Multivariate volatility estimation with high frequency data using Fourier method

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Motivation

- Computation of **volatility/covariance** of financial asset returns plays a central role for many issues in finance: risk management, hedging strategies, forecasting...

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- Black&Scholes model - constant volatility does not account for: heteroschedasticity, predictability, volatility smile, covariance between asset returns and volatility (leverage effect) $\Rrightarrow$
stochastic volatility models proposed to model asset price evolution and to price options (adding **risk factors** represented by Brownian motions [Heston, 1993, Hull and White, 1987, Stein and Stein, 1991, Meddahi, 2001], **jumps** [Bates, 1996], or introducing **memory** [Hobson and Rogers, 1998])

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  stochastic volatility models proposed to model asset price evolution and to price options (adding **risk factors** represented by Brownian motions [Heston, 1993, Hull and White, 1987, Stein and Stein, 1991, Meddahi, 2001], **jumps** [Bates, 1996], or introducing **memory** [Hobson and Rogers, 1998])
- Availability of **high frequency data** have the potential to improve the capability of computing volatility/covariances in an efficient way to many extend [Andersen et al., 2006] (forecasting), [Bollerslev and Zhang, 2003] (risk factor models), [Fleming et al., 2003] (asset allocation)....

Outline

- Definition of Fourier estimator of spot and integrated volatility/covariance
- Properties of Fourier estimator with high frequency data
- Potentiality of Fourier estimator for some applications:
  - Volatility of Volatility and Leverage
  - Forecasting and Asset Allocation
  - Quarticity

Non-parametric and model free context

Model: continuous Brownian semimartingale

$$\textbf{(B)} \qquad dp^j(t) = \sum_{i=1}^{d} \sigma_i^j(t)\ dW^i + b^j(t)\ dt, \qquad j = 1, \ldots, n,$$

$W = (W^1, \ldots, W^d)$ are independent Brownian motions and σ_*^* and b^* are adapted random processes satisfying

$$E[\int_0^{2\pi} (b^j(t))^2 dt] < \infty, \quad E[\int_0^{2\pi} (\sigma_i^j(t))^4 dt] < \infty \qquad i = 1, \ldots, d,\ j = 1, \ldots, m$$

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Objective: estimation of the time dependent *volatility matrix*:

$$\Sigma^{jk}(t) = \sum_{i=1}^{d} \sigma_i^j(t)\sigma_i^k(t) \qquad j, k = 1, \ldots, n$$

Main Issues

$p^*(t)$ asset log-price Brownian semimartingale $\Rightarrow$ integrated volatility/covariance

$$\int_0^t \Sigma^{ik}(s)ds = P-\lim_{n\to\infty} \sum_{0\le j<t2^n} \left(p^i((j+1)2^{-n}) - p^i(j2^{-n})\right)\left(p^k((j+1)2^{-n}) - p^k(j2^{-n})\right).$$

Nevertheless, when sampling high frequency returns, three **difficulties** arise:

1) the distortion from efficient prices due to the **market microstructure** noise such as price discreteness, infrequent trading,...[Roll, 1984].
2) instantaneous volatility computation involves a sort of numerical derivative, which gives rise to numerical instabilities
[Foster and Nelson, 1996, Comte and Renault, 1998, Mykland and Zhang, 2006]
In the multivariate case also:
3) the **non-synchronicity** of the arrival times of trades across markets leads to a bias towards zero in correlations among stocks as the sampling frequency increases
[Epps, 1979]

Mean covariance [Malliavin and M. 2002, 2009]

Theorem

Consider a process p satisfying the assumption **(B)**. Then we have:

$$\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij}) = \mathcal{F}(dp^i) *_B \mathcal{F}(dp^j). \tag{1}$$

The convergence of the convolution product (1) is attained in probability

where, for $k \in \mathbf{Z}$

$$\mathcal{F}(dp^i)(k) := \frac{1}{2\pi}\int_0^{2\pi} \exp(-\mathrm{i}k\vartheta)\, dp^i(\vartheta)$$

$$(\Phi *_B \Psi)(k) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{s=-N}^{N} \Phi(s)\Psi(k-s)$$

$$\mathcal{F}(\Sigma^{ij})(k) := \frac{1}{2\pi}\int_0^{2\pi} \exp(-\mathrm{i}k\vartheta)\, \Sigma^{ij}(\vartheta)\, d\vartheta$$

Fourier instantaneous covariance computation

By the theorem we gather all the Fourier coefficients of the volatility matrix by means of the Fourier transform of the log-returns. Then reconstruct the **cross-volatility functions** $\Sigma^{ij}(t)$ from its Fourier coefficients by the Fourier-Fejer summation:
let for $i, j = 1, 2$ and for any $|k| \leq N$,

$$c_N^{ij}(k) := \frac{1}{2N+1} \sum_{|s| \leq N} \mathcal{F}(dp^i)(s) \mathcal{F}(dp^j)(k-s),$$

then

$$\Sigma^{ij}(t) = \lim_{N \to \infty} \sum_{|k|<N} (1 - \frac{|k|}{N}) c_N^{ij}(k) \exp(\mathrm{i}kt)$$

Consistency

$I_i^1 := [t_i^1, t_{i+1}^1[$, $J_j^2 := [t_j^2, t_{j+1}^2[$, $\rho(n) := \rho^1(n_1) \vee \rho^2(n_2)$ and $\rho^i(n_i) = \max_{t_j^i} |t_{j+1}^i - t_j^i|$,

$$c_k(dp_{n_1}^1) := \frac{1}{2\pi} \sum_{i=0}^{n_1-1} \exp(-\mathrm{i}kt_i^1)(p^1(t_{i+1}^1) - p^1(t_i^1))$$

$$c_k(dp_{n_2}^2) := \frac{1}{2\pi} \sum_{j=0}^{n_2-1} \exp(-\mathrm{i}kt_j^2)(p^2(t_{j+1}^2) - p^2(t_j^2))$$

$$c_k(\Sigma^{12}) := \frac{1}{2\pi} \int_0^{2\pi} e^{-\mathrm{i}kt} \Sigma^{12}(t) dt$$

Consistency

Define for any $|k| \leq N$

$$\alpha_k(N, p^1_{n_1}, p^2_{n_2}) = \frac{2\pi}{2N+1} \sum_{|s| \leq N} c_s(dp^1_{n_1}) c_{k-s}(dp^2_{n_2}). \tag{2}$$

Suppose that $N\rho(n) \to 0$ as $N, n \to \infty$. Then, for any k, in probability

$$\alpha_k(N, p^1_{n_1}, p^2_{n_2}) \to c_k(\Sigma^{12})$$

In probability, uniformly in t,

$$\widehat{\Sigma}^{12}_{n_1,n_2,N}(t) := \sum_{|k| \leq N} (1 - \frac{|k|}{N}) \alpha_k(N, p^1_{n_1}, p^2_{n_2}) e^{ikt} \to \Sigma^{12}(t) \tag{3}$$

Consistency of the estimator of integrated covariance

In particular the Fourier estimator of integrated covariance $\int_0^{2\pi} \Sigma^{12}(t)dt$ is

$$\widehat{\Sigma}^{12}_{N,n_1,n_2} := \frac{(2\pi)^2}{2N+1} \sum_{|s|\leq N} c_s(dp^1_{n_1})c_{-s}(dp^2_{n_2})$$

Theorem

If $\rho(n)N \to 0$, the following convergence in probability holds:

$$\lim_{n_1,n_2,N\to\infty} \widehat{\Sigma}^{12}_{N,n_1,n_2} = \int_0^{2\pi} \Sigma^{12}(t)dt.$$

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In the application we consider also the following version which preserves **definite positiveness of the covariance matrix**

$$\widehat{\Sigma}^{12}_{N,n_1,n_2} := \frac{(2\pi)^2}{N+1} \sum_{|s|\leq N} (1 - \frac{|s|}{N}) c_s(dp^1_{n_1}) c_{-s}(dp^2_{n_2}).$$

Model with microstructure

Consider the following model for the observed log-returns

$$\widetilde{p}^i(t) := p^i(t) + \eta^i(t) \quad \text{for } i = 1, 2,$$

Moreover the following assumptions hold:

(M)

M1. $p := (p^1, p^2)$ and $\eta := (\eta^1, \eta^2)$ are independent processes, moreover $\eta(t)$ and $\eta(s)$ are independent for $s \neq t$ and $E[\eta(t)] = 0$ for any t.
M2. $E[\eta^i(t)\eta^j(t)] = \omega_{ij} < \infty$ for any t, $i, j = 1, 2$.

or **(MD)**

the microstructure noise is correlated with the price process and there is also a temporal dependence in the noise components

Quadratic covariation type

The following estimators are based on the choice of a synchronization procedure, which gives the observations times $\{0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq 2\pi\}$ for both assets

Realized covariation $$RC^{12} := \sum_{i=1}^{n-1} \delta_i(p^1)\delta_i(p^2),$$

Realized covariation with leads and lags $$RCLL^{12} := \sum_{i} \sum_{h=-l}^{L} \delta_{i+h}(p^1)\delta_i(p^2),$$

Realized covariance kernels estimator $$RCLLW^{12} := \sum_{i} \sum_{h=-l}^{L} w(h)\delta_{i+h}(p^1)\delta_i(p^2),$$

where $\delta_i(p^*) = p^*(\tau_{i+1}) - p^*(\tau_i)$, and $w(h)$ is a kernel.
inconsistent for asynchronous observations and **inconsistent under (i.i.d) noise**, the MSE diverges as the number of observations increases; $RCLL^{1,2}$, $RCLLW^{1,2}$ more robust to microstructure noise, but they are much biased by dependent noise contaminations [Griffin and Oomen, 2010]

Refresh times consistent estimators

• [Barndorff-Nielsen and al., 2008a] Realized covariance kernels with refresh times consistent for asynchronous observations/robust to some kind of noise

$$K^{12} := \sum_{h=-n}^{n} k\left(\frac{h}{H+1}\right)\Gamma_h^{12},$$

Γ_h^{12} is h-th realised autocovariance of the two assets, $k(\cdot)$ belongs to a suitable class of kernel functions (Parzen).

• [Kinnebrock and Podolskij, 2008] Modulated Realised Covariation
pre-averaging technique to reduce the microstructure effects (if one averages a number of observed log-prices, one is closer to the latent process $p(t)$)

Consistent estimator

- [Hayashi and Yoshida, 2005] All-overlapping estimator

$$AO^{12} := \sum_{i,j} \delta_{I_i^1}(p^1)\delta_{I_j^2}(p^2) I_{(I_i^1 \cap I_j^2 \neq \emptyset)},$$

where $\delta_{I_i^*}(p^*) := p^*(t_{i+1}^*) - p^*(t_i^*)$. Consistent for asynchronous observations, not robust to noise.

- [Voev et Lunde, 2007] Sub-sampled All-overlapping estimator

$$AO_{sub}^{12} := \frac{1}{S} \sum_{s=1}^{S} AO^{12}(s),$$

where the $AO^{12}(s)$'s are computed on different non overlapping subgrids using only the skip-S returns for the base asset.

MSE

regular asynchronous trading:
the asset 1 trades at regular points: $\Pi^1 = \{t_i^1 : i = 1, \ldots, n_1 \text{ and } t_{i+1}^1 - t_i^1 = \frac{2\pi}{n_1}\}$; also asset 2 trades at regular points: $\Pi^2 = \{t_j^2 : j = 1, \ldots, n_2 \text{ and } t_{j+1}^2 - t_j^2 = \frac{4\pi}{n_1}\}$, but no trade of asset 1 occurs at the same time of a trade of asset 2

$$MSE_{AO} = o(1)$$

$$MSE_{AOm} = o(1) + 2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt] + 2\omega_{22} \sum_{i=1}^{n-1} E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{11}(t)dt] +$$
$$+2(n-1)\omega_{11}\omega_{22}$$

MSE

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$$+2(n-1)\omega_{11}\omega_{22}$$

$$MSE_F = o(1)$$

$$MSE_{Fm} = o(1) + 2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} D_N^2(t_{n-1}^1 - t_j^2) E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt] +$$

$$+2\omega_{22} \sum_{i=1}^{n-1} D_N^2(t_i^1 - t_{\frac{n}{2}-1}^2) E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{11}(t)dt] + 4\omega_{11}\omega_{22} D_N^2(t_{n-1}^1 - t_{\frac{n}{2}-1}^2)$$

where $D_N(t) := \frac{1}{2N+1} \frac{\sin[(N+\frac{1}{2})t]}{\sin \frac{t}{2}}$

Fourier estimator properties

1) uses all the available observations, no synchronization of the original data: it is based on the **integration of the time series of returns** rather than on its differentiation
2) it is designed specifically **for high frequency data**: by cutting the highest frequencies, it uses as much as possible of the sample path without being more sensitive to market frictions
3) later...

Montecarlo Analysis

We simulate discrete data from the continuous time bivariate GARCH model

$$\begin{bmatrix} dp^1(t) \\ dp^2(t) \end{bmatrix} = \begin{bmatrix} \beta_1\sigma_1^2(t) \\ \beta_2\sigma_4^2(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \\ \sigma_3(t) & \sigma_4(t) \end{bmatrix} \begin{bmatrix} dW_5(t) \\ dW_6(t) \end{bmatrix}$$

$$d\sigma_i^2(t) = (\omega_i - \theta_i\sigma_i^2(t))dt + \alpha_i\sigma_i^2(t)dW_i(t), \quad i = 1, \ldots, 4,$$

The logarithmic noises $\eta^1(t), \eta^2(t)$ are i.i.d. Gaussian, possibly contemporaneously correlated and independent from p.

We generate second-by-second return and variance paths over a daily trading period of $h = 6$ hours. Then we sample the observations according to different scenarios: *regular synchronous trading* with durations $\rho_1 = \rho(n_1)$ and $\rho_2 = 2\rho_1$; *regular non-synchronous trading* with durations ρ_1 and $\rho_2 = 2\rho_1$ and displacement $\delta \cdot \rho_1$; *Poisson trading* with durations between trades drawn from an exponential distribution with means λ_1, λ_2.

	Reg-NS		Reg-S + Unc		Reg-NS + Unc		Reg-NS + Cor	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	5.72e-4	-9.88e-3	3.35e-4	-6.09e-3	7.29e-4	-1.12e-2	4.73e-4	-8.82e-3
$RC^{12}_{0,5min}$	2.96e-2	-1.68e-1	1.06e-3	8.80e-4	3.45e-2	-1.80e-1	3.20e-2	-1.74e-1
RC^{12}_{1min}	9.14e-3	-8.44e-2	2.08e-3	2.70e-3	1.12e-2	-9.16e-2	9.74e-3	-8.65e-2
RC^{12}_{5min}	1.16e-2	-1.80e-2	1.14e-2	5.00e-3	1.44e-2	-2.33e-2	1.13e-2	-1.68e-2
$RCLL^{12}_{0,5min}$	2.88e-3	-1.68e-3	3.34e-3	2.94e-3	3.71e-3	-2.43e-3	3.15e-3	-1.55e-3
$RCLL^{12}_{1min}$	6.40e-3	-3.13e-3	6.42e-3	5.04e-3	8.00e-3	-3.37e-4	6.13e-3	3.09e-3
$RCLL^{12}_{5min}$	3.35e-2	1.11e-2	3.12e-2	3.15e-4	4.23e-2	-7.22e-3	3.61e-2	6.79e-3
AO^{12}	4.72e-4	-1.20e-3	4.47e-4	-1.08e-3	6.88e-4	9.45e-4	5.98e-4	-5.91e-4
K^{12}	9.33e-4	-8.13e-3	9.13e-4	-5.22e-4	1.28e-3	-6.32e-3	1.09e-3	-7.18e-3
MRC^{12}	2.80e-3	-3.27e-2	2.57e-3	-2.55e-2	3.38e-3	-3.01e-2	2.91e-3	-2.87e-2

	Reg-NS + Dep		Poisson + Unc		Poisson + Cor		Poisson + Dep	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	3.96e-4	-6.32e-3	1.07e-3	-1.38e-2	1.18e-3	-1.53e-2	1.00e-3	-1.43e-2
$RC^{12}_{0,5min}$	3.02e-2	-1.66e-1	3.33e-2	-1.76e-1	3.11e-2	-1.70e-1	2.91e-2	-1.64e-1
RC^{12}_{1min}	9.97e-3	-8.17e-2	1.08e-2	-8.95e-2	1.05e-2	-8.85e-2	1.03e-2	-8.62e-2
RC^{12}_{5min}	1.47e-2	-1.70e-2	1.28e-2	-2.50e-2	1.36e-2	-2.06e-2	1.23e-2	-2.64e-2
$RCLL^{12}_{0,5min}$	4.42e-3	3.20e-3	3.81e-3	-7.98e-3	3.40e-3	-6.84e-3	3.73e-3	-9.08e-3
$RCLL^{12}_{1min}$	8.06e-3	-9.21e-4	6.81e-3	-3.41e-3	7.23e-3	1.26e-3	7.80e-3	3.78e-3
$RCLL^{12}_{5min}$	3.59e-2	-1.60e-2	3.31e-2	-3.59e-3	3.74e-2	6.35e-3	3.67e-2	-1.47e-2
AO^{12}	7.42e-3	7.46e-2	1.29e-3	-8.75e-4	1.24e-3	9.32e-3	8.10e-3	7.49e-2
K^{12}	5.25e-3	5.43e-2	5.88e-3	-6.35e-2	4.57e-3	-5.46e-2	2.85e-3	-1.95e-2
MRC^{12}	3.93e-3	-1.59e-2	4.19e-3	-3.00e-2	3.71e-3	-2.71e-2	4.72e-3	-2.24e-2

Tabella: Comparison of integrated volatility estimators. The noise variance is 90% of the total variance for 1 second returns. $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

	Reg-S + Unc		Reg-NS + Unc		Reg-NS + Cor		Reg-NS + Dep	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	3.41e-4	-6.07e-3	6.14e-4	-9.26e-3	4.84e-4	-8.06e-3	3.50e-4	-4.88e-3
$RC^{12}_{0,5min}$	2.00e-3	4.20e-4	3.65e-2	-1.81e-1	3.41e-2	-1.78e-1	5.71e-2	-1.68e-1
RC^{12}_{1min}	2.69e-3	-2.10e-3	1.22e-2	-9.36e-2	1.09e-2	-8.71e-2	2.34e-2	-8.50e-2
RC^{12}_{5min}	1.10e-2	-2.29e-3	1.61e-2	-1.84e-2	1.38e-2	-1.92e-2	1.87e-2	-1.59e-2
$RCLL^{12}_{0,5min}$	3.95e-3	-3.33e-3	5.03e-3	-1.63e-4	4.24e-3	7.19e-4	1.83e-2	-1.31e-3
$RCLL^{12}_{1min}$	6.94e-3	1.29e-3	9.24e-3	-3.85e-3	8.14e-3	4.15e-3	1.88e-2	3.26e-3
$RCLL^{12}_{5min}$	2.98e-2	6.90e-3	4.56e-2	-7.88e-4	3.93e-2	7.38e-4	4.14e-2	1.37e-2
AO^{12}	1.95e-3	7.56e-4	2.18e-3	1.54e-3	2.23e-3	4.78e-3	4.42e-2	7.40e-2
K^{12}	1.71e-3	-1.01e-3	2.18e-3	-1.90e-3	2.18e-3	2.87e-6	2.39e-2	5.62e-2
MRC^{12}	3.19e-3	-1.54e-2	4.33e-3	-1.71e-2	3.82e-3	-1.45e-2	6.56e-3	-1.17e-2

	Poisson + Unc		Poisson + Cor		Poisson + Dep		
	MSE	bias	MSE	bias	MSE	bias	
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	1.26e-3	-1.68e-2	1.10e-3	-1.40e-2	5.36e-4	-8.04e-3	
$RC^{12}_{0,5min}$	3.50e-2	-1.80e-1	3.00e-2	-1.63e-1	4.95e-2	-1.57e-1	
RC^{12}_{1min}	1.24e-2	-9.53e-2	1.09e-2	-8.32e-2	2.58e-2	-8.45e-2	
RC^{12}_{5min}	1.27e-2	-2.36e-2	1.48e-2	-5.81e-3	1.98e-2	-7.07e-3	
$RCLL^{12}_{0,5min}$	4.62e-3	-1.01e-2	5.61e-3	-6.71e-3	2.16e-2	-4.23e-3	
$RCLL^{12}_{1min}$	7.61e-3	-1.24e-3	9.35e-3	7.58e-3	1.90e-2	3.88e-3	
$RCLL^{12}_{5min}$	3.83e-2	-1.34e-2	4.21e-2	1.83e-2	4.43e-2	1.59e-3	
AO^{12}	2.58e-3	-3.12e-3	1.38e-2	1.06e-1	4.42e-2	8.25e-2	
K^{12}	6.95e-3	-6.62e-2	2.88e-3	-5.89e-3	2.13e-2	-9.48e-3	
MRC^{12}	5.03e-3	-2.23e-2	5.26e-3	-1.55e-2	7.78e-3	-1.13e-2	

Tabella: Comparison of integrated volatility estimators. The noise is ten times the one in Table 1. $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

	Reg-S + Unc		Reg-NS + Unc		Reg-NS + Cor		Reg-NS + Dep	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	3.42e-4	-3.28e-3	3.93e-4	-4.93e-3	4.37e-4	-3.86e-3	8.67e-4	-4.90e-3
$RC^{12}_{0,5min}$	3.81e-2	4.01e-3	6.92e-2	-1.66e-1	8.73e-2	-1.81e-1	2.00e+0	-1.47e-1
RC^{12}_{1min}	2.26e-2	-4.08e-3	3.35e-2	-8.09e-2	4.31e-2	-8.67e-2	1.14e+0	-1.19e-1
RC^{12}_{5min}	1.93e-2	-4.05e-3	2.21e-2	-1.48e-2	2.67e-2	-8.87e-3	2.84e-1	-5.89e-2
$RCLL^{12}_{0,5min}$	2.77e-2	5.92e-3	3.46e-2	-1.57e-3	4.28e-2	2.48e-3	1.37e+0	-3.36e-2
$RCLL^{12}_{1min}$	2.29e-2	-1.27e-3	2.59e-2	-9.86e-4	3.45e-2	-8.57e-3	6.82e-1	1.37e-2
$RCLL^{12}_{5min}$	4.47e-2	1.02e-3	4.46e-2	1.02e-3	4.91e-2	1.48e-2	2.22e-1	-6.84e-4
AO^{12}	9.76e-2	5.38e-3	7.71e-2	2.49e-2	9.23e-2	-7.94e-3	4.40e+0	-8.95e-3
K^{12}	3.69e-2	-2.57e-3	3.80e-2	1.67e-2	4.94e-2	-7.48e-3	2.14e+0	2.44e-2
MRC^{12}	6.42e-3	-1.66e-2	7.74e-3	-1.40e-2	8.04e-3	-9.84e-3	1.25e-2	-2.21e-2

	Poisson + Unc		Poisson + Cor		Poisson + Dep			
	MSE	bias	MSE	bias	MSE	bias		
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	1.14e-3	-1.26e-2	5.35e-4	-5.62e-3	5.24e-4	-3.54e-3		
$RC^{12}_{0,5min}$	9.50e-2	-2.10e-1	5.11e-2	-4.78e-2	1.82e+0	-1.44e-1		
RC^{12}_{1min}	4.71e-2	-1.04e-1	3.00e-2	-1.54e-2	1.03e+0	-6.62e-2		
RC^{12}_{5min}	2.79e-2	-3.07e-2	2.39e-2	-1.75e-2	3.01e-1	-3.93e-2		
$RCLL^{12}_{0,5min}$	4.13e-2	-1.00e-2	3.70e-2	3.25e-4	1.43e+0	6.61e-2		
$RCLL^{12}_{1min}$	3.18e-2	1.08e-2	2.87e-2	-8.09e-3	6.96e-1	-3.81e-2		
$RCLL^{12}_{5min}$	5.88e-2	1.61e-2	4.39e-2	-2.27.e-3	2.40e-1	-3.03e-2		
AO^{12}	8.83e-2	5.85e-3	1.27e+0	1.07e+0	2.91e+0	1.12e-1		
K^{12}	4.87e-2	-5.59e-2	2.63e-1	4.70e-1	1.61e+0	1.83e-3		
MRC^{12}	1.23e-2	-2.12e-2	9.94e-3	-2.22e-2	1.58e-2	-2.66e-2		

Tabella: Comparison of integrated volatility estimators. Increased Noise (as in [Griffin and Oomen, 2010]). $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

Again: Fourier estimator properties

1) uses all the available observations, no synchronization of the original data: it is based on the **integration of the time series of returns** rather than on its differentiation
2) it is designed specifically **for high frequency data**: by cutting the highest frequencies, it uses as much as possible of the sample path without being more sensitive to market frictions

Focus

3) it allows to reconstruct the volatility/covariance as a **stochastic function of time**: we can handle the volatility function as an observable variable. This property makes possible to iterate the volatility functor.

Stochastic Volatility Model

[Barucci and M., 2010]

$$\begin{cases} dp(t) = \sigma(t)dW_0(t) + a(t)dt \\ dv(t) = \gamma(t)dZ(t) + b(t)dt \end{cases}$$

$v(t) := \sigma^2(t)$ is the variance process, W_0 and Z correlated Brownian motions

The second equation can be rewritten as

$$dv(t) = b(t)dt + \alpha(t)dW_0(t) + \beta(t)dW_1(t)$$

where W_0 and W_1 are independent Brownian motions.

$\eta(t)dt = dW_0(t) * dZ(t) \Rightarrow dZ(t) = \eta(t)dW_0(t) + \sqrt{1 - \eta^2(t)}dW_1(t)$.

Method

Compute pathwise the diffusion coefficients $\sigma(t)$, $\gamma(t)$ and the covariance between the price and the instantaneous variance, $\varrho(t)$, given the observation of the asset price trajectory $p(t)$, $t \in [0, T]$

1. compute the Fourier coefficients of the unobservable instantaneous variance process $v(t)$, $t \in [0, T]$ in terms of the Fourier coefficients of $p(t) \Rrightarrow v(t)$ is reconstructed from its Fourier coefficients by the Fourier-Fejer summation method

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3. finally compute $\eta(t)$ by to the identity $\varrho(t) = \eta(t)\sigma(t)\gamma(t)$ with $\sigma(t)$ and $\gamma(t)$ a.s. positive

Volatility of volatility

- Derive an estimator for Fourier coefficients $(c_k(\gamma^2))$ of $\gamma^2(t)$ given the observations of the variance process:
  By parts
  $$c_k(dv_{n,M}) = \mathrm{i}kc_k(v_{n,M}) + \frac{1}{2\pi}(v_{n,M}(2\pi) - v_{n,M}(0)),$$
  where $c_k(v_{n,M})$ were computed from dp
- Let
  $$c_k(\gamma^2_{n,N,M}) := \frac{2\pi}{2N+1} \sum_{|j|\le N} c_j(dv_{n,M})c_{k-j}(dv_{n,M})$$
- If $\frac{N^4}{M} \to 0$ and $M^{\frac{5}{4}}\rho(n) \to 0$ for $n, N, M \to \infty$
  $$P - \lim_{n,N,M,L\to\infty} c_k(\gamma^2_{n,N,M,L}) = c_k(\gamma^2)$$

Leverage

To compute the instantaneous covariance $\varrho(t)$ we exploit the multivariate version of Fourier estimator.

- obtain a consistent estimator of the k-th Fourier coefficient of $\varrho(t)$ starting from the Fourier coefficients of the observed asset returns.

$$c_k(\varrho_{n,N,M}) = \frac{2\pi}{2N+1} \sum_{|j| \leq N} c_j(dp_n) c_{k-j}(dv_{n,M})$$

- If $\frac{N^2}{M} \to 0$, $N^2 \rho(n) \to 0$ and $M\rho(n) \to 0$ for $n, N, M \to \infty$, then

$$P - \lim_{n,N,M \to \infty} c_k(\varrho_{n,N,M}) = c_k(\varrho)$$

(Preliminary) Montecarlo Analysis

Replicate [Bollerslev and Zhou, 2002] estimating ξ, $\xi\eta(=\varrho)$ using Fourier approach and square root process:

$$dp(t) = \sqrt{v(t)}dW_0(t)$$

$$dv(t) = k(\theta - v(t))dt + \xi\sqrt{v(t)}dZ(t)$$

k=mean reversion
θ=long run
ξ= volatility of variance

W_0, Z are standard Brownian motions $dW_0(t) * dZ(t) = \eta dt$

Montecarlo Analysis

We consider the three parameter scenarios suggested in [Bollerslev and Zhou, 2002]:

$$\text{Scenario A}: \quad \mathrm{k} = 0.03, \; \theta = 0.25, \; \xi = 0.1,$$

$$\text{Scenario B}: \quad \mathrm{k} = 0.1, \; \theta = 0.25, \; \xi = 0.1,$$

$$\text{Scenario C}: \quad \mathrm{k} = 0.1, \; \theta = 0.25, \; \xi = 0.2,$$

Two values of η: $\eta = -0.2$ and $\eta = -0.7$
We reports average and median estimates of ξ, $\xi\eta (= \varrho)$ and the corresponding standard deviation obtained with the Fourier approach for the three sets of parameters (daily observations).

True values	Mean		Median		Standard Deviation	
	T=1000	T=4000	T=1000	T=4000	T=1000	T=4000
Panel A						
$\xi\eta = -0.02$	-0.0220	-0.0221	-0.0125	-0.0262	0.2157	0.1474
$\xi = 0.1$	0.1040	0.1014	0.1040	0.1014	0.0890	0.0768
Panel A						
$\xi\eta = -0.07$	-0.0706	-0.0729	-0.0622	-0.0730	0.2201	0.2106
$\xi = 0.1$	0.1075	0.1048	0.1075	0.1048	0.0856	0.0138
Panel B						
$\xi\eta = -0.02$	-0.0181	-0.0282	-0.0177	-0.0201	0.2865	0.2488
$\xi = 0.1$	0.1012	0.1069	0.1012	0.1069	0.0699	0.0695
Panel B						
$\xi\eta = -0.07$	-0.0717	-0.0737	-0.1314	-0.0711	0.2828	0.2560
$\xi = 0.1$	0.1330	0.1075	0.1331	0.1075	0.1188	0.0753
Panel C						
$\xi\eta = -0.04$	-0.0469	-0.0409	-0.1394	-0.0373	0.2707	0.1987
$\xi = 0.2$	0.2023	0.2066	0.2341	0.2165	0.1474	0.0892
Panel C						
$\xi\eta = -0.14$	-0.1263	-0.1569	-0.1442	-0.1561	0.3380	0.0616
$\xi = 0.2$	0.1994	0.2006	0.1984	0.2130	0.1571	0.0926

Tabella: Average value, median value and standard deviation of ξ and of $\xi\eta$ for three parameter scenarios, two correlation values and two choices of the size of the simulation sample.

Simulation results are satisfactory. The mean and the median of the parameters obtained in Table 4 are similar to those obtained in [Bollerslev and Zhou, 2002], only the standard deviation is slightly higher. Note that the two methodologies are different: the methodology in [Bollerslev and Zhou, 2002] exploits the knowledge of the square root model that generates the asset price observations, our methodology instead is model free and is able to recover the parameters of the data generating process without making a parametric assumption.

The performance of Fourier method is comparable to the one of the parametric method proposed in [Bollerslev and Zhou, 2002]. This exercise is only an illustrative example to show the efficiency of the method: as a matter of fact, parametric methods exploiting the assumption of a model, are expected to outperform non parametric methods. Further analysis on going...

Portfolio choice

Evaluation of the economic benefit applying different methods of high frequency estimation of covariance from the perspective of an **asset-allocation** decision problem [Mancino et Sanfelici, 2010]:

Compare the utility obtained by virtue of covariance forecasts based on the Fourier estimator to the utility obtained through covariance forecasts constructed using the other estimators

Remark: we focus on covariance estimators, the results are fully justified by considering the properties of the different estimators for both the variance and the covariance measures

Portfolio choice

R^f=risk-free return, R_{t+1}=return vector on k risky assets over a day $[t, t+1]$, $\mu_t = E_t[R_{t+1}]$ and $\Sigma_t = E_t[(R_{t+1} - \mu_t)(R_{t+1} - \mu_t)']$

Consider a mean-variance investor who solves the problem

$$\min_{w_t} w_t' \Sigma_t w_t \quad \text{subject to} \quad w_t' \mu_t + (1 - w_t' \mathbf{1}_k) R^f = \mu_p,$$

where w_t=k-vector of portfolio weights, μ_p=target expected return on the portfolio,$\mathbf{1}_k = (1, \ldots, 1)^T$. The solution to this program is

$$w_t = \frac{(\mu_p - R^f)\Sigma_t^{-1}(\mu_t - R^f \mathbf{1}_k)}{(\mu_t - R^f \mathbf{1}_k)' \Sigma_t^{-1}(\mu_t - R^f \mathbf{1}_k)}.$$

We estimate Σ_t using one-day-ahead forecasts $\hat{C}_t$ given a time series of daily covariance estimates, obtained using some different estimators.

Portfolio

Given R^f, μ_p and μ_t, each one-day-ahead forecast leads to a daily portfolio weight w_t: the time series of daily portfolio weights then leads to daily portfolio returns. Choose $\mu_t = E_t[R_{t+1}]$= sample means of the returns on the risky assets over the forecasting horizon.
We employ the investor's long-run mean-variance utility as a metric to evaluate the economic benefit of alternative covariance forecasts $\hat{C}_t$:

$$U^* = \bar{R}^p - \frac{\lambda}{2}\frac{1}{m}\sum_{t=1}^{m}(R^p_{t+1} - \bar{R}^p)^2,$$

$R^p_{t+1} = R^f + w_t'(R_{t+1} - R^f \mathbf{1}_k)$ =return on portfolio with estimated weights w_t,

$\bar{R}^p = \frac{1}{m}\sum_{t=1}^{m} R^p_{t+1}$=sample mean of portfolio returns across $m \leq n$ days,

λ=coefficient of risk-aversion.

Portfolio

[Engle and Colacito, 2006] in order to avoid contaminations induced by noisy first moment estimation, consider the variance component of U^*

$$U = \frac{\lambda}{2}\frac{1}{m}\sum_{t=1}^{m}(R^p_{t+1} - \bar{R}^p)^2,$$

$U^A - U^B$ the fee that the investor would be willing to pay to switch from covariance forecasts based on estimator A to covariance forecasts based on estimator B

Implementation: R_f=0.03 (converted to daily values by dividing by 250), three targets μ_p, namely 0.09, 0.12, 0.15. For all times t, the conditional covariance matrix is computed as an out-of-sample forecast based on the different variance/covariance estimates

Portfolio

Method	$\mu_p = 0.09$			$\mu_p = 0.12$			$\mu_p = 0.15$		
λ	2	7	10	2	7	10	2	7	10
RC^{1min}	1.907	6.675	9.536	4.291	15.019	21.456	7.629	26.701	38.144
RC^{5min}	0.361	1.262	1.803	0.811	2.839	4.056	1.442	5.048	7.211
RC^{10min}	1.801	6.303	9.004	4.052	14.181	20.258	7.203	25.210	36.014
$RCLL^{1min}$	-1.817	-6.359	-9.084	-4.088	-14.308	-20.439	-7.267	-25.436	-36.337
$RCLL^{5min}$	3.245	11.359	16.227	7.302	25.557	36.510	12.981	45.435	64.906
$RCLL^{10min}$	8.587	30.056	42.937	19.321	67.625	96.607	34.349	120.222	171.746
RC^{opt}	0.110	0.385	0.551	0.248	0.867	1.239	0.441	1.542	2.203
AO	5.236	18.326	26.180	11.781	41.133	58.905	20.944	73.304	104.720
KER	-1.169	-4.090	-5.844	-2.630	-9.204	-13.148	-4.675	-16.362	-23.374
SUB	-0.980	-3.429	-4.898	-2.204	-7.714	-11.020	-3.918	-13.714	-19.592

Tabella: Annualized fees $U^{\hat{C}} - U^{Fourier}$ that a mean-variance investor would be willing to pay to switch from $\hat{C}$ to Fourier estimates. Case $\omega_{ii}^{1/2} = 0.002$.

Note: a positive number is evidence in favor of of better performance of Fourier estimator over $\hat{C}$

Portfolio

Method	$\mu_p = 0.09$			$\mu_p = 0.12$			$\mu_p = 0.15$		
λ	2	7	10	2	7	10	2	7	10
RC^{1min}	4.673	16.355	23.364	10.514	36.799	52.570	18.691	65.420	93.457
RC^{5min}	2.803	9.811	14.015	6.307	22.074	31.535	11.212	39.243	56.061
RC^{10min}	3.505	12.268	17.526	7.887	27.603	39.433	14.020	49.072	70.103
$RCLL^{1min}$	0.747	2.613	3.733	1.680	5.880	8.399	2.986	10.452	14.932
$RCLL^{5min}$	5.145	18.009	25.727	11.577	40.520	57.886	20.582	72.036	102.909
$RCLL^{10min}$	5.247	18.363	26233	11.805	41.317	59.024	20.986	73.452	104.931
RC^{opt}	2.168	7.588	10.840	4.878	17.073	24.390	8.672	30.352	43.360
AO	4.206	14.722	21.032	9.464	33.125	47.322	16.826	58.889	84.128
KER	3.088	10.808	15.440	6.948	24.318	34.740	12.352	43.232	61.760
SUB	1.644	5.755	8.221	3.700	12.948	18497	6.577	23.018	32.883

Tabella: Annualized fees $U^{\hat{C}} - U^{Fourier}$ that a mean-variance investor would be willing to pay to switch from $\hat{C}$ to Fourier estimates. Case $\omega_{ii}^{1/2} = 0.004$.

Portfolio

Method	$\mu_p = 0.09$			$\mu_p = 0.12$			$\mu_p = 0.15$		
λ	2	7	10	2	7	10	2	7	10
RC^{1min}	4.944	17.305	24.722	11.125	38.937	55.624	19.778	69.221	98.887
RC^{5min}	1.805	6.316	9.023	4.060	14.211	20.301	7.218	25.264	36.091
RC^{10min}	2.311	8.090	11.557	5.201	18.202	26.002	9.245	32.359	46.227
$RCLL^{1min}$	-0.66	-0.232	-0.332	-0.149	-0.522	-0.746	-0.265	-0.929	-1.327
$RCLL^{5min}$	2.823	9.880	14.115	6.352	22.231	31.758	11.292	39521	56.458
$RCLL^{10min}$	4.689	16.412	23.446	10.551	36.927	52.753	18.757	65.649	93.784
RC^{opt}	1.555	5.442	7.774	3.498	12.243	17.491	6.219	21.766	31.094
AO	8.509	29.782	42.546	19.146	67.010	95.728	34.037	119.128	170.183
KER	0.918	3.213	4.590	2.066	7.229	10.328	3.672	12.852	18.360
SUB	-0.417	-0.461	-2.087	-0.939	-3.287	-4.695	-1.669	-5.843	-8.347

Tabella: Annualized fees $U^{\hat{C}} - U^{Fourier}$ that a mean-variance investor would be willing to pay to switch from $\hat{C}$ to Fourier estimates. Dependent noise, with $\omega_{ii}^{1/2} = 0.004$.

In order to produce **feasible central limit theorems** for all the estimators, and as a consequence feasible confidence intervals, it is necessary to obtain **efficient estimators of the so called quarticity**, which appears as conditional variance in the central limit theorems.

Nevertheless, the studies about estimation of quarticity are still few: [Barndorff-Nielsen and al., 2008a] remark that *estimating integrated quarticity reasonably efficiently is a tougher problem than estimating the integrated volatility*, as the effect of noise is magnified up.

Let p be a semi-martingale p satisfying assumption **(B)**

$$\mathcal{F}(\sigma^2)(k) = \lim_{N\to\infty} \frac{2\pi}{2N+1} \sum_{|s|\leq N} \mathcal{F}(dp)(s)\,\mathcal{F}(dp)(k-s), \quad \textit{for all } k \in \mathbf{Z}.$$

The second step consists in the computation of the k-th Fourier coefficient of $\sigma^4(t)$.

Theorem

The k-th Fourier coefficient of the function $\sigma^4(t)$ is obtained as the following limit in probability

$$\mathcal{F}(\sigma^4)(k) = \lim_{M\to\infty} \sum_{|s|<M} (1 - \frac{|s|}{M})\mathcal{F}(\sigma^2)(s)\,\mathcal{F}(\sigma^2)(k-s).$$

Consistency

Fourier estimator of quarticity by

$$\sigma^4_{n,N,M} := 2\pi \sum_{|s|<M} (1 - \frac{|s|}{M}) c_s(\sigma^2_{n,N}) c_{-s}(\sigma^2_{n,N}). \tag{4}$$

Theorem

Let $\sigma^4_{n,N,M}$ defined in (4). If $NM\rho(n) \to 0$ and $\frac{M^2}{N} \to 0$ as $M, N, n \to \infty$, then the following convergence in probability holds

$$\lim_{n,N,M\to\infty} \sigma^4_{n,N,M} = \int_0^{2\pi} \sigma^4(t)dt.$$

We simulate second-by-second return and variance paths over a daily trading period of $T = 6$ hours, for a total of 252 trading days and $n = 21600$ observation per day
CIR square-root model :

$$\begin{aligned} dp(t) &= \sigma(t)\ dW_1(t) \\ d\sigma^2(t) &= \alpha(\beta - \sigma^2(t))dt + \nu\sigma(t)\ dW_2(t), \end{aligned} \tag{5}$$

where W_1, W_2 are independent Brownian motions.
(Parameters $\alpha = 0.01$, $\beta = 1.0$, $\nu = 0.05$. The initial value of σ^2 is set equal to one, while $p(0) = \log 100$)

$$RQ := \frac{n}{3T} \sum_{i=0}^{n-1} \delta_i(p)^4 \quad \text{[Barndorff-Nielsen and Shephard, 2002]}$$

$$RQ_{sub} := \frac{1}{S} \sum_{s=1}^{S} RQ^{(s)} \quad \text{[Ghysels and Sinko, 2007]}$$

where the $RQ^{(s)}$'s are computed on different non overlapping subgrids using skip-S returns

$$BQ := \frac{n}{T} \sum_{i=1}^{n-1} |\delta_i(p)|^2 |\delta_{i-1}(p)|^2 \quad \textit{realized bipower quarticity} \text{ [Barndorff-Nielsen and Shephard, 2004a]}$$

$$TQ_1 := \mu_{4/3}^{-3} \frac{n^2}{(n-2)T} \sum_{i=2}^{n-1} |\delta_i(p)|^{4/3} |\delta_{i-1}(p)|^{4/3} |\delta_{i-2}(p)|^{4/3} \quad \textit{realized tripower quarticity}$$

($\mu_p = E(|Z|^p)$, Z is a standard normally distributed random variable)

$$TQ^{(k)} := \mu_{4/3}^{-3} \frac{n^2}{(n-2-2k)T} \sum_{i=2+2k}^{n-1} |\delta_i(p)|^{4/3} |\delta_{i-(1+k)}(p)|^{4/3} |\delta_{i-2(1+k)}(p)|^{4/3} \quad \text{[Andersen et al., 2006]}$$

$$QQ := \mu_1^{-4} \frac{n}{T} \sum_{i=3}^{n-1} |\delta_i(p)||\delta_{i-1}(p)||\delta_{i-2}(p)||\delta_{i-3}(p)| \quad \textit{realized quadpower quarticity} \text{ [Barndorff-Nielsen and Shephard, 2006]}$$

$$Q_{av} = \frac{1}{3\theta^2 \psi_2^2} \sum_{i=0}^{n-k_n+1} (\bar{p}_i^n)^4 - \frac{\rho(n)\psi_1}{\theta^4 \psi_2^2} \sum_{i=0}^{n-2k_n+1} (\bar{p}_i^n)^2 \sum_{j=i+k_n}^{i+2k_n-1} (\delta_j(p))^2 + \frac{\rho(n)\psi_1^2}{4\theta^4 \psi_2^2} \sum_{i=0}^{n-3} (\delta_i(p))^2 (\delta_{i+2}(p))^2,$$

[Jacod et al., 2009], where the pre-averaged price process is

$$\bar{p}_i^n = \frac{1}{k_n} \left(\sum_{j=k_n/2}^{k_n-1} p_{i+j} - \sum_{j=0}^{k_n/2-1} p_{i+j} \right), \quad \theta = k_n \sqrt{\rho(n)},\ \psi_1 = 1,\ \psi_2 = 1/12.$$

	NotF - average		F - average		NotF - day by day		F - day by day	
	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS
Fourier	5.95e-004	1.42e-004	6.62e-004	5.44e-003	1.47e-006	5.54e-005	9.38e-004	2.96e-002
RQ	9.07e-004	3.42e-004	7.90e-004	-7.11e-004	3.97e-005	-2.74e-004	7.84e-004	-1.58e-003
BQ	1.02e-003	-1.40e-003	8.12e-004	-1.91e-003	5.84e-005	-5.47e-004	9.76e-004	-2.27e-003
Q	1.29e-003	1.21e-003	1.19e-003	-1.09e-004	3.89e-005	-3.70e-004	1.19e-003	-1.23e-003
TQ_1	1.05e-003	-2.61e-003	9.51e-004	-1.94e-003	4.68e-005	-3.80e-004	1.22e-003	-8.90e-004
TQ_2	1.05e-003	-3.28e-003	9.49e-004	-2.57e-003	4.78e-005	-4.40e-004	1.22e-003	-1.52e-003
$TQ^{(k)}$	1.10e-003	-1.10e-003	1.06e-003	-1.76e-003	5.23e-005	-6.76e-004	1.16e-003	-3.32e-003
QQ	1.09e-003	-3.94e-003	1.07e-003	-2.60e-003	4.35e-005	-2.89e-004	1.36e-003	-1.89e-003
RQ_{sub}	8.15e-004	-1.31e-003	4.84e-004	-1.20e-003	5.66e-004	-2.79e-003	4.89e-004	-2.11e-003
BQ_{sub}	7.24e-004	-1.36e-002	8.67e-004	-2.02e-002	5.15e-004	-1.60e-002	1.04e-003	-2.51e-002
Q_{av}	3.04e-004	-6.47e-003	3.22e-004	-1.12e-002	1.35e-004	-5.16e-003	3.12e-004	-1.07e-002

Tabella: No noise. F stand for feasible optimization; average means that we optimize the MSE averaged over the 252 trading days, while day by day means that the optimization is performed on each day separately.

	NotF - average		F - average		NotF - day by day		F - day by day	
	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS
Fourier	6.71e-004	6.72e-003	8.87e-004	1.45e-002	3.13e-006	1.14e-004	1.45e-003	3.73e-002
RQ	5.30e-003	2.60e-002	5.44e-003	3.18e-002	1.07e-004	1.65e-003	4.63e-003	2.71e-002
BQ	5.26e-003	3.37e-002	5.67e-003	3.01e-002	8.70e-005	1.18e-003	5.51e-003	2.87e-002
Q	6.83e-003	2.59e-002	7.45e-003	3.26e-002	7.71e-005	-2.71e-004	6.00e-003	2.63e-002
TQ_1	6.27e-003	3.65e-002	7.34e-003	3.75e-002	9.94e-005	6.48e-004	6.38e-003	3.05e-002
TQ_2	5.98e-003	3.37e-002	6.99e-003	3.44e-002	9.01e-005	3.73e-005	6.10e-003	2.74e-002
$TQ^{(k)}$	6.87e-003	3.39e-002	8.41e-003	3.90e-002	1.31e-004	4.66e-004	7.22e-003	2.78e-002
QQ	6.23e-003	3.13e-002	7.21e-003	3.34e-002	7.74e-005	-3.37e-004	6.47e-003	2.70e-002
RQ_{sub}	3.16e-003	2.85e-002	3.17e-003	2.78e-002	2.11e-003	3.00e-002	3.24e-003	2.68e-002
BQ_{sub}	7.59e-004	-1.43e-002	2.41e-003	-9.56e-003	5.09e-004	-1.59e-002	2.36e-003	-1.01e-002
Q_{av}	3.39e-004	-6.81e-003	4.36e-004	-3.37e-003	1.23e-004	-4.28e-003	4.42e-004	-3.65e-003

Tabella: Microstructure effects. $\xi = 4.6341e-005$.

Conclusion

We have proved that the Fourier estimator of covariance is:
(i) consistent under asynchronous trading,
(ii) positive definite,
(iii) asymptotically unbiased in the presence of various types of microstructure noise,
(iv) inconsistent in the presence of microstructure noise, nevertheless the MSE of the Fourier estimator converges to a constant as the number of observations increases
(v) further it allows to treat volatility as an observable variable
$\Rightarrow$ a very interesting alternative especially when microstructure effects are particularly relevant in the available data

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