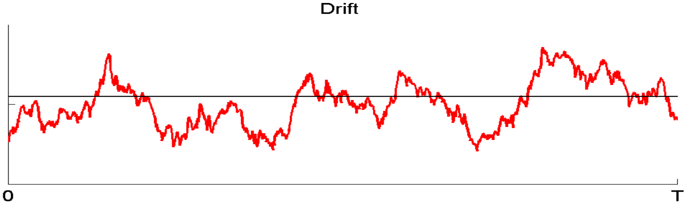
Drift Models

- Bayesian Model: μ is random but time-independent
  KARATZAS/XUE (1991)
- Linear Gaussian Model (LGM) or Kim-Omberg model

  $$d\mu_t = \kappa(\overline{\mu} - \mu_t)dt + \delta dW_t^{\mu}$$

  leads to Kalman filter
  LAKNER (1998), BRENDLE (2006)
- Hidden Markov Model (HMM)
  μ as a continuous-time Markov chain
  leads to Wonham or HMM filter
  SASS, HAUSSMANN (2004), RIEDER, BÄUERLE (2005)

Linear Gaussian Model (LGM)



Drift is a Gaussian mean-reversion (Ornstein-Uhlenbeck) process

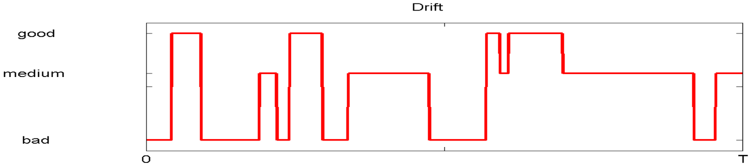
$$d\mu_t = \kappa(\overline{\mu} - \mu_t)dt + \delta dW_t^{\mu}$$

where W_t^{μ} is a Brownian motion (in)dependent of W_t^R

Closed-form solution available

Stationary distribution for $t \to \infty$ is $\mathcal{N}(\overline{\mu}, \frac{\delta^2}{2\kappa})$

Hidden Markov Model (HMM)



Drift $\mu_t = \mu(Y_t)$ is a finite-state Markov chain, independent of W_t^R

Y has state space $\{e_1, \ldots, e_d\}$, unit vectors in $\mathbb{R}^d$

$\mu(Y_t) = MY_t$ where $M = (\mu_1, \ldots, \mu_d)$ contains states of drift

generator or rate matrix $Q \in \mathbb{R}^{d \times d}$

diagonal: $Q_{kk} = -\lambda_k$ exponential rate of leaving state k

conditional transition prob. $P(Y_t = e_l \mid Y_{t-} = k, Y_t \neq Y_{t-}) = Q_{kl}/\lambda_k$

initial distribution $(\rho^1, \ldots, \rho^d)^\top$

Expert Opinions

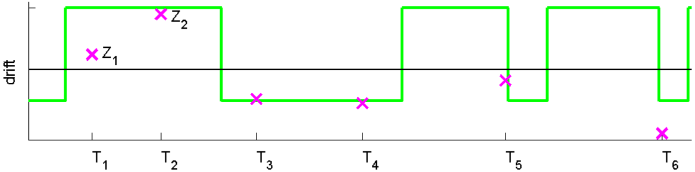
- Motivation: static Black-Litterman model
  Practitioners apply Bayesian updating to combine
  subjective views (such as "asset 1 will grow by 5%")
  with empirical or implied drift estimates.
- Present paper includes such views or expert opinions
  into dynamic models with partial observation.
- Investor receives noisy signals about current drift
  - at fixed and known points in time e.g. analysts, company reports
  - at random (unknown) points in time e.g. news, ratings
    timing does not carry any useful information
    jump times of a Poisson process
  - in continuous time (limiting case)

Discrete-Time Expert Opinions

- At times T_k investor observes r.v. $Z_n \in \mathcal{Z}$ (views)
- Z_k depends on current drift μ_{T_k}, density $f(z, \mu_{T_n})$
  (Z_k) cond. independent given $\mathcal{F}_T^\mu = \sigma(\mu_s : s \in [0, T])$

Examples

- Absolute view: $Z_k = \mu_{T_k} + \sqrt{\Gamma_k}\, \varepsilon_k$, (ε_k) i.i.d. $N(0, 1)$
  The view "*S* will grow by 5%" is modelled by $Z_k = 0.05$
  Γ_k models confidence of expert



- Relative view (2 assets): $Z_k = \mu^1_{T_k} - \mu^2_{T_k} + \sqrt{\Gamma_k}\, \varepsilon_k$
- $T_k = t_k$ fixed or T_k jump times of a Poisson process with intensity λ
  $\Rightarrow$ marked point process

Literature on Expert Opinions

- Discrete-time expert opinions
  - LGM: Gabih/Kondakji/Sass/W.(2014)
  - HMM: Frey/Gabih/W. (2012, 2014)
- Continuous-time expert opinions
  - LGM: Davis/Lleo (2013)
  - HMM: Seifried/Sass/W. (working paper)
- Diffusion approximations

Maximizing Log-Utility in a Model with Gaussian Drift

- We consider one stock (n = 1) with

$$\begin{aligned} \text{returns} \quad dR_t &= \mu_t dt + \sigma dW_t^R \\ \text{and drift} \quad d\mu_t &= \kappa(\overline{\mu} - \mu_t)dt + \delta dW_t^{\mu}, \quad \mu_0 \sim \mathcal{N}(m_0, \eta_0) \end{aligned}$$

- N expert opinions arrive at fixed times $0 = t_0 < t_1 < \ldots < t_{N-1} < T$.
  Views are modeled as Gaussian unbiased estimates Z_k of the current drift.

$$Z_k = \mu_{t_k} + \sqrt{\Gamma_k}\varepsilon_k, \quad \text{for i.i.d. } \varepsilon_1, \ldots, \varepsilon_N \sim \mathcal{N}(0,1).$$

  $\Gamma_k > 0$ describes the confidence of the expert.

- We distinguish four information regimes for the investor

$\mathbb{F}^R$	observing	returns only
$\mathbb{F}^E$		expert opinions only
$\mathbb{F}^C$		both returns and expert opinions
$\mathbb{F}^F$	having	full information

- We have to compute filters $\widehat{\mu}_t^H = E[\mu_t | \mathcal{F}_t^H]$
  and conditional variances $q_t^H = E[(\mu_t - \widehat{\mu}_t^H)^2 | \mathcal{F}_t^H]$ for $H = R, E, C, F$.

Filtering: Returns Only ($H = R$)

- For $H = R$, we are in the classical Kalman filter case and get

$$d\widehat{\mu}_t^R = \kappa(\overline{\mu} - \widehat{\mu}_t^R)dt + \sigma^{-2} q_t^R \left(dR_t - \widehat{\mu}_t^R dt\right), \quad \widehat{\mu}_0^R = m_0,$$

  and deterministic conditional variance satisfying the Riccati equation

$$\frac{d}{dt} q_t^R = \delta^2 - 2\kappa q_t^R - \sigma^{-2} (q_t^R)^2, \quad q_0^R = \eta_0.$$

  For $n = 1$ we have a closed-form solution for q_t^R.
- For $t \to \infty$ we have $q_t^R \to q_\infty^R := \kappa\sigma^2 \left(\sqrt{1 + (\frac{\delta}{\kappa\sigma})^2} - 1\right)$
- q_t^R is decreasing if $\eta_0 > q_\infty^R$ and
  increasing if $\eta_0 < q_\infty^R$.

Filtering: Returns and Expert Opinions ($H = C$)

- Between the information dates the filter and the conditional variance evolve as in regime $H = R$ (Kalman filter).
- At the information dates t_k the expert opinion $Z_k \sim \mathcal{N}(\mu_{t_k}, \Gamma_k)$ leads to a Bayesian update

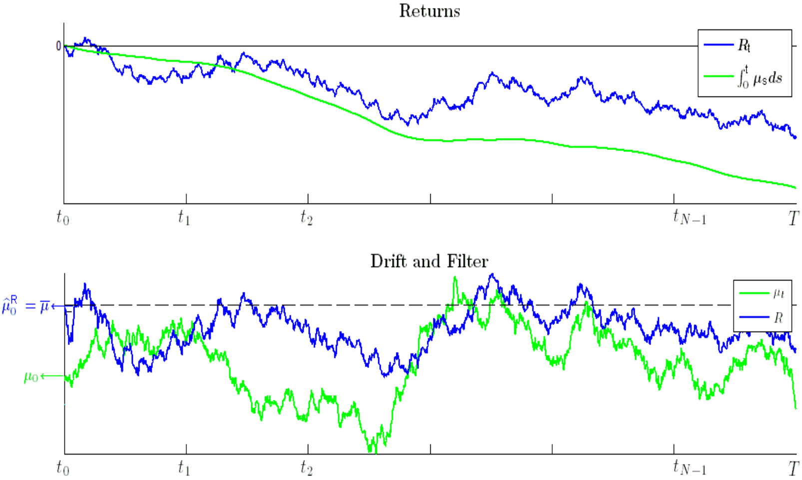
$$\begin{aligned} \widehat{\mu}^C_{t_k} &= \rho_k \, \widehat{\mu}^C_{t_k-} + (1-\rho_k)\, Z_n \\ q^C_{t_k} &= \rho_k q^C_{t_k-} \end{aligned}$$

  with the factor

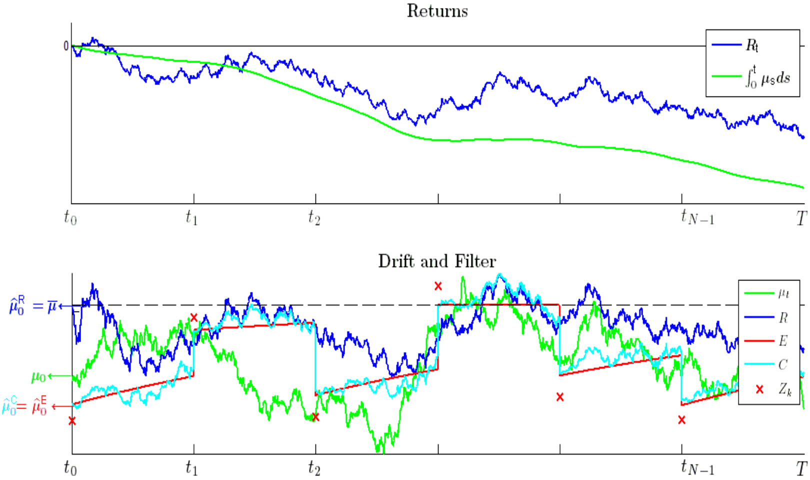
$$\rho_k = \frac{\Gamma_k}{\Gamma_k + q_{t_k-}} \in (0,1)$$

- For $H = E$ we get corresponding updating formulas and between the information dates we can consider the limiting case $\sigma = \infty$ in regime $H = C$.
- For $H = F$ we have full information and thus $\widehat{\mu}^F_t = \mu_t$ and $q^F_t = 0$.

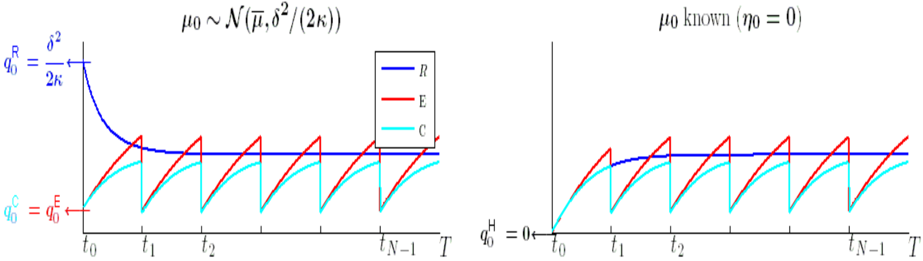
Example: Filter $\widehat{\mu}_t^H$



Example: Filter $\widehat{\mu}_t^H$



Example: Conditional Variance q_t^H



Properties of Conditional Variance

- The conditional variances in all cases $H = R, E, C, F$ are deterministic, thus

$$\begin{aligned} q_t^H &= E[(\mu_t - \widehat{\mu}_t^H)^2 | \mathcal{F}_t^H] = E[(\mu_t - \widehat{\mu}_t^H)^2] \\ &= E[\mu_t^2] - E[(\widehat{\mu}_t^H)^2] = Var(\mu_t) - Var(\widehat{\mu}_t^H) \end{aligned}$$

- Small q_t^H means that the filter is close to the true state.

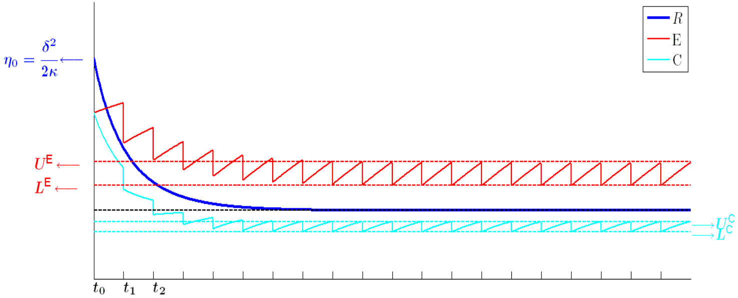
  Therefore q_t^H may serve as good performance measure.
- $q_t^C \leq q_t^R$ and $q_t^C \leq q_t^E$

Asymptotics of Conditional Variance for $t \to \infty$

- Let $T = \infty$
- $q_t^R \to q_\infty^R$ for $t \to \infty$
- For equidistant expert opinions with $\Gamma_k = \Gamma > 0$ we get for $H = E, C$

$$\limsup_{t\to\infty} q_t^H = U^H \quad \text{and} \quad \liminf_{t\to\infty} q_t^H = L^H,$$

where $U^H > L^H > 0$ can be computed explicitly.



Asymptotics of Conditional Variance for $N \to \infty$

- Let $T < \infty$ fixed
- Expert opinions arrive more frequent and have some minimum confidence:

  For $N \to \infty$ and $H = E, C$ it holds $\quad q_t^{H,N} \to q_t^F = 0 \quad$ (full info, LLN)
- Expert opinions arrive more frequent and become "less confident":

  Consider equidistant expert opinions with $\quad t_k = k\Delta_N \qquad$ with $\Delta_N = T/N$

  $$\Gamma_k = \Gamma = \frac{\sigma_J^2}{\Delta_N} \quad \text{with } \sigma_J > 0$$
- Diffusion process $\quad dJ_t = \mu_t dt + \sigma_J dW_t^J \quad$ models "continuous-time expert"

  An estimator for the drift in $[t_k, t_k + \Delta_N]$ is $\quad Z_k = \frac{1}{\Delta_N}(J_{t_k+\Delta_N} - J_{t_k})$

  For constant $\mu_t = \mu_{t_k}$ we have $\quad Z_k \sim \mathcal{N}(\mu_{t_k}, \sigma_J^2/\Delta_N)$
- Let $\widehat{\mu}^J$ and q^J denote Kalman filter and cond. variance from observing J

Diffusion approximation

For $N \to \infty$ it holds
$$|q_t^{E,N} - q_t^J| \quad \to \quad 0 \quad \text{uniformly for all } t \in [0, T]$$
$$\int_0^T E[\, |\widehat{\mu}_t^{E,N} - \widehat{\mu}_t^J|^2 \,]dt \quad \to \quad 0$$

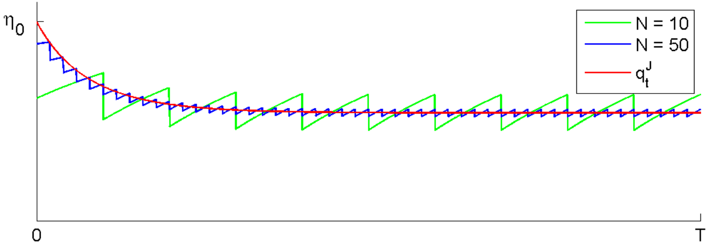
Example: Diffusion approximation

Diffusion approximation

For $N \to \infty$ it holds

$$\begin{aligned} |q_t^{E,N} - q_t^J| &\to 0 \text{ uniformly for all } t \in [0, T] \\ \int_0^T E[\, |\widehat{\mu}_t^{E,N} - \widehat{\mu}_t^J|^2 \,]dt &\to 0 \end{aligned}$$

Conditional variances $q_t^{E,N}$ and q_t^J



Optimal Expected Logarithmic Utility

$$V(x_0) = \sup_{\pi \in \mathcal{A}^H} E[\log X_T^*]$$

Already known: for $U = U_0 = \log$ and information $\mathbb{F}^H$ the optimal strategy is

$$\pi_t^* = \sigma^{-2} \widehat{\mu}_t^H$$

Therefore,
$$V^H(x_0) = E[\log X_T^{\pi^*}] = \log x_0 + E\Big[\int_0^T \Big(\pi_t^* \, \widehat{\mu}_t^H - \frac{1}{2}(\sigma \pi_t^*)^2\Big) dt\Big]$$
$$= \log x_0 + \frac{1}{2\sigma^2} \int_0^T \underbrace{E\big[(\widehat{\mu}_t^H)^2\big]}_{E[\mu_t^2] - q_t^H} dt$$

Theorem (Gabih/Kondakji/Sass/W. 2014)

$$V^H(x_0) = \log x_0 + \frac{1}{2\,\sigma^2}\Big(\int_0^T E[\mu_t^2]\, dt - \int_0^T q_t^H\, dt\Big)$$

where the integrals (in all four cases) can be computed explicitly.

Properties derived for q_t^H allow to derive corresponding properties for $V^H(x_0)$.

Efficiency

We want to quantify the monetary value of the information.

Investor	F	$H \ (= R, E, C)$
Information (observations)	$\mathbb{F}^F$	$\mathbb{F}^H$
Initial capital	$x_0^F = 1$	x_0^H
Opt. terminal wealth	X_T^F	X_T^H

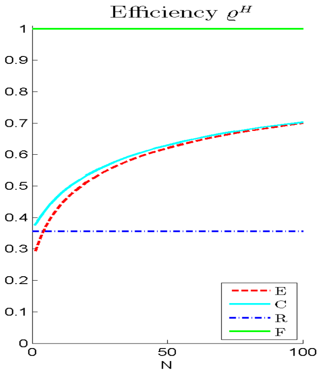
How much initial capital x_0^H needs H to obtain the same expected utility as F?

$$\text{Solve} \quad V^F(1) = V^H(x_0^H):$$
$$x_0^H = \exp\left(\frac{1}{2\sigma^2}\int_0^T q_t^H\, dt\right)$$

Loss of information $\quad x_0^H - 1$

Efficiency $\quad \varrho^H = 1/x_0^H$

for (non-fully informed) H-investor .



Efficiency: Numerical Example

Value $V^H(1)$ and efficiency ϱ^H in % for various numbers N

	$V^H(1)$		ϱ^H	
R	0.3213		35.63	
N	E	C	E	C
10	0.5208	0.6008	43.49	47.12
100	0.9957	1.0017	69.94	70.36
1.000	1.2297	1.2299	88.37	88.39
10.000	1.3134	1.3134	96.09	96.09
100.000	1.3407	1.3407	98.74	98.74
1.000.000	1.3493	1.3493	99.60	99.60
10.000.000	1.3521	1.3521	99.87	99.87
F	1.3533		100.00	

Maximizing Power Utility in an HMM Model

- We consider n stocks with returns $\quad dR_t = \mu(Y_t)dt + \sigma dW_t^R$

  and drift driven by a finite-state Markov chain Y.
- expert opinions arrive at jump times of a Poisson process with intensity λ and modeled by a marked point process (T_k, Z_k)
- Z_n depends on current state Y_{T_n}, $\quad$ density $f(z, Y_{T_n})$

  (Z_n) cond. independent given $\mathcal{F}_T^Y = \sigma(Y_s : s \in [0, T])$
- We are mainly interested in the information regimes

$\mathbb{F}^R$	observing	returns only
$\mathbb{F}^C$		both returns and expert opinions

  and want to maximize $E[U(X_T^\pi)]$ for power utility $U_\theta(x) = \frac{x^\theta}{\theta}, \;\; \theta < 1, \; \theta \neq 0$
- For $H = F$ (full info) the problem is solved in RIEDER & BÄUERLE (2004)

HMM Filtering: Returns Only ($H = R$)

Returns $dR_t = \frac{dS_t}{S_t} = \mu(Y_t)\,dt + \sigma\,dW_t$ observations

Drift $\mu(Y_t) = M\,Y_t$ non-observable (hidden) state

Investor filtration $\mathbb{F}^R = (\mathcal{F}_t^R)_{t\in[0,T]}$ with $\mathcal{F}_t^R = \sigma(R_u:\ u \le t) \subset \mathcal{G}_t$

Filter $p_t^k := P(Y_t = e_k | \mathcal{F}_t^R)$

$$\widehat{\mu(Y_t)} := E[\mu(Y_t)|\mathcal{F}_t^R] = \mu(p_t) = \sum_{j=1}^{d} p_t^j \mu_j$$

Innovations process $\widetilde{W}_t^R := \sigma^{-1}\big(R_t - \int_0^t \widehat{\mu(Y_s)}ds \big)$ is an $\mathbb{F}^R$-BM

HMM filter LIPTSER, SHIRYAEV (1974), WONHAM (1965), ELLIOTT (1993)

$$\begin{aligned} p_0^k &= \rho^k \\ dp_t^k &= \sum_{j=1}^{d} Q^{jk} p_t^j dt + \beta_k(p_t)^\top d\widetilde{W}_t^R \end{aligned}$$

$$\text{where} \quad \beta_k(p) = p^k \sigma^{-1}\Big(\mu_k - \sum_{j=1}^{d} p^j \mu_j\Big)$$

HMM Filtering: Returns and Expert Opinions ($H = C$)

Extra information has no impact on filter p_t between 'information dates' T_n

Bayesian updating at $t = T_n$:

$$p^k_{T_n} \propto p^k_{T_n-}\, f(Z_n, e_k) \qquad \text{recall: } f(\cdot, Y_{T_n}) \text{ is density of } Z_n \text{ given } Y_{T_n}$$

$$\text{with normalizer} \quad \sum_{j=1}^{d} p^j_{T_n-} f(Z_n, e_j) \;=:\; \bar{f}(Z_n, p_{T_n-})$$

HMM filter

$$\begin{aligned} p^k_0 &= \rho^k \\ dp^k_t &= \sum_{j=1}^{d} Q^{jk} p^j_t dt + \beta_k(p_t)^\top d\widetilde{W}^R_t + p^k_{t-} \int_{\mathcal{Z}} \left(\frac{f(z,e_k)}{\bar{f}(z,p_{t-})} - 1 \right) \widetilde{I}(dt \times dz) \end{aligned}$$

Compensated measure $\widetilde{I}(dt \times dz) := I(dt \times dz) - \underbrace{\lambda dt \sum_{k=1}^{d} p^k_{t-} f(z, e_k)\, dz}_{\text{compensator}}$

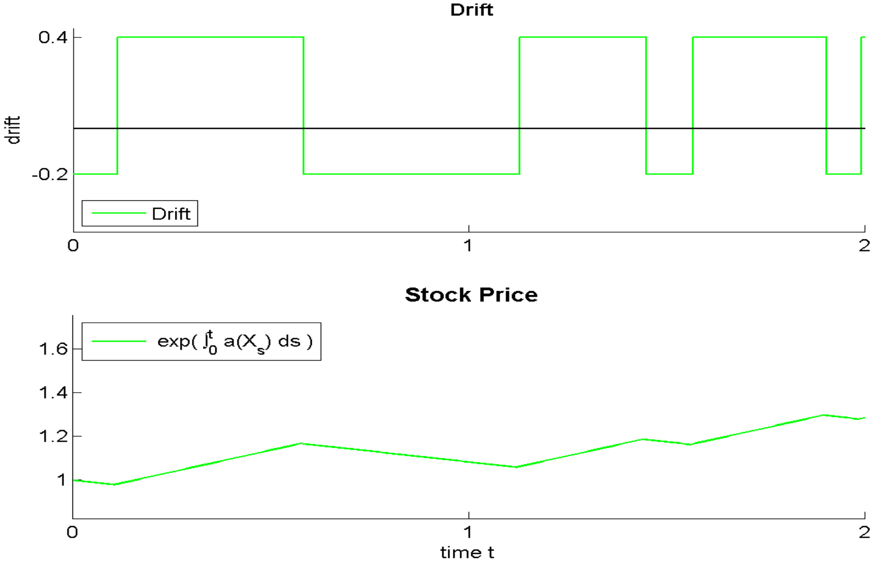
Zakai equation for the unnormalized filter & robust filter

see ELLIOTT, SIU, YANG (2010), KONDAKJI (2012)

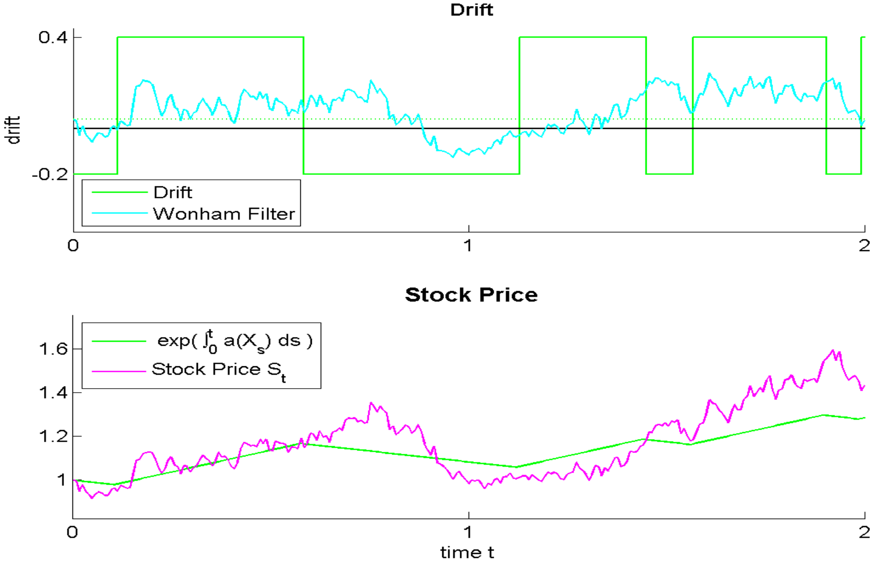
Filter: Example

Start animation

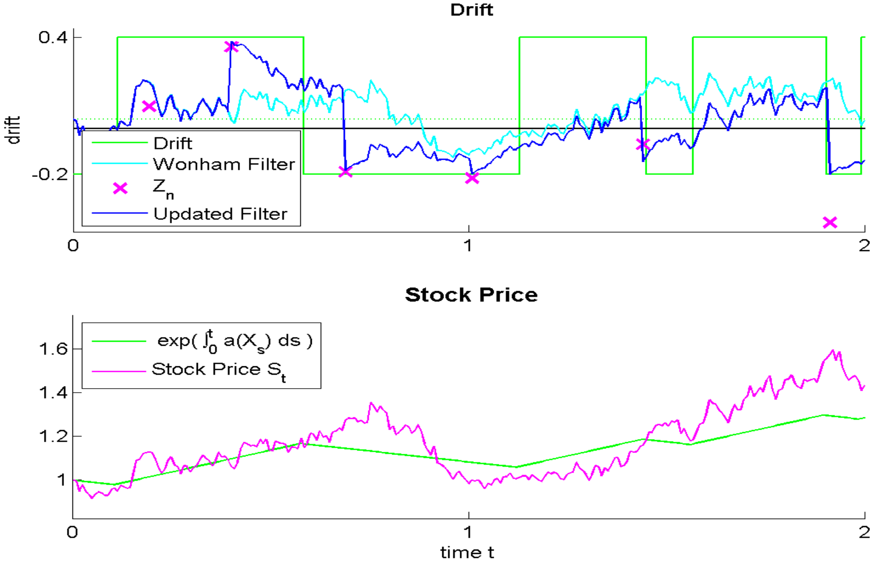
Filter: Example



Filter: Example



Filter: Example



Optimization Problem Under Partial Information

Wealth $dX_t^{(\pi)} = X_t^{(\pi)}\,\pi_t^\top\,(\mu(Y_t)\,dt + \sigma dW_t), \quad X_0^{(\pi)} = x_0$

Admissible strategies $\mathcal{A}^H = \{(\pi_t)_{t\in[0,T]} \mid \pi_t \in K \subset \mathbb{R}^n$ with K compact

π is $\mathbb{F}^H$-adapted $\}$

Reward function $v(t,x,\pi) = E_{t,x}[\,U(X_T^{(\pi)})\,]$ for $\pi \in \mathcal{A}^H$

Value function $V(t,x) = \sup_{\pi\in\mathcal{A}^H} v(t,x,\pi)$

Find optimal strategy $\pi^* \in \mathcal{A}^H$ such that $V(0,x_0) = v(0,x_0,\pi^*)$

Reduction to an OP Under Full Information

Consider augmented state process (X_t, p_t)

Wealth $$dX_t^{(\pi)} = X_t^{(\pi)}\, \pi_t^\top \big(\underbrace{\widehat{\mu(Y_t)}}_{=M\,p_t}\, dt + \sigma\, d\widetilde{W}_t^R \big), \qquad X_0^{(\pi)} = x_0$$

Filter $$dp_t^k = \sum_{j=1}^{d} Q^{jk} p_t^j dt + \beta_k(p_t)^\top d\widetilde{W}_t^R$$
$$+ p_{t-}^k \int_{\mathcal{Z}} \left(\frac{f(z,e_k)}{\bar{f}(z,p_{t-})} - 1 \right) \widetilde{I}(dt \times dz), \qquad p_0^k = \rho^k$$

Reward function $v(t,x,p,\pi) = E_{t,x,p}[\, U(X_T^{(\pi)})\,]$ for $\pi \in \mathcal{A}^H$

Value function $V(t,x,p) = \sup_{\pi \in \mathcal{A}^H} v(t,x,p,\pi)$

Find $\pi^* \in \mathcal{A}^H(0)$ such that $V(0,x_0,\rho) = v(0,x_0,\rho,\pi^*)$

Dynamic Programming Approach

Transformation to risk-sensitive control problem

NAGAI & RUNGGALDIER (2008), DAVIS & LLEO (2012)

Change of measure: $P^{(\pi)}(A) = E[Z^{\pi} 1_A]$ for $A \in \mathcal{F}_T$

$$\text{where} \quad Z^{\pi} := \exp\left\{\theta \int_0^T \pi_s^{\top} \sigma \, d\widetilde{W}_s^R - \frac{\theta^2}{2} \int_0^T \pi_s^{\top} \sigma \sigma^{\top} \pi_s ds\right\}$$

Reward function

$$E_{t,x,p}[U(X_T^{(\pi)})] = \frac{x^{\theta}}{\theta} \underbrace{E_{t,p}^{(\pi)}\left[\exp\left\{-\int_t^T b(p_s, \pi_s) ds\right\}\right]}_{=: \, v(t,p,\pi)} \quad \text{independent of } x$$

$$\text{where} \quad b(p,\pi) := -\theta\left(\pi^{\top} M p - \frac{1-\theta}{2} \pi^{\top} \sigma \sigma^{\top} \pi\right)$$

Value function $V(t,p) = \sup_{\pi \in \mathcal{A}^H} v(t,p,\pi)$ for $0 < \theta < 1$

Find $\pi^* \in \mathcal{A}^H$ such that $V(0,\rho) = v(0,\rho,\pi^*)$

Dynamic Programming Equation (DPE)

State $\quad dp_t = \alpha(p_t, \pi_t)dt + \beta^\top(p_t)dB_t + \int_{\mathcal{Z}} \gamma_I(p_t, z)\widetilde{I}(dt \times dz)$

Generator $\quad \mathcal{L}^a g(p) = \frac{1}{2} tr\left[\beta^\top(p)\beta(p)D^2 g\right] + \alpha^\top(p, a)\nabla g + \lambda \int_{\mathcal{Z}} \{g(p + \gamma_I(p, z)) - g(p)\}\bar{f}(z, p)dz$

DPE (Generalized Hamilton-Jacobi-Bellman Equation)

$$V_t(t, p) + \sup_{a \in K} \left\{ \mathcal{L}^a V(t, p) - b(p, \pi)V(t, p) \right\} = 0$$

$$\text{terminal condition} \quad V(T, p) = 1$$

Candidate for the Optimal Strategy

$$\pi^* = \pi^*(t, p) = \underbrace{\frac{1}{(1-\theta)}(\sigma\sigma^\top)^{-1}}_{} \left\{ Mp + \frac{1}{V(t, p)} \sigma\beta(p)\, \nabla_p V(t, p) \right\}$$

myopic strategy $+$ correction

Certainty equivalence principle does not hold

Justification and regularization of DPE FREY, GABIH, W. (2014)

Computation of the Optimal Strategy

Dynamic Programming Equation

$$V_t(t,p) + \sup_{a \in K} \left\{ \mathcal{L}^a V(t,p) - b(p,a) V(t,p) \right\} = 0$$

$$\text{terminal condition} \quad V(T,p) = 1$$

$$\text{Generator} \quad \mathcal{L}^a g(p) = \frac{1}{2} tr\left[\beta^\top(p)\beta(p)D^2 g\right] + \alpha^\top(p,a)\nabla g + \lambda \int_{\mathcal{Z}} \{g(p+\gamma(p,z)) - g(p)\} \overline{f}(z,p)dz$$

Plugging in the optimal strategy

$$\pi^* = \pi^*(t,p) = \frac{1}{(1-\theta)} (\sigma\sigma^\top)^{-1} \left\{ Mp + \frac{1}{V(t,p)} \sigma\beta(p)\, \nabla_p V(t,p) \right\}$$

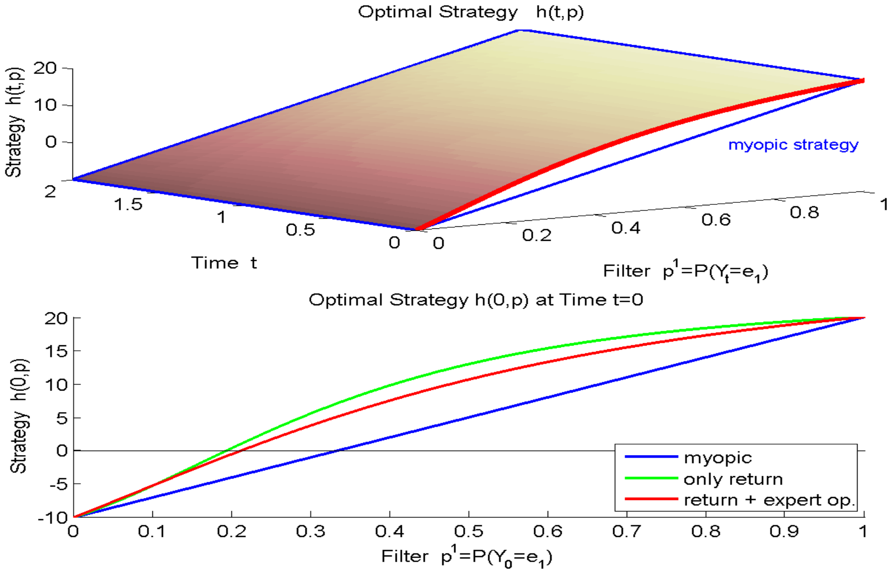
yields a nonlinear partial integro-differential equation (PIDE)

Normalization of p: reduction to $d-1$ ”spatial” variables

For $d=2$ states: only one ”spatial” variable, ellipticity condition is satisfied

Solve PIDE numerically using an explicit finite difference scheme.

Computation of the Optimal Strategy (cont.)



Monetary Value of Expert Opinions

We want to quantify the value of the extra information.

Investor	R	C
Information (observations)	only returns	returns + expert opinions
Initial capital	x_0^R	$x_0^C = 1$
Opt. terminal wealth	X_T^R	X_T^C

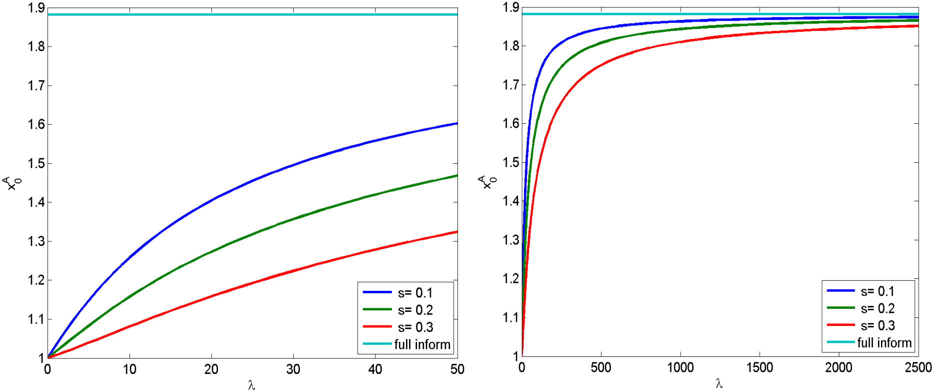
How much initial capital needs R to obtain the same expected utility as C?

$$\begin{aligned} E[U(X_T^R)] &= E[U(X_T^C)] \\ \frac{(x_0^R)^\theta}{\theta} V^R(0,p) &= \frac{(x_0^C)^\theta}{\theta} V^C(0,p) \qquad (x_0^C = 1) \\ x_0^R &= \left(\frac{V^C(0,p)}{V^R(0,p)} \right)^{\frac{1}{\theta}} \end{aligned}$$

Difference $x_0^R - x_0^C$ measures information gain of investor C.

Monetary Value of Expert Opinions (cont.)

Expert opinions $Z_k \sim \mathcal{N}(\mu(Y_{T_k}), \Gamma)$ with "confidence" $\Gamma = s^2$ arrive with intensity λ



Limit case $\lambda \longrightarrow \infty$ full information on the drift

state of Markov chain is observable

see RIEDER & BÄUERLE (2004)

Diffusion Approximations

For intensity $\lambda \to \infty$ and expert's variance Γ which is

bounded	$\Rightarrow$	full information	(LLN)
increasing (properly scaled)	$\Rightarrow$	**?**	(CLT)

- $d = 2$ states of the drift μ_1 and μ_2
- conditional distributions of Z_n: truncated Gaussian $\mathcal{N}(\mu_j, \Gamma)$, $j = 1, 2$
- expert's variance $\Gamma = c\lambda$ grows linearly with intensity λ
- truncate to "κ–sigma–interval" $[m - \kappa\sqrt{c\lambda}, m + \kappa\sqrt{c\lambda}]$

Updated HMM filter $p = (p_1, p_2)^\top = (\nu, 1-\nu)^\top$, $\nu = \nu^\lambda$ satisfies

$$
\begin{aligned}
d\nu_t^\lambda &= \alpha(\nu_t^\lambda)dt + \nu_t^\lambda(1-\nu_t^\lambda)\,\frac{\mu_1 - \mu_2}{\sigma}\,d\widetilde{W}_t^R + \int_{\mathcal{Z}} \gamma_I^\lambda(\nu_{t-}^\lambda, z)\,\widetilde{I}(dt \times dz) \\
&\downarrow \quad \lambda \to \infty \\
d\xi_t &= \alpha(\xi_t)dt + \xi_t(1-\xi_t)\,\frac{\mu_1 - \mu_2}{\sigma}\,d\widetilde{W}_t^R + \underbrace{\xi_t(1-\xi_t)\,\frac{\mu_1 - \mu_2}{\sigma_J}\,d\widetilde{W}_t^J}_{(*)}
\end{aligned}
$$

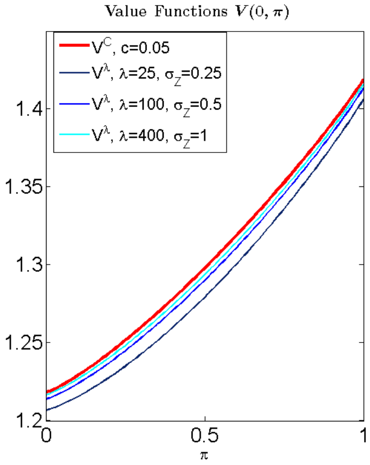
$(*)$ corresponds to observation of a Markov-modulated Brownian motion

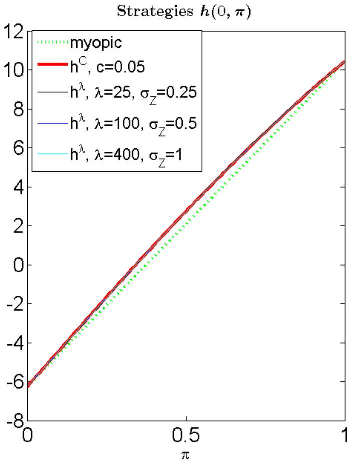
$$dJ_t = \mu(Y_t)dt + \sigma_J\,dW_t^J \qquad \text{where } \sigma_J = c + o(\kappa) \quad \text{for } \kappa \to \infty$$

continuous-time expert DAVIS & LLEO (2013b)

Diffusion Approximations: Example

Compare value functions $V^\lambda(0,\nu)$ and optimal strategies $\pi^\lambda(0,\nu)$ at time $t = 0$ with corresponding values for the limiting case $\lambda = \infty$ for $\Gamma = c\lambda$, $c = 0.05$





Conclusion

- Portfolio optimization under partial information on the drift
- Investor observes stock prices and expert opinions
- Closed-form solutions for log-utility and LGM
- For HMM and power utility:
  non-linear dynamic programming equation with a jump part
- Computation of the optimal strategy

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