Evaluating CDF and PDF of the Sum of Lognormals by Monte Carlo Simulation

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May 24, 2018

Research Seminar of Institute for Statistics and Mathematics, WU Wien

Outline

- Problem Definition: Sum of Lognormals
- Efficient Monte Carlo simulation of the cumulative distribution function (CDF) of sum of lognormals
- Simulation of probability density function (PDF)
- Sum of i.i.d. lognormals
- Conclusions and possible extensions

Problem Definiton

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  - Large sample $100(1-\alpha)\%$ Confidence Interval:

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  - Probabilistic error bound: $z_{\alpha/2}\ \frac{s}{\sqrt{n}}$

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- Tail probability $p = P\left(\sum_{i=1}^{d} e^{X_i} < \gamma\right) = \mathrm{E}\left[\mathbf{1}_{\{\sum_{i=1}^{d} e^{X_i} < \gamma\}}\right]$

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- Rare-event setting: For small p, naive Monte Carlo becomes impractical.

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  - Multiply the output with Likelihood Ratio

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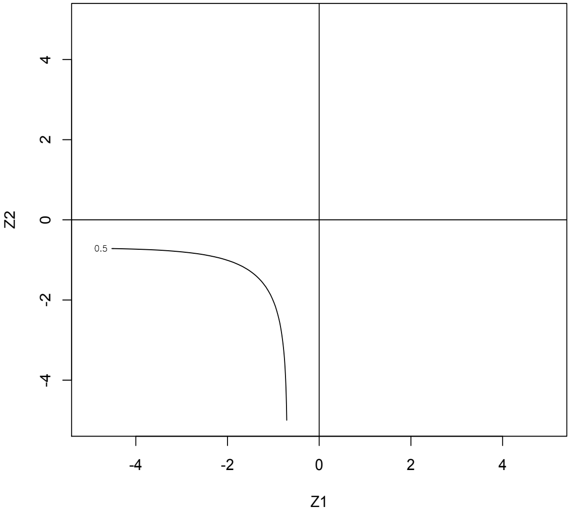

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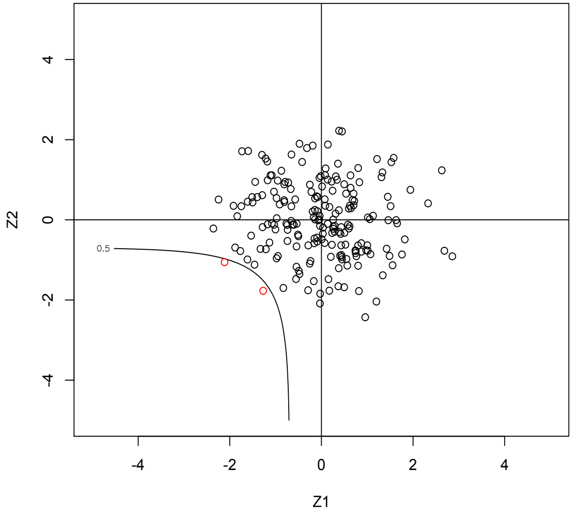

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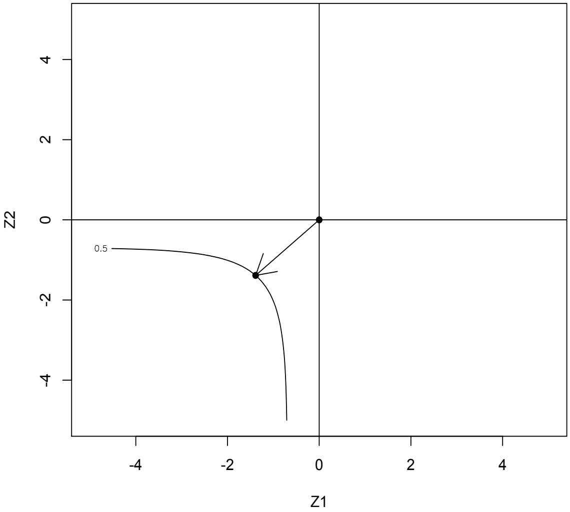

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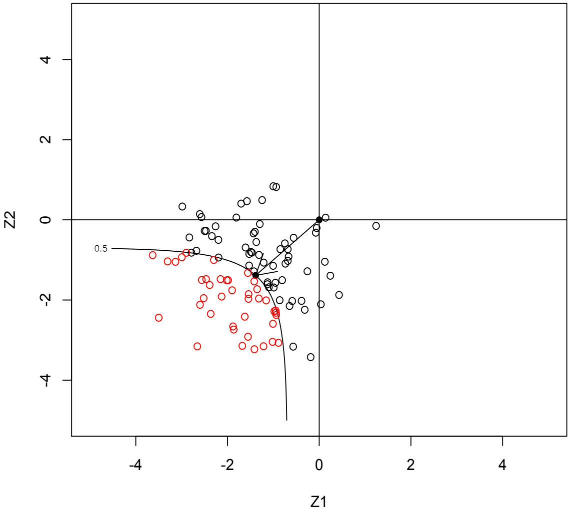

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$$S(Z) \equiv \sum_{i=1}^{d} e^{\nu_i + \sigma_i \sum_{j=1}^{i} L_{ij} Z_j}$$

denote the lognormal sum as a function of standard normal vector $Z \sim N(0, I)$.
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- In this problem, the distance between the origin and the set $\{z | S(z) = \gamma\}$ is minimized.

Finding optimal mean shift

- Sak et al. (2010), A numerical method for the solution of (P1)

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- Cont and Tankov (2013) Finding a shift μ that guarantees asymptotic optimality (logarithmic efficiency)

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$$\mathrm{E}\left[q(Z_1, Z_2)|Z_2\right] = \mathrm{E}\left[\mathbf{1}_{\{e^{Z_1}+e^{Z_2}<\gamma\}}|Z_2\right] = P(e^{Z_1} + e^{Z_2} < \gamma | Z_2)$$

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$$\begin{aligned}\mathrm{Var}(q(Z_1,Z_2)) &= \mathrm{Var}(\mathrm{E}\left[q(Z_1,Z_2)|Z_2\right]) + \mathrm{E}\left[\mathrm{Var}(q(Z_1,Z_2)|Z_2\right] \\ &\leq \mathrm{Var}(\mathrm{E}\left[q(Z_1,Z_2)|Z_2\right])\end{aligned}$$

    CMC always yields some variance reduction

NEW IDEA

- Lognormal sum

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- Our proposal: The first column of A is selected as

$$A_1 = \mu / ||\mu||$$

μ is the mean shift of IS.

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- The root r can be calculated in closed form for sum of i.i.d. lognormals.

A simple example: $P(e^{Z_1} + e^{Z_2} < \gamma)$

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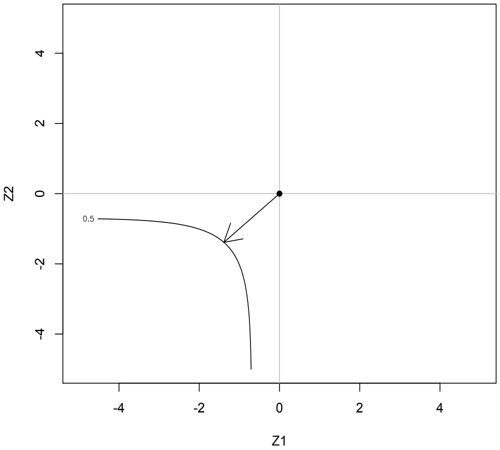

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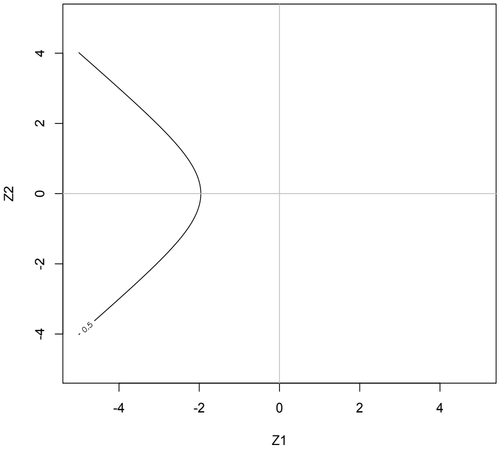
$$P(e^{Z_1} + e^{Z_2} < 0.5) = P(e^{(Z_1+Z_2)/\sqrt{2}} + e^{(Z_1-Z_2)/\sqrt{2}} < 0.5)$$



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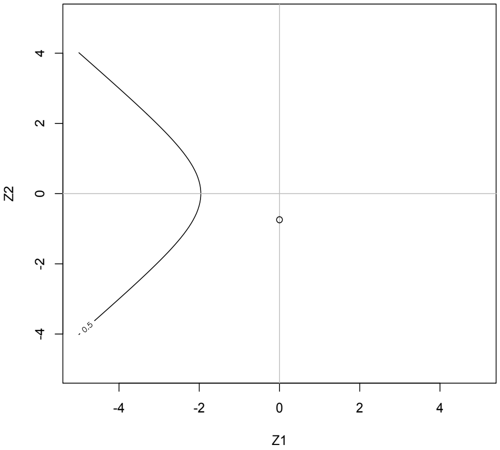

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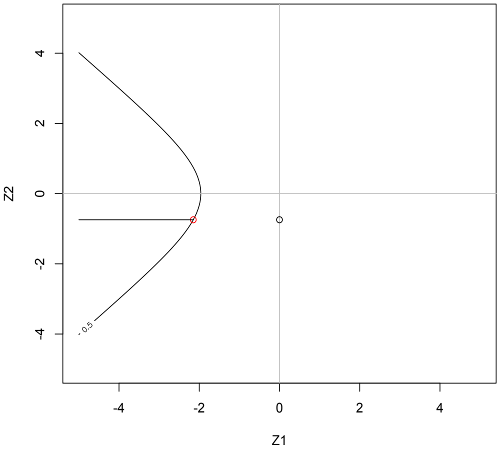

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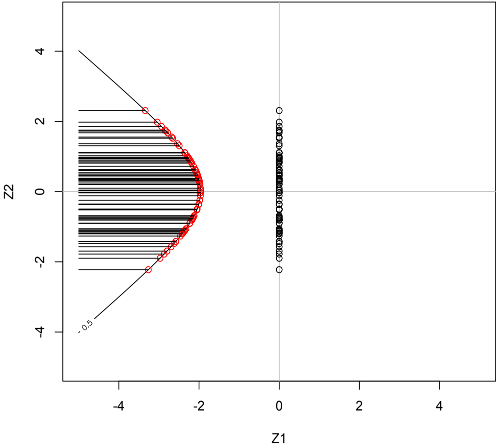

CMC or IS?

- Simple algebra shows that variance of mean shift IS is greater than (or equal to) the variance of CMC using the same mean shift as direction.
- Numerical Results for CDF: Sum of $d = 10$ independent lognormals, $\sigma_k^2 = k, \nu_k = k - d$ for $k = 1, \ldots, d$.
  Sample size: $n = 10^6$

	IS-OPT		CMC-OPT		
γ	Estimate	RE(%)	Estimate	RE(%)	VRF
1	1.25E-01	0.23	1.25E-01	0.11	4.7
1E-01	2.75E-03	0.44	2.73E-03	0.19	5.2
1E-02	7.05E-07	1.03	7.08E-07	0.39	6.9
1E-03	8.90E-14	3.31	8.72E-14	0.88	14.0
1E-04	9.50E-26	5.35	1.03E-25	1.88	8.1
1E-05	1.06E-43	12.10	1.06E-43	3.59	11.4
1E-06	5.42E-68	25.15	4.50E-68	5.63	19.9

Slow-down factor ≈ 6

PDF Estimation

- PDF : $f(\gamma) = \frac{\mathrm{d}F}{\mathrm{d}\gamma}$
  Smooth simulation output with respect to γ
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- PDF estimator

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\mathrm{E}\left[\mathbf{1}_{\{S(Z)<\gamma\}}\right] = \frac{\mathrm{d}}{\mathrm{d}\gamma}\mathrm{E}\left[\mathrm{E}\left[\mathbf{1}_{\{S(Z)<\gamma\}}\,|Z_2,\ldots,Z_d\right]\right]$$
$$= \mathrm{E}\left[\frac{\mathrm{d}}{\mathrm{d}\gamma}\mathrm{E}\left[\mathbf{1}_{\{S(Z)<\gamma\}}\,|Z_2,\ldots,Z_d\right]\right]$$

IID case

- Sum of IID lognormals: $X_i \sim N(\nu, \sigma^2)$, for $i = 1, \ldots, d$ and $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$

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- The first column of orthonormal matrix A is $\frac{1}{\sqrt{d}}(1, \ldots, 1)$

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- Logarithmically efficient

$$\lim_{\gamma \to 0} \frac{\log \mathrm{E}\left[\hat{\ell}^2\right]}{\log \mathrm{E}\left[\hat{\ell}\right]} = 2.$$

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- The multivariate optimal IS density of $(Z_2, \ldots, Z_d)$ is

$$g(z) \propto \Phi\left[\frac{\log(\gamma/d) - \nu}{\sigma/\sqrt{d}} - \frac{\sqrt{d}}{\sigma}\log\left(\frac{1}{d}\sum_{i=1}^{d} e^{\frac{\sigma}{\sqrt{d}}\sum_{j=2}^{d} H_{ij} z_j}\right)\right] e^{-\frac{1}{2}\sum_{j=2}^{d} z_j^2}$$

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- The jth one-dimensional conditional density is

$$g_j(z) \propto \Phi\left[\frac{\log(\gamma/d) - \nu}{\sigma/\sqrt{d}} - \frac{\sqrt{d}}{\sigma}\log\left(\frac{1}{d}\left[e^{\frac{\sigma}{\sqrt{d}}H_{1j}z} + \cdots + e^{\frac{\sigma}{\sqrt{d}}H_{dj}z}\right]\right)\right]\phi(z)$$

$$= \Phi\left[\frac{\log(\gamma/d) - \nu}{\sigma/\sqrt{d}} - \frac{\sqrt{d}}{\sigma}\log\left(\frac{1}{2}[e^{\frac{\sigma}{\sqrt{d}}z} + e^{-\frac{\sigma}{\sqrt{d}}z}]\right)\right]\phi(z)$$

$$= \Phi\left(\frac{\log(\gamma/d) - \nu}{\sigma/\sqrt{d}} - \frac{\sqrt{d}}{\sigma}\log\cosh\left[\frac{\sigma}{\sqrt{d}}z\right]\right)\phi(z)$$

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- Random variate generation from one dimensional density $g_j(z)$ PINV (Polynomial Inversion), TDR (Transformed density rejection)
- The CMC+IS estimator is

$$\mu^{d-1} \frac{\Phi\left(\frac{\log\left(\frac{\gamma}{d}\right)-\nu}{\sigma/\sqrt{d}} - \frac{\sqrt{d}}{\sigma}\log\left[\frac{1}{d}\sum_{i=1}^{d} e^{\frac{\sigma}{\sqrt{d}}\sum_{j=2}^{d} H_{ij}Z_j}\right]\right)}{\prod_{j=2}^{d}\Phi\left(\frac{\log\left(\frac{\gamma}{d}\right)-\nu}{\sigma/\sqrt{d}} - \frac{\sqrt{d}}{\sigma}\log\cosh\left[\frac{\sigma}{\sqrt{d}}Z_j\right]\right)}, \quad Z_j \sim g,\ j = 2, \ldots, d,$$

where

$$\mu \equiv \int_{-\infty}^{+\infty} \Phi\left(\frac{\log\left(\frac{\gamma}{d}\right)-\nu}{\sigma/\sqrt{d}} - \frac{\sqrt{d}}{\sigma}\log\cosh\left[\frac{\sigma}{\sqrt{d}}z\right]\right)\phi(z)\mathrm{d}z$$

IID case

- Moreover, since $\log\cosh(\cdot)$ is an even function, antithetic variates (AV) can be used easily

$$\mu^{d-1}\ \frac{1}{\prod_{j=2}^{d}\Phi\left(t-\frac{\sqrt{d}}{\sigma}\log\cosh\left[\frac{\sigma}{\sqrt{d}}Z_j\right]\right)}$$
$$\times\frac{1}{2}\left\{\Phi\left(t-\frac{\sqrt{d}}{\sigma}\log\left[\frac{1}{d}\sum_{i=1}^{d}e^{\frac{\sigma}{\sqrt{d}}\sum_{j=2}^{d}H_{ij}Z_j}\right]\right)\right.$$
$$\left.+\Phi\left(t-\frac{\sqrt{d}}{\sigma}\log\left[\frac{1}{d}\sum_{i=1}^{d}e^{-\frac{\sigma}{\sqrt{d}}\sum_{j=2}^{d}H_{ij}Z_j}\right]\right)\right\}$$

where $Z_j\sim g,\ j=2,\ldots,d$, and

$$t=\frac{\log\left(\frac{\gamma}{d}\right)-\nu}{\sigma/\sqrt{d}}$$

IID case

- We propose to use the same estimator even for the case that d is not a multiple of 4

$$
\mu^{d-1} \, \frac{1}{\prod_{j=2}^{d} \Phi\left(t - \frac{\sqrt{d}}{\sigma} \log\cosh\left[\frac{\sigma}{\sqrt{d}} Z_j\right]\right)}
\times \frac{1}{2}\left\{ \Phi\left(t - \frac{\sqrt{d}}{\sigma} \log\left[\frac{1}{d}\sum_{i=1}^{d} e^{\sigma \sum_{j=2}^{d} A_{ij} Z_j}\right]\right) + \Phi\left(t - \frac{\sqrt{d}}{\sigma} \log\left[\frac{1}{d}\sum_{i=1}^{d} e^{-\sigma \sum_{j=2}^{d} A_{ij} Z_j}\right]\right)\right\}
$$

Numerical results

$d = 5, \sigma = 1, \nu = \log(1/d), n = 10^5,$

		CMC+IS			CMC+IS+AV			
d	γ	Estimate	RE (%)	VRF	Estimate	RE (%)	VRF	VRF-Total
5	0.5	1.61E-02	0.13	17.6	1.61E-02	0.04	11.0	194
	0.4	4.50E-03	0.13	21.3	4.50E-03	0.04	11.0	236
	0.3	6.36E-04	0.14	25.9	6.35E-04	0.05	6.7	172
	0.2	2.16E-05	0.15	31.6	2.16E-05	0.05	10.7	339
	0.1	1.17E-08	0.15	51.0	1.17E-08	0.05	7.5	382
10	0.7	1.52E-02	0.21	11.1	1.53E-02	0.11	3.9	43
	0.6	4.52E-03	0.21	14.0	4.52E-03	0.10	4.9	68
	0.5	8.34E-04	0.22	17.9	8.34E-04	0.09	6.4	115
	0.4	7.20E-05	0.24	21.9	7.19E-05	0.08	8.6	189
	0.3	1.60E-06	0.25	31.3	1.60E-06	0.09	7.7	242

Implementation using PINV is about 30 times slower than pure CMC.
Speed-up is possible if TDR is used

Why is AV useful?

- Let's consider the simulation output of CMC estimator as function of $Z = (Z_2, \ldots, Z_d) \sim N(0, I_{d-1})$

$$q(Z) \equiv \Phi\left(t - \frac{\sqrt{d}}{\sigma}\log\left[\frac{1}{d}\sum_{i=1}^{d} e^{\frac{\sigma}{\sqrt{d}}\sum_{j=2}^{d} A_{ij}Z_j}\right]\right)$$

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$$q_{\text{AV}}(Z) = \frac{1}{2}[q(Z) + q(-Z)]$$

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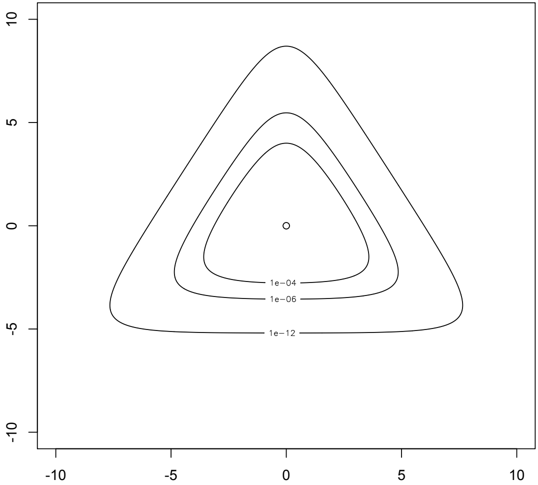
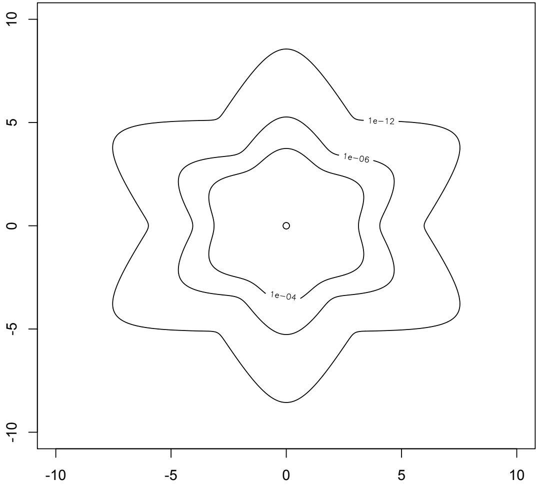
$$q_{\mathrm{AV}}(Z) = \frac{1}{2}[q(Z) + q(-Z)]$$

- The contour plots of $q(Z)$ and $q_{\mathrm{AV}}(Z)$ for $d = 3, \sigma = 1, \nu = \log(1/3)$, and $\gamma = 0.4$

The contour plot of $q(Z_2, Z_3)$



The contour plot of $q_{\mathrm{AV}}(Z_2, Z_3)$



In progress: Reducing variance coming from the Radius

- Let's write the simulation output as a function of the radius R and the direction $\Theta = (\Theta_2, \ldots, \Theta_d) \in \mathbb{S}^{d-2}$

$$Q(R,\Theta) \equiv q(R\Theta) = \Phi\left(t - \frac{\sqrt{d}}{\sigma}\log\left[\frac{1}{d}\sum_{i=1}^{d} e^{\frac{\sigma}{\sqrt{d}}R\sum_{j=2}^{d}A_{ij}\Theta_j}\right]\right)$$

and

$$Q_{\mathrm{AV}}(R,\Theta) = \frac{1}{2}[Q(R,\Theta) + Q(R,-\Theta)] = \frac{1}{2}[q(R\Theta) + q(-R\Theta)]$$

In progress: Reducing variance coming from the Radius

- Let's write the simulation output as a function of the radius R and the direction $\Theta = (\Theta_2, \ldots, \Theta_d) \in \mathbb{S}^{d-2}$

$$Q(R,\Theta) \equiv q(R\Theta) = \Phi\left(t - \frac{\sqrt{d}}{\sigma}\log\left[\frac{1}{d}\sum_{i=1}^{d} e^{\frac{\sigma}{\sqrt{d}} R \sum_{j=2}^{d} A_{ij}\Theta_j}\right]\right)$$

and

$$Q_{\mathrm{AV}}(R,\Theta) = \frac{1}{2}[Q(R,\Theta) + Q(R,-\Theta)] = \frac{1}{2}[q(R\Theta) + q(-R\Theta)]$$

- The best possible method to reduce the variance coming from R is CMC

$$\mathrm{E}\left[Q(R,\Theta)|\Theta\right] = \int_0^\infty \Phi\left(t - \frac{\sqrt{d}}{\sigma}\log\left[\frac{1}{d}\sum_{i=1}^{d} e^{\frac{\sigma}{\sqrt{d}} r \sum_{j=2}^{d} A_{ij}\Theta_j}\right]\right) f_R(r)\,\mathrm{d}r$$

However, it is difficult calculate the integral for each sample of Θ.

In progress: Reducing variance coming from the Radius

- Instead, an IS can be used by changing the distribution of R

$$Q(R,\Theta)\frac{f(R)}{g(R)}, \qquad R \sim g(R)$$

In progress: Reducing variance coming from the Radius

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- In progress:
  Finding a good IS density for R
  Random variate generation from that density

Conclusions

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- Possible extension: Sum of log-spherical random variables

Thank You