On the dual of the solvency cone

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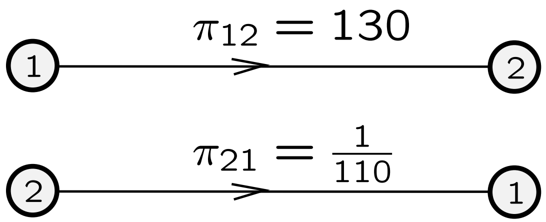
Simplest solvency cone example

Exchange between:

Currency 1: Nepalese Rupee

Currency 2: Euro



$\begin{pmatrix} \text{Rupee} \\ \text{Euro} \end{pmatrix}$-portfolios:

$$\begin{pmatrix} 130 \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -110 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$K = \text{cone}\left\{ \begin{pmatrix} 130 \\ -1 \end{pmatrix}, \begin{pmatrix} -110 \\ 1 \end{pmatrix} \right\}$$

price systems:

$$\begin{pmatrix} 1 \\ 130 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 110 \end{pmatrix}$$

$$\begin{pmatrix} 1/130 \\ 1 \end{pmatrix} \begin{pmatrix} 1/110 \\ 1 \end{pmatrix}$$

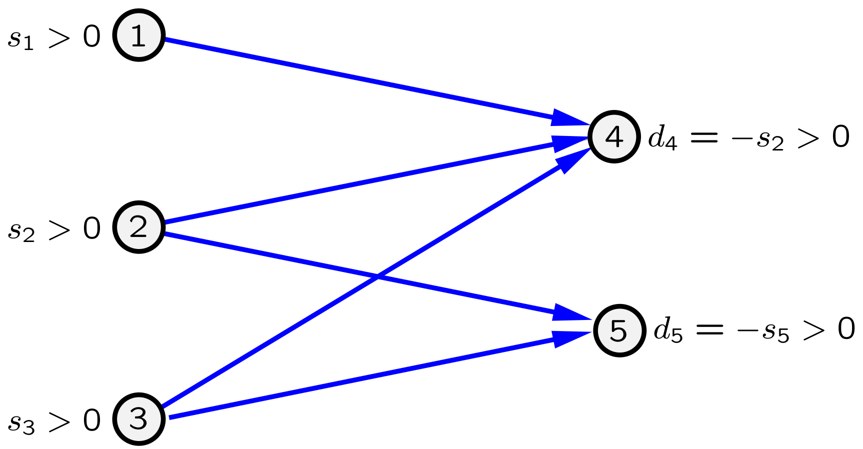
$$K^+ = \text{cone}\left\{ \begin{pmatrix} 1 \\ 130 \end{pmatrix}, \begin{pmatrix} 1 \\ 110 \end{pmatrix} \right\}$$

Solve a problem stated in

Bouchard, B., Touzi, N. (2000): Explicit solution to the multivariate super-replication problem under transaction costs, Ann. Appl. Probab.

"provide explicitly a generating family for the polar [or dual] cone [of K_d for $d > 2$]"

Basic facts about transportation problem



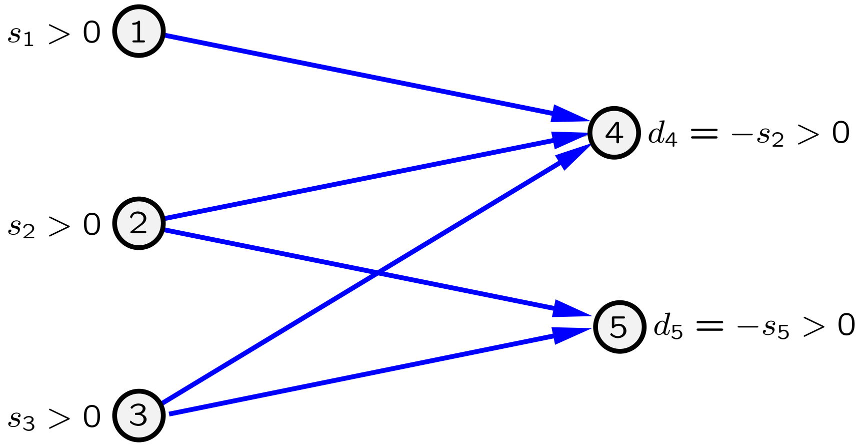
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

Variables: $x = \begin{pmatrix} x_{14} \\ x_{24} \\ x_{25} \\ x_{34} \\ x_{35} \end{pmatrix}$ Neg. cost: $c = \begin{pmatrix} c_{14} \\ c_{24} \\ c_{25} \\ c_{34} \\ c_{35} \end{pmatrix}$ Supply $s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix}$

$$\max c^T x \quad \text{s.t.} \quad Ax = s, \quad x \geq 0$$

Dual transportation problem

$$\min s^T y \quad \text{s.t.} \quad A^T y \geq c$$



$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

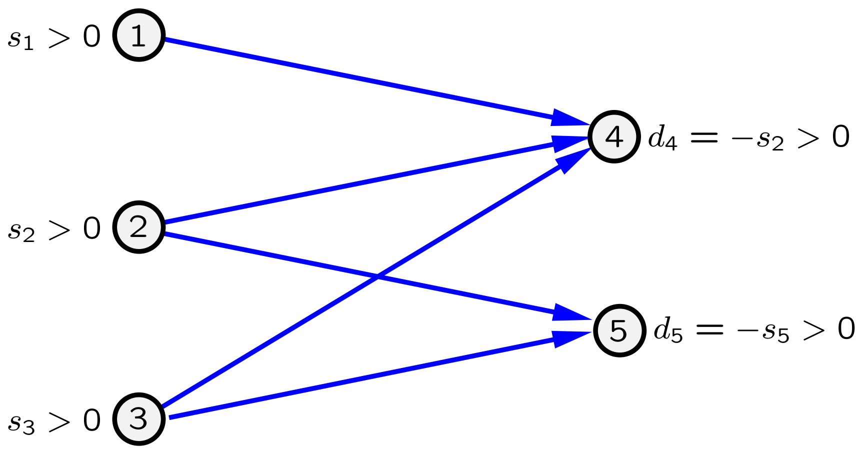
$$y_1 \geq y_4 + c_{14}$$
$$y_2 \geq y_4 + c_{24}$$
$$y_2 \geq y_5 + c_{25}$$
$$y_3 \geq y_4 + c_{34}$$
$$y_3 \geq y_5 + c_{35}$$

$$c = 0$$

(primal problem = feasibility problem)

$$y_1 \geq y_4$$
$$y_2 \geq y_4$$
$$y_2 \geq y_5$$
$$y_3 \geq y_4$$
$$y_3 \geq y_5$$
$$s^T y = 0$$

Modified transportation problem



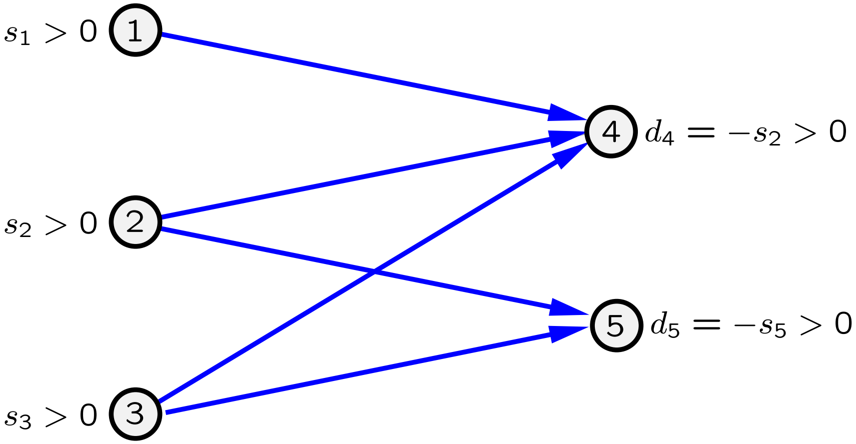
$$A = \begin{pmatrix} \pi_{14} & 0 & 0 & 0 & 0 \\ 0 & \pi_{24} & \pi_{25} & 0 & 0 \\ 0 & 0 & 0 & \pi_{34} & \pi_{35} \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

Variables: $x = \begin{pmatrix} x_{14} \\ x_{24} \\ x_{25} \\ x_{34} \\ x_{35} \end{pmatrix}$ Neg. cost: $c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ Supply $s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix}$

$$\max c^T x \quad \text{s.t.} \quad Ax = s, \quad x \geq 0$$

Modified transportation problem (dual)

$$\min s^T y \quad \text{s.t.} \quad A^T y \geq c$$



$$A = \begin{pmatrix} \pi_{14} & 0 & 0 & 0 & 0 \\ 0 & \pi_{24} & \pi_{25} & 0 & 0 \\ 0 & 0 & 0 & \pi_{34} & \pi_{35} \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

$$\pi_{14} \cdot y_1 \geq y_4$$

$$\pi_{24} \cdot y_2 \geq y_4$$

$$\pi_{25} \cdot y_2 \geq y_5$$

$$\pi_{34} \cdot y_3 \geq y_4$$

$$\pi_{35} \cdot y_3 \geq y_5$$

$$s^T y = 0$$

Definition (solvency cone)

Let $d \in \{2, 3, \ldots\}$, $V = \{1, \ldots, d\}$ and let $\Pi = (\pi_{ij})$ be a $(d \times d)$-matrix such that

$$\forall i \in V : \quad \pi_{ii} = 1, \tag{1}$$

$$\forall i, j \in V : \quad 0 < \pi_{ij}, \tag{2}$$

$$\forall i, j, k \in V : \quad \pi_{ij} \leq \pi_{ik}\pi_{kj}, \tag{3}$$

$$\exists i, j, k \in V : \quad \pi_{ij} < \pi_{ik}\pi_{kj}. \tag{4}$$

Sometimes, (3) and (4) is replaced by

$$\forall i, j \in V,\ \forall k \in V \setminus \{i, j\} : \quad \pi_{ij} < \pi_{ik}\pi_{kj}. \tag{5}$$

The polyhedral convex cone

$$K_d := \text{cone} \left\{ \pi_{ij} e^i - e^j |\ ij \in V \times V \right\}$$

is called solvency cone induced by Π.

The dual cone

$K_d^+ := \left\{ y \in \mathbb{R}^d \,|\, \forall x \in K_d : \; x^T y \geq 0 \right\}$ ... (positive) dual cone of K_d

Proposition 1. One has $K_d^+ = \left\{ y \in \mathbb{R}^d \,|\, \forall i,j \in V : \pi_{ij} y_i \geq y_j \right\}$.

Proof: obvious
Recall: $K_d :=$ cone $\left\{ \pi_{ij} e^i - e^j \,|\, ij \in V \times V \right\}$

Proposition 2. One has $\mathbb{R}_+^d \setminus \{0\} \subseteq \operatorname{int} K_d$ and $K_d^+ \setminus \{0\} \subseteq \operatorname{int} \mathbb{R}_+^d$.

Proof: Follows from (1) to (4), a separation argument is used.

Proposition 3. One has $K_d \cap -\mathbb{R}_+^d = \{0\}$.

Proof: Elementary.

Feasible tree solution

$V = \{1, \ldots, d\}$
(P, N) ... bi-partition of V, i.e., $\emptyset \neq P \subsetneq V$, $N = V \setminus P$
$G(P, N)$ ... bi-partite digraph with arc set $E = P \times N$

$y \in \mathbb{R}^d$ is called generated by a tree T if T is a spanning tree of $G(P, N)$ such that

$$\forall ij \in E(T) \subseteq P \times N : \ \pi_{ij} y_i = y_j > 0. \tag{6}$$

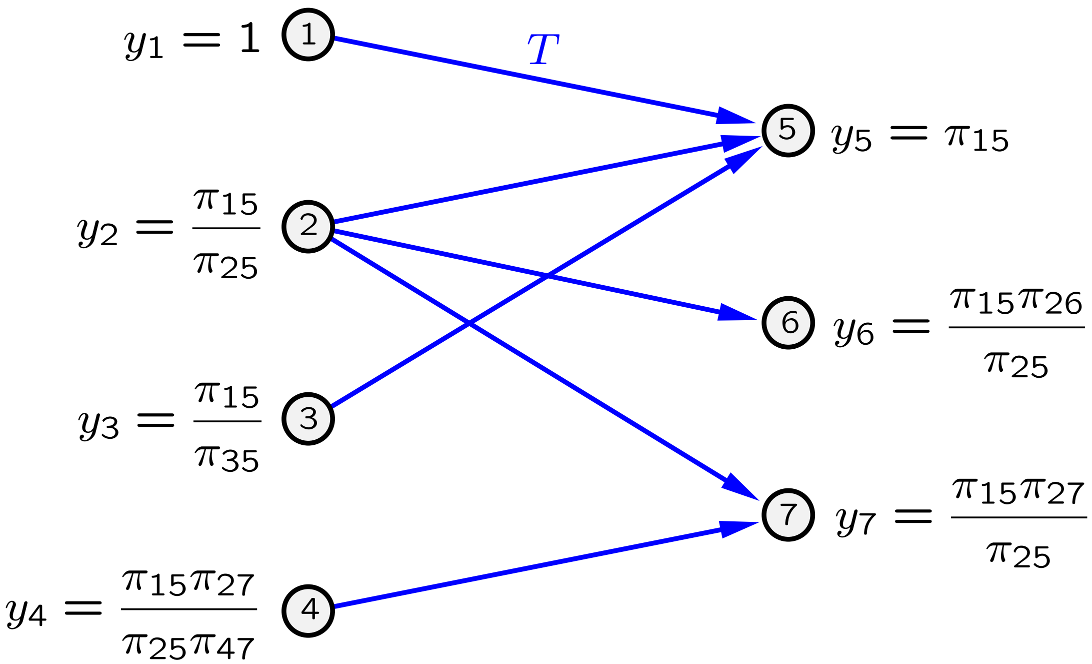
$y \in \mathbb{R}^d$ is called feasible with respect to (P, N) if

$$\forall ij \in P \times N : \pi_{ij} y_i \geq y_j > 0. \tag{7}$$

y is called feasible tree solution w.r.t (P, N) if both properties hold.

Feasible tree solution

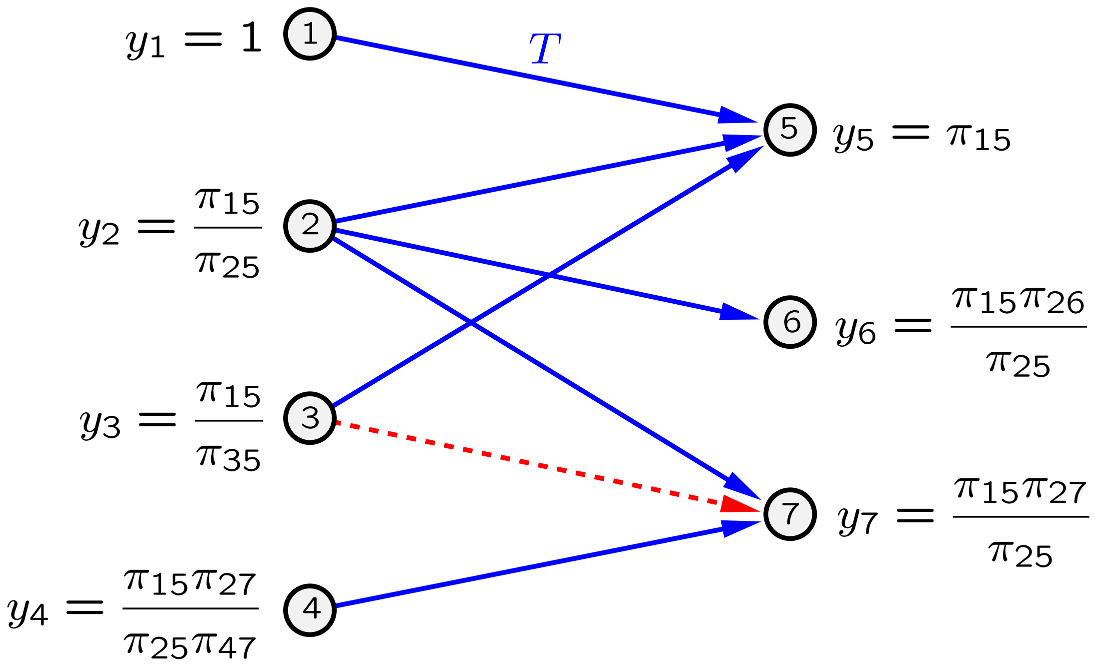
$$V = \{1,2,3,4,5,6,7\},\ P = \{1,2,3,4\},\ N = \{5,6,7\}$$



Tree solution: $\pi_{ij} y_i = y_j$ for $ij \in E(T)$

Feasible tree solution

$$V = \{1,2,3,4,5,6,7\},\ P = \{1,2,3,4\},\ N = \{5,6,7\}$$



Feasibility: e.g. $\pi_{37} y_3 \geq y_7$

Characterization of K_d^+

Theorem 1. For $y \in \mathbb{R}^d$, the following statements are equivalent.

(i) y is an extremal direction of K_d^+;

(ii) y is a feasible tree solution w.r.t. some bipartition (P, N) of V.

Questions:

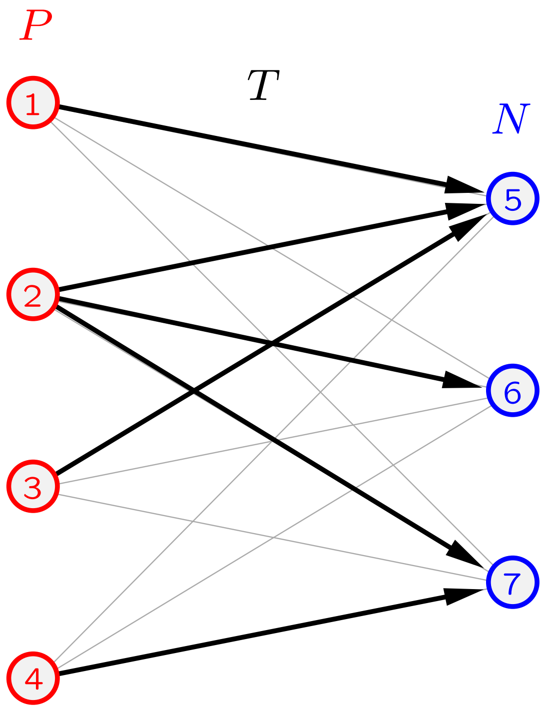
Existence of extremal directions/feasible tree solutions

Construction of extremal directions/feasible tree solutions

Structure of extremal directions/feasible tree solutions

Degree vectors

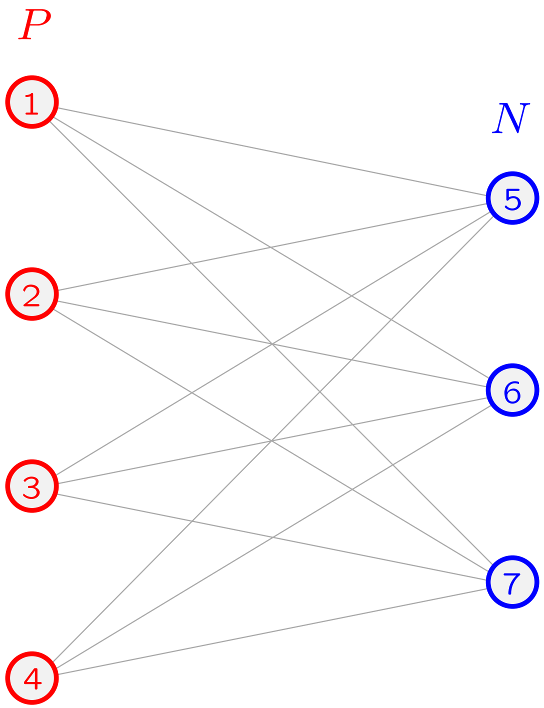
$$\deg_T(P) = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$



$$\deg_T(N) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

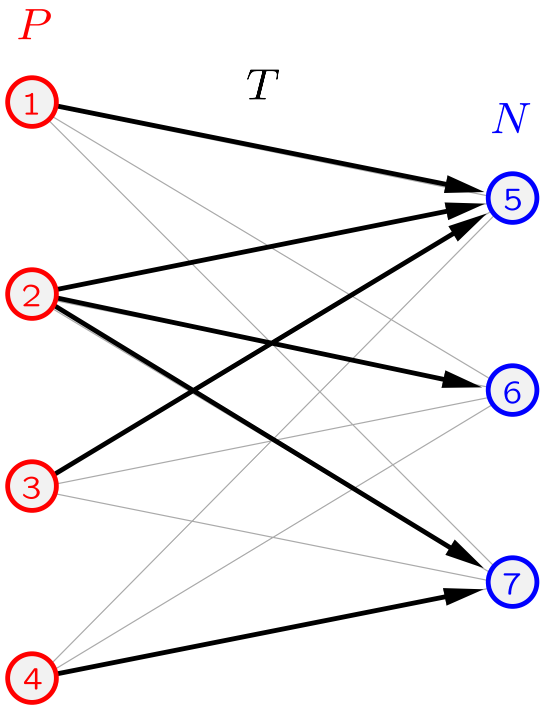
$c \in \mathbb{N}^P$ is called P-configuration if $\sum_{i \in P} c_i = d - 1$

$b \in \mathbb{N}^N$ is called N-configuration if $\sum_{i \in N} b_i = d - 1$

$\mathbb{N} = \{1, 2, \ldots\}$

Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

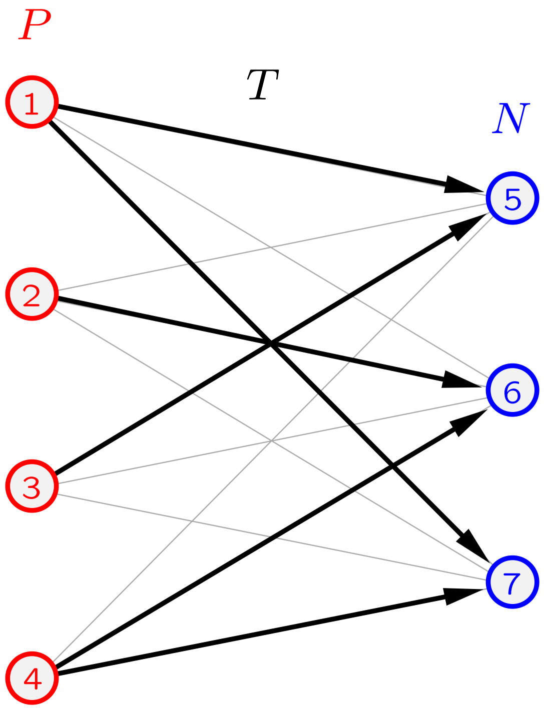
$c \in \mathbb{N}^P$ is called P-configuration if $\sum_{i \in P} c_i = d - 1$

$b \in \mathbb{N}^N$ is called N-configuration if $\sum_{i \in N} b_i = d - 1$

$\mathbb{N} = \{1, 2, \ldots\}$

Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

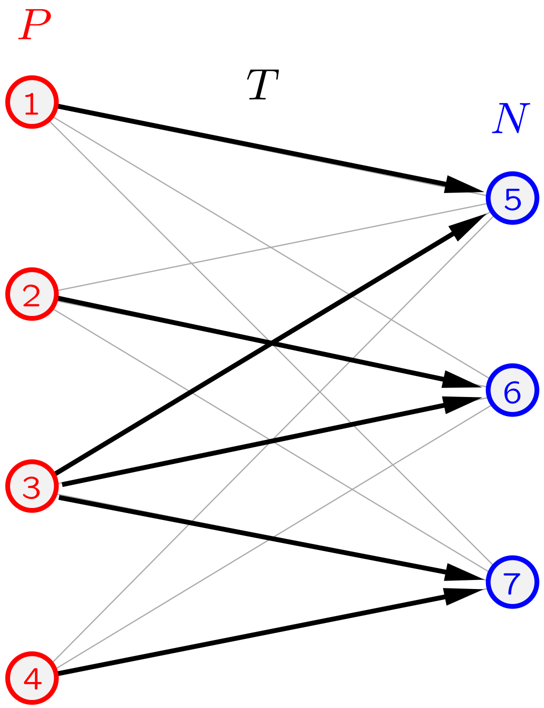
$c \in \mathbb{N}^P$ is called P-configuration if $\sum_{i \in P} c_i = d - 1$

$b \in \mathbb{N}^N$ is called N-configuration if $\sum_{i \in N} b_i = d - 1$

$\mathbb{N} = \{1, 2, \ldots\}$

Degree vectors of spanning trees

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 1 & 1 & 3 & 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 4 & 1 & 2 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

$c \in \mathbb{N}^P$ is called P-configuration if $\sum_{i \in P} c_i = d - 1$

$b \in \mathbb{N}^N$ is called N-configuration if $\sum_{i \in N} b_i = d - 1$

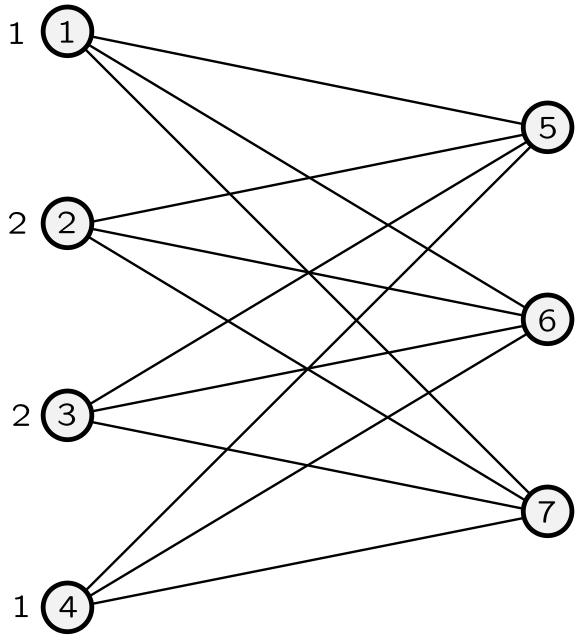
$$\mathbb{N} = \{1, 2, \ldots\}$$

Existence of feasible tree solutions

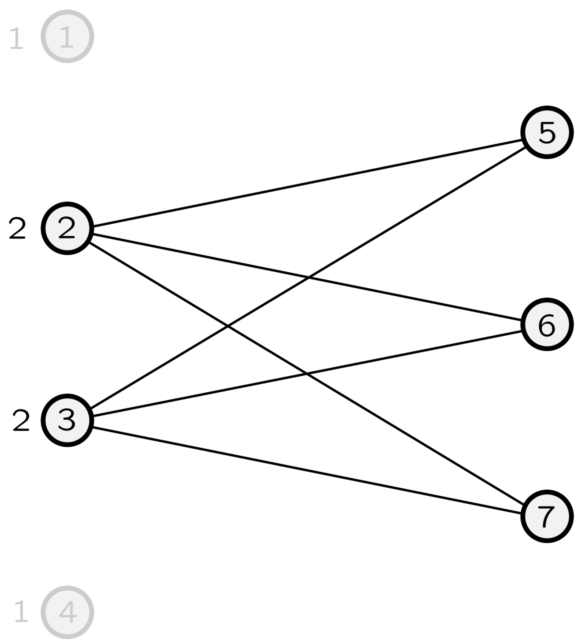
Theorem 2. For every bi-partition (P, N) of V and every P-configuration $c \in \mathbb{N}^P$ there exists a feasible tree solution $y \in \mathbb{R}^d$ generated by a spanning tree T of the bi-partite graph $G(P, N)$ with $\deg_T(P) = c$.

An analogous statement holds if an N-configuration is given.

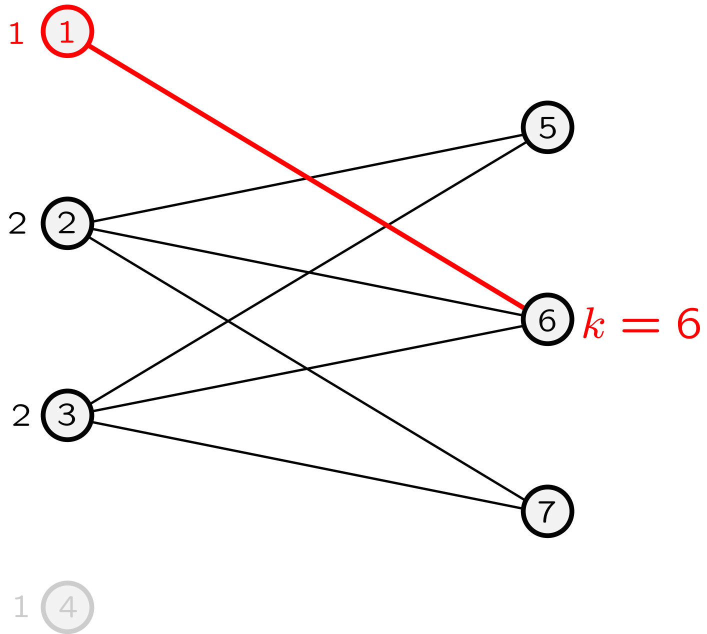
Towards a proof of Theorem 2



Towards a proof of Theorem 2

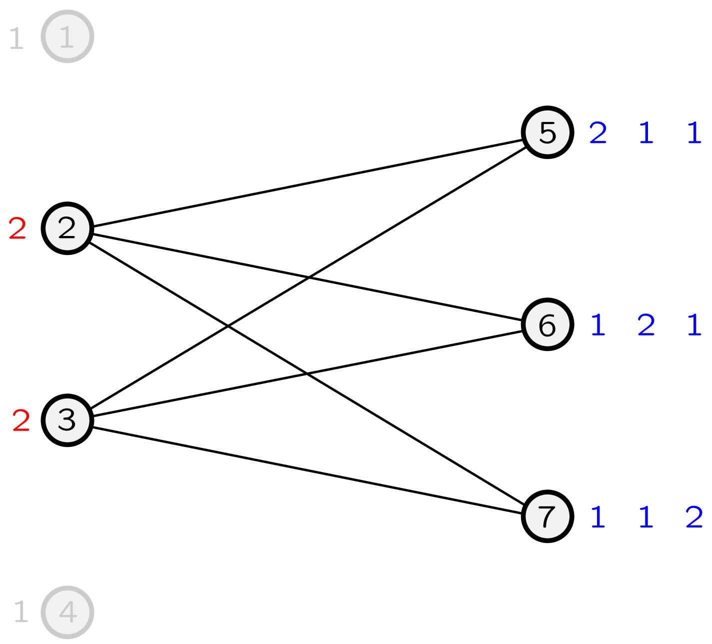


Towards a proof of Theorem 2



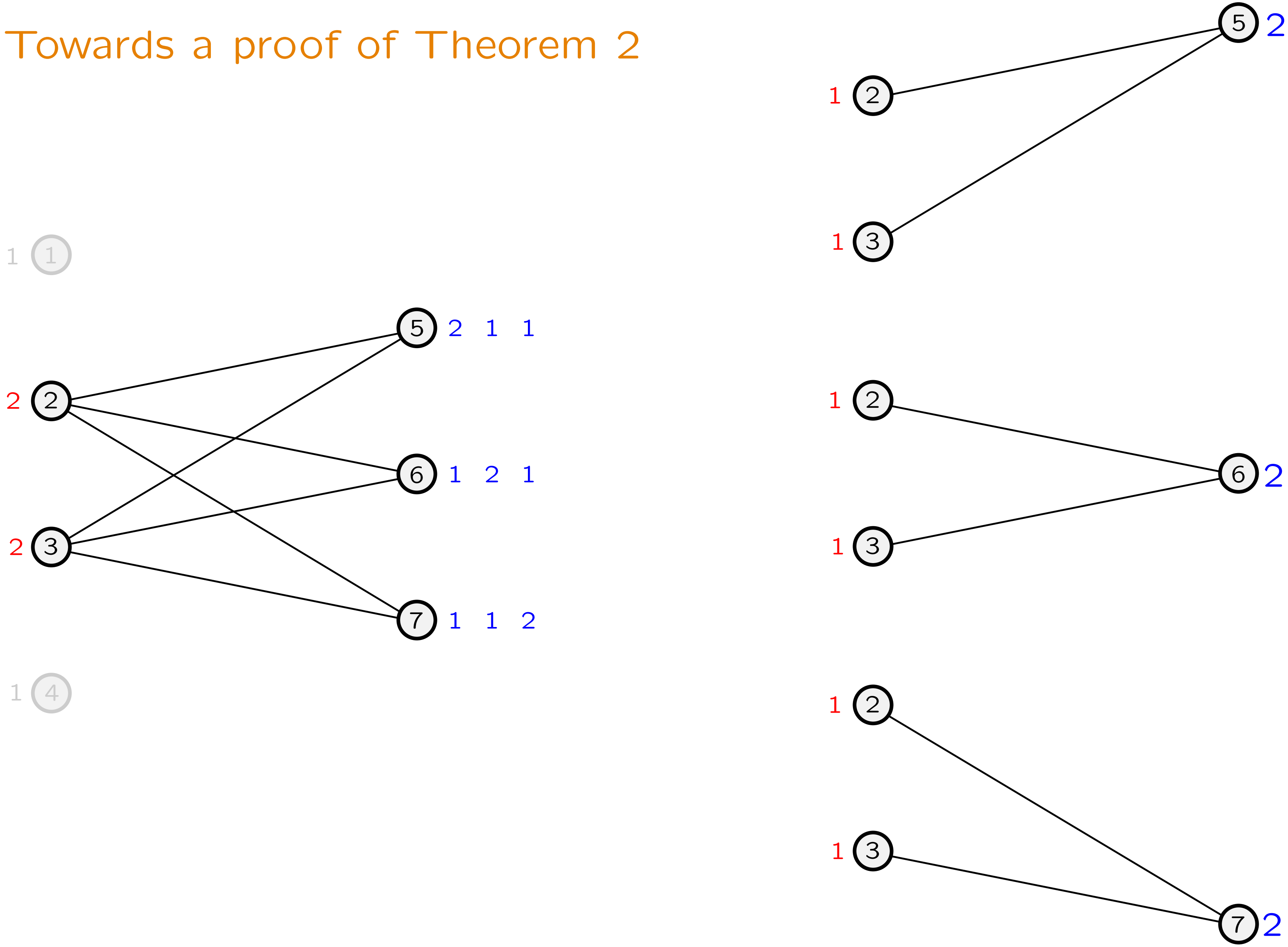
$k \in \arg\max\{y_j/\pi_{1j} \mid j \in N\}$

Towards a proof of Theorem 2

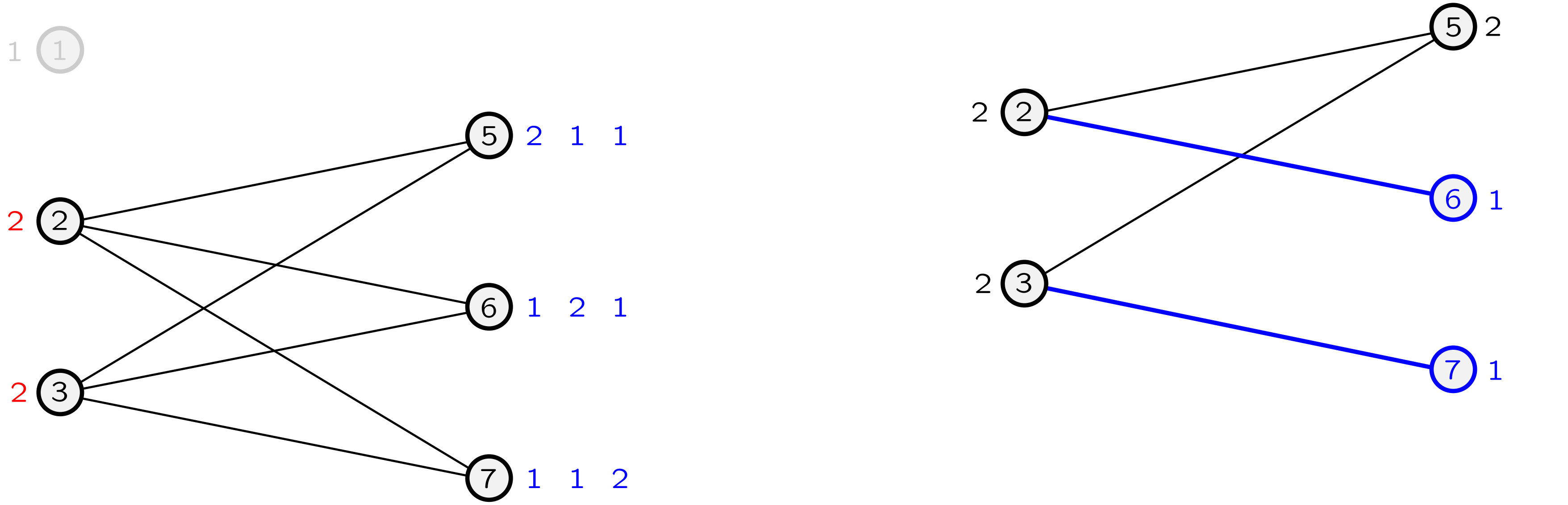


Is there an N-configuration $b \in \mathbb{N}^N$ and a feasible tree solution y generated by T such that $b = \deg_T(N)$ and $c = \deg_T(P)$?

Towards a proof of Theorem 2

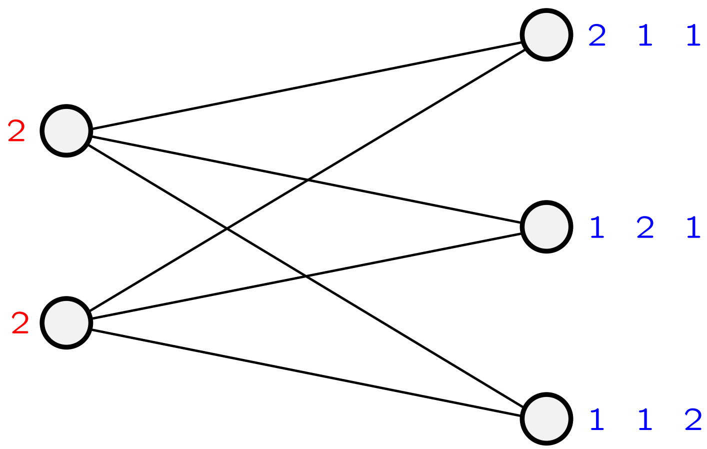


Towards a proof of Theorem 2



$$k \in \arg\min\{y_i \cdot \pi_{ij} \mid i \in P\}$$

Remaining question:



Given a P-configuration $c \in \mathbb{N}^P$.

Is there an N-configuration $b \in \mathbb{N}^N$ and a feasible tree solution y generated by T such that $b = \deg_T(N)$ and $c = \deg_T(P)$?

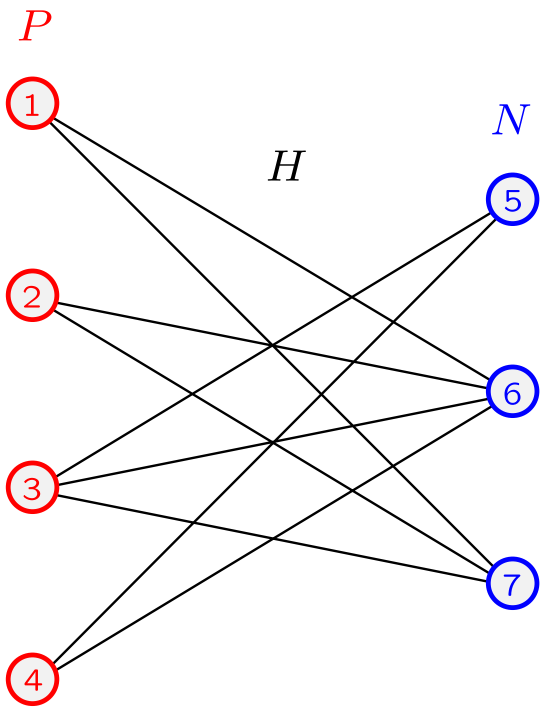
Towards a proof of Theorem 2

$\mathcal{T}(H)$ ... set of all spanning trees of a graph H

Lemma 1. Let $H = H(P, N)$ be a bi-partite graph. Then

$$|\{\deg_T(P) \mid T \in \mathcal{T}(H)\}| = |\{\deg_T(N) \mid T \in \mathcal{T}(H)\}|.$$

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 \\ 3 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 4 & 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}$$

Sang-Il Oum $\longrightarrow$ Postnikov 2009 (about generalized permutohedra)

Toward a proof of Theorem 2

For a feasible tree solution y, define subgraph $H(y)$ of $G = G(P, N)$

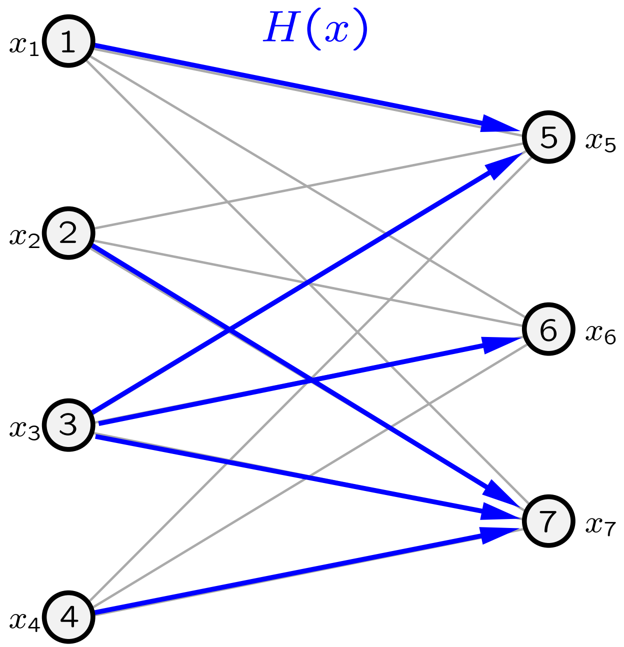
$$V(H(y)) := V(G),\ E(H(y)) := \left\{ ij \in P \times N | \ \pi_{ij} y_i = y_j \right\}$$

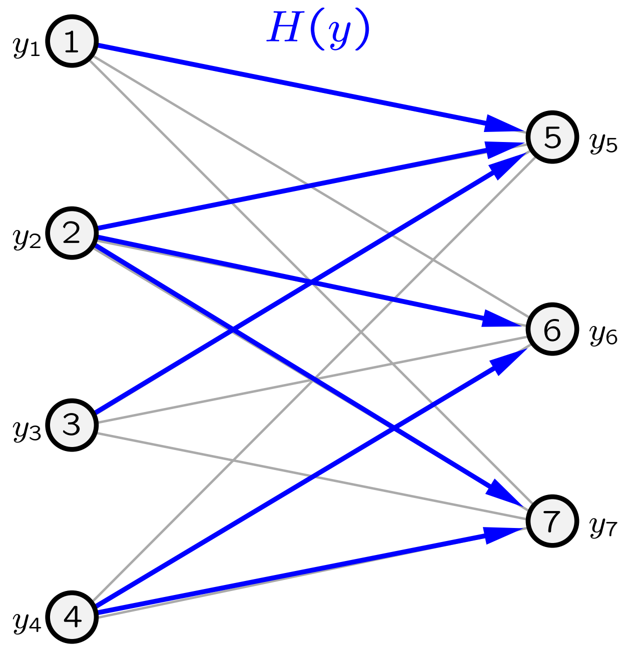
$$\mathcal{P}(y) := \{\text{deg}_T(P) |\ T \in \mathcal{T}(H(y))\}$$
$$\mathcal{N}(y) := \{\text{deg}_T(N) |\ T \in \mathcal{T}(H(y))\}$$

Lemma 2. Let x, y be two feasible tree solutions such that $x \neq \alpha y$ for all $\alpha > 0$. Then

$$\mathcal{P}(x) \cap \mathcal{P}(y) = \emptyset \quad \text{and} \quad \mathcal{N}(x) \cap \mathcal{N}(y) = \emptyset.$$

Illustration of Lemma 1 and Lemma 2





$\pi_{ij}x_i = x_j$, $\pi_{ij}x_i > x_j$

$\pi_{ij}y_i = y_j$, $\pi_{ij}y_i > y_j$

$$\mathcal{P}(x) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{P}(y) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{N}(x) = \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\}$$

$$\mathcal{N}(y) = \left\{ \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Consequences of Theorem 1 and 2

Corollary 1. Assume that also (5) holds. Let x, y be two feasible tree solutions with respect to bi-partitions (P_x, N_x) and (P_y, N_y) of V, respectively. Then $(P_x, N_x) \neq (P_y, N_y)$ implies $x \neq \alpha y$ for all $\alpha > 0$. Moreover, K_d^+ has at least $2^d - 2$ extremal directions.

Corollary 2. K_d^+ has at most $\sum_{p=1}^{d-1} \binom{d-2}{p-1}\binom{d}{p}$ extremal directions.

Example. The upper bound in Corollary 2 cannot be improved.

Let the non-diagonal entries be pairwise different prime numbers such that

$$\left(\min\left\{\pi_{ij} \mid ij \in V \times V, i \neq j\right\}\right)^2 > \max\left\{\pi_{ij} \mid ij \in V \times V, i \neq j\right\}$$

Example. $d = 20$, $\pi_i i = 1$, $\pi_{12} = 59$, $\pi_{12} = 61$ ... $\pi_{20,19} = 2713$

$$59^2 > 2713 \implies (5)$$

K_{20}^+ has exactly $\sum_{p=1}^{19} \binom{18}{p-1}\binom{20}{p} = 35.345.263.800$ extremal directions.

$P = \{5, 6, 7, 8, 9, 10, 11\}$, $N = \{1, \ldots, 4, 12, \ldots, 20\}$.

$\binom{d-2}{p-1} = \binom{18}{6} = 18564$ P-configurations for this bi-partition ($p := |P|$).

$c = (3, 2, 4, 2, 2, 2, 4)^T \in \mathbb{N}^P$

Algorithm (Matlab, about 15 minutes):

$$y = \Big(\tfrac{487\cdot757}{503\cdot859}, \tfrac{491\cdot757}{503\cdot859}, \tfrac{619\cdot947\cdot1367}{677\cdot953\cdot1427}, \tfrac{757}{859}, \tfrac{757}{503\cdot859}, \tfrac{947\cdot1367}{677\cdot953\cdot1427}, \tfrac{1}{859}, \tfrac{1367}{953\cdot1427},$$
$$\tfrac{1}{1117}, \tfrac{839}{859\cdot1237}, \tfrac{1}{1427}, \tfrac{1327}{1427}, \tfrac{947\cdot1367}{953\cdot1427}, \tfrac{1367}{1427}, \tfrac{1373}{1427}, \tfrac{829}{859}, \tfrac{839}{859}, \tfrac{839\cdot1249}{859\cdot1237}, \tfrac{1109}{1117}, 1\Big)^T$$

$b = (1, 1, 1, 2, 1, 2, 2, 1, 1, 2, 1, 1, 3)^T \in \mathbb{N}^N$

Special case 1

$$\left.\begin{array}{l}\pi_{ii} := 1 \text{ and } \pi_{ij} := a_j/b_i \ (i \neq j), \\ 0 < b_i \leq a_i \text{ for all } i \in V, \\ 0 < b_k < a_k \text{ for at least one } k \in V\end{array}\right\} \quad \Rightarrow (1) \text{ to } (4)$$

Recursion formula

$$Y_2 = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix} \qquad Y_d = \begin{pmatrix} & & & b_1 & & & & a_1 \\ & Y_{d-1} & & \vdots & & Y_{d-1} & & \vdots \\ & & & b_{d-1} & & & & a_{d-1} \\ a_d & \ldots & a_d & a_d & b_d & \ldots & b_d & b_d \end{pmatrix}.$$

Direct description

$$K_d^+ = \text{cone}\left\{y \in \mathbb{R}^d | \ (P, N) \text{ bi-part. of } V, \ \forall i \in P : \ y_i = b_i, \ \forall j \in N : \ y_j = a_j\right\}$$

Consequence

K_d^+ has at most $2^d - 2$ extremal directions.

Special case 2

$$\left.\begin{array}{l}\pi_{ii} := 1 \text{ and } \pi_{ij} := a_j/b_i \ (i \neq j), \\ 0 < b_i < a_i \text{ for all } i \in V,\end{array}\right\} \quad \Rightarrow \text{(1) to (5)}$$

The same as in special case 1, but now

K_d^+ has exactly $2^d - 2$ extremal directions.

Special case 3

$$\left.\begin{array}{l}\pi_{ii} := 1 \text{ and } \pi_{ij} := a_j/b_i \ (i \neq j),\\ 0 < b_i \leq a_i \text{ for all } i \in V,\\ 0 < b_k < a_k \text{ for at least one } k \in V\end{array}\right\} \quad \Rightarrow \text{(1) to (4)}$$

$b_k = a_k$ for some $k \in V$

Recursion formula (w.l.o.g. $a_1 = b_1 = 1$)

$$Y_2 = \begin{pmatrix} 1 & 1 \\ a_2 & b_2 \end{pmatrix} \qquad Y_d = \begin{pmatrix} & Y_{d-1} & & & Y_{d-1} & \\ a_d & \ldots & a_d & b_d & \ldots & b_d \end{pmatrix}.$$

Direct description

$$K_d^+ = \text{cone}\left\{ y \in \mathbb{R}^d | \ Q \subseteq V \setminus \{k\}, \ \forall i \in Q: \ y_i = b_i, \ \forall j \in V \setminus Q: \ y_j = a_j \right\}.$$

Consequence

K_d^+ has at most 2^{d-1} extremal directions.

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