Measuring systemic risk via model uncertainty

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based on a joint work with Birgit Rudloff

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- Interconnected financial system
- Failures affecting multiple entities
  - e.g. chain of defaults
- Important in the event of financial crisis
- Systemic vs. institutional risk

Measuring systemic risk

1. Aggregation function Λ
2. Acceptance set $\mathcal{A}$
3. Systemic risk measure R^{sys}

1. Aggregation function

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  - Like a utility function but multivariate!

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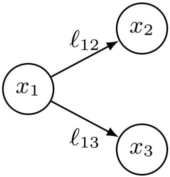

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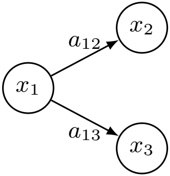
- Total liability of entity i: $\bar{p}_i = \sum_{j=0}^{d} \ell_{ij}$
- Relative liabilitiy of i to j: $a_{ij} = \frac{\ell_{ij}}{\bar{p}_i}$



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- Clearing/realized payment vector (equilibrium): $p(x) = (p_1(x), \dots, p_d(x))$
- $p(x) \in \mathbb{R}_+^d$ is the solution of the fixed point problem

$$p_i(x) = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^{d} p_j(x) a_{ji} \right), \quad i \in \{1, \dots, d\}.$$

- Equilibrium: Pay either what you owe or what you have.
- There exists a unique $p(x)$ under mild conditions.

Example: Eisenberg, Noe ('01) (cont'd)

- Equity/loss of entity i after clearing:

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- Aggregation function: equity of the society

$$\Lambda(x) := e_0(x) = \sum_{j=1}^{d} p_j(x) a_{j0}.$$

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- The impact of wealth vector X on society is $\Lambda(X)$.
- More features could be modeled:
  - Liquid and illiquid assets (e.g. Cifuentes, Shin, Ferrucci ('05))
  - Random liability matrix (e.g. Amini, Filipovic, Minca ('15))
  - Impact on a group of entities: $\Lambda \colon \mathbb{R}^d \to \mathbb{R}^m$ with $\Lambda(x) = (e_1(x), \ldots, e_m(x))$, e.g. $\{1, \ldots, m\}$ are the small banks

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- Chen, Iyengar, Moallemi ('13), Kromer, Overbeck, Zilch ('14)
- More naive choices as they ignore the network structure

- When is $\Lambda(X) \in L^\infty$ acceptable?
- $\mathcal{A} \subseteq L^\infty$ acceptance set of a scalar convex risk measure ρ
- $\mathcal{A} = \{Y \in L^\infty \mid \rho(Y) \leq 0\}$
- $\rho(Y) = \inf\{y \in \mathbb{R} \mid Y + y \in \mathcal{A}\}$
- If $\mathcal{A}$ is weak*-closed and convex, then ρ admits the dual representation

$$\rho(Y) = \sup_{\mathbb{S} \in \mathcal{M}(\mathbb{P})} \left(\mathbb{E}^{\mathbb{S}}[-Y] - \alpha(\mathbb{S}) \right),$$

where α can be chosen as

$$\alpha(\mathbb{S}) = \sup_{Y \in L^\infty} \left(\mathbb{E}^{\mathbb{S}}[-Y] - \rho(Y) \right).$$

3. Systemic risk measure

- A measure of systemic risk is the set of all capital allocations that make the impact to the society acceptable.

1. Aggregation mechanism insensitive to capital levels

$$R^{\text{ins}}(X) = \left\{ z \in \mathbb{R}^d \mid \Lambda(X) + \sum_{i=1}^{d} z_i \in \mathcal{A} \right\}$$

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  - Set-valued version of the systemic risk measure of Chen, Iyengar, Moallemi ('13):

$$\rho^{\text{ins}}(X) = \rho(\Lambda(X)) = \inf\{y \in \mathbb{R} \mid \Lambda(X) + y \in \mathcal{A}\}$$

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   - Feinstein, Rudloff, Weber ('15): grid search algorithm
   - Biagini, Fouque, Fritelli, Meyer-Brandis ('15): similar structure

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- ... under mild assumptions:
  - ρ is a convex weak*-lsc risk measure.
  - Λ is concave and increasing (with respect to componentwise ordering).
  - $\rho(0) \in \operatorname{int} \Lambda(\mathbb{R}^d)$.

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- Recall scalar case:

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Dual representations for systemic risk measures

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$$R^{\text{sen}}(X) = \bigcap_{\mathbb{Q} \in \mathcal{M}_d(\mathbb{P}),\, w \in \mathbb{R}^d_+ \setminus \{0\}} \mathbb{E}^{\mathbb{Q}}\left[-X\right] + \left\{z \in \mathbb{R}^d \mid w^{\mathsf{T}} z \geq -\alpha^{\text{sys}}(\mathbb{Q}, w)\right\},$$

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- $\mathbb{S} \in \mathcal{M}^e(\mathbb{P})$ equivalent probability measure
- $g(z) = \sup_{x \in \mathbb{R}^d} \left(\Lambda(x) - x^{\mathsf{T}} z \right)$ Legendre-Fenchel conjugate of $x \mapsto -\Lambda(-x)$
- $x \cdot z = (x_1 z_1, \ldots, x_d z_d)^{\mathsf{T}}$

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- $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d) \in \mathcal{M}_d(\mathbb{P})$ vector probability measure with $\mathbb{Q}_i \ll \mathbb{P}$ for each i
- $\mathbb{E}^{\mathbb{Q}}\left[X\right] = \left(\mathbb{E}^{\mathbb{Q}_1}\left[X_1\right], \ldots, \mathbb{E}^{\mathbb{Q}_d}\left[X_d\right]\right)$
- $w \in \mathbb{R}^d_+ \setminus \{0\}$

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- Penalty for using $(\mathbb{Q}, w)$ relative to the society's probability measure $\mathbb{S}$:

$$\mathbb{E}^{\mathbb{S}}\left[g\left(w_1 \frac{d\mathbb{Q}_1}{d\mathbb{S}}, \dots, w_d \frac{d\mathbb{Q}_d}{d\mathbb{S}}\right)\right].$$

- "Weighted distance" of the vector probability measure $\mathbb{Q}$ to the society's probability measure $\mathbb{S}$ (multivariate version of the well-known f-divergence).

- In addition: Penalty for assuming $\mathbb{S}$ as the society's probability measure is $\alpha(\mathbb{S})$.

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- Then, we minimize the sum of these two quantities over all choices for $\mathbb{S}$.

$$\alpha^{\text{sys}}(\mathbb{Q}, w) = \inf_{\mathbb{S} \in \mathcal{M}^e(\mathbb{P})} \left(\alpha(\mathbb{S}) + \mathbb{E}^{\mathbb{S}} \left[g \left(w \cdot \frac{d\mathbb{Q}}{d\mathbb{S}} \right) \right] \right).$$

Dual representation: Interpretation

- A capital allocation vector $z \in \mathbb{R}^d$ is considered feasible with respect to the model $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$ and weight vector $w \in \mathbb{R}^d_+ \backslash \{0\}$ if its *weighted sum* exceeds a certain thresold threshold, precisely, if

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- Derived by set-valued convex analysis + a conjugation result for the composition of convex functions (Boț, Grad, Wanka ('13)).

Dual representation: Example (Entropic model)

- Exponential aggregation: $\Lambda(x) = -\sum_{i=1}^{d} e^{-x_i - 1}$
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- Systemic penalty function becomes

$$\alpha^{\text{sys}}(\mathbb{Q}, w) = \inf_{\mathbb{S} \in \mathcal{M}^e(\mathbb{P})} \left(\mathcal{H}\left(\mathbb{S} \| \mathbb{P}\right) + \sum_{i=1}^{d} w_i \mathcal{H}\left(\mathbb{Q}_i \| \mathbb{S}\right) \right) + c(w).$$

- $\mathcal{H}(\mathbb{Q}_i \| \mathbb{S}) = \mathbb{E}^{\mathbb{S}}\left[\frac{d\mathbb{Q}_i}{d\mathbb{S}} \log\left(\frac{d\mathbb{Q}_i}{d\mathbb{S}}\right)\right]$ (relative entropy)

Dual representation: Example (Eisenberg-Noe model)

- Realized wealth: $x \in \mathbb{R}_+^d$
- Clearing payments: $p(x) = (p_1(x), \ldots, p_d(x))$
- Aggregation function: $\Lambda(x) = \sum_{j=1}^d a_{j0} p_j(x)$
- Multivariate divergence function takes the form

$$\mathbb{E}^{\mathbb{S}}\left[g\left(w \cdot \frac{d\mathbb{Q}}{d\mathbb{S}}\right)\right] = \sum_{i=1}^d \mathbb{E}^{\mathbb{S}}\left[\left(\sum_{j=0}^d \ell_{ij}\left(w_j \frac{d\mathbb{Q}_j}{d\mathbb{S}} - w_i \frac{d\mathbb{Q}_i}{d\mathbb{S}}\right)\right)^+\right].$$

A model uncertainty interpretation

- Aggregation function Λ is a multivariate utility function.
  - Campi, Owen ('11): same type of utility function for utility maximization
  - A., Hamel, Rudloff ('15): vector-valued versions for shortfall risk measures

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- Use the dual representation of ρ:

$$
\begin{aligned}
R^{\text{sen}}(X) &= \left\{z \in \mathbb{R}^d \mid \Lambda(X+z) \in \mathcal{A}\right\} \\
&= \left\{z \in \mathbb{R}^d \mid \rho(\Lambda(X+z)) \leq 0\right\} \\
&= \left\{z \in \mathbb{R}^d \mid \sup_{\mathbb{S} \in \mathcal{M}(\mathbb{P})} \left(\mathbb{E}^{\mathbb{S}}[-\Lambda(X+z)] - \alpha(\mathbb{S})\right) \leq 0\right\} \\
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&= \bigcap_{\mathbb{S} \in \mathcal{M}(\mathbb{P})} \left\{z \in \mathbb{R}^d \mid \mathbb{E}^{\mathbb{S}}[\ell(-X-z)] \leq \alpha(\mathbb{S})\right\} \\
&= \bigcap_{\mathbb{S} \in \mathcal{M}(\mathbb{P})} R^{\mathbb{S}}(X),
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where $R^{\mathbb{S}}$ is a multivariate utility-based shortfall risk measure with threshold value $x^0 = \alpha(\mathbb{S})$ under the model $(\Omega, \mathcal{F}, \mathbb{S})$.

Thank you!

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