

Impulse Control: Recent Progress and Applications

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Joint work with **Sören Christensen** (Kiel), **Lukas Mich** (Trier), and **Frank Seifried** (Trier).



Research Seminar

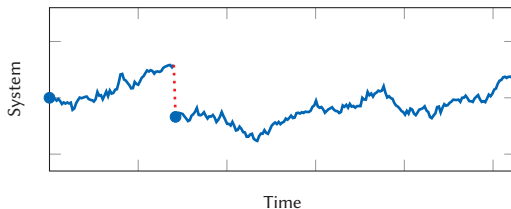
WU Vienna, November 08, 2019

Stochastic Impulse Control

Stochastic control problems with a strictly positive lower bound on the cost per control action.

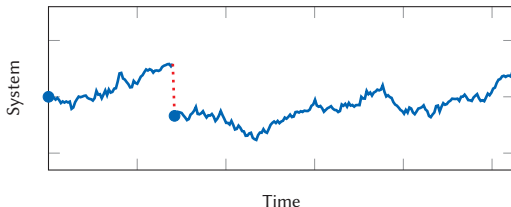
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Applications: Harvesting, inventory control, real options, control of exchange rates, optimal investment with transaction costs, ...

Outline

- (1) Impulse Control: General Formulation
- (2) Superharmonic Functions and Stochastic Perron
- (3) Optimal Investment with Transaction Costs
- (4) Numerical Results

Impulse Control: General Formulation

The General Impulse Control Problem

Consider an \mathbb{R}^n -valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k \in \mathbb{N}}$ as follows:

$$\begin{aligned}dX(t) &= \mu(X(t))dt + \sigma(X(t)) dW(t), & t \in [\tau_k, \tau_{k+1}), \\X(\tau_k) &= \Gamma(X(\tau_k-), \Delta_k),\end{aligned}$$

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where

- ▶ the stopping times τ_k are **increasing** and **do not accumulate** in that

$$\mathbb{P}[\lim_{k \rightarrow \infty} \tau_k > T] = 1,$$

- ▶ the impulses Δ_k are chosen from a **state-dependent** set $Z(X(\tau_k-)) \subset \mathbb{R}^m$.

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The **objective** is to find a maximizer of

$$\mathcal{V}(t, x) = \sup_{\Lambda \in \mathcal{A}(t, x)} \mathbb{E} \left[\sum_{k \in \mathbb{N}} K(X_{t,x}^\Lambda(\tau_k-), \Delta_k) \mathbf{1}_{\{\tau_k \leq T\}} + g(X_{t,x}^\Lambda(T)) \right].$$

The Quasi-Variational Inequalities

Classical Theory: Compute the value function \mathcal{V} by solving

$$\begin{aligned} \min\{-\partial_t \mathcal{V}(t, x) - \mathcal{L}\mathcal{V}(t, x), \mathcal{V}(t, x) - \mathcal{M}\mathcal{V}(t, x)\} &= 0, \\ \mathcal{V}(T, x) &= g(x), \end{aligned}$$

Notation: We refer to the PDE as the **Quasi-Variational Inequalities** (QVIs).

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$$\mathcal{L}\mathcal{V}(t, x) \triangleq \mu(x)^\top D_x \mathcal{V}(t, x) + \frac{1}{2} \text{tr}[\sigma(x)\sigma(x)^\top D_x^2 \mathcal{V}(t, x)],$$

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and \mathcal{M} is the **maximum operator** given by

$$\mathcal{M}\mathcal{V}(t, x) \triangleq \sup_{\Delta \in Z(x)} [\mathcal{V}(t, \Gamma(x, \Delta)) + K(x, \Delta)].$$

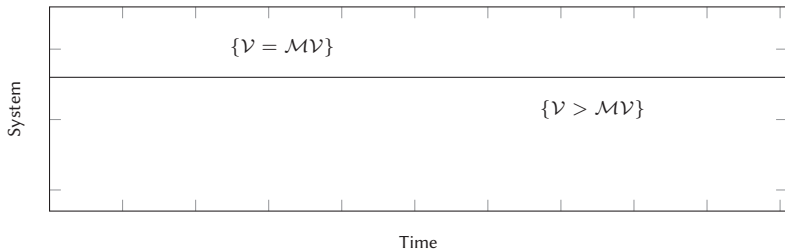
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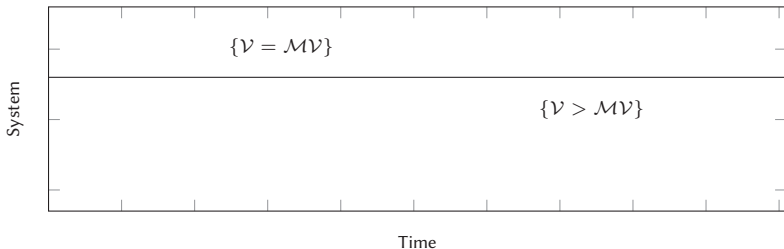
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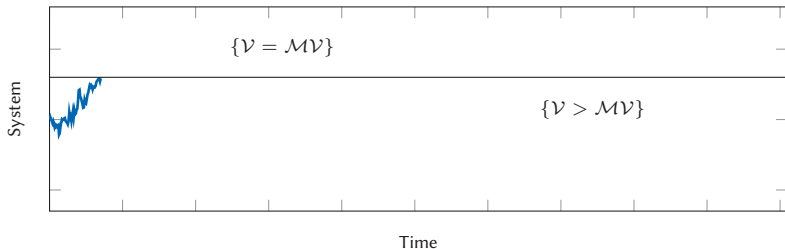
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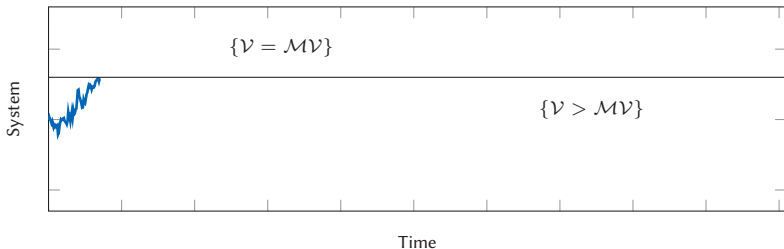
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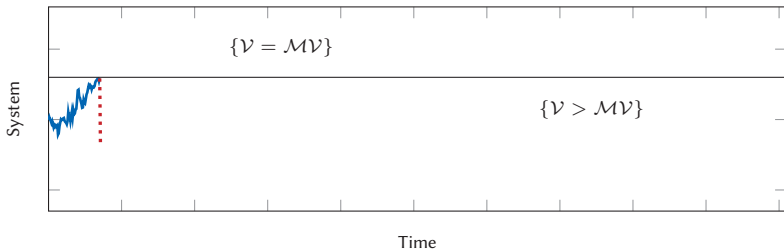
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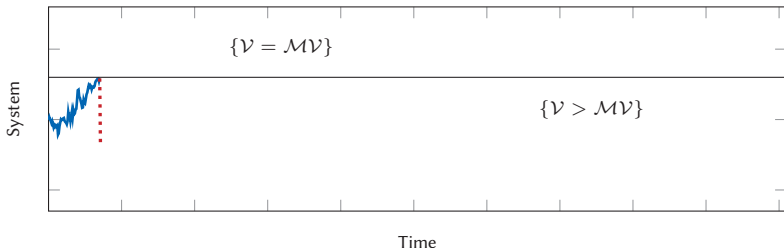
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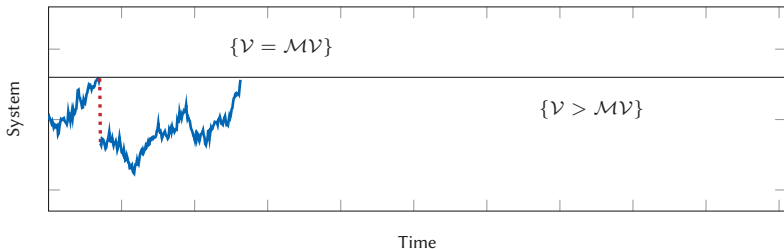
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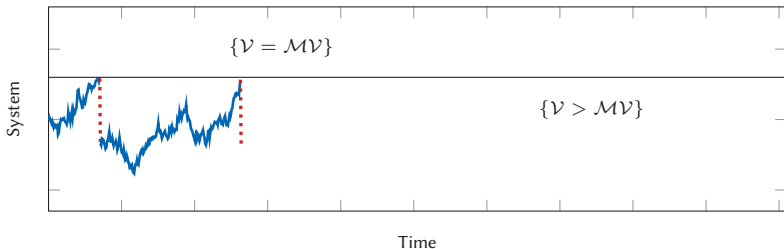
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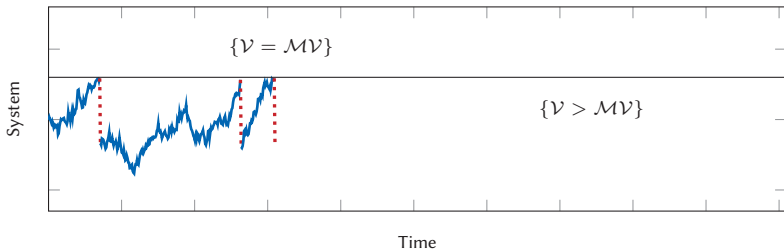
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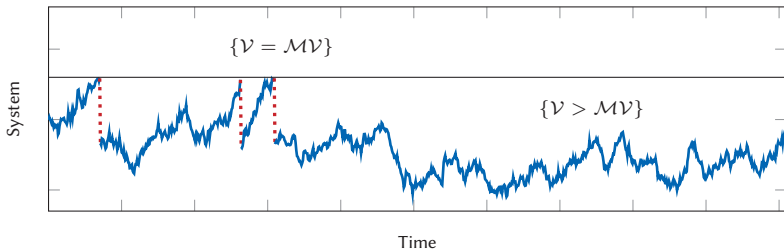
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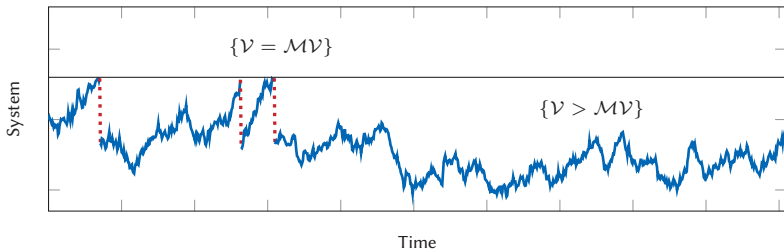
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Problem: Verification requires a solution of the QVIs which is sufficiently smooth to apply Itô's formula.

Superharmonic Functions and Stochastic Perron

An Implicit Optimal Stopping Problem

By the (heuristic!) **Bellman principle**, we expect that

$$\mathcal{V}(t, x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}[\mathcal{M}\mathcal{V}(\tau, \bar{X}(\tau))]$$

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Typically: $\mathcal{M}\mathcal{V}$ is upper semicontinuous if \mathcal{V} is upper semicontinuous. So we essentially need \mathcal{V} to be continuous.

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Proof: Iteratively solve the implicit optimal stopping problem. The argument adapts classical optimal stopping techniques and uses the fact that \mathcal{V} is the pointwise minimum of \mathbb{H} .

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- (3) Then $\mathfrak{V} \leq \mathcal{V} \leq \mathbb{V}$. Now apply viscosity comparison (if it holds) so that $\mathbb{V} \leq \mathfrak{V}$ and hence

$$\mathcal{V} = \mathbb{V} = \mathfrak{V}$$

is **continuous**, the **unique viscosity solution** of the QVIs, and the **pointwise infimum** of \mathbb{H} .

Discussion of the Approach

Our procedure is based on **three ingredients**:

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Challenges when applying the approach:

- ▷ The bottleneck is viscosity comparison, needed for continuity of \mathcal{V} ;
- ▷ Admissibility of the candidate optimal control has to be checked on a case-by-case basis. This is a general problem though.

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We assume that the \mathbb{R}^2 -valued **portfolio process** X evolves as

$$dX_1(t) = rX_1(t)dt, \quad t \in [\tau_k, \tau_{k+1}),$$

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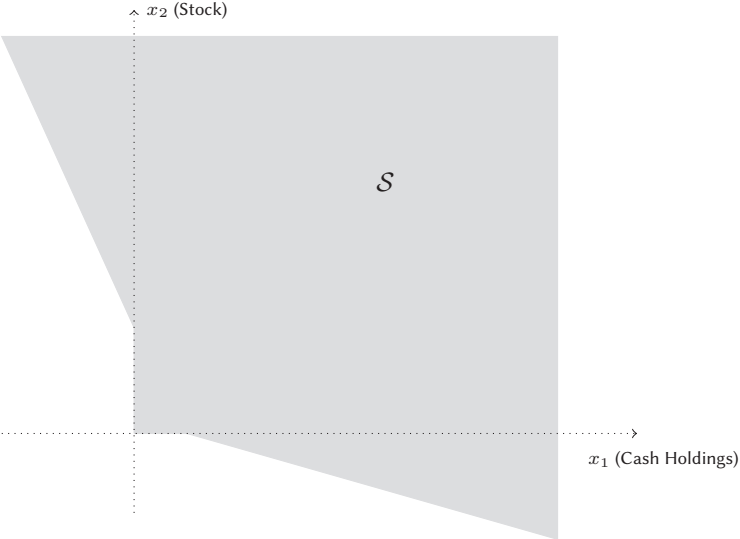
Capped Proportional

A portfolio $x \in \mathbb{R}^2$ is **solvent** if it has a positive liquidation value $L(x) \geq 0$, where

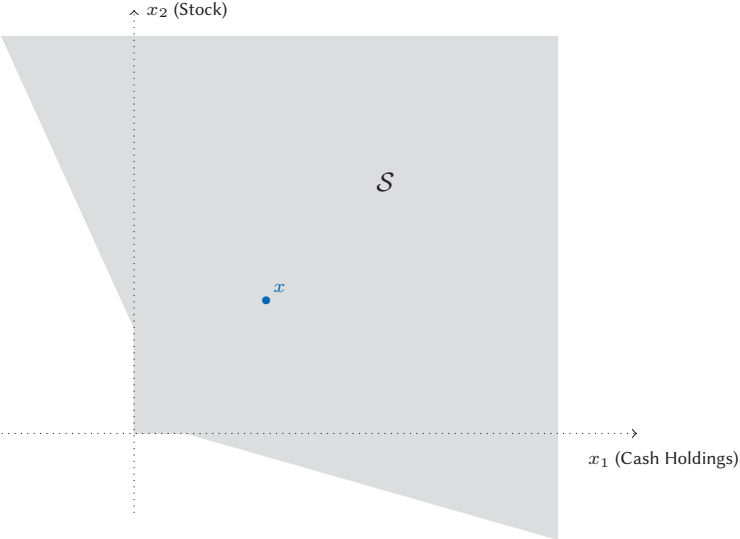
$$L(x) \triangleq \begin{cases} x_1 + x_2 - C(-x_2) & \text{if } x_2 < 0, \\ x_1 + (x_2 - C(-x_2))^+ & \text{otherwise.} \end{cases}$$

The set $\mathcal{S} \subset \mathbb{R}^2$ of solvent portfolios is called the **solvency region**.

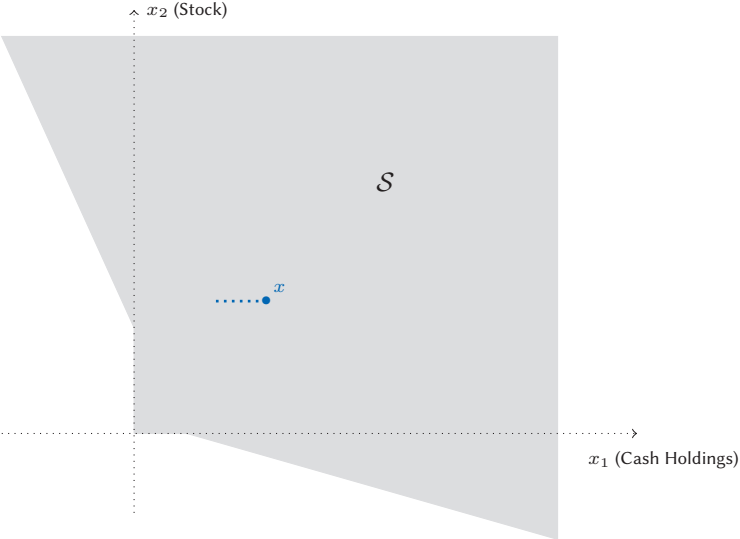
The Solvency Region for Constant + Proportional Costs



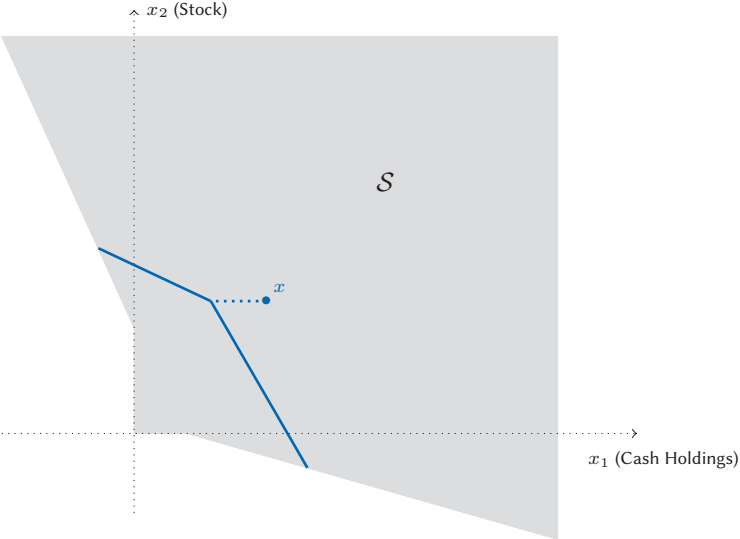
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The Optimization Criterion

For simplicity, we restrict to (positive) **power utility**

$$U : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad \ell \mapsto U(\ell) \triangleq \frac{1}{p} \ell^p \quad \text{with } p \in (0, 1).$$

The objective is to **maximize expected utility of terminal wealth**, i.e.

$$\mathcal{V}(t, x) = \sup_{\Lambda \in \mathcal{A}(t, x)} \mathbb{E} \left[U \left(L(X_{t,x}^\Lambda(T)) \right) \right],$$

where $\mathcal{A}(t, x)$ denotes the set of **admissible strategies** Λ for the initial state (t, x) , i.e. the set of strategies Λ for which

$$L(X_{t,x}^\Lambda) \geq 0 \quad \text{on } [t, T].$$

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Idea: Localize the viscosity argument by splitting the solvency region as follows:

- ▷ $x_1 \geq 0$ and $x_2 \geq 0$: **Long Portfolios**,
- ▷ $x_1 \geq 0$ and $x_2 < 0$: **Short Portfolios**,
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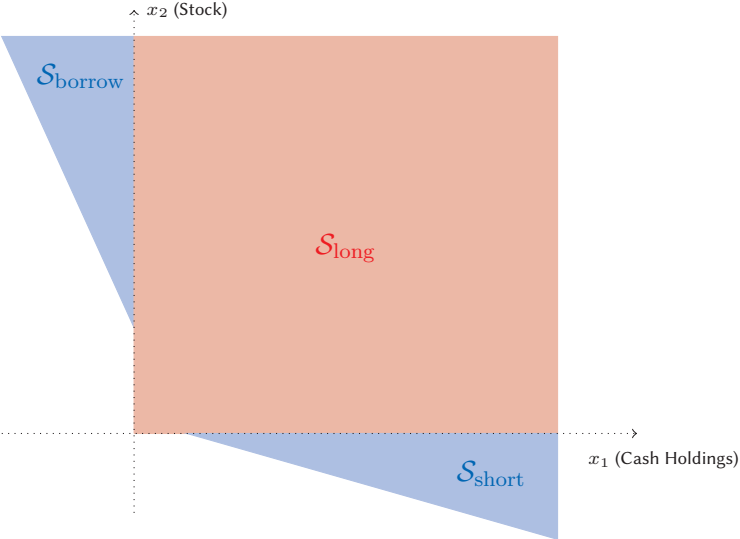
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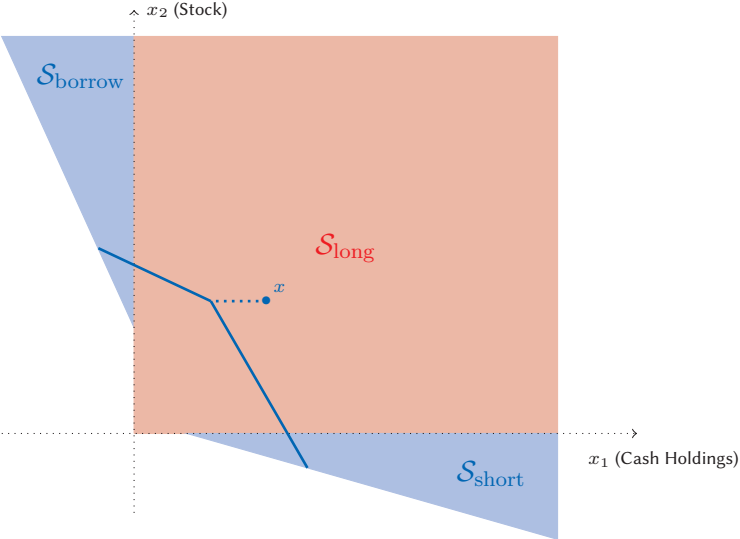
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Difficulty: The QVIs have a **non-local** term: $\mathcal{V}(t, x) - \mathcal{M}\mathcal{V}(t, x)$.

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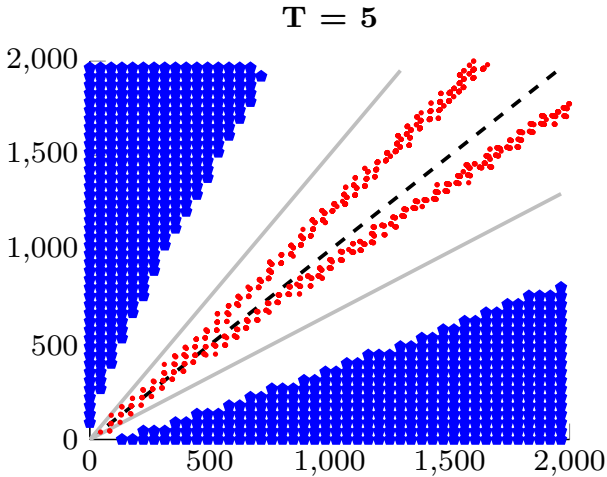
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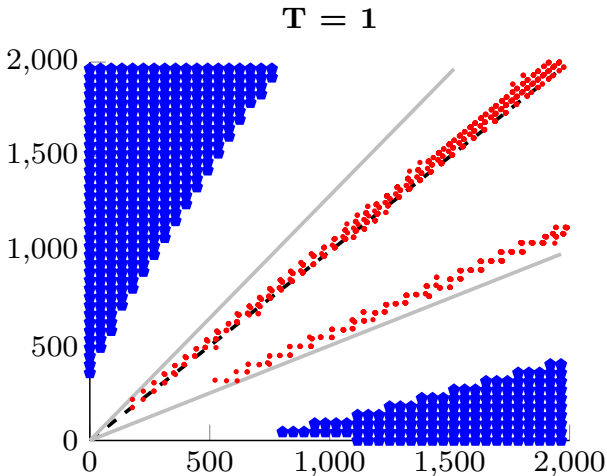
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- (3) The candidate **optimal strategy** does not accumulate and is indeed optimal.

Numerical Results: Constant + Proportional Costs

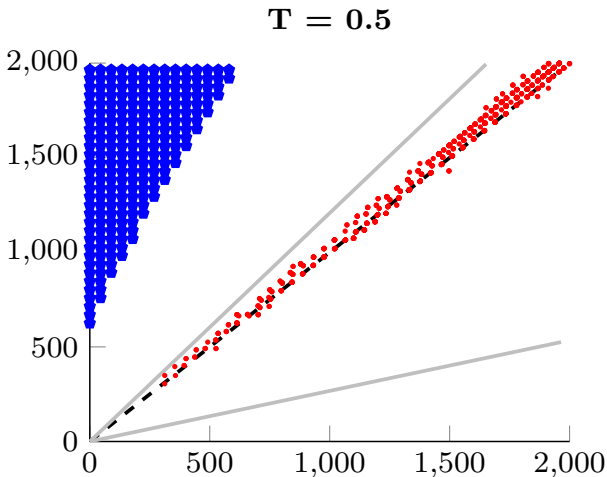
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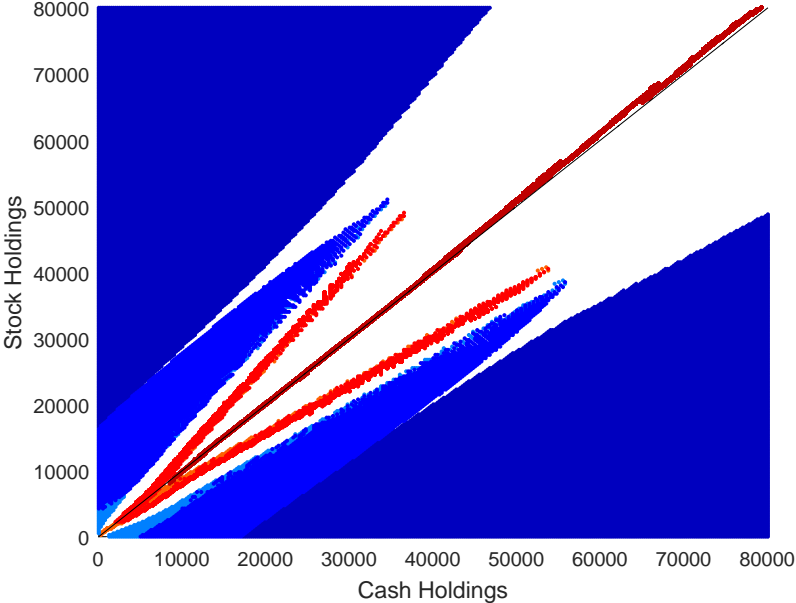


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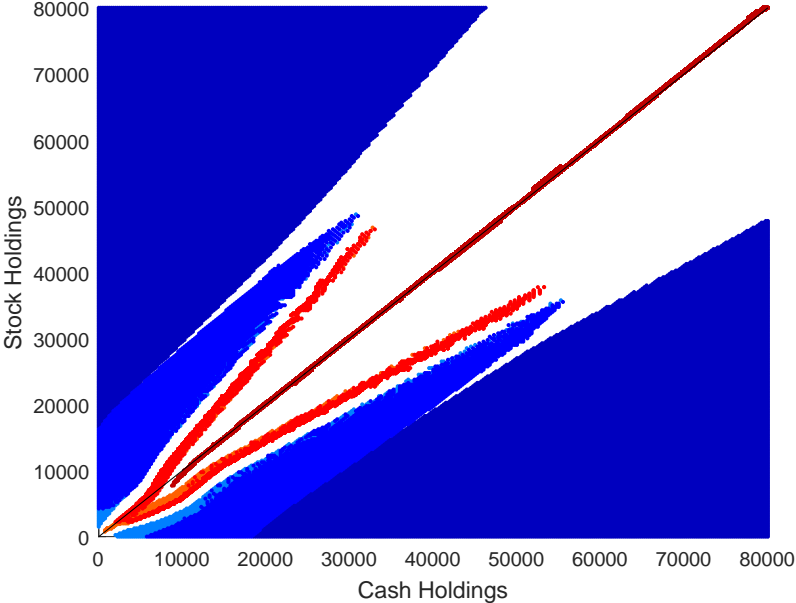


Numerical Results: Capped Proportional Costs

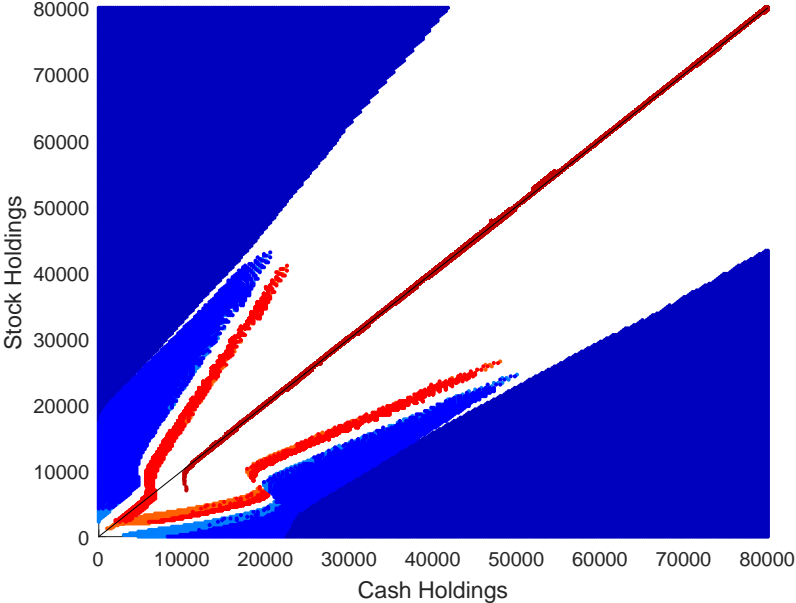
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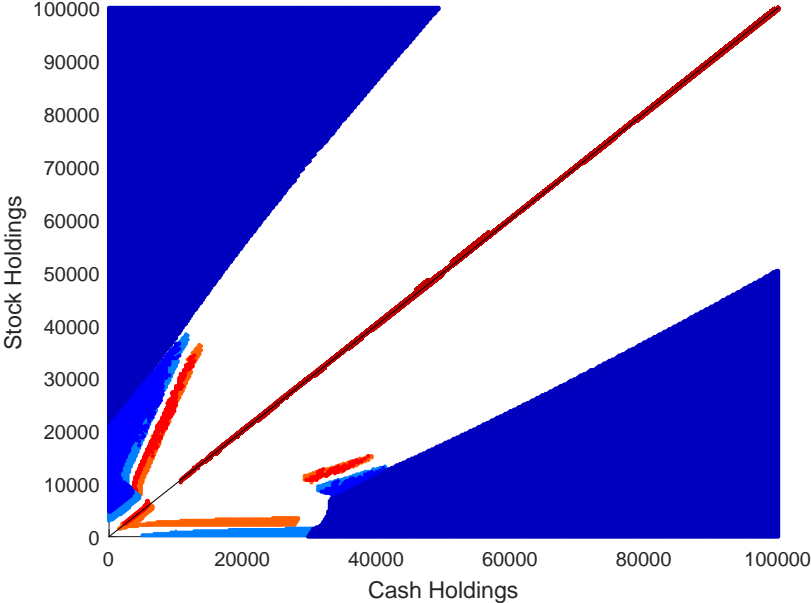
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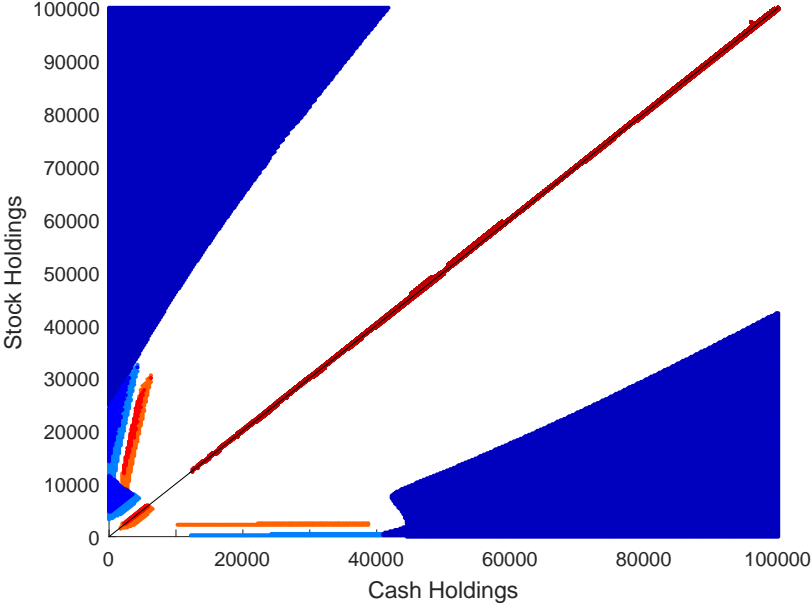
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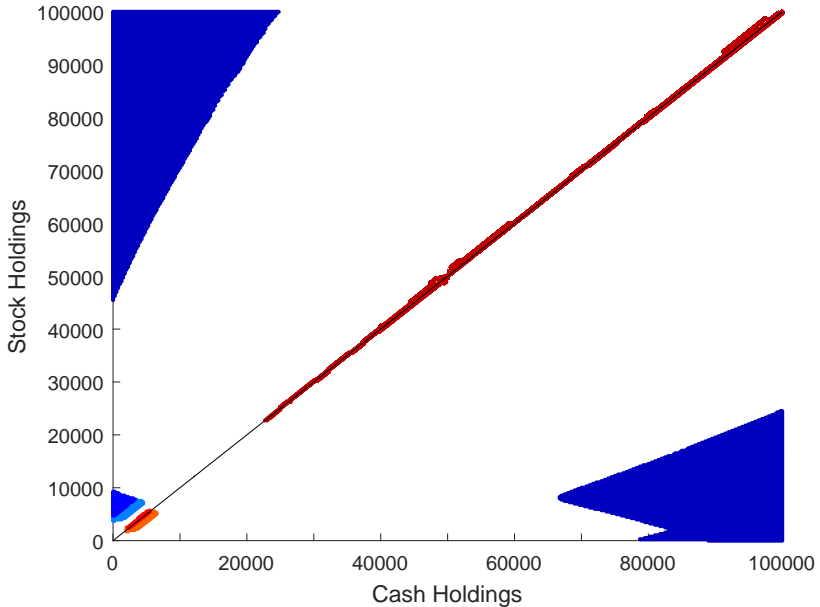
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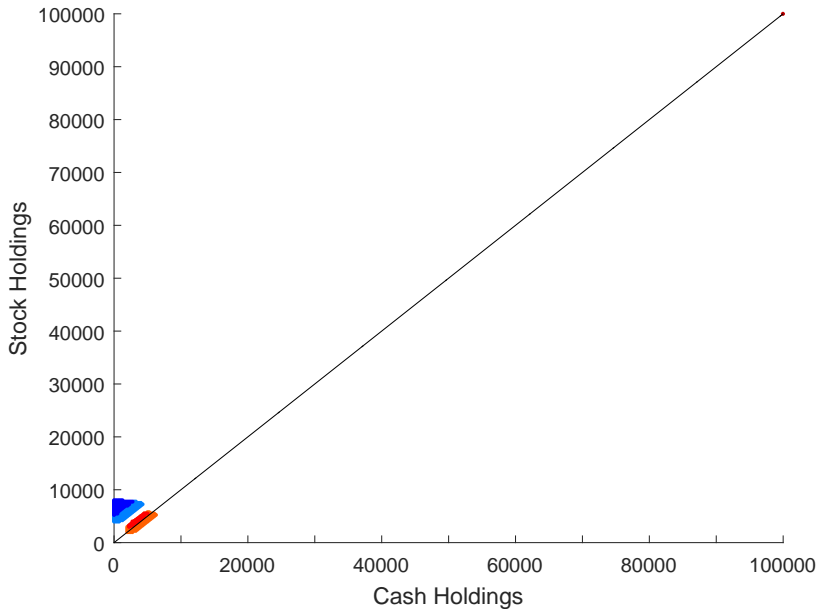
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Thank you for the attention!