Wavelet Analysis of Generalized Fractional Process

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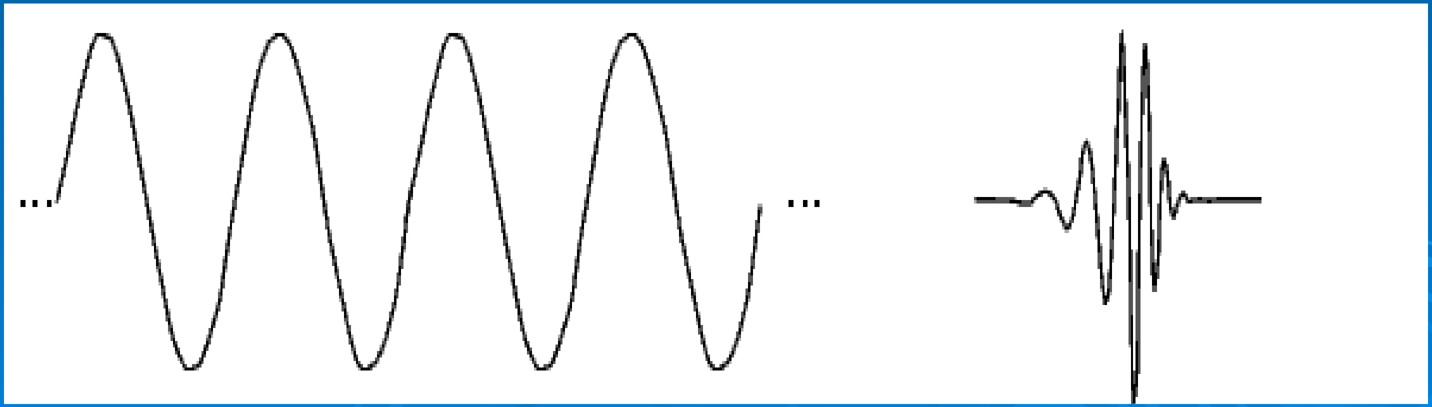
What is a wavelet?

- A wavelet is a waveform of effectively limited duration that has an average value of zero.

- Sine wave Wavelet



What is a wavelet?

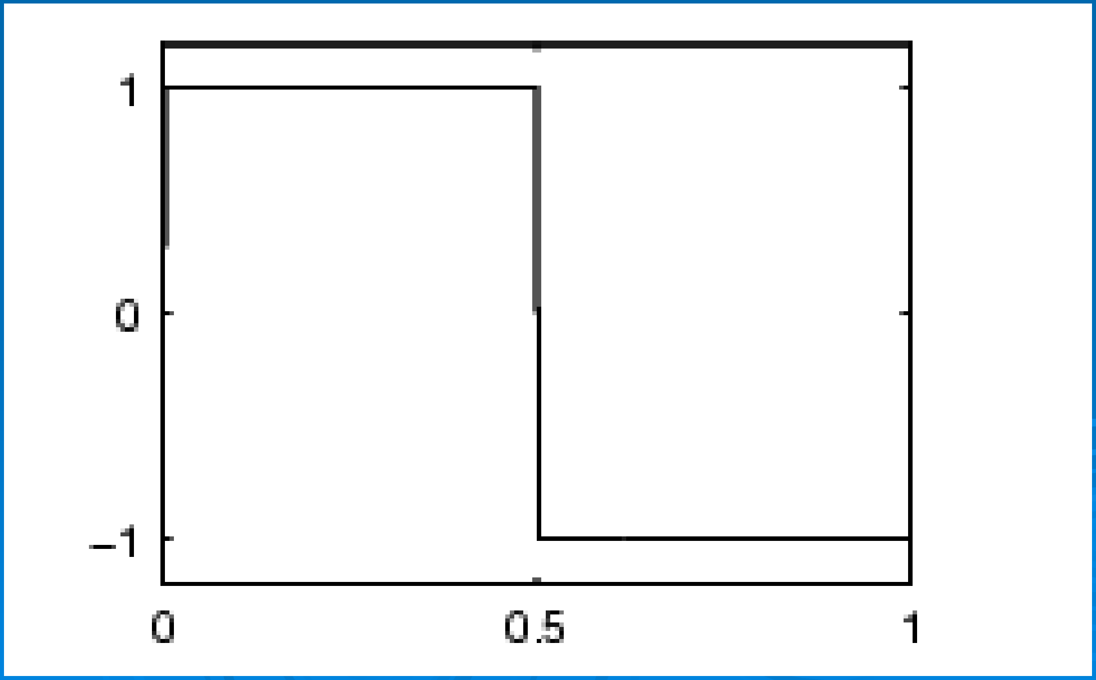
- Admissibility condition:
- The function $\psi(t) \in L^2(R)$ is often referred to as the *mother wavelet* and must satisfy the admissibility condition given by
- $$\int_R |\Psi(w)|^2 |w|^{-1} \, dw < \infty,$$
- where $\Psi(w)$ is the Fourier transform of $\psi(t)$.

- If $\psi(t)$ has sufficient decay, then this condition is equivalent to
- $$\Psi(0) = \int_R \psi(t) \, dt = 0.$$

What is a wavelet?

- Example: Haar wavelet

$$\psi(t) = \begin{cases} 1 & 0 \le t \le 1/2 \\ -1 & 1/2 \le t \le 1 \\ 0 & otherwise \end{cases}.$$



What can wavelets do?

1) Performing local analysis
2) Analyzing nonstationary signals
3) Denoising Signals
4) Data Compression
5) Decorrelating time series

Discrete Wavelet Transform

As discretized cwt:

$$C(a,b) = \int_R s(t) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) dt$$

$$a = 2^j, b = k2^j, (j,k) \in Z^2$$

Discrete Wavelet Transform

wavelet coefficients:

$$d_{j,t} = \sum_{l=0}^{L-1} h_l c_{j-1,2t-1 \bmod N_j - 1}$$

scaling coefficients:

$$c_{j,t} = \sum_{l=0}^{L-1} g_l c_{j-1,2t-1 \bmod N_j - 1}$$

$$t = 0, \ldots, N_j - 1$$

Discrete Wavelet Transform

Let $c_{0,t} = X_t$ (the time series), then the wavelet and scaling coefficients are:

$$d_{j,t} = \sum_{l=0}^{L_j - 1} h_{j,l} X_{2^j (t+1) - 1 - l \bmod N}$$

$$c_{j,t} = \sum_{l=0}^{L_j - 1} g_{j,l} X_{2^j (t+1) - 1 - l \bmod N}$$

$$L_j = (2^j - 1)(L - 1) + 1$$

Long-memory Process

Definition:

A *long-memory process* is commonly defined as a stationary process for which the autocorrelation function at lag k satisfies

$$\rho(k) \sim C_\rho k^{2d-1}$$

as $k \to \infty$, where $C_\rho \neq 0$ and $0<d<0.5$.

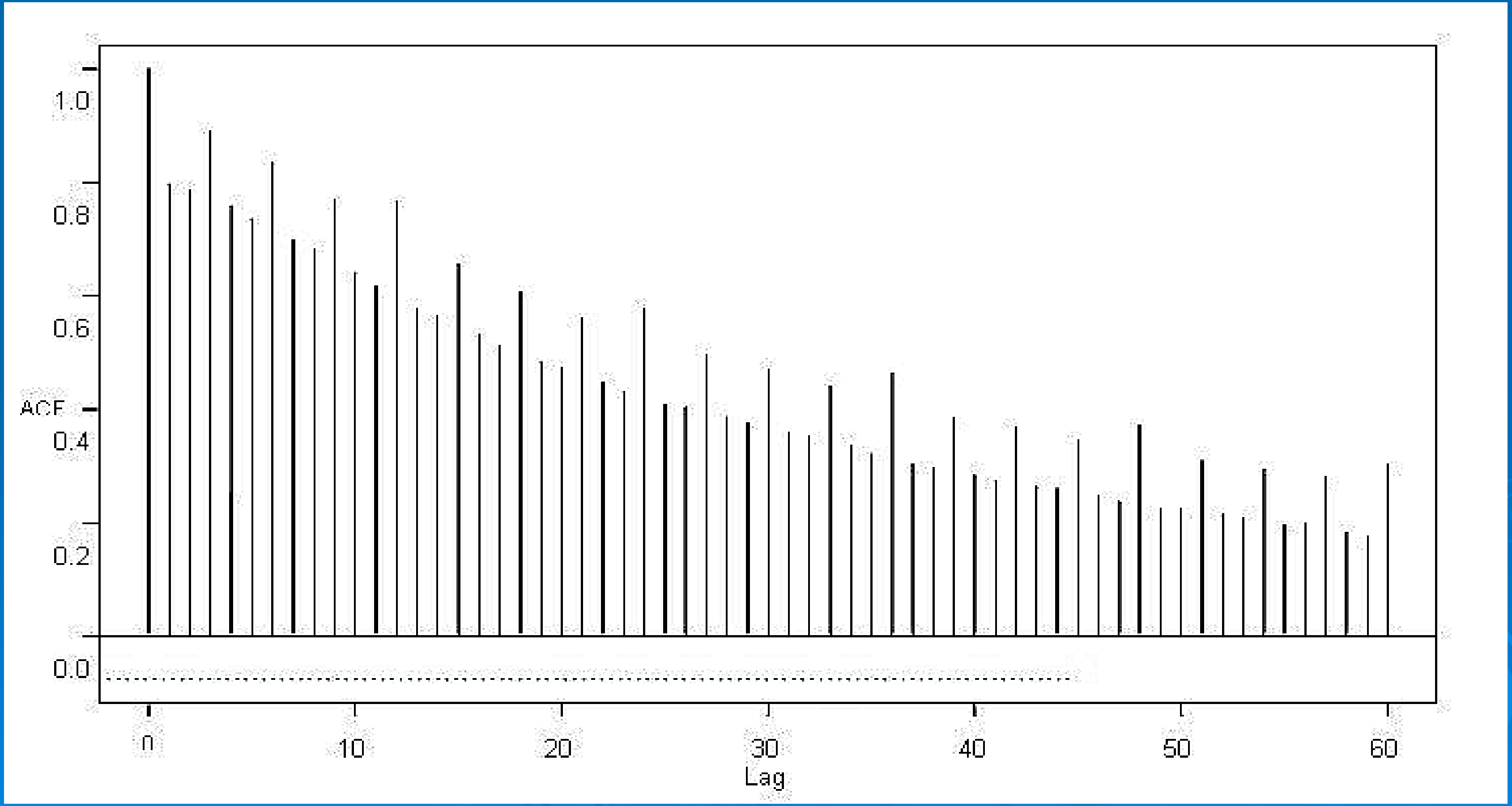
An equivalent statement for the power spectrum is given by

$$f(\omega) \sim C_f |\omega|^{-2d}, \quad \text{as } \omega \to 0,$$

which has a pole at the origin when $0<d<0.5$.

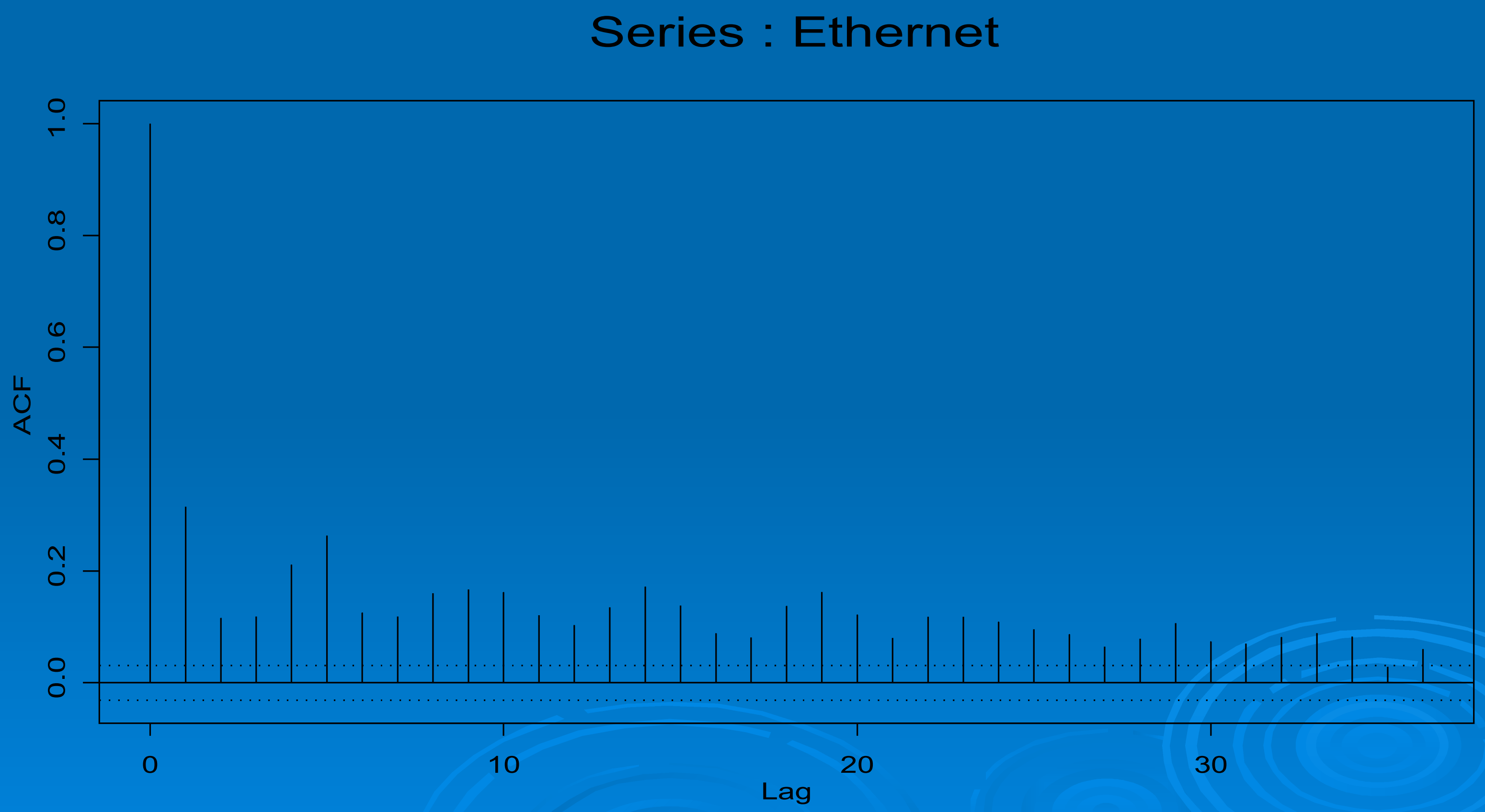
Long-memory Process

Example: Autocorrelations of video traffic data



Long-memory Process

Example: Autocorrelations of Ethernet data



Generalized Fractional Process

The Model

A *Gegenbauer autoregressive moving average GARMA(p,d,u,q) process* is the output of the system function

$$H(z) = \frac{\Theta(z)}{\Phi(z)}(1 - 2uz^{-1} + z^{-2})^{-d}$$

driven by a stationary white noise input with mean 0 and variance σ^{2}.

Generalized Fractional Process

The Model

The rational function

$$\frac{\Theta(z)}{\Phi(z)} = \frac{1+\theta_1 z^{-1} + \ldots + \theta_q z^{-q}}{1-\phi_1 z^{-1} - \ldots - \phi_p z^{-p}}$$

is the *autoregressive moving average, ARMA(p,q) system*, such that $z^p\Theta(z)$ and $z^p\Phi(z)$ have no common zeros and the zeros lie outside the unit circle.

Generalized Fractional Process

The Model

The *Gegenbauer system* is defined by

$$\left(1-2uz^{-1}+z^{-2}\right)^{-d}=\sum_{n=0}^{\infty}C_n^d(u)z^{-n}$$

$$C_n^d(u) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(2u)^{n-2k}\Gamma((d)-k+n)}{k!(n-2k)!\Gamma(d)}$$

When the input is a stationary white noise, the output is called a *Gegenbauer process*, which is stationary if $d<0.5$ and $|u|<1$ If or if $d<0.25$ and $|u|=1$; it is invertible if $-0.5<d$ and $|u|<1$ or $-0.25<d$ and $|u|=1$. If u=1, we have ARFIMA(p,d,q) process.

Generalized Fractional Process

The Model

The power spectrum of a GARMA(*p,d,u,q*) process is given by

$$f(\omega) = \sigma_z^2 \left| \frac{\Theta(e^{-i2\pi\omega})}{\Phi(e^{-i2\pi\omega})} \right|^2 \left[4\left(cos(2\pi\omega) - u \right)^2 \right]^{-d}$$

where ωε(-0.5,0.5] and $v = cos^{-1}(u)/2\pi$ is called the *Gegenbauer frequency* at which the power spectrum becomes unbounded when 0<*d*<0.5.

Generalized Fractional Process

The Model

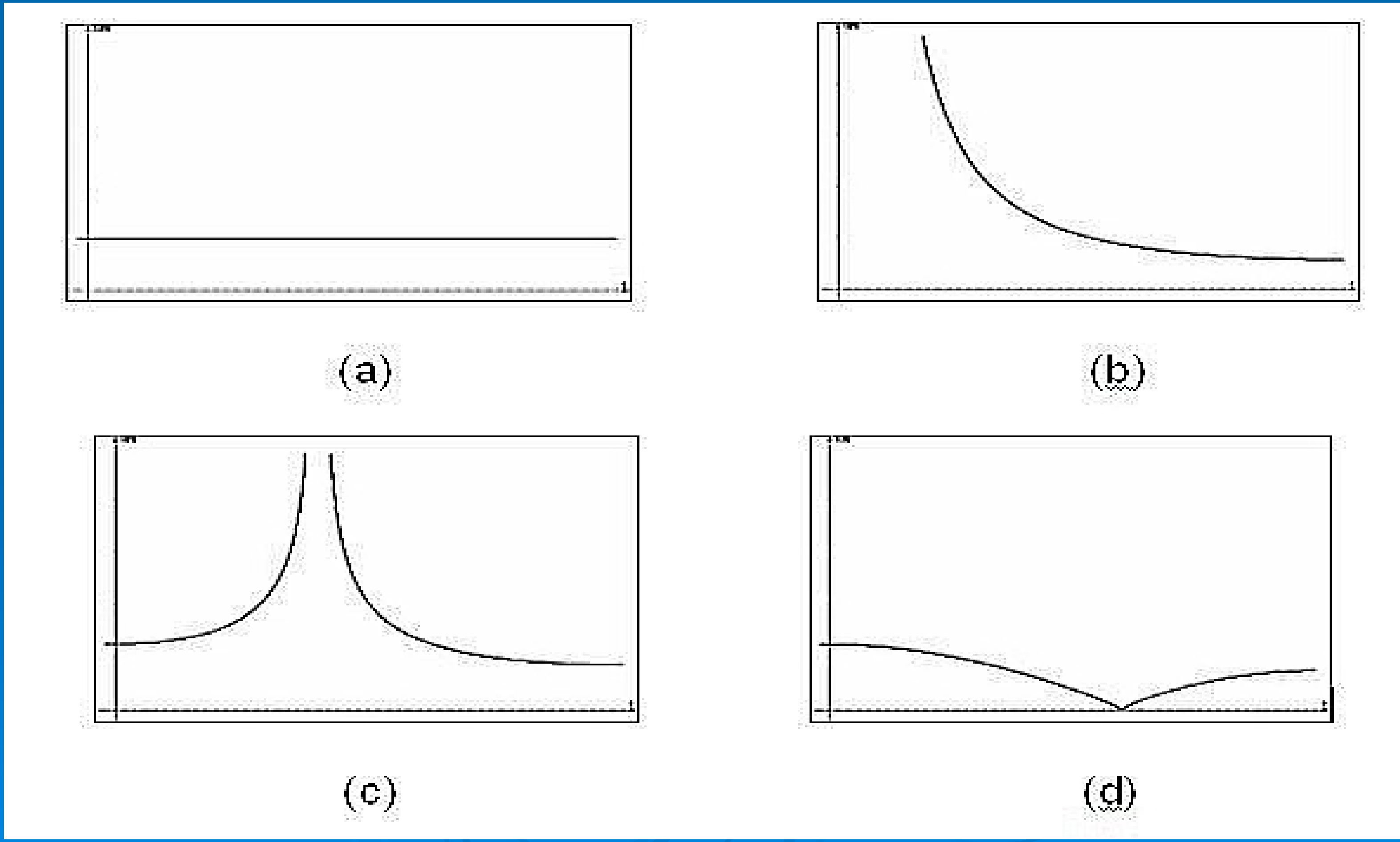
The autocovariance function of a GARMA(*p,d,u,q*) process is given by

$$\gamma(k)=\frac{\sigma^2}{2\sqrt{\pi}}\Gamma(1-2d)[2\sin(v)]^{0.5-2d}\cdot\left[P_{k-0.5}^{2d-0.5}(u)+(-1)^k P_{k-0.5}^{2d-0.5}(-u)\right]$$

where are the *associated Legendre functions of the first kind*.

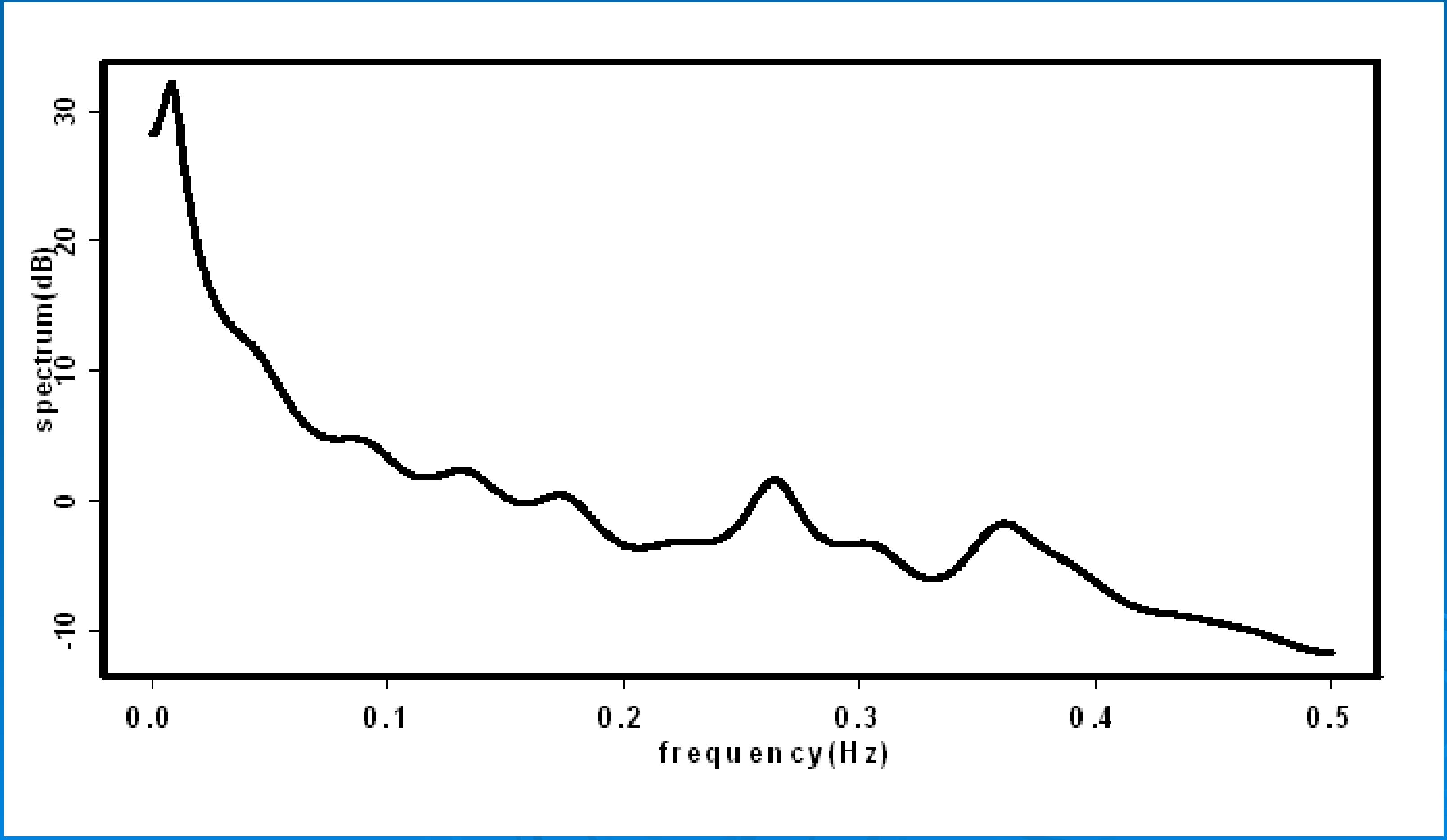
Generalized Fractional Process

Spectrum of Garma(0,d,u,0) Process



Generalized Fractional Process

Example: Spectrum of heart rate data



Generalized Fractional Process

GARMA(p,d,u,q) Process generalizes the following:

1) Gegenbauer process

2) Fractionally Integrated process

3) ARMA Process

4) AR Process

5) MA Process

Covariance Structure of Wavelet Coefficients

Covariance of wavelet coefficients

$$Cov(d_{jt}, d_{j't'}) = \int_{-1/2}^{1/2} e^{i2\pi f(2^{j'}(t'+1)-2^{j}(t+1))} H_j(f) H^*_{j'}(f) S_Y(f)\, df$$

$$Cov(d_{jt}, d_{j(t+s)}) = \int_{-1/2}^{1/2} e^{i2\pi fs} |H_j(f)|^2 S_Y(f)\, df$$

where $H_{j,L}(f)$ is the Fourier transform of the level j Daubechies wavelet filter $\{h_{j,l}\}$

Covariance Structure of Wavelet Coefficients

Note that:

$$H_{j,L}(f) = H_{1,L}(2^{j-1} f) \prod_{l=0}^{j-2} G_{1,L}(2^l f)$$

$$G_{j,L}(f) = \prod_{l=0}^{j-1} G_{1,L}(2^l f)$$

Covariance Structure of Wavelet Coefficients

Lemma 1 (Gonzaga and Kawanaka) Let $\{h_{\ell,1},\ \ell = 0,1,\ldots,L-1\}$ the orthonormal Daubechies wavelet filter of length L, then as $L \to \infty$,

$$\left|H_{j',L}(f)\right|^2 \left|G_{1,L}\left(2^{j'-1} f\right)\right|^2 \to 0 \text{ a.e. and}$$

$$\left|H_{1,L}\left(2^{j'-1} f\right)\right|^2 \prod_{m=0}^{j-2} \left|G_{1,L}\left(2^m f\right)\right|^2 \to 0 \text{ a.e.} \qquad \text{(A1)}$$

on [0,0.5]

Covariance Structure of Wavelet Coefficients

Theorem 2 (Gonzaga and Kawanaka). Let $\{Y_t\}$ be a generalized fractional process and $\{h_{\ell,1},\ \ell = 0,1,\dots,L-1\}$ the orthonormal Daubechies wavelet filter of length L, then for $j > j'$ and d<0,

$$|Cov(d_{jt}, d_{j't'})| = O\left(\frac{1}{L^{3/4}}\right). \qquad \text{(A7)}$$

Covariance Structure of Wavelet Coefficients

Theorem 3 (Gonzaga and Kawanaka)**.** Let $\{Y_t\}$ be a generalized fractional process and $\{h_{\ell,1},\ \ell = 0,1,\ldots,L-1\}$ the orthonormal Daubechies wavelet filter of length L, then for $j > j'$ and d>0,

$$|Cov(d_{jt}, d_{j't'})| = O\left(\frac{1}{L^{3/4}}\right), \qquad \text{if } v \in \left[0, 2^{-j-1}\right], \tag{A24}$$

and

$$|Cov(d_{jt}, d_{j't'})| = O\left(\frac{1}{L^{1/4}}\right), \qquad \text{if } v \notin \left[0, 2^{-j-1}\right]. \tag{A25}$$

Covariance Structure of Wavelet Coefficients

Theorem 4 (Gonzaga and Kawanaka) Let $\{Y_t\}$ be a generalized fractional process and $\{h_{\ell,1},\ \ell = 0,1,\ldots,L\text{-}1\}$ the orthonormal Daubechies wavelet filter of length L, then

$$\lim_{L\to\infty} cov(d_{jt}, d_{j(t+s)}) = 2^{j+1} \int_{2^{-j-1}}^{2^{-j}} cos(2^{j+1}\pi f s) S_Y(f)\,df, \qquad \text{(A52)}$$

which exists for $j \in Z^+$.

Covariance Structure of Wavelet Coefficients

Theorem 5 (Gonzaga and Kawanaka). Let $\{Y_t\}$ be a generalized fractional process and $\{h_{\ell,1},\ \ell = 0,1,\ldots,L-1\}$ the orthonormal Daubechies wavelet filter of length L, then if u=-1 and j>1

$$|cov(d_{jt}, d_{j(t+s)})| = O\left([2^j s]^{-[L-4d]-1}\right) \text{ as } 2^j s \to \infty. \qquad \text{(A59)}$$

Covariance Structure of Wavelet Coefficients

Lemma 6. (Gonzaga and Kawanaka) Let $\{h_{\ell,1},\ \ell = 0,1,\ldots,L-1\}$ be the orthonormal Daubechies wavelet filter of length L and $|H_{j,L}(f)|^2$ its energy spectrum at level j and frequency f. Then

$$|H_{j,L}(f)|^2 \leq \frac{1}{2^j}\left(\frac{2\sin^2(2^{j-1}\pi f)}{\sin(\pi f)}\right)^{L-2}. \qquad \text{(A66)}$$

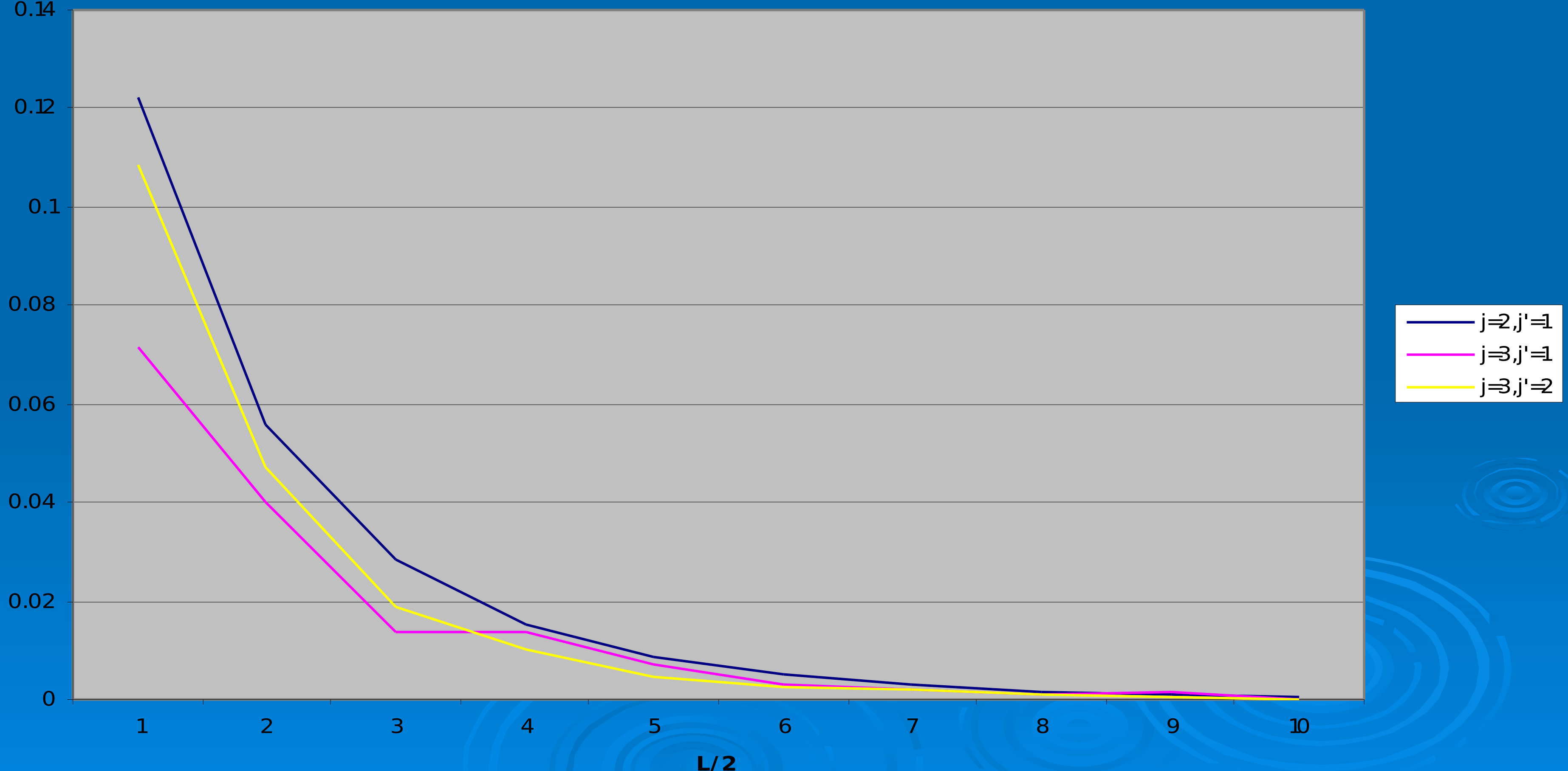
Covariance Structure of Wavelet Coefficients

Theorem 7 (Gonzaga and Kawanaka). Let $\{Y_t\}$ be a generalized fractional process and $\{h_{\ell,1},\ \ell = 0,1,\ldots,L-1\}$ the orthonormal Daubechies wavelet filter of length L, then if $u \in (-1,1)$

$$|Cov(d_{jt}, d_{j(t+s)})| = O\left(2^{j(2d-2)} s^{2d-1}\right) \text{ as } 2^j \to \infty \text{ and } s \to \infty. \quad \text{(A78)}$$

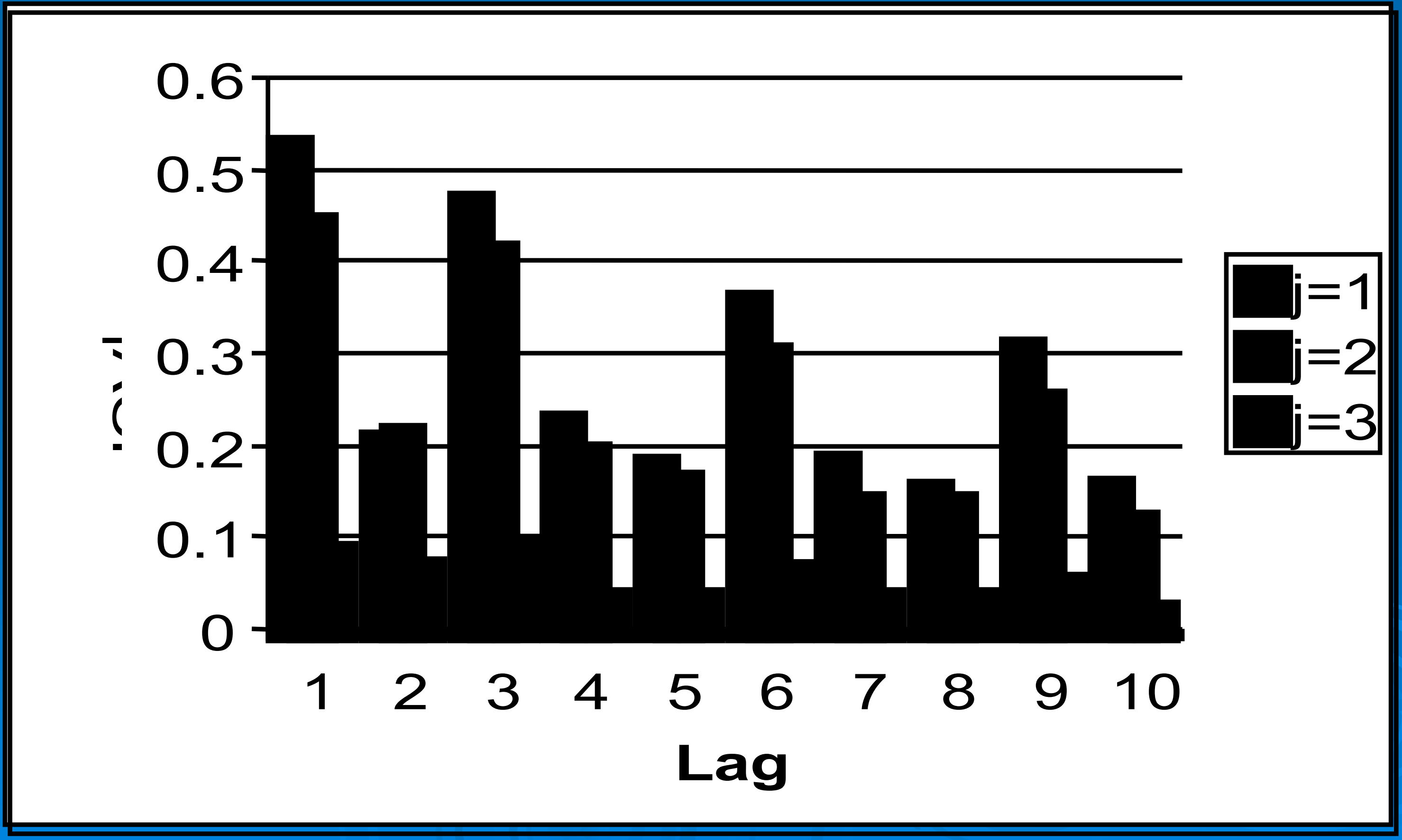
Covariance Structure of Wavelet Coefficients

Absolute maximum values of correlations



Covariance Structure of Wavelet Coefficients

Within-scale correlations



Weighted Least Square Estimation

Wavelet variance

$$v_Y^2(\lambda) = \frac{E(W_{t,\lambda}^2)}{2\lambda}$$

Maximal overlap estimator

$$\hat{v}_Y^{\,2}(\lambda) = \frac{1}{2\lambda N_{W_\lambda}} \sum_{t=L_\lambda}^{N} w_{t,\lambda}^2$$

Note:

$$\log \hat{v}^2(\lambda) \xrightarrow{d} N\left(\log v^2(\lambda), A_{W_\lambda} / (2\lambda^2 N_{W_\lambda} v^4(\lambda))\right)$$

Weighted Least Square Estimation

Regression equation:

$$\log\left(v^2(\lambda_j)\right) \approx -2d\log\left(2\left|\cos(2\pi\mu_j) - \cos(2\pi v\right|\right)$$

Weighted Least Square Estimation

Estimator of the long-memory parameter

$$\hat{d} = -\frac{1}{2}\left[\frac{\sum_{j=1}^{J} u_j x_j y_j - \left(\sum_{j=1}^{J} u_j y_j\right)\left(\sum_{j=1}^{J} u_j x_j\right)}{\sum_{j=1}^{J} u_j x_j^2 - \left(\sum_{j=1}^{J} u_j x_j\right)^2}\right]$$

$$x_j = \log\left(2\left|\cos(2\pi u_j) - \cos(2\pi v)\right|\right) \qquad y_j = log\left(\hat{v}^2(2^j)\right)$$

Weighted Least Square Estimation

Estimation of short-memory parameters:

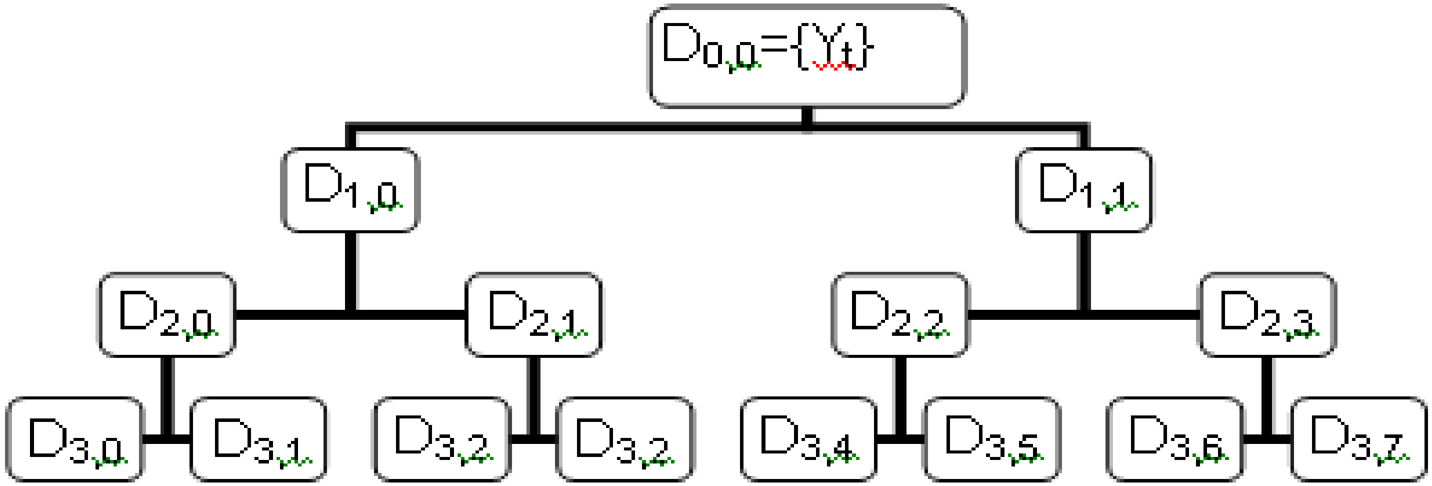
$$J_X(f_j) = (1 - 2ue^{-if_j} + e^{-i2f_j})^{\hat{d}} J_Y(f_j)$$

DFT of ARMA(p,q)

DFT of GARMA(p,d,u,q)

Likelihood Estimation

- We use wavelet packet wavelet transform



Wavelet packet decomposition

Likelihood Estimation

Basis Selection Algorithm

For $j<J$, we test the vector $D_{j,n}$ for white noise. If the test fails to reject, we retain $D_{j,n}$. If the test rejects, we split $D_{j,n}$ into $D_{j+1,2n}$ and $D_{j+1,2n+1}$, and test both the resulting subbands for white noise. We repeat this process until $j=J$ in which we retain $D_{J,n}$. We denote the resulting vector of DWPT coefficients by $\mathbf{D}=\left(D_{j,n},(j,n)\in B\right)$, which is approximately uncorrelated.

Likelihood Estimation

The approximate likelihood can be written as a univariate density from which e.g. the posterior density can be obtained and an MCMC algorithm be implemented:

$$L(D \mid \Psi) = \left(2\pi\sigma_{\varepsilon}^{2}\right)^{-N/2} \left(\prod_{(j,n)\in B} \left(\sigma_{jn}^{2}\right)^{-N_{jn}/2} \right) \exp\left[\frac{-1}{2\sigma_{\varepsilon}^{2}} \sum_{(j,n)\in B} \frac{D_{j,n}^{T} D_{j,n}}{\sigma_{jn}^{2}} \right]$$

End