Particle Filter-Based On-Line Estimation of Spot (Cross-)Volatility with Nonlinear Market Microstructure Noise Models

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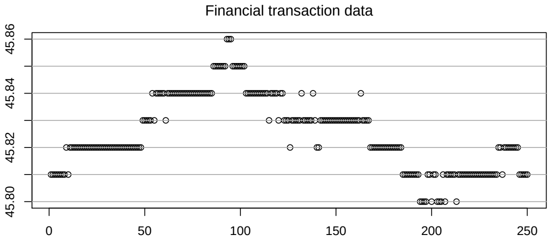
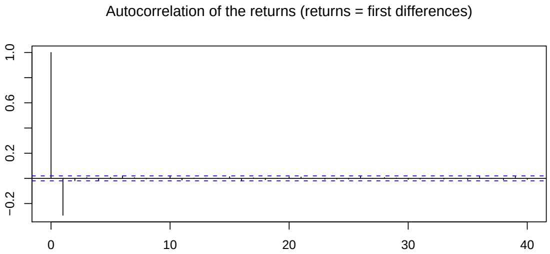
Market microstructure features

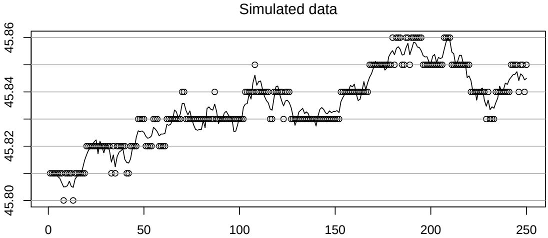


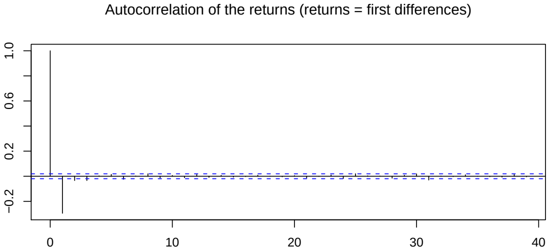


Main features

- discreteness of prices
- many zero returns
- bid-ask bounce

Assumption: unobserved efficient price process





A nonlinear state-space model in transaction time

Notation:

- t_j, $j = 1, 2, \ldots$, transaction times (strictly increasing)
- Y_{t_j} observed transaction prices
- X_{t_j} unobserved efficient log-prices

Nonlinear state-space model:

$$
\begin{aligned}
Y_{t_j} &= g_{t_j; Y_{t_{1:j-1}}}(\exp[X_{t_j}]) && \text{(observation equation)} \\
X_{t_j} &= X_{t_{j-1}} + Z_{t_j} && \text{(state equation)}
\end{aligned}
$$

where $Z_{t_j} \sim \mathcal{N}(0, \sigma^2_{t_j})$ with constant or time-varying volatility σ_{t_j}.

Key quantity: $p(x_{t_j} | y_{t_{1:j}})$ filtering distribution[1]

[1] $y_{t_{1:j}} = \{y_{t_1}, \ldots, y_{t_j}\}$

A class of nonlinear market microstructure noise models

Aim: A model for the nonlinear time-varying (possibly stochastic) dependence of the observed transaction prices and the latent efficient prices

$$Y_{t_j} = g_{t_j;Y_{t_{1:j-1}}}\left(\exp[X_{t_j}]\right)$$

Model assumptions:

1. The distribution of $Y_{t_j} = g_{t_j;Y_{t_{1:j-1}}}\big(\exp[X_{t_j}]\big)$ is discrete.
2. The conditional distribution of Y_{t_j} given X_{t_j} and previous observations is of the form

$$p\big(y_{t_j}\big|y_{t_{1:j-1}}, \exp[x_{t_j}]\big) \propto \mathbf{1}_{A_{t_j}}\big(\exp[x_{t_j}]\big) \qquad \text{(a.s.)}$$

   where A_{t_j} depends on y_{t_j} and on the past observations $y_{t_1}, \ldots, y_{t_{j-1}}$.

Remarks:

- If $g_{t_j} = g_{t_j;Y_{t_{1:j-1}}}$ is deterministic then $A_{t_j} := g_{t_j}^{-1}(y_{t_j}) = \{z : g_{t_j}(z) = y_{t_j}\}$ is the inverse image of y_{t_j} under g_{t_j}.
- Concrete specifications of the set A_{t_j} give quite different models

Examples

1. Rounding to the nearest cent
   - Deterministic: $A_{t_j} = [y_{t_j} - 0.5, y_{t_j} + 0.5)$ and $y_{t_j} = \text{round}(\exp[x_{t_j}])$
   - Stochastic: e.g. rounding up/down with probability 1/2, $A_{t_j} = (y_{t_j} - 1, y_{t_j} + 1)$
2. Rounding to the nearest order book level
   - $A_{t_j} = \{z \in \mathbb{R} : \text{argmin}_{\gamma \in \mathcal{M}_{t_j-}} |z - \gamma| = y_{t_j}\}$ and $g_{t_j;y_{t_{1:j-1}}}(z) = \text{argmin}_{\gamma \in \mathcal{M}_{t_j-}} |z - \gamma|$
3. Rounding to the nearest market maker quote
4. A general rounding for transaction data
   - $A_{t_j} = [y_{t_j} - \Delta_{t_j}, y_{t_j} + \Delta_{t_j})$ with $\Delta_{t_j} = \begin{cases} 0.5\,|y_{t_j} - y_{t_{j-1}}| & \text{if } y_{t_j} \neq y_{t_{j-1}}, \\ \Delta_{t_{j-1}} & \text{else} \end{cases}$

Example: order book data

Economic intuition: The efficient price should be closer to the observed price than to any other order book level.

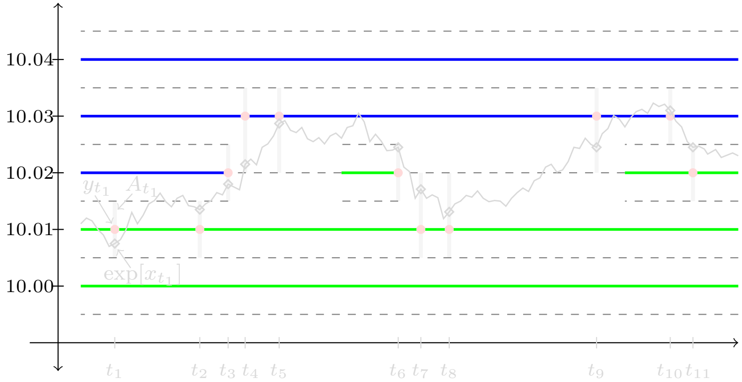


Figure: Bid and ask levels of the order book (green and blue lines), supports of the filtering distributions (gray vertical lines), transaction prices (red circles), efficient prices in transaction time (diamonds), the efficient price process in clock time (black line)

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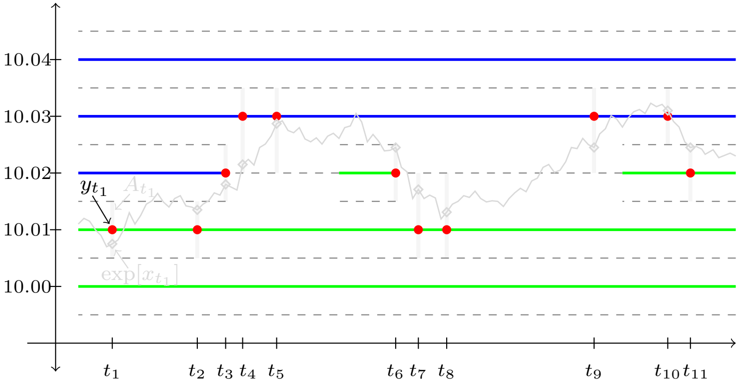

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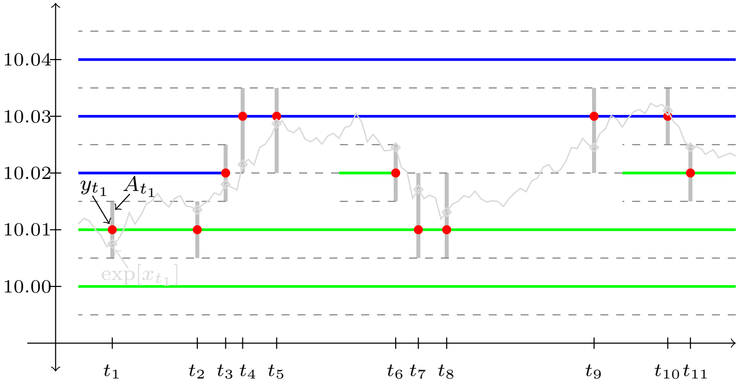

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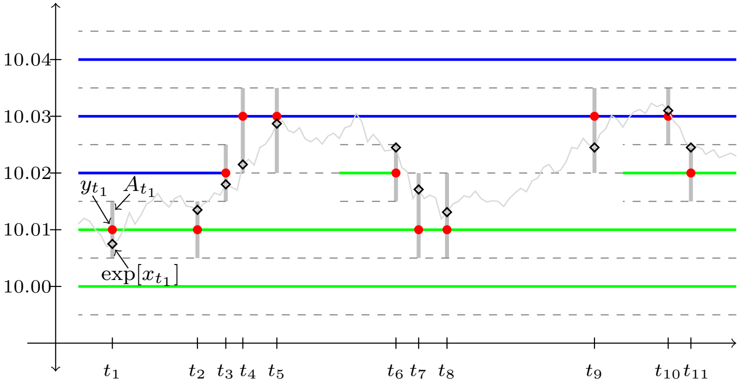

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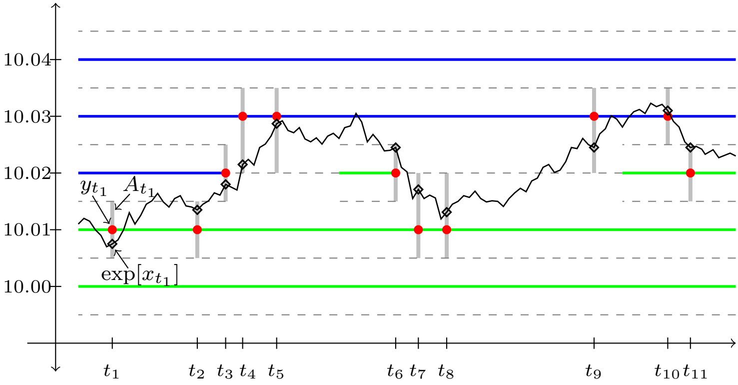

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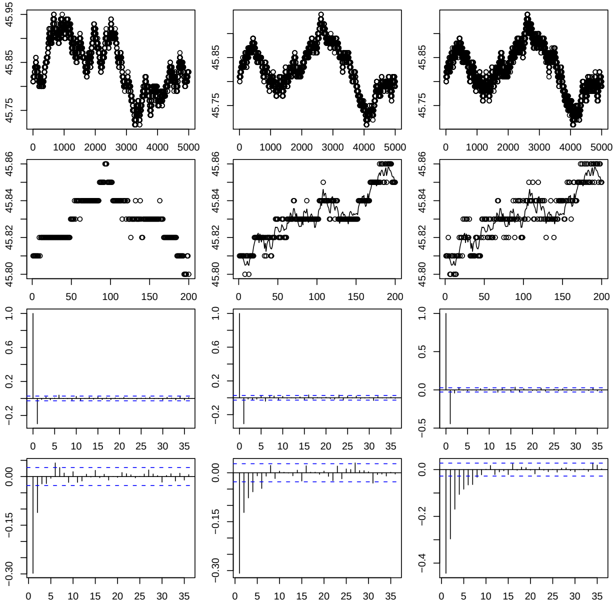
Real data vs. deterministic rounding vs. stochastic rounding



⇒ Deterministic rounding reproduces the major microstructure features

Outline

The synchronous trading case

- Consider S securities: efficient log-prices $X_{t,s}$ and transaction prices $Y_{t,s}$, $s = 1, \ldots, S$

$$\begin{aligned} \mathbf{Y}_{t_j} &= g_{t_j}\left(\exp[\mathbf{X}_{t_j}]\right), \\ \mathbf{X}_{t_j} &= \mathbf{X}_{t_{j-1}} + \mathbf{Z}_{t_j}, \end{aligned}$$

where

$$\mathbf{Z}_{t_j} \sim \mathcal{N}(\mathbf{0}, \Sigma_{t_j}),$$

$$g_{t_j}(\exp[\mathbf{X}_{t_j}]) = (g_{t_j^{(1)}}(\exp[X_{t_j,1}]), \ldots, g_{t_j^{(S)}}(\exp[X_{t_j,S}]))^T, \; \mathbf{X}_t = (X_{t,1}, \ldots, X_{t,S})^T.$$

- **Aim:** On-line estimation of covariance matrix Σ_t (spot cross-volatilities)

- **Remarks:**
  - Assumption: Σ_t is slowly varying (or constant)
  - Spot volatility estimation: $S = 1$ and $\Sigma_t = \sigma_t^2$

An efficient particle filter

Assume weighted particles $\{\mathbf{x}^i_{t_{1:j-1}}, \omega^i_{t_{j-1}}\}_{i=1}^N$ approximating $p(\mathbf{x}_{t_{1:j-1}}|\mathbf{y}_{t_{1:j-1}})$ are given.

1. For $i = 1, \ldots, N$:
   - Sample $\mathbf{x}^i_{t_j} \sim p(\mathbf{x}_{t_j}|\mathbf{y}_{t_{1:j}}, \mathbf{x}^i_{t_{j-1}}) \propto \mathcal{N}(\mathbf{x}_{t_j}|\mathbf{x}^i_{t_{j-1}}, \Sigma_{t_j})|_{\log A_{t_j,1}\times\cdots\times\log A_{t_j,S}}$
   - Compute importance weights $\breve{\omega}^i_{t_j} \propto \omega^i_{t_{j-1}} \int_{\log \mathbf{A}_{t_j}} \mathcal{N}(\mathbf{x}_{t_j}|\mathbf{x}^i_{t_{j-1}}, \Sigma_{t_j}) d\mathbf{x}_{t_j}$
2. For $i = 1, \ldots, N$: Normalize importance weights $\omega^i_{t_j} = \breve{\omega}^i_{t_j}/(\sum_{k=1}^N \breve{\omega}^k_{t_j})$
3. Resample particles

Result: Particles $\{\mathbf{x}^i_{t_{1:j}}, \omega^i_{t_j}\}_{i=1}^N$ approximating $p(\mathbf{x}_{t_{1:j}}|\mathbf{y}_{t_{1:j}})$.

Remarks:

- The particle filter is very efficient because the particles are sampled from the optimal proposal $p(\mathbf{x}_{t_j}|\mathbf{y}_{t_{1:j}}, \mathbf{x}_{t_{j-1}})$.
- In practice, Σ_{t_j} is replaced by an estimate $\hat{\Sigma}^{\mathsf{pf}}_{t_j}$.

EM algorithms

- **The standard EM algorithm:**
  - Maximization of

$$\mathcal{Q}(\Sigma|\hat{\Sigma}^{(m)}) = \text{const} + \sum_{j=2}^{T} \mathbf{E}_{\hat{\Sigma}^{(m)}}[\log p_{\Sigma}(\mathbf{X}_{t_j}|\mathbf{X}_{t_{j-1}})|\mathbf{y}_{t_{1:T}}]$$

    with respect to Σ
  - Depends on smoothing distributions ($\Rightarrow$ off-line algorithm)

- **Our sequential EM algorithm:**
  - Recursive maximization of

$$\mathcal{Q}_{t_j}(\Sigma|\hat{\Sigma}_{t_{1:j-1}}) = \{1-\lambda_j\}\,\mathcal{Q}_{t_{j-1}}(\Sigma|\hat{\Sigma}_{t_{1:j-2}})+\lambda_j\,\mathbf{E}_{\hat{\Sigma}_{t_{1:j-1}}}[\log p_{\Sigma}(\mathbf{X}_{t_j}|\mathbf{X}_{t_{j-1}})|\mathbf{y}_{t_{1:j}}]$$

    with respect to Σ
  - Depends on filtering distributions ($\Rightarrow$ on-line algorithm)

Our sequential EM-type algorithm

- E-step: (based on particles $\{\mathbf{x}^i_{t_{j-1:j}}, \omega^i_{t_j}\}_{i=1}^N$)

$$\hat{\mathcal{Q}}_{t_j}(\Sigma|\hat{\Sigma}_{t_{1:j-1}}) = \{1-\lambda_j\}\, \hat{\mathcal{Q}}_{t_{j-1}}(\Sigma|\hat{\Sigma}_{t_{1:j-2}}) - \lambda_j\, \frac{1}{2}\sum_{i=1}^N \omega^i_{t_j}\left[S\log 2\pi + \log|\Sigma| + \mathrm{tr}\left\{\Sigma^{-1}(\mathbf{x}^i_{t_j} - \mathbf{x}^i_{t_{j-1}})(\mathbf{x}^i_{t_j} - \mathbf{x}^i_{t_{j-1}})^T\right\}\right]$$

- M-step:

$$\hat{\Sigma}_{t_j} = \{1-\lambda_j\}\,\hat{\Sigma}_{t_{j-1}} + \lambda_j\, \breve{\Sigma}_{t_j}(\omega_{t_j})$$

with

$$\breve{\Sigma}_{t_j}(\omega_{t_j}) = \sum_{i=1}^N \omega^i_{t_j}(\mathbf{x}^i_{t_j} - \mathbf{x}^i_{t_{j-1}})(\mathbf{x}^i_{t_j} - \mathbf{x}^i_{t_{j-1}})^T$$

Step size selection:

- Simple approach: constant step size $\lambda_j = \lambda$
- Advanced approach: adaptive time-varying step size
  e.g. spatially aggregated exponential smoothing (SAGES) developed by Chen and Spokoiny (2009)

Back and forth between particle filter and seq. EM algorithm

Time	Particle Filter		Sequential EM algorithm
t_{j-1}		$\hat{\Sigma}_{t_{j-1}}$	
t_j	- Simulate transition $\mathbf{x}^i_{t_{j-1}} \rightsquigarrow \mathbf{x}^i_{t_j}$ using $\mathbf{y}_{t_j}$ and $\hat{\Sigma}^{\mathbf{pf}}_{t_j} = \hat{\Sigma}_{t_{j-1}}$ - Obtain particles $\{\mathbf{x}^i_{t_{j-1:j}}, \omega^i_{t_j}\}$	$\{\mathbf{x}^i_{t_{j-1:j}}, \omega^i_{t_j}\}$	
		$\hat{\Sigma}_{t_j}$	- Compute update $\breve{\Sigma}_{t_j}(\omega_{t_j})$ using $\{\mathbf{x}^i_{t_{j-1:j}}, \omega^i_{t_j}\}$ - Obtain estimate $\hat{\Sigma}_{t_j}$
t_{j+1}			

Clock time estimation

Until now we considered the estimation of $\Sigma_{t_j} = \Sigma(t_j)$ in a transaction time model.

A simple clock time estimator:

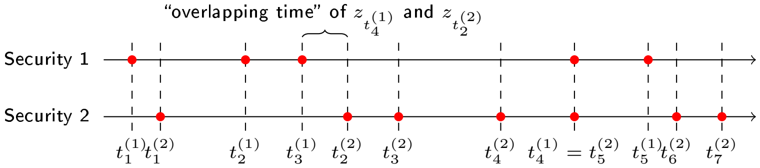
- Consider
$$d\mathbf{X}(t) = \Gamma(t)\, d\mathbf{W}(t) \qquad \text{where} \quad \Gamma(t)\,\Gamma^T(t) = \Sigma^c(t).$$
$\mathbf{W}(t)$ is a multivariate Brownian motion and $\Sigma^c(t)$ denotes the "volatility per time unit" (while $\Sigma(t)$ denotes "volatility per transaction")

- We obtain
$$\hat{\Sigma}^c_{t_j} = (1-\lambda_j)\hat{\Sigma}^c_{t_{j-1}} + \lambda_j \sum_{i=1}^{N} \omega^{ci}_{t_j} \frac{\left(\mathbf{x}^{ci}_{t_j} - \mathbf{x}^{ci}_{t_{j-1}}\right)\left(\mathbf{x}^{ci}_{t_j} - \mathbf{x}^{ci}_{t_{j-1}}\right)^T}{|t_j - t_{j-1}|}$$
with modified particles $\{\mathbf{x}^{ci}_{t_{j-1:j}}, \omega^{ci}_{t_j}\}_{i=1}^{N}$.

An alternative clock time estimator: see our paper

Non-synchronous trading case

Let $Z_{t_j} = X_{t_j} - X_{t_{j-1}}$ be the returns of the efficient log-prices.



Facts:

- Trading times are non-synchronous
- Each security s evolves in an individual transaction time $\{t_j^{(s)}\}_{j=1}^{T_s}$
- "Overlapping time" is the key quantity for cross-volatility estimation

Our model: a non-standard state-space model

Our model:

$$\begin{aligned} Y_{t_j^{(s)}} &= g_{t_j^{(s)}}\left(\exp[X_{t_j^{(s)}}]\right) && \text{(observation equation)} \\ X_{t_j^{(s)}} &= X_{t_{j-1}^{(s)}} + Z_{t_j^{(s)}} && \text{(state equation)} \end{aligned}$$

for $s = 1, \ldots, S$ and $Z_{t_j^{(s)}} \sim \mathcal{N}(0, (\Sigma_{t_j^{(s)}})_{ss})$,

$$\mathrm{Cov}(Z_{t_j^{(s_1)}}, Z_{t_k^{(s_2)}}) = \frac{|[t_{j-1}^{(s_1)}, t_j^{(s_1)}) \cap [t_{k-1}^{(s_2)}, t_k^{(s_2)})|}{|t_j^{(s_1)} - t_{j-1}^{(s_1)}|^{1/2} |t_k^{(s_2)} - t_{k-1}^{(s_2)}|^{1/2}} (\Sigma_{\min\{t_j^{(s_1)}, t_k^{(s_2)}\}})_{s_1 s_2}.$$

Properties:

- Nonlinear observation equation
- Components evolve non-synchronously in different times (non-standard state-space model)
- Standard particle filters cannot be applied
- Synchronous trading is a special case

The recursive covariance estimator

Let t_v denote the joint transaction time. Our EM-type estimator is given, componentwise, by

$$(\hat{\Sigma}_{t_v})_{s_1 s_2} = (1 - \lambda_{v,s_1,s_2})(\hat{\Sigma}_{t_{v-1}})_{s_1 s_2} + \lambda_{v,s_1,s_2}(\breve{\Sigma}_{t_v})_{s_1 s_2},$$

where

$$(\breve{\Sigma}_{t_v})_{s_1 s_2} = \begin{cases} \sum_{i=1}^{N} \omega^i_{\max\{t^{(s_1)}_{h_1^v}, t^{(s_2)}_{h_2^v}\}} z^i_{t^{(s_1)}_{h_1^v}} z^i_{t^{(s_2)}_{h_2^v}} \frac{|t^{(s_1)}_{h_1^v} - t^{(s_1)}_{h_1^v - 1}|^{1/2} |t^{(s_2)}_{h_2^v} - t^{(s_2)}_{h_2^v - 1}|^{1/2}}{|[t^{(s_1)}_{h_1^v - 1}, t^{(s_1)}_{h_1^v}) \cap [t^{(s_2)}_{h_2^v - 1}, t^{(s_2)}_{h_2^v})|} & \text{if } t^{(s_1)}_{h_1^v} = t_v \\ & \text{or } t^{(s_2)}_{h_2^v} = t_v \\ (\hat{\Sigma}_{t_{v-1}})_{s_1 s_2} & \text{else} \end{cases}$$

with $h_s^v = \min\{h_s : t^{(s)}_{h_s} \geq t_v\}$.

Remark:

- The number of updates for each component is different

Outline

Real data: volatility estimation in transaction time

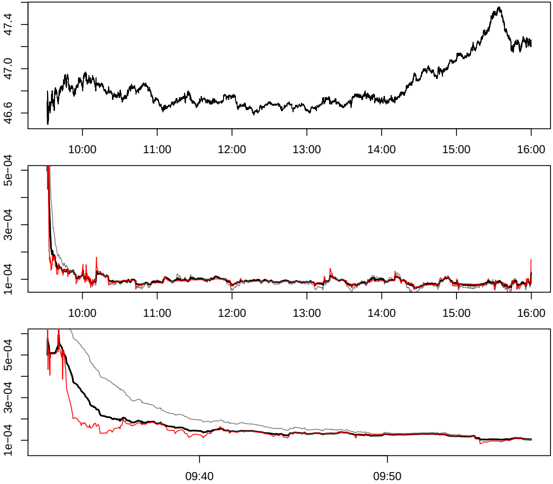


Figure: The upper plot shows transaction data of Citigroup for the 3rd September 2007. The middle and the lower plot give the volatility estimators $\hat{\Sigma}_{t_j}$ (black line), $\hat{\Sigma}^S_{t_j}$ (red line), and the benchmark estimator $\hat{\Sigma}^{\mathrm{B}}_{t_j}$ (gray line).

Real data: volatility estimation in clock time

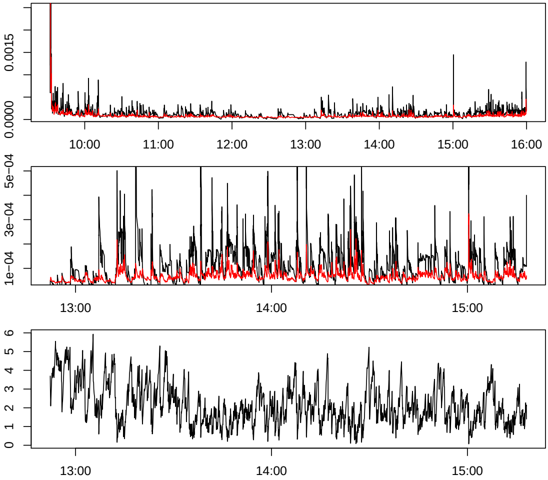


Figure: Estimation of time-varying spot volatility in clock time based on the transactions data of Citigroup for the 3rd September 2007. Simple clock time estimator $\hat{\Sigma}^{cS}_{t_j}$ (black line) and alternative clock time estimator $\hat{\Sigma}^{cS}_{\textbf{alt}}(t_j) = \hat{\Sigma}^{S}_{t_j}/\bar{\delta}_j$ (red line). Lower plot: averaged duration times $\bar{\delta}_j$ (y-axis shows seconds).

Real data: cross-volatility estimation

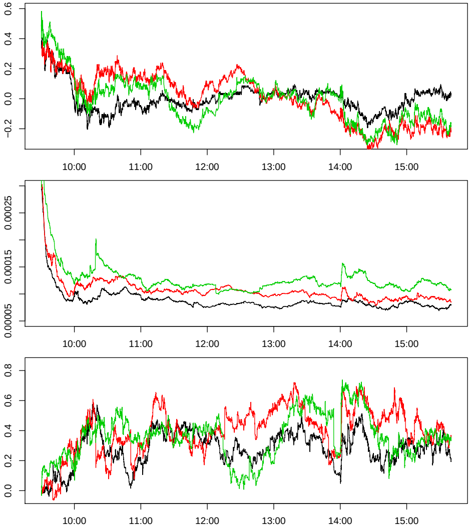


Figure: Upper plot: Transaction data of BAC (black line), C (red line), and JPM (green line) plotted with offsets. Middle plot: Volatility estimates. Lower plot: Correlation estimates for BAC/C (black line), BAC/JPM (green line), and C/JPM (red line).

Outline

Conclusion

New modeling and estimation aspects:

- A class of nonlinear market microstructure noise models
- A method for spot volatility estimation based on a particle filter and a new sequential EM-type algorithm
- Estimation in transaction time and clock time
- Our method works on-line and is computationally very efficient

Empirical results:

- Deterministic rounding schemes reproduce the major microstructure features
- The volatility in transaction time is often rougly constant (in contrast to volatility in clock time)
- The correlations of real stock returns vary significantly over the trading day

Thank you for your attention!

- Dahlhaus, R. and Neddermeyer, J.C. (2010), "Particle Filter-Based On-Line Estimation of Spot Volatility with Nonlinear Market Microstructure Noise Models," under revision.

- Neddermeyer, J.C. (2010), "Importance Sampling-Based Monte Carlo Methods with Applications to Quantitative Finance," PhD dissertation, University of Heidelberg.

Questions?