Stochastic numerics and issues in the stability analysis of numerical methods

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joint work with
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Outline

1. **Introduction: Stochastic Differential Equations**
2. **Introduction: Some numerical methods, notions of convergence**
3. **Analysis of numerical methods for SDEs beyond convergence**

Introduction: Stochastic Differential Equations

Problem:
Mathematical formulation of systems with 'noise', randomness, insufficient knowledge of parameter values, ...

Applications

- Population dynamics;
- Molecular dynamics;
- Chemical kinetics;
- Finance;
- Electrical circuit simulation;
- Polymer physics;
- Neuroscience;
- PDE simulation;

Applications

- Population dynamics;
- Molecular dynamics;
- Chemical kinetics;
- Finance;
- Electrical circuit simulation;
- Polymer physics;
- Neuroscience;
- PDE simulation;
- .... practically everywhere ....

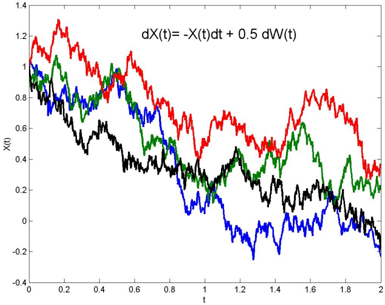
Stochastic differential equations (SDEs)

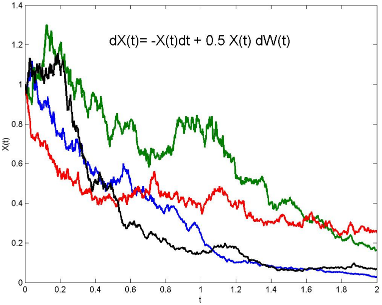
$$\boxed{\mathrm{d}X(t) = F(t, X(t))\,\mathrm{d}t + G(t, X(t))\,\mathrm{d}W(t),\ t \in [0, T],\ X(0) = x_0}$$

- coefficients: (globally Lipschitz) $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $G = (G_1, \ldots, G_m) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$;
- Wiener process: $W = \{W(t, \omega), t \in [0, T], \omega \in \Omega\}$ is an m-dim. Wiener process on probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$.
- F is called 'drift coefficient', G is called 'diffusion coefficient'.
- if G does not depend on X, the SDE 'has additive noise', otherwise it 'has multiplicative noise'.

Solutions of (Itô) stochastic differential equations

- $\mathrm{d}X(t) = aX(t)\mathrm{d}t + b\mathrm{d}W(t)$, $\triangleright X(t) = e^{at}(1 + b\int_0^t e^{-as}\mathrm{d}W(s))$
- $\mathrm{d}X(t) = aX(t)\mathrm{d}t + bX(t)\mathrm{d}W(t)$,
  $\triangleright X(t) = \exp((a - \frac{1}{2}b^2)t + bW(t))$





Introduction: Some numerical methods, notions of convergence

Introduction: Some numerical methods

$I_r^{t_n,t_{n+1}} = W_r(t_{n+1}) - W_r(t_n) \sim \sqrt{h}\mathcal{N}(0,1)$, $\quad I_{r_1,r_2}^{t_n,t_{n+1}} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_{r_1}(u)\,\mathrm{d}W_{r_2}(s)$

and $t_n = n \cdot h$, $n = 0, 1, \ldots$,

Euler-Maruyama-method (G. Maruyama 1955):

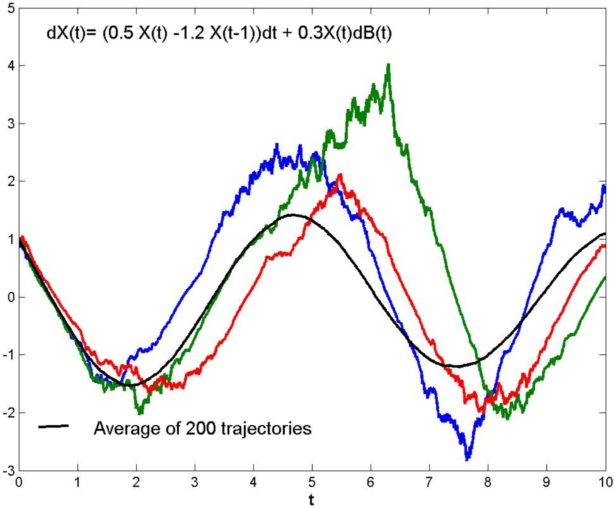
$$X_{n+1} = X_n + h\,F(t_n, X_n) + \sum_{r=1}^{m} G_r(t_n, X_n)\, I_r^{t_n,t_{n+1}}$$

θ-Milstein-method (G. Milstein 1974):

$$X_{n+1} = X_n + h\,(\theta F(t_{n+1}, X_{n+1}) + (1-\theta) F(t_n, X_n))$$
$$+ \sum_{r=1}^{m} G_r(t_n, X_n)\, I_r^{t_n,t_{n+1}} + \sum_{r_1,r_2=1}^{m} (G_{r_1})'_x\, G_{r_2}(t_n, X(t_n))\; I_{r_1,r_2}^{t_n,t_{n+1}}$$

BDF2-Maruyama-method (E.B., R.Winkler 2006):

$$X_n - \tfrac{4}{3} X_{n-1} + \tfrac{1}{3} X_{n-2} = h\; \tfrac{2}{3} F(t_n, X_n)$$
$$+ \sum_{r=1}^{m} G_r(t_{n-1}, X_{n-1})\; I_r^{t_{n-1},t_n} - \tfrac{1}{3} \sum_{r=1}^{m} G_r(t_{n-2}, X_{n-2})\; I_r^{t_{n-2},t_{n-1}}$$


dX(t)= (0.5 X(t) -1.2 X(t-1))dt + 0.3X(t)dB(t)
Average of 200 trajectories
t
5
4
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6
7
8
9
10

Two Objectives - Two Modes of Convergence

Strong Approximations: Compute (several to many) single paths, strong convergence criterion (mean-square convergence), p order of method:

$$\boxed{\max_{1\le n\le N} \left(\mathbb{E}|X(t_n) - X_n|^2\right)^{\frac{1}{2}} \quad \le \quad C\ h^p, \quad \text{for} \ \ h \to 0\,.}$$

Weak Approximations: Compute (using many paths) the expectation of a function Ψ of the solution, weak convergence criterion, p order of method:

$$\boxed{\max_{1\le n\le N} |\mathbb{E}\Psi(X(t_n)) - \mathbb{E}\Psi(X_n)| \quad \le \quad C\ h^p, \quad \text{for} \ \ h \to 0\,.}$$

approx. $\mathbb{E}(\Psi(X_n))$ by M realisations $\frac{1}{M}\sum_{i=1}^{M} \Psi(X_n^{(i)})$

$\rightarrow$ 'Monte Carlo Method' computes $\mathbb{E}\Psi(X(t))$, full error and efficiency depend on step-size h **and** number of paths M!

Modes of Convergence

$\boxed{\text{Euler(BDF2)-Maruyama method: strong order } \frac{1}{2}\text{, weak order 1.}}$

$\boxed{\theta\text{-Milstein method: strong order 1, weak order 1.}}$

Note: inclusion of iterated Wiener integrals

$$I^{t,t+h}_{r_1,r_2,\ldots,r_j} = \int_t^{t+h} \int_t^{s_1} \ldots \int_t^{s_{j-1}} \mathrm{d}W_{r_1}(s_j) \ldots \mathrm{d}W_{r_j}(s_1),$$

where $r_i \in \{0, 1, \ldots, m\}$ and $\mathrm{d}W_0(s) = \mathrm{d}s$ determines order of convergence, higher order integrals difficult to simulate!

State-of-the-Art (not in the least complete!)

- Development and (finite time) weak and strong convergence analysis of 'standard' classes (Taylor-type, Runge-Kutta, Linear multi-step methods) of numerical methods for Itô or Stratonovich SODEs. ✓
  Results e.g., by Kloeden, Platen, Milstein, Tretyakov, Talay, Rößler, Komori, Buckwar & Winkler.
- Efficiency and reduction of complexity for Monte Carlo Methods by Multi-level Monte Carlo. Current.
  Results e.g., by Heinrich, Kebaier, Giles, Higham.
- Structure preserving numerical methods, such as Lie groups methods or methods for stochastic Hamiltonian systems. Current.
  Results e.g., by Mizawa, Wiese, Talay, Milstein, Tretyakov, Bou-Rabee.
- Linear and nonlinear stability analysis of numerical methods for SODEs. Current.
  Results e.g., by Higham, Buckwar, Mao, Mitsui, Abdulle.
- Development and (finite time) weak and strong convergence analysis of, e.g., Finite Element/Difference methods, Spectral methods, Galerkin methods for SPDEs. Current.
  Results e.g., by Győngy, Hausenblas, Debussche, Larsson, Kruse, Lang, Lord, Shardlow.

- Less developed: Efficiency, stability, robustness of methods, esp. for SPDEs

Analysis of numerical methods for SDEs beyond convergence

Analysis of numerical methods for SDEs beyond convergence

Motivation:

Provide the knowledge of what an algorithm actually **does** in the stochastic case when implemented on a computer: Convergence is a limit procedure, whereas running a simulation means fixing a step-size/number of paths and dealing with the dynamics of a discrete system!

Goal:

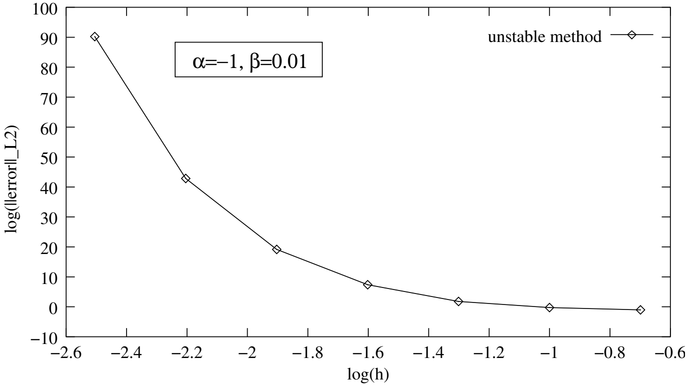
Develop a systematic dynamic analysis of numerical methods, justifying the choice of test equations/systems, gaining insight into deterministic/stochastic features relevant for stability and other issues, identifying benchmark problems, develop appropriate analytical techniques......

A standard first step: Linear stability analysis of numerical methods

Disambiguation: Stability

- Numerical stability, Zero-stability, Dahlquist stability, Lax stability : robustness of a numerical scheme wrt perturbations such as round-off error, 'measured' over finite interval for step-size to zero, necessary for convergence!
- Lyapunov stability: characterises qualitative behaviour of equilibria wrt perturbations in the i.v., fundamental problem 'does the (convergent) numerical method have the same stability behaviour as the continuous problem and if under which conditions on the step-size?', 'measured' for 'fixed step-size' and t going to infinity.

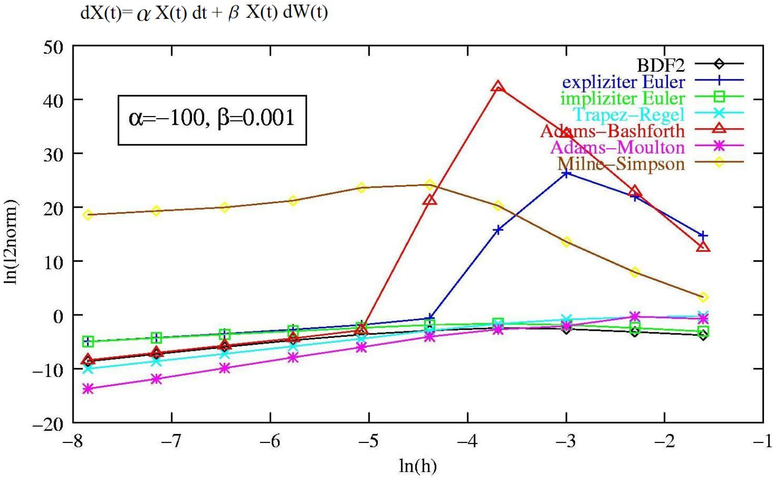
An illustration of a numerically unstable, thus not converging scheme



$\mathrm{d}X(t) = \alpha X(t)\mathrm{d}t + \beta X(t)\mathrm{d}W_1(t)$ using the numerically unstable scheme

$$X_n - 3X_{n-1} + 2X_{n-2} = h\alpha(\frac{1}{2}X_{n-1} - \frac{3}{2}X_{n-2}) + \beta(X_{n-1}I_1^{t_{n-1},t_n} - 2X_{n-2}I_1^{t_{n-2},t_{n-1}})$$

In contrast, convergent schemes and a different 'problem'



Linear stability analysis of numerical methods for ODEs

- Question: given an ODE $x'(t) = f(x(t))$ and a numerical method, does the (convergent) method share the qualitative properties of the ODE and if so, under which restrictions on the step-size?
- (Usually) first step: linear stability analysis, using the test equation $x'(t) = \lambda x(t)$, $\lambda \in \mathbb{C}$. This means: apply the method to the test equation, determine its stability behaviour and compare with that of the test equation.
- Based on: linearisation and centering of nonlinear ODE around an equilibrium, the resulting linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ (A the Jacobian of f evaluated at equilibrium) is then diagonalised and the system thus decoupled, justifying the use of the scalar test equation.

Linear stability analysis of numerical methods for SODEs

- Question: given an SODE as above and a numerical method, does the (convergent) method share the qualitative properties of the SODE and if so, under which restrictions on the step-size?
- (Usually) first step: linear stability analysis, now with which test equation?
- Further questions: Stability in which sense, i.e. in the a.s. sense or in mean-square? What effect does the r-dim noise have?
- Still holding: linearisation and centering of nonlinear SODE around an equilibrium, the resulting linear system is now $\mathrm{d}X(t) = (AX(t))\mathrm{d}t + \sum_{j=1}^{r} B_j X(t)\mathrm{d}W_j(t)$ (A, B_j the Jacobians of F, G_j evaluated at equilibrium). Simultaneously diagonalisable?

Linear stability analysis of numerical methods for SODEs, the set-up (1)

Consider autonomous (SDEs)

$$\mathrm{d}X(t) = F(X(t))\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t)\,, \tag{1}$$

where $X \in \mathbb{R}^d$ and F and G as stated above, we denote a solution to (1) by $X(t)$, with initial conditions $X(0)$.
Equilibria X_e (or equilibrium points, fixed points or stationary points), are **constant solutions**

$$X(t) \equiv X_e \qquad \text{with} \qquad \mathrm{d}X(t) = F(X_e) = G(X_e) = 0\,, \tag{2}$$

Note: In general, it is known from stochastic/random dynamical systems theory, that equilibria in a stochastic setting do not need to be **deterministic constants**. In particular, the appropriate notion of equilibrium for an SDE with additive noise is a 'stationary process'.

Linear stability analysis of numerical methods for SODEs, the set-up (2)

Definition

Lyapunov-stability

1. The equilibrium X_e of an SODE (1) is mean-square stable/a.s. stable if and only if, for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that

$$\mathbb{E}|X(t) - X_e|^2 < \epsilon, \quad t \geq 0, \quad / \quad |X(t) - X_e| < \epsilon, \quad t \geq 0, \quad a.s.$$

   whenever $\mathbb{E}|X(0) - X_e|^2 < \delta$ / $|X(0) - X_e| < \delta$;

2. The equilibrium X_e is asymptotically mean-square stable/a.s. stable if and only if it is mean-square stable/a.s. stable, and for all $X(0) - X_e \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}|X(t) - X_e|^2 = 0 \quad / \quad \lim_{t\to\infty} X(t) - X_e = 0 \quad a.s.$$

Linear stability analysis of numerical methods for SODEs, Example

Linear equation, for $t \geq 0$, with $X(0) = X_0, \quad \lambda, \mu, X_0 \in \mathbb{R}$,

$$\boxed{\mathrm{d}X(t) = \lambda\, X(t)\mathrm{d}t \;+\; \mu\, X(t)\mathrm{d}W(t), \qquad (1)}$$

with the geometric Brownian motion $X(t) = \exp((\lambda - \frac{1}{2}\mu^2)t + \mu W(t))$ as exact solution.

Thm.: (e.g. in Arnold 1974, Khasminskii 1980, 2011)
The zero solution of (1) is asymptotically mean-square stable iff

$$\boxed{\lambda + \frac{1}{2}\, |\mu|^2 \, < 0}$$

and asymptotically a.s. stable iff

$$\boxed{\lambda - \frac{1}{2}\, |\mu|^2 \, < 0}$$

Linear stability analysis of numerical methods for SODEs, Example

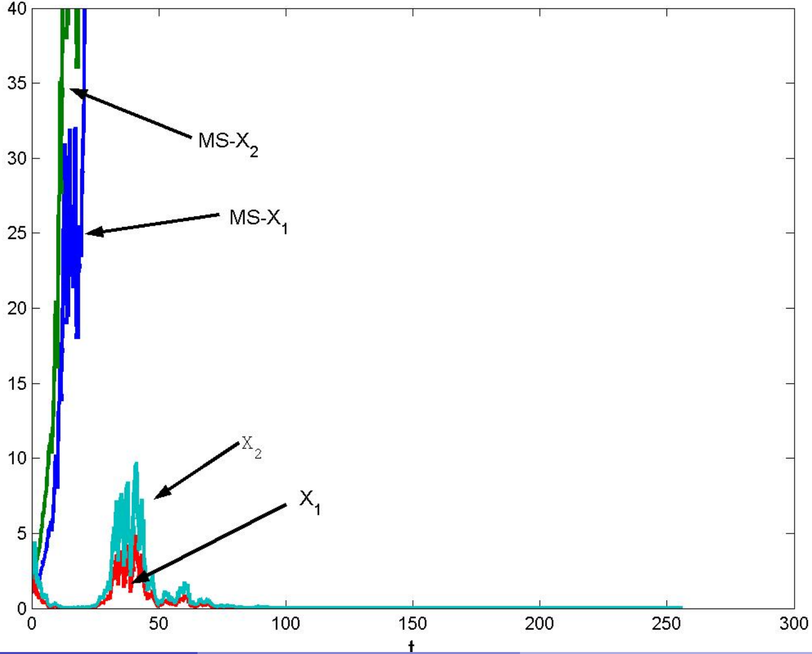
Consider

$$\boxed{\mathrm{d}X(t) = 0.1\, X(t)\mathrm{d}t \; + \; 0.5\, X(t)\mathrm{d}W(t), \qquad (1)}$$

then

$$\lambda + \frac{1}{2}\sigma^2 = 0.225 > 0, \quad \lambda - \frac{1}{2}\sigma^2 = -0.025 < 0,$$

and therefore the equilibrium solution is simultaneously mean-square unstable and a.s. asymptotically stable.


40
35
30
25
20
15
10
5
0
0
50
100
150
200
250
300
t
MS-X2
MS-X1
X2
X1

Linear mean-square stability analysis for Geometric Brownian Motion (D. Higham 2000)

Consider: scalar linear test equation, geometric Brownian Motion:

$$\boxed{\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W_1(t),\ X(0) = X_0}$$

θ-Maruyama-method:

$$\boxed{X_{i+1} = X_i + h\left(\theta\lambda X_{i+1} + (1-\theta)\lambda X_i\right) + \sqrt{h}\mu\, X_i\, \xi_{1,i}}$$

Rewrite as a recurrence equation

$$X_{i+1} = (\widetilde{a} + \widetilde{b}\,\xi_i)\ X_i, \quad \text{where} \quad \widetilde{a} = \frac{1+(1-\theta)\lambda h}{1-\theta\lambda h}, \quad \widetilde{b} = \frac{\mu h^{\frac{1}{2}}}{1-\theta\lambda h}.$$

Mean-square stability analysis consists of 'Squaring and taking the expectation' $\Rightarrow$ *exact* one-step recurrence for $\mathbb{E}|X_i|^2$

$$\mathbb{E}|X_{i+1}|^2 = (|\widetilde{a}|^2 + 2|\widetilde{a}|\,|\widetilde{b}|\,|\mathbb{E}\,\xi_i| + |\widetilde{b}|^2\,|\mathbb{E}\,\xi_i^2|)\,\mathbb{E}|X_i|^2 = (|\widetilde{a}|^2 + |\widetilde{b}|^2)\,\mathbb{E}|X_i|^2$$

Mean-square Stability for Linear Systems of SODEs

EB & T Sickenberger, APNUM 2012

$$\mathrm{d}X(t) = FX(t)\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\mathrm{d}W_r(t), \quad t \geq t_0 \geq 0, \quad X(t_0) = X_0\,. \tag{3}$$

Here, the drift and diffusion matrices are given by $F \in \mathbb{R}^{d\times d}$ and $G_1, \ldots, G_m \in \mathbb{R}^{d\times d}$, respectively, and $W = (W_1, \ldots, W_m)^T$ is an m-dimensional Wiener process.

Notation

(i) The vectorisation vec(A) of an $m \times n$ matrix A transforms the matrix A into an $mn \times 1$ column vector obtained by stacking the columns of the matrix A on top of one another.

(ii) The Kronecker product of an $m \times n$ matrix A and a $p \times q$ matrix B is the $mp \times nq$ matrix defined by $A \otimes B = \left(\, a_{ij} \cdot B \, \right)_{i,j=1,\ldots,n}$.

(iii) $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$, when A, B and C are three matrices, such that the matrix product ABC is defined;

(iv) A special case of (iii) is given by
$\mathrm{vec}(AB) = (B^T \otimes \mathrm{Id}_m)\mathrm{vec}(A) = (\mathrm{Id}_q \otimes A)\mathrm{vec}(B)$, where A is an $m \times n$ matrix, B a $n \times q$ matrix, and Id_s is the s-dimensional identity matrix for any $s \in \mathbb{N}$.

(v) The spectral abscissa $\alpha(A)$ of a matrix A is defined by $\alpha(A) = \max_i \Re(\lambda_i)$, where $\Re$ is the real part of the real or complex eigenvalues λ_i of the matrix A.

(vi) The spectral radius $\rho(A)$ of a matrix A is defined by $\rho(A) = \max_i |\lambda_i|$, where again λ_i are the real or complex eigenvalues of the matrix A.

Equation for the second moment

The expectation of the matrix-valued process $P(t) = X(t)X(t)^T$ with i.v. $P(t_0) = X_0 X_0^T$ is given by

$$\mathrm{d}\mathbb{E}(P(t)) = \Big(F\mathbb{E}(P(t)) + \mathbb{E}(P(t))F^T + \sum_{r=1}^{m} G_r \mathbb{E}(P(t))\, G_r^T \Big)\, \mathrm{d}t\,,$$

$$\begin{aligned} \mathrm{vec}(P(t)) &= Y(t) = (Y_1(t), Y_2(t), \ldots, Y_{d^2}(t))^T \\ &= (X_1^2(t), X_2(t)X_1(t), \ldots, X_d(t)X_1(t), \\ &\quad X_1(t)X_2(t), X_2^2(t), X_3(t)X_2(t), \ldots, X_d(t)X_2(t), \ldots, X_d^2(t))^T. \end{aligned}$$

Arrive at the deterministic linear system of ODEs for the d^2-dimensional vector $\mathbb{E}(Y(t))$

$$\mathrm{d}\mathbb{E}(Y(t)) = S\,\mathbb{E}(Y(t))\,\mathrm{d}t\,, \tag{4}$$

where S is given by

$$S = \mathrm{Id}_d \otimes F + F \otimes \mathrm{Id}_d + \sum_{r=1}^{m} G_r \otimes G_r\,.$$

Classical result

Lemma

The zero solution of the deterministic ODE system (4) *is asymptotically stable if and only if*

$$\alpha(S) < 0\,. \tag{5}$$

Discrete equation

Explicit one-step recurrence equation involving a sequence $\{\mathfrak{A}_i\}_{i\geq 0}$ of independent random matrices

$$X_{i+1} = \mathfrak{A}_i X_i, \qquad i = 0, 1, \ldots\,. \tag{6}$$

The second moments of the discrete approximation process $\{X_i\}_{i\in\mathbb{N}_0}$ are given by

$$\mathbb{E}(Y_{i+1}) = \mathbb{E}(\mathfrak{A}_i \otimes \mathfrak{A}_i)\mathbb{E}(Y_i)\,, \qquad i \in \mathbb{N}_0\,, \tag{7}$$

where the d^2-dimensional discrete process $\{Y_i\}_{i\in\mathbb{N}_0}$ is given by $Y_i = \mathrm{vec}(X_i X_i^T)$.

$$\mathcal{S} = \mathbb{E}(\mathfrak{A} \otimes \mathfrak{A}) \tag{8}$$

Lemma

The zero solution of the system of linear difference equations (7) *is asymptotically stable in mean-square if and only if*

$$\rho(\mathcal{S}) < 1\,.$$

Discrete equation, Example

For a simple system of SODEs

$$d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dW_1(t) + \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dW_2(t), \quad t > 0, \tag{9}$$

We obtain an explicit one-step recurrence equation from applying the θ-Maruyama method involving a sequence $\{\mathfrak{A}_i\}_{i\geq 0}$ of independent random matrices

$$X_{i+1} = \mathfrak{A}_i X_i, \qquad i = 0, 1, \ldots . \tag{10}$$

as

$$\begin{pmatrix} X_{1,n+1} \\ X_{2,n+1} \end{pmatrix} = \begin{pmatrix} \frac{1+(1-\theta)h\lambda}{1-\theta h\lambda} + \frac{\sqrt{h}\sigma\xi_{1,n+1}}{1-\theta h\lambda} & \frac{-\sqrt{h}\varepsilon\xi_{2,n+1}}{1-\theta h\lambda} \\ \frac{\sqrt{h}\varepsilon\xi_{2,n+1}}{1-\theta h\lambda} & \frac{1+(1-\theta)h\lambda}{1-\theta h\lambda} + \frac{\sqrt{h}\sigma\xi_{1,n+1}}{1-\theta h\lambda} \end{pmatrix} \begin{pmatrix} X_{1,n} \\ X_{2,n} \end{pmatrix} \tag{11}$$

Analysis of general matrices

$$\mathrm{d}X(t) = FX(t)\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\mathrm{d}W_r(t), \quad t \geq 0, \quad X(t_0) = X_0 \,.$$

We have studied

- θ-Maruyama method applied to the SDE above
- θ-Milstein method applied to the SDE above with a single noise
- θ-Milstein method applied to the SDE above with commutative noise
- θ-Milstein method applied to the SDE above with non-commutative noise

Analysis of general matrices, example

$$X_{i+1} = \mathfrak{A}_i \, X_i \quad \text{with} \quad \mathfrak{A}_i = \bar{A} + \sum_{r=1}^{m} B_r \, \xi_{r,i} + \sum_{r_1,r_2=1}^{m} C_{r_1,r_2} \, \xi_{r_1,i} \, \xi_{r_2,i} \tag{12}$$

where $\bar{A}$, B, and C are deterministic matrices determined by

$$A = (\mathrm{Id} - h\theta F)^{-1}(\mathrm{Id} + h(1-\theta)F) \tag{13}$$

$$\bar{A} = A - (\mathrm{Id} - h\theta F)^{-1}\left(\sum_{r=1}^{m} \frac{1}{2} h \, G_r^2 \right) = A - \sum_{r=1}^{m} C_{r,r} \, ,$$

$$B_r = (\mathrm{Id} - h\theta F)^{-1}\left(\sqrt{h} \, G_r \right), \tag{14}$$

$$C_{r_1,r_2} = (\mathrm{Id} - h\theta F)^{-1}\left(\frac{1}{2} h \, G_{r_1} G_{r_2} \right).$$

Analysis of general matrices, example

Theorem

The mean-square stability matrix of the θ-Milstein method applied to the system (3) *with commutative noise is given by*

$$\mathcal{S} = (A \otimes A) + \sum_{r=1}^{m} (B_r \otimes B_r) + 2 \sum_{r=1}^{m} (C_{r,r} \otimes C_{r,r}) + \left(\sum_{\substack{r_1, r_2 = 1 \\ r_1 \neq r_2}}^{m} C_{r_1, r_2} \otimes \sum_{\substack{r_1, r_2 = 1 \\ r_1 \neq r_2}}^{m} C_{r_1, r_2} \right).$$

The test equations

Considering

$$\mathrm{d}X(t) = FX(t)\mathrm{d}t + \sum_{r=1}^{m} G_r X(t)\mathrm{d}W_r(t), \quad t \geq t_0 \geq 0, \quad X(t_0) = X_0\,.$$

with full matrices F and G_r has two problems: a) Maple (or similar) waves the white flag when it comes to computing eigenvalues, b) even if it could one would be 'drowning in parameters', in particular there are too many parameters around to get any insight of the effect of each paramter.
Solution: **choose** a few parameters **wisely** and set the remaining ones to 0.

The test equations based on ideas from EB & Kelly, SINUM 2010

$$\mathrm{d}X(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X(t)\mathrm{d}t + \begin{pmatrix} \sigma & \epsilon \\ \epsilon & \sigma \end{pmatrix} X(t)\mathrm{d}W_1(t)\,; \tag{15}$$

the second test system is a two-dimensional system with two commutative noise terms:

$$\mathrm{d}X(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X(t)\mathrm{d}t + \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} X(t)\mathrm{d}W_1(t) + \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} X(t)\mathrm{d}W_2(t)\,; \tag{16}$$

and the third one has two non-commutative noise terms:

$$\mathrm{d}X(t) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} X(t)\mathrm{d}t + \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} X(t)\mathrm{d}W_1(t) + \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} X(t)\mathrm{d}W_2(t)\,. \tag{17}$$

The test equations, stability conditions

Corollary

Stability conditions for θ-Maruyama method

$$\text{for (15):}\qquad \lambda + \frac{1}{2}(\sigma^2 + \epsilon^2 + 2|\sigma\epsilon|) + \frac{1}{2}h(1-2\theta)\lambda^2 < 0\,,$$

$$\text{for (16) and (17):}\qquad \lambda + \frac{1}{2}(\sigma^2 + \epsilon^2) + \frac{1}{2}h(1-2\theta)\lambda^2 < 0\,.$$

Stability conditions for θ-Milstein method

$$\text{for (15):}\qquad \lambda + \frac{1}{2}(\sigma + |\epsilon|)^2 + \frac{1}{2}h(1-2\theta)\lambda^2 + \frac{1}{4}h(\sigma + |\epsilon|)^4 < 0\,,$$

$$\text{for (16):}\qquad \lambda + \frac{1}{2}(\sigma^2 + \epsilon^2) + \frac{1}{2}h(1-2\theta)\lambda^2 + \frac{1}{4}h(\sigma^2 + \epsilon^2)^2 < 0\,,$$

and for (17):

$$\lambda + \frac{1}{2}(\sigma^2 + \epsilon^2) + \frac{1}{2}h(1-2\theta)\lambda^2 + \frac{1}{4}h(\sigma^2 + \epsilon^2)^2 + (K(p) - 1)h\sigma^2\epsilon^2 < 0\,.$$

Summary and further projects

- We have suggested a structural framework to perform a linear mean-square stability analysis of numerical methods for systems of SDEs with multiplicative noise. The main points are:
- Test equations for this type of analysis require some justification and some thought!
- 'Matrix analysis' approach allows to work more efficiently with systems of equations.
- Interaction between drift and diffusion terms, as well as dimension of Wiener process and SDE system play a role!
- Characterising stiffness in a stochastic setting. (Questions: Can one get a system of SDEs that is only stiff due to the diffusion? Is stiffness in mean-square different from almost sure stiffness?)
- Stability issues and Multi-level Monte-Carlo methods.
- Stability issues for space discretised SPDEs, in particular for additive noise SPDEs.

Thank you for your attention