Estimating the Spot Covariation of Asset Prices – Statistical Theory and Empirical Evidence

Nikolaus Hautsch
University of Vienna, CFS

joint work with

Markus Bibinger (Humboldt University Berlin)
Peter Malec (University of Cambridge)
Markus Reiss (Humboldt University Berlin)

Introduction

Covariance estimation is crucial for

- risk management
- portfolio management
- strategic asset allocation
- asset pricing
- hedging
- quantification of systemic risk
- ...

$\Rightarrow$ Benefit from high-frequency data!

- Recent literature shows strong empirical evidence for distinct time variations in daily and long-term correlations between asset prices.
- But: Surprisingly little is known about intraday variations of asset return covariances.

Questions:

- Do covariances, correlations and betas systematically vary within a day
  $\Rightarrow$ Is there intraday correlation risk?
- How do covariances, correlations and betas behave in extreme market periods?

Why Important?

- Intraday risk management:
  Assess intraday correlation risks.
- Market microstructure research:
  Studies on HF trading, impact of market fragmentation, benefits of circuit breakers.
- Analysis of days with distinct information & "Flash Crashes":
  Asymmetry of correlation behaviour during bull/bear markets at lower frequencies (e.g., De Santis & Gerard, 1997).
  $\Rightarrow$ Similar effects during intraday intervals?
- Crucial for co-jump tests (e.g. Bibinger & Winkelmann, 2014).

In a perfect world ...

- Consider a d-dimensional continuous martingale price process,

$$X_t = X_0 + \int_0^t \Sigma^{1/2}(s)\, dB_s\,, t \in [0,1],$$

  where B_t denotes a standard Brownian motion.

- Objects of interest: $\int_0^t \Sigma(s)ds$ and $\Sigma(s)$.

- If X_t is discretely observed with $X_{i/n}, i = 0, \ldots, n$, a natural estimator for $\int_0^t \Sigma(s)ds$ is

$$\mathrm{RC}_n = \sum_{i=1}^{n} (X_{i/n} - X_{(i-1)/n})(X_{i/n} - X_{(i-1)/n})^\top$$

  with

$$\mathrm{vec}\left(n^{1/2}\left(\mathrm{RC}_n - \int_0^1 \Sigma(t)\, dt\right)\right) \xrightarrow{\mathcal{L}} N\left(0, \int_0^1 \left(\Sigma(t) \otimes \Sigma(t) dt\right) \mathcal{Z}\right).$$

Example

- For $d = 1$:

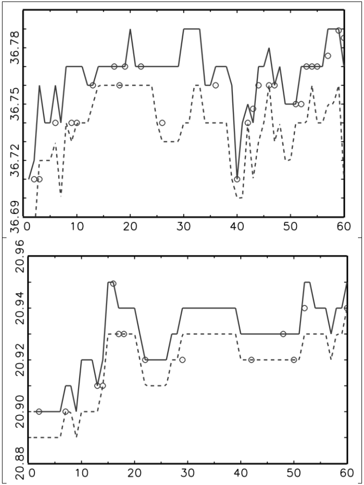
$$n^{1/2}\left(\mathrm{RC}_n - \int_0^1 \sigma^2(s)\,ds\right) \xrightarrow{\mathcal{L}} \mathbf{N}\Big(0, 2\int_0^1 \sigma^4(s)\,ds\Big).$$

- For $d = 2$:

$$\Sigma \otimes \Sigma = \begin{pmatrix} \Sigma_{11}\,\Sigma & \Sigma_{12}\,\Sigma \\ \Sigma_{12}\,\Sigma & \Sigma_{22}\,\Sigma \end{pmatrix},\ \mathcal{Z} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

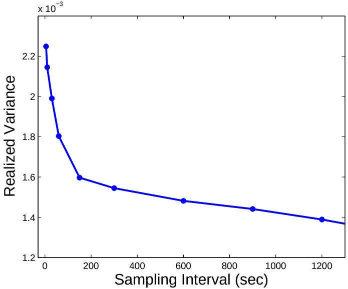
$$\big(\Sigma \otimes \Sigma\big)\mathcal{Z} = \begin{pmatrix} 2\sigma_1^4 & 2\rho\sigma_1^3\sigma_2 & 2\rho\sigma_1^3\sigma_2 & 2\rho^2\sigma_1^2\sigma_2^2 \\ 2\rho\sigma_1^3\sigma_2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 \\ 2\rho\sigma_1^3\sigma_2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & (1+\rho^2)\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 \\ 2\rho^2\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 & 2\rho\sigma_1\sigma_2^3 & 2\sigma_2^4 \end{pmatrix}$$

Real Intraday Price Path

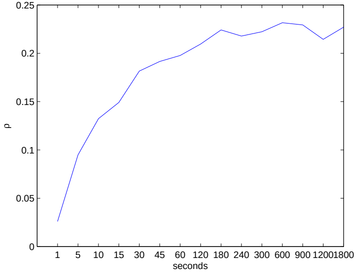


Realized Covariances in Practice

Signature Plot



Epps Effect

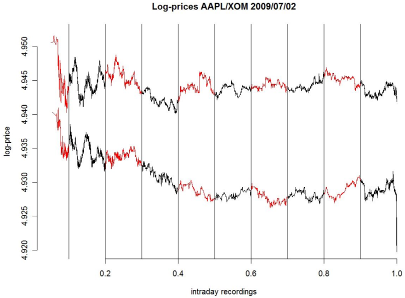


- Challenges:
  - Market microstructure noise
  - Asynchronicity of observations
  - Efficiency
  - Positive definiteness

- Approaches:
  - Hayashi/Yoshida (2011)
  - Realized kernels (Barndorff-Nielsen et al, 2011)
  - Pre-averaging (Christensen et al, 2012)
  - QML (Ait-Sahalia et al, 2010)
  - Spectral estimation (Bibinger/Reiss, 2013)

- Open questions:
  - How to optimally deal with asynchronicity and different speeds in observation frequencies?
  - How to construct spot covariance estimators?

This Paper



Extend and adapt Local Method of Moments (LMM) approach by Bibinger et al. (2014) to spot covariance matrix estimation.
⇒ Build on locally constant approximations of the process
⇒ Robust to microstructure noise and asynchronicity.

- Allow for autocorrelated noise and propose consistent autocorrelation estimators.
  - ⇒ Can use tick-by-tick data.
- Derive stable central limit theorem.
  - ⇒ Prove rate optimality of estimator.
- Simulation study shows optimal implementation of estimator.
- First empirical evidence on spot covariances & correlations.

Relation to Literature

- Integrated **covariance matrix estimation**:
  - Hayashi/Yoshida (2011);
  - Barndorff-Nielsen et al (2011);
  - Christensen et al (2012);
  - Ait-Sahalia et al (2010);
  - Bibinger et al. (2014).

- **Spot** volatility estimation:
  - Foster & Nelson (1996);
  - Kristensen (2010);
  - Mancini et al. (2012);
  - Bos et al. (2012);
  - Zu & Boswijk (2014).

Outline

1. Introduction
2. LMM: Univariate Case
3. Estimation of Spot Covariances
4. Empirical Results
5. Conclusions

2. Local Method of Moments: Univariate Setting

Univariate Setting

- Consider equi-distantly observed (log) price process:

$$Y_{i/n} = X_{i/n} + \varepsilon_{i/n}, \qquad i = 1, \ldots, n, \qquad (\mathcal{E}_0)$$
$$dX_t = \sigma(t) dB_t, \quad \varepsilon_{i/n} \overset{iid}{\sim} N(0, \eta^2),$$

  where $\varepsilon_{i/n}$ denotes microctructure noise with variance η^2.

- Experiment $(\mathcal{E}_0)$ is asymptotically equivalent to the "continuous-time white noise" process

$$dY_t = X_t dt + \psi dW_t, \qquad (\mathcal{E}_1)$$

  where $X_t \perp W_t$ and $\psi := \eta/\sqrt{n}$.

- Asymptotic equivalence (in the Le Cam sense) for $n \to \infty$ provided a certain Hölder-regularity of σ_t (Reiss, 2011).

Local Parametric Approximation

- Consider blocks $[kh, (k+1)\,h]\,, k = 0, \dots, h^{-1} - 1$.
- Assume that block lengths shrink sufficiently fast with increasing n: $h^{\alpha} = o\Big(n^{-1/4}\Big)$ for $\alpha \in (1/2, 1]$.
- Observing ($\mathcal{E}_0$) is asymptotically equivalent to observing

$$dY_t = X_t^h dt + \psi dW_t, \qquad (\mathcal{E}_2)$$

with the efficient (log-) price process

$$dX_t^h = \lfloor \sigma(t) \rfloor_h dB_t, \qquad \lfloor t \rfloor_h = \lfloor t/h \rfloor h,$$

where $\lfloor \sigma(t) \rfloor_h$ denotes the block h-specific constant volatility.

- On block k, we have

$$\tilde{Y}^k_{i^*} = \tilde{X}^k_{i^*} + \varepsilon_{i^*}, \quad i^* = i - khn,$$

with

$$d\tilde{X}^k_{t^*} = \sigma_k \, dB_{t^*}, \quad t^* = t - kh, \quad t \in [kh, (k+1)\, h],$$

σ_k: spot volatility at the beginning of block k.

- Observed returns:

$$\Delta\tilde{Y}^k_{i^*} := \tilde{Y}^k_{i^*} - \tilde{Y}^k_{i^*-1} = \Delta\tilde{X}^k_{i^*} + \varepsilon_{i^*} - \varepsilon_{i^*-1},$$

with $\Delta\tilde{X}^k_{i^*} \overset{\text{i.i.d.}}{\sim} N(0, \sigma_k^2/n)$, $\varepsilon_{i^*} \overset{\text{i.i.d.}}{\sim} N(0, \eta^2)$ and $i^* = 1, \ldots, nh$.

- $\Delta\tilde{Y}^k_{i^*}$ follow MA(1) process with $\mathbb{E}\left[\Delta\tilde{Y}^k_{i^*}\right] = 0$ and

$$\mathbb{C}\text{ov}\left[\Delta\tilde{Y}^k_{i^*}, \Delta\tilde{Y}^k_{i^*-l}\right] = \begin{cases} \sigma_k^2/n + 2\eta^2 & \text{if} \quad l = 0 \\ -\eta^2 & \text{if} \quad l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Spectral Statistics

- Idea: Constructing a statistic in the spectral domain which yields maximal information about $\lfloor \sigma(t) \rfloor_h$.
- Define a set of block-specific functions $\varphi_{jk}(t)$ which form an othornomal system in $L^2([0,1])$.
- Defining $\Phi_{jk}(t) := \int \varphi_{jk}(t)dt$ and setting $\Phi_{jk}(kh) = \Phi_{jk}((k+h)h) = 0$, yields

$$\begin{aligned}\int_{kh}^{(k+1)h} \varphi_{jk}(t)dY_t &= \int_{kh}^{(k+1)h} \varphi_{jk}(t)X_t^h dt + \psi \int_{kh}^{(k+1)h} \varphi_{jk}(t)dW_t, \\ &= -\int_{kh}^{(k+1)h} \Phi_{jk}(t)\lfloor \sigma(t) \rfloor_h dB_t + \psi \int_{kh}^{(k+1)h} \varphi_{jk}(t)dW_t \\ &\overset{d}{=} \left(\int_{kh}^{(k+1)h} \Phi_{jk}^2(t)\lfloor \sigma^2(t) \rfloor_h dt + \psi^2 \right)^{1/2} \xi_{jk},\end{aligned}$$

where $(\xi_{jk})_{j\geq 1}$ is $N(0,1)$ and independent across j.

- Maximizing information load of $\int_{kh}^{(k+1)h} \varphi_{jk}(t)dY_t$ wrt to $\lfloor \sigma^2(t) \rfloor_h$ yields

$$\varphi_{jk} = \sqrt{2/h} \cos\left[\frac{(t-kh)}{h} j\pi\right] \mathbf{1}_{\{kh,(k+1)h\}}$$

with antiderivative given by

$$\Phi_{jk} = \frac{\sqrt{2h}}{jh} \sin\left[\frac{(t-kh)}{h} j\pi\right] \mathbf{1}_{\{kh,(k+1)h\}}.$$

- Then, for the statistics $S_{jk} = \int_{kh}^{(k+1)h} \varphi_{jk}(t)dY_t$, we have

$$\begin{aligned} S_{jk} &\sim N\left(0, \int_{kh}^{(k+1)h} \Phi_{jk}^2 \lfloor \sigma(t)^2 \rfloor_h dt + \psi^2\right) \\ &= N\left(0, \sigma(kh)^2 \int_{kh}^{(k+1)h} \Phi_{jk}^2 dt + \psi^2\right) \end{aligned}$$

where $\sigma(kh) = \lfloor \sigma(t) \rfloor_k$ for $t \in [kh, (k+1)h]$.

- Thus: $S_{jk} \sim N\left(0, \frac{h^2}{j^2\pi^2}\sigma^2(kh) + \psi^2\right)$

Non-Equidistant Observations

- Consider the process

$$Y_i = X_{t_i} + \varepsilon_i, \qquad (\mathcal{E}_0^*)$$

where $t_i = F^{-1}(i/n)$, where $F : [0,1] \to [0,1]$ is a differentiable cdf with $F'() > 0$ denoting the local observation density.

- Then, $(\mathcal{E}_0^*)$ is asymptotically equivalent to

$$dY_t = X_t dt + \psi(t)\, dW_t, \qquad (\mathcal{E}_1^*)$$

where $\psi(t) := \eta/\sqrt{nF'(t)}$.

- Locally constant approximation:

$$dY_t = X_t^h dt + \lfloor \psi(t) \rfloor_h dW_t, \qquad (\mathcal{E}_2^*)$$

with $\lfloor \psi(t) \rfloor_h = \frac{\eta}{\sqrt{n}} \lfloor \frac{1}{F'(t)} \rfloor_h$.

- Then, under $(\mathcal{E}_2^*)$, we have

$$\int_{kh}^{(k+1)h} \varphi_{jk}(t) dY_t = \int_{kh}^{(k+1)h} \varphi_{jk} X_t^h dt + \int_{kh}^{(k+1)h} \varphi_{jk} \lfloor \psi(t) \rfloor_h dW_t$$
$$\stackrel{d}{=} (||\Phi_{jk}||^2 \sigma(kh)^2 + \psi(kh)^2)^{1/2} \xi_{jk},$$

where $\xi_{jk} \sim N(0,1)$.

- Hence:

$$S_{jk} \sim N\left(0, ||\Phi_{jk}||^2 \sigma(kh)^2 + \frac{\eta^2}{nF'(kh)}\right).$$

with $||\Phi_{jk}||^2 := \int_{kh}^{(k+1)h} \Phi_{jk}^2(t) dt = h^2/j^2\pi^2$.

Local Method of Moments Estimation

- $nh-1$ independent moment estimators of σ_k^2:

$$\hat{\sigma}_{jk}^2 := \|\Phi_{jk}\|^{-2} \left(S_{jk}^2 - \frac{\eta^2}{nF'(kh)} \right), \qquad j = 1, \ldots, nh-1.$$

- Combine them to:

$$\hat{\sigma}_k^2 = \sum_{j=1}^{nh-1} w_{jk}\, \hat{\sigma}_{jk}^2 \qquad \text{with} \qquad \sum_{j=1}^{nh-1} w_{jk} = 1.$$

- Minimize variance by choosing weights prop. to Fisher inf. of $\hat{\sigma}_{jk}^2$:

$$w_{jk} = \frac{I_{jk}}{\sum_{l=1}^{nh-1} I_{lk}}, \quad I_{jk} = \frac{1}{2} \left(\sigma_k^2 + \|\Phi_{jk}\|^{-2} \frac{\eta^2}{nF'(kh)} \right)^{-2}.$$

Estimation of Integrated Variance

- Estimator of $\int_0^1 \sigma_t^2 dt$:

$$\widehat{\mathsf{IV}}^{\mathsf{LMM}} := h \sum_{k=0}^{h^{-1}} I_k^{-1} \sum_{j=1}^{nh-1} I_{jk}\, \hat{\sigma}_{jk}^2, \quad I_k := \sum_{j=1}^{nh-1} I_{jk}.$$

- CLT with $n^{1/4}$ rate and $AVAR = 8\eta \int_0^1 \sigma_t^3 dt$ (Reiss, 2011).

3. Estimation of Spot Covariances

Setup

- Efficient log-price X_t follows continuous Itô semi-martingale:

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dB_s, \; t \in [0,1], \tag{1}$$

  where B_s is d-dimensional standard Brownian motion.

- $(d \times d)$ spot covariance matrix: $\Sigma_s = \sigma_s \sigma_s^\top$.

- Observations are non-synchronous and noisy:

$$Y_i^{(p)} = X_{t_i^{(p)}}^{(p)} + \epsilon_i^{(p)}\,, \; i = 0, \ldots, n_p, \; p = 1, \ldots, d\,, \tag{2}$$

  with observation times $t_i^{(p)}$ and observation errors $\epsilon_i^{(p)}$.

- Let $n = \min_{1 \leq p \leq d} n_p$ denote number of obs. of "slowest" asset.
  $\Rightarrow$ HF asymptotics with $n/n_p \to \nu_p$ for $0 < \nu_p < 1$.

Assumption 1

$(b_s)_{s\in[0,1]}$ is a càdlàg process with $b_s \in C^{\nu,R}([0,1],\mathbb{R}^d)$ for some $R<\infty$ and some $\nu>0$.

Assumption 2

(i) $(\sigma_s)_{s\in[0,1]}$ follows a càdlàg process with $\Sigma_s = \sigma_s\sigma_s^\top \geq \underline{\Sigma}$ uniformly for some strictly positive definite matrix $\underline{\Sigma}$.
(ii) For $\sigma_s \in C^{\alpha,R}\big([0,1],\mathbb{R}^{d\times d'}\big)$ with $R<\infty$ and $\alpha\in(0,1/2]$,
$\sigma_s = f\big(\sigma_s^{(1)},\sigma_s^{(2)}\big)$ with $f:\mathbb{R}^{2d\times 2d'}\to\mathbb{R}^{d\times d'}$ continuously differentiable, where
$\sigma_s^{(1)}$ is a continuous Itô semi-martingale and
$\sigma_s^{(2)} \in C^{\alpha,R}\big([0,1],\mathbb{R}^{d\times d'}\big)$ with $R<\infty$.
(iii) For $\sigma_s \in C^{\alpha,R}\big([0,1],\mathbb{R}^{d\times d'}\big)$ with $R<\infty$ and $\alpha\in(1/2,1]$, $\sigma^{(1)}$ vanishes.

Assumption 3

(i) $\epsilon = \{\epsilon_i^{(p)}, i = 0, \ldots, n_p, p = 1, \ldots, d\}$ *is independent of* X *and* $\epsilon_i^{(p)}$ *is independent of* $\epsilon_j^{(q)}$ $\forall\ i, j$ *and* $p \neq q$.
(ii) At least first eight moments of $\epsilon_i^{(p)}, i = 0, \ldots, n_p$, *exist for* $p = 1, \ldots, d$.
(iii) $\mathbb{C}\mathrm{ov}\big(\epsilon_i^{(p)}, \epsilon_{i+u}^{(p)}\big) = 0$ *for* $u > R$, $R < \infty$ *and* $p = 1, \ldots, d$.

Define:

$$\eta_p = \eta_0^{(p)} + 2\sum_{u=1}^{R} \eta_u^{(p)}, \quad \text{with} \quad \eta_u^{(p)} := \mathbb{C}\mathrm{ov}\big(\epsilon_i^{(p)}, \epsilon_{i+u}^{(p)}\big), u \leq R,$$

with $\eta_u^{(p)}, 0 \leq u \leq R$, *constant for all* $0 \leq i \leq n - u$.

Impose $\eta_p > 0$ *for all* p.

Assumption 4

There exist differentiable c.d.f.s F_p, $p = 1, \ldots, d$, *such that observations satisfy* $t_i^{(p)} = F_p^{-1}(i/n_p)$, $0 \le i \le n_p, p \in \{1, \ldots, d\}$, *where* $F_p' \in C^{\alpha,R}\big([0,1], [0,1]\big), p = 1, \ldots, d$, *with* α *being the smoothness exponent in Assumption 2 for* $R < \infty$.

Definition 1

In the asymptotic framework with $n/n_p \to \nu_p$, *where* $0 < \nu_p < \infty, p = 1, \ldots, d$, *for* $n \to \infty$, *define the continuous-time noise level matrix*

$$H_s = \operatorname{diag}\left(\left(\eta_p \nu_p (F_p^{-1})'(s)\right)^{1/2}\right)_{1 \le p \le d} . \tag{3}$$

Local Method of Moments Estimation

- Estimation using LMM approach by Bibinger et al. (2014).
- Partition interval $[0, 1]$ into blocks $[kh_n, (k+1)h_n], k = 0, \dots, h_n^{-1} - 1$ with $h_n \to 0$ as $n \to \infty$.
- Approximate original process by process with block-wise constant covariance matrices Σ_{kh_n} and noise levels H_k^n.

$\Rightarrow$ Estimation error can be asymptotically neglected for sufficient smoothness of Σ_t and F_p and block sizes h_n shrinking sufficiently fast.

- Bibinger et al. (2014) propose **integrated** covariance matrix estimator in **simplified** setting.

$\Rightarrow$ Here: estimate **spot** covariance matrix in **generalized** setting.

- Local spectral statistics:

$$S_{jk} = \pi j h_n^{-1} \left(\sum_{i=1}^{n_p} \left(Y_i^{(p)} - Y_{i-1}^{(p)} \right) \Phi_{jk} \left(\frac{t_{i-1}^{(p)} + t_i^{(p)}}{2} \right) \right)_{1 \le p \le d},$$

where

$$\Phi_{jk}(t) = \frac{\sqrt{2h_n}}{j\pi} \sin\left(j\pi h_n^{-1} \left(t - kh_n\right)\right) \mathbf{1}_{[kh_n,(k+1)h_n)}(t), j \ge 1\,.$$

- Can show that

$$\mathrm{Cov}(S_{jk}) = (\Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n)(1 + \mathcal{O}(1)),$$

where $\mathbf{H}_k^n$ has entries

$$\left(\mathbf{H}_k^n\right)^{(pp)} = n_p^{-1} \eta_p (F_p^{-1})'(kh_n)\,,$$

$\Rightarrow$ Estimate Σ_{kh_n} by $S_{jk} S_{jk}^{\top} - \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n$!

An Initial Spot Covariance Matrix Estimator

- Average across frequencies $j = 1, \ldots, J_n^p$ and adjacent blocks:

$$\operatorname{vec}\left(\hat{\Sigma}_{kh_n}^{pre}\right) = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} (J_n^p)^{-1} \sum_{j=1}^{J_n^p} \operatorname{vec}\Big(S_{jk} S_{jk}^\top - \pi^2 j^2 h_n^{-2} \hat{\mathbf{H}}_k^n\Big),$$

where $L_{s,n} = \max\{\lfloor s h_n^{-1} \rfloor - K_n, 0\}$,
$U_{s,n} = \min\{\lfloor s h_n^{-1} \rfloor + K_n, \lceil h_n^{-1} \rceil - 1\}$

- $\hat{\mathbf{H}}_k^n$ is a $\sqrt{n}$-consistent estimator of $\mathbf{H}_k^n$ with diagonal element

$$\left(\hat{\mathbf{H}}_k^n\right)^{(pp)} = \frac{\hat{\eta}_p}{h_n} \sum_{kh_n \le t_i^{(p)} \le (k+1)h_n} \left(t_i^{(p)} - t_{i-1}^{(p)}\right)^2,$$

with $\hat{\eta}_p$ being long-run noise variance estimator.

LMM Spot Covariance Matrix Estimator

- Equal weights for frequencies $j = 1, \ldots, J_n^p$ in general not optimal.
- Increase efficiency: obtain pre-estimated spot covariance matrices using $\operatorname{vec}\left(\hat{\Sigma}_{kh_n}^{pre}\right)$ and derive estimated optimal weight matrices $\hat{W}_j$.

$\Rightarrow$ LMM spot covariance matrix estimator:

$$\operatorname{vec}\left(\hat{\Sigma}_s\right) = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} \sum_{j=1}^{J_n} \hat{W}_j\left(\hat{\mathbf{H}}_k^n, \hat{\Sigma}_{kh_n}^{pre}\right) \times \operatorname{vec}\left(S_{jk} S_{jk}^{\top} - \pi^2 j^2 h_n^{-2} \hat{\mathbf{H}}_k^n\right).$$

- Optimal weights proportional to local Fisher info matrices:

$$
\begin{aligned}
W_j\big(\mathbf{H}_k^n, \Sigma_{kh_n}\big) &= \Big(\sum_{u=1}^{J_n} \big(\Sigma_{kh_n} + \pi^2 u^2 h_n^{-2} \mathbf{H}_k^n\big)^{-\otimes 2}\Big)^{-1} \\
&\qquad\qquad \times \big(\Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n\big)^{-\otimes 2} \\
&= I_k^{-1} I_{jk},
\end{aligned}
$$

  with

$$
I_{jk} = \big(\Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} \mathbf{H}_k^n\big)^{-\otimes 2},
$$

  and $I_k = \sum_{j=1}^{J_n} I_{jk}$.

- Note: $\hat{\Sigma}_s$ symmetric, but not necessarily positive semi-definite.

⇒ E.g., project on space of positive semi-definite matrices.

Pointwise Central Limit Theorem

Theorem 1

Assume a setup with observations of type (2), *a signal* (1) *and validity of Assumptions 1-4.*

Then, for $h_n = \kappa_1 \log(n) n^{-1/2}$, $K_n = \kappa_2 n^{\beta} (\log(n))^{-1}$ *with constants* κ_1, κ_2 *and* $0 < \beta < \alpha(2\alpha + 1)^{-1}$, *for* $J_n \to \infty$ *and* $n/n_p \to \nu_p$ *with* $0 < \nu_p < \infty, p = 1, \ldots, d$, *as* $n \to \infty$, $\hat{\Sigma}_s$ *satisfies:*

$$n^{\beta/2} \operatorname{vec}\left(\hat{\Sigma}_s - \Sigma_s\right) \stackrel{d-(st)}{\longrightarrow} \mathbf{N}\Big(0, 2\big(\Sigma \otimes \Sigma_H^{1/2} + \Sigma_H^{1/2} \otimes \Sigma\big)_s \mathcal{Z}\Big), s \in [0,1]\,,$$

where $\Sigma_H = H\left(H^{-1} \Sigma H^{-1}\right)^{1/2} H$ *with noise level* H *from* (3) *and*

$\mathcal{Z} = \mathbb{C}\mathrm{OV}(\operatorname{vec}(ZZ^{\top}))$ *for* $Z \sim \mathbf{N}(0, E_d)$ *being a standard normally distributed random vector.*

Feasible Central Limit Theorem

Corollary 1

Under the assumptions of Theorem 1, $\hat{\Sigma}_s$ satisfies

$$(U_{s,n} - L_{s,n} + 1)^{1/2} \left(\hat{\mathbb{V}}_s^n\right)^{-1/2} \operatorname{vec}\left(\hat{\Sigma}_s - \Sigma_s\right) \xrightarrow{d} \mathbf{N}\Big(0, \mathcal{Z}\Big),\ s \in [0,1]\,,$$

where $$\mathbb{V}_s^n = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} \left(\sum_{j=1}^{J_n} I_{jk} \right)^{-1}.$$

Spot Correlations and Betas

- Spot correlation estimator: $\hat{\rho}_s^{(pq)} = \hat{\Sigma}_s^{(pq)} / \sqrt{\hat{\Sigma}_s^{(pp)} \hat{\Sigma}_s^{(qq)}}$.
- Spot beta estimator: $\hat{\beta}_s^{(pq)} = \hat{\Sigma}_s^{(pq)} / \hat{\Sigma}_s^{(pp)}$.
- Delta method yields:

$$n^{\beta/2} \operatorname{vec}\left(\hat{\rho}_s^{(pq)} - \rho_s^{(pq)}\right) \stackrel{d-(st)}{\longrightarrow} \mathbf{N}\Big(0, \mathbb{AV}_{\rho,s}\Big),\ s \in [0,1]\,,$$
$$n^{\beta/2} \operatorname{vec}\left(\hat{\beta}_s^{(pq)} - \beta_s^{(pq)}\right) \stackrel{d-(st)}{\longrightarrow} \mathbf{N}\Big(0, \mathbb{AV}_{\beta,s}\Big),\ s \in [0,1]\,.$$

$\Rightarrow$ Analogously for feasible CLTs.

Estimating Noise Autocovariances

- Estimation of long-run noise variance $\eta_p, p = 1, \ldots, d$, only requires component-wise autocovariance estimates.

  $\Rightarrow$ Restrict analysis to $d = 1$: $n + 1$ observations of $Y_i = X_{t_i} + \epsilon_i, i = 0, \ldots, n$.

- Fix $R \geq 0$ and successively estimate autocovariances by

$$\hat{\eta}_R = (2n)^{-1} \sum_{i=1}^{n} \big(\Delta_i Y\big)^2 + n^{-1} \sum_{r=1}^{R} \sum_{i=1}^{n-r} \Delta_i Y \Delta_{i+r} Y \,,$$

$$\hat{\eta}_r - \hat{\eta}_{r+1} = (2n)^{-1} \sum_{i=1}^{n} \big(\Delta_i Y\big)^2 + n^{-1} \sum_{u=1}^{r} \sum_{i=1}^{n-u} \Delta_i Y \Delta_{i+u} Y \,, \quad 0 \leq r \leq R - 1 \,.$$

- The variance of $\hat{\eta}_r, 0 \leq r \leq R$, is consistently estimated by

$$\widehat{\mathbb{V}\mathsf{ar}}(\hat{\eta}_r) = n^{-1} \left(V^n_{r+1} + V^n_r + 2C^n_{r,r+1} \right),$$

with

$$C^n_{r,r+1} = \left(\frac{\hat{\Gamma}^{00}_0}{4} + \frac{1}{2} \sum_{u=1}^{r} \hat{\Gamma}^{00}_u + \sum_{u=0}^{r} \sum_{u'=1}^{r+1} \left(\hat{\Gamma}^{uu'}_0 + 2 \sum_{q=1}^{R} \hat{\Gamma}^{uu'}_q \right) \right),$$

and $V^n_r = C^n_{r,r}$, where $\hat{\Gamma}^{rr'}_q$, $q, r, r' \in \{0, \ldots, R\}$ is the fourth sample moment of $\Delta_i Y$.

- In particular, for $r = R$, $\widehat{\mathbb{V}\mathsf{ar}}(\hat{\eta}_R) = n^{-1} V^n_R$.

Theorem 2

Under Assumption 3 and $\mathbb{H}_0^Q : \eta_u = 0$ *for all* $u \geq Q$*,* $Q = R + 1$*, we have*

$$T_Q^n(Y) = \sqrt{n/V_Q^n}\,\hat{\eta}_Q \xrightarrow{d} \mathbf{N}(0,1)\,.$$

Suitable strategy for selecting R:

- Compute $T_Q^n(Y)$ for $Q \leq \tilde{Q} = \tilde{R} + 1$ "large".
- Incorporate all autocovariances until first hypothesis of zero autocovariance cannot be rejected.

$\Rightarrow$ Using $\hat{R}$, compute long-run noise variance estimate as

$$\hat{\eta} = \hat{\eta}_0 + 2\sum_{u=1}^{\hat{R}} \hat{\eta}_u .$$

4. Empirical Results

Data

- Mid-quotes and transaction prices for 30 most liquid NASDAQ100 constituents and PowerShares QQQ ETF.
- Sample period from May 2010 to April 2014.
- Data sampled from LOBSTER database: `https://lobster.wiwi.hu-berlin.de/`
- Handle (few) errors in the trade and mid-quote samples using cleaning procedures by Barndorff-Nielsen et al. (2009).
- Preliminary analysis: huge share of zero returns in quote data.

$\Rightarrow$ Focus on quote revisions to reduce computational burden.

Choice of Inputs and Implementation

- Theory requires:
  $h_n = \mathcal{O}\big(\log(n) n^{-1/2}\big)$, $J_n = \mathcal{O}\big(\log(n)\big)$,
  J_n^p fixed at a value not "too large" (e.g., $J_n^p = 5$) and $K_n = \mathcal{O}\big(n^{1/4-\varepsilon}\big)$ for $\varepsilon > 0$ "small".
- Introduce proportionality parameters:
  $h_n = \theta_h \log(n) n^{-1/2}$, $J_n = \lfloor \theta_J \log(n) \rfloor$ and $K_n = \lceil \theta_K n^{1/4-\delta} \rceil$, where $\theta_h, \theta_J, \theta_K > 0$.

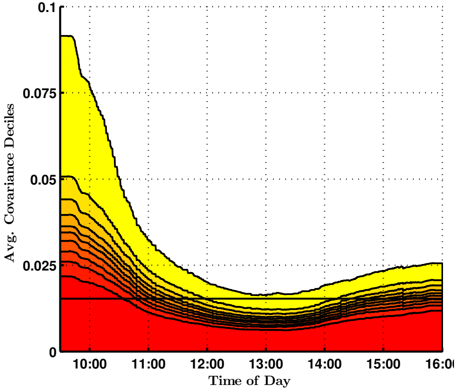
$\Rightarrow$ Based on simulations: $\theta_h = 0.2$, $\theta_J = 8$, $\theta_K = 0.4$, $J_n^p = 5$.

- Estimate
  - 30×30 spot covariance matrices for NASDAQ100 constituents: spot covariances and correlations, volatilities.
  - 31×31 spot covariance matrices including QQQ ETF: spot betas with QQQ as market proxy.

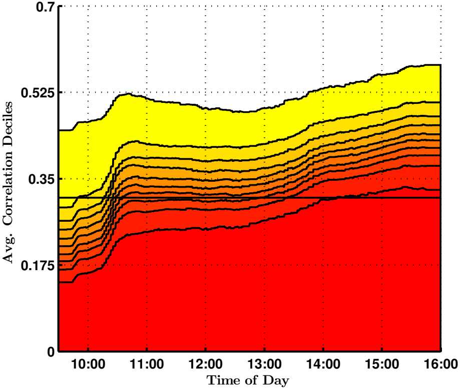
Summary Statistics of Input Values

Input	$q_{0.05}$	Mean	$q_{0.95}$	Std.
$\lceil h_n^{-1} \rceil$	18.000	22.516	29.000	3.922
J_n	48.000	53.532	60.000	3.672
K_n	2.000	2.435	3.000	0.300

Cross-Sectional Deciles of Avg. Covariance and Correlation



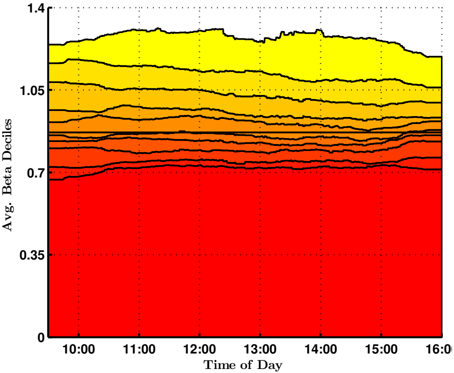
(a) Spot Covariances



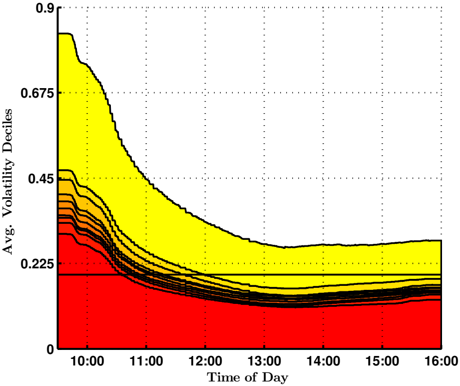
(b) Spot Correlations

Spot estimates are averaged across days. Then, cross-sectional sample deciles of across-day averages are computed.

Cross-Sectional Deciles of Avg. Beta and Volatility



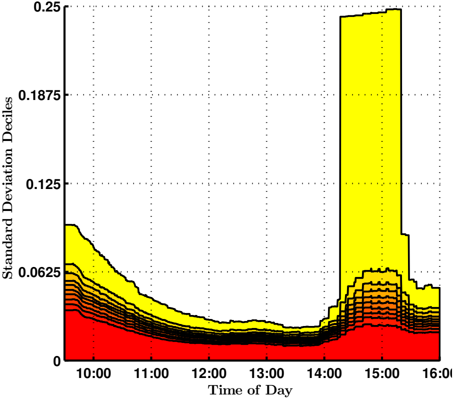
(a) Spot Betas



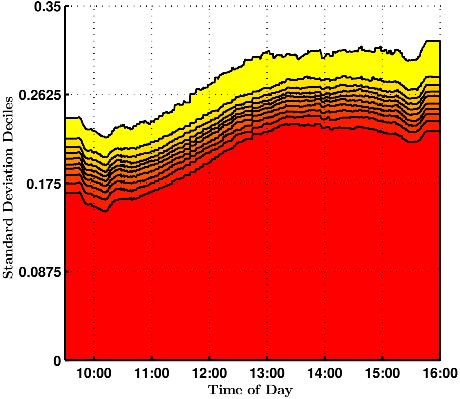
(b) Spot Volatilities

Spot estimates are averaged across days. Then, cross-sectional sample deciles of across-day averages are computed.

Cross-Sectional Deciles of Std. Dev. of Covariance and Correlation



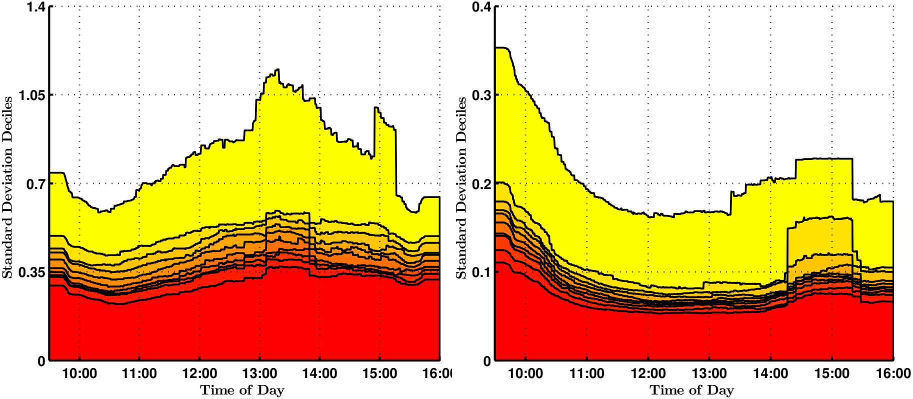
(a) Spot Covariances



(b) Spot Correlations

Sample standard deviations of spot estimates are computed across days. Then, cross-sectional sample deciles of across-day standard deviations are computed.

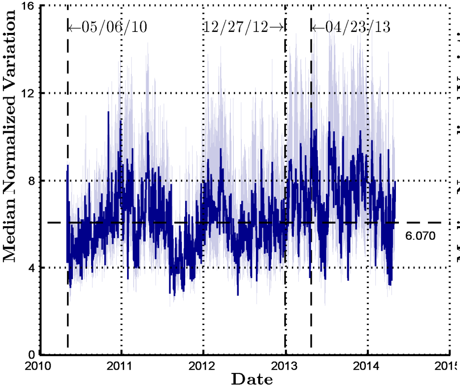
Cross-Sectional Deciles of Std. Dev. of Beta and Volatility



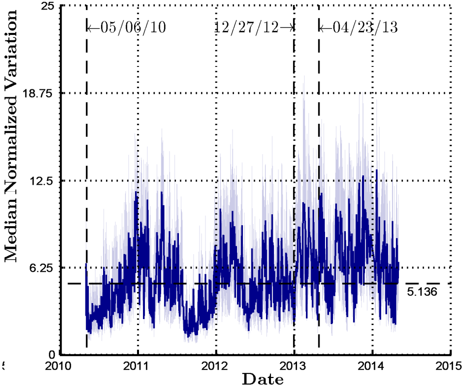
(a) Spot Betas (b) Spot Volatilities

Sample standard deviations of spot estimates are computed across days. Then, cross-sectional sample deciles of across-day standard deviations are computed.

Cross-Sectional Medians of Intraday Variation Proxy for Covariance and Correlation



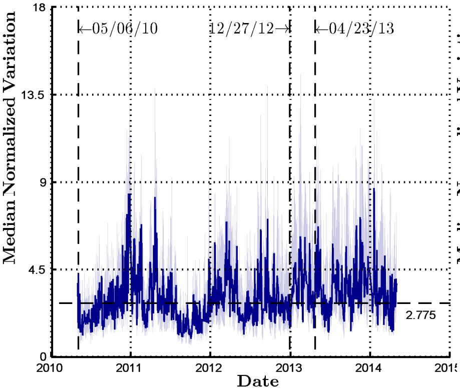
(a) Spot Covariances



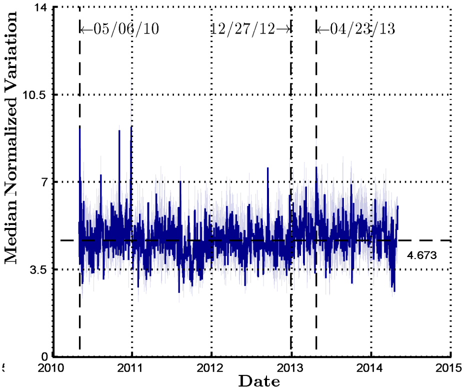
(b) Spot Correlations

Total intraday variation proxy: $\sum_{i=1}^{n_g} |f(t_i) - f(t_{i-1})| \left[\sum_{i=1}^{n_g} |f(t_i)| \Delta t_i\right]^{-1}$.

Cross-Sectional Medians of Intraday Variation Proxy for Beta and Volatility



(a) Spot Betas



(b) Spot Volatilities

Total intraday variation proxy: $\sum_{i=1}^{n_g} |f(t_i) - f(t_{i-1})| \left[\sum_{i=1}^{n_g} |f(t_i)| \Delta t_i\right]^{-1}$.

Event I: “Flash Crash” (05/06/10)

(1) Protests in Athens trigger Euro down movement vs. Yen.
U.S. fund managers short-sell E-Mini contracts in vast amounts.

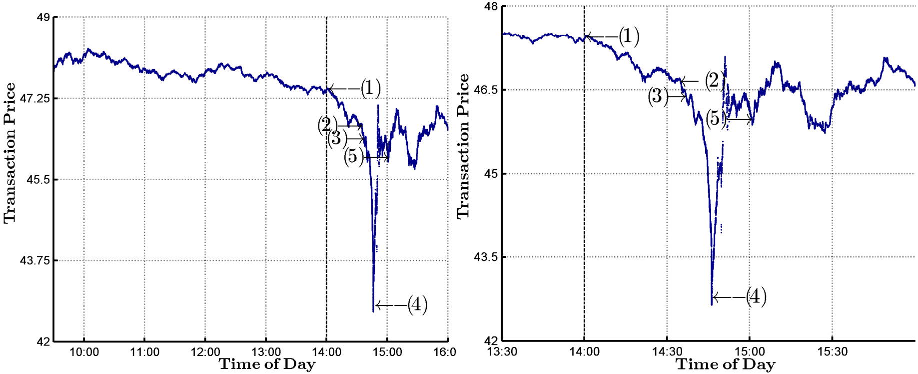
(2) E-Mini market makers cut back trading.

(3) NASDAQ stops order routing to ARCA.

(4) Rumors suggesting that decline occurred due to “fat-finger” error, and not bad news.

(5) NASDAQ resumes routing to ARCA.

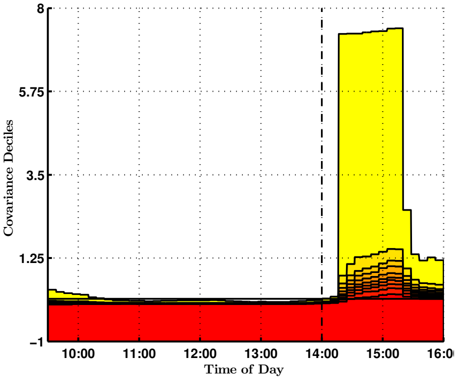
05/06/10: QQQ Transaction Prices



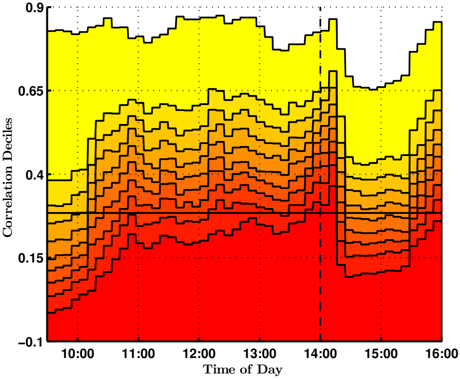
(a) Entire trading day

(b) 1:30 pm – 4:00 pm

05/06/10: Cross-Sectional Deciles of Covariance and Correlation

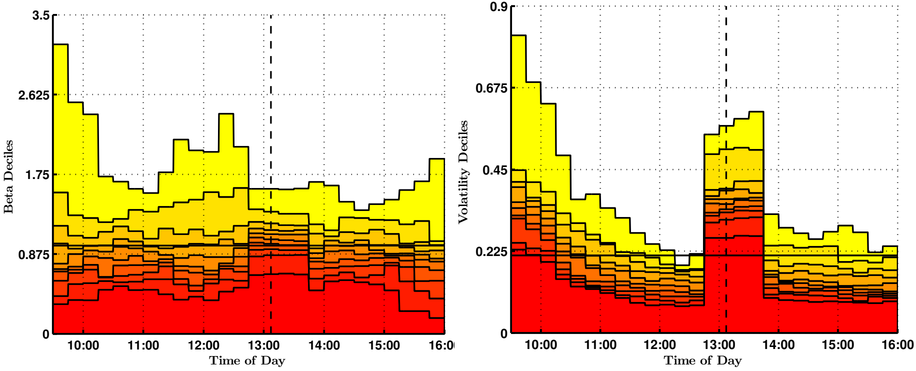


(a) Spot Covariances



(b) Spot Correlations

05/06/10: Cross-Sectional Deciles of Beta and Volatility



(a) Spot Betas (b) Spot Volatilities

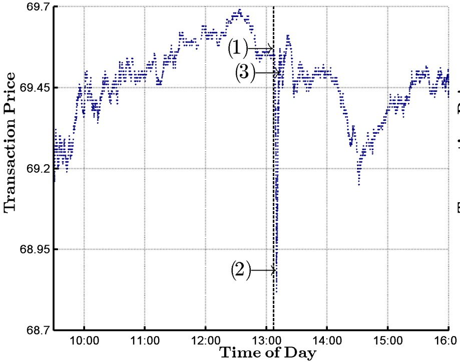
Event II: “Twitter Flash Crash” (04/23/13)

(1) Fake tweet from the account of AP stating “Breaking: Two Explosions in the White House and Barack Obama is injured”.

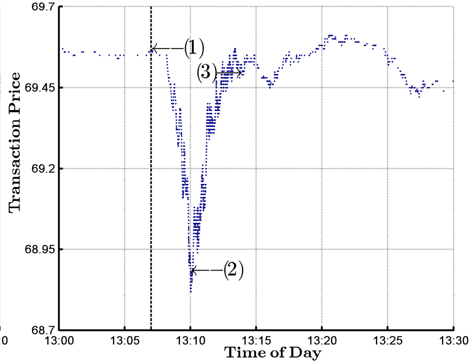
(2) Official denial by AP.

(3) AP’s twitter account suspended.

04/23/13: QQQ Transaction Prices

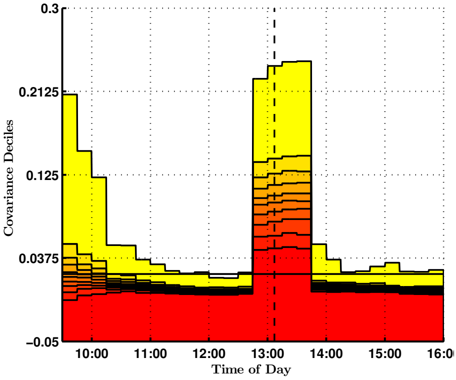


(a) Entire trading day

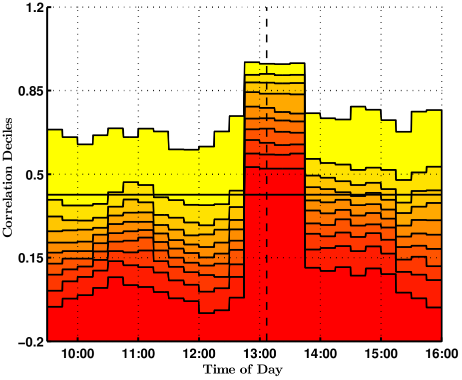


(b) 1:00 pm – 1:30 pm

04/23/13: Cross-Sectional Deciles of Covariance and Correlation

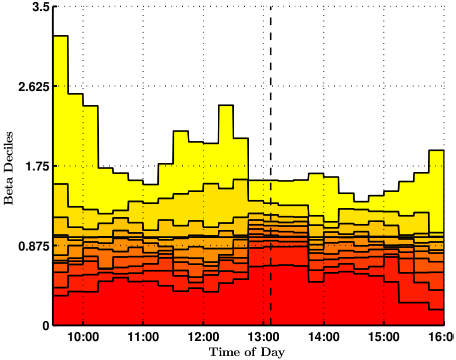


(a) Spot Covariances

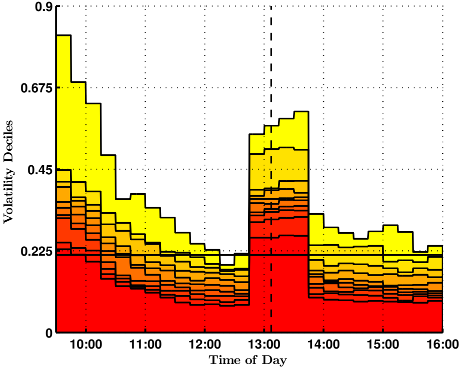


(b) Spot Correlations

04/23/13: Cross-Sectional Deciles of Beta and Volatility



(a) Spot Betas



(b) Spot Volatilities

5. Conclusions

Conclusions

- Introduce spot covariance matrix estimator relying on LMM approach by Bibinger et al. (2014).
- Extend LMM to allow for autocorrelated noise and provide method for choosing order of dependence.
- Derive stable CLT along with feasible version.
- Simulation study demonstrates how to implement estimator.
- Emprical evidence based on NASDAQ100 stocks:
  - Spot covariances, correlations & volatilities exhibit considerable intraday seasonality.
  - Distinct intraday changes of (co-)volatilities in periods of extreme market movements.