Existence, uniqueness and stability of optimal portfolios of eligible assets

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Introducing optimal value functionals

Throughout the talk we work under the following specifications:

- $\mathcal{X}$ is a topological vector space with partial order $\geq$
- $\mathcal{A}$ is a closed subset of $\mathcal{X}$ such that $0 \in \mathcal{A}$ and

$$X \in \mathcal{A},\ Y \geq X \implies Y \in \mathcal{A}$$

- $V_0 : \mathbb{R}^N \to \mathbb{R}$ is a linear functional
- $V_1 : \mathbb{R}^N \to \mathcal{X}$ is a linear operator

We focus on functionals $\rho : \mathcal{X} \to [-\infty, \infty]$ defined by

$$\rho(X) = \inf\{V_0(\lambda)\,;\ \lambda \in \mathbb{R}^N,\ X + V_1(\lambda) \in \mathcal{A}\}$$

Motivating examples

The setup. We consider a one-period economy where:

- future uncertainty is modeled by a probability space

$$(\Omega, \mathcal{F}, \mathbb{P})$$

- the market consists of N frictionless and liquid assets

$$S^i = (S^i_0, S^i_1)$$

- the value of a portfolio $\lambda \in \mathbb{R}^N$ at time t is

$$V_t(\lambda) = \sum_{i=1}^{N} \lambda^i S^i_t.$$

We denote by $\mathcal{X}$ a set of random variables of interest.

Motivating example (1)

Capital Adequacy. Assume that X represents the capital position of a financial institution at time 1. Then

$$\rho(X) = \inf\{V_0(\lambda)\,;\ \lambda \in \mathbb{R}^N,\ X + V_1(\lambda) \in \mathcal{A}\}$$

where

$$\mathcal{A} = \begin{cases} \{X \in \mathcal{X}\,;\ \mathrm{VaR}_\alpha(X) \leq 0\} & \text{(Value at Risk)} \\ \{X \in \mathcal{X}\,;\ \mathrm{ES}_\alpha(X) \leq 0\} & \text{(Expected Shortfall)} \end{cases}$$

can be interpreted as a capital requirement for X.

Reference: Artzner, Delbaen, Eber, Heath (1999), Föllmer, Schied (2002), Frittelli, Scandolo (2006), Artzner, Delbaen, Koch-Medina (2009), Farkas, Koch-Medina, Munari (2014), Liebrich, Svindland (2107), ...

Motivating example (2)

Pricing/Hedging. Assume X represents a payoff at time 1. Then

$$\rho(-X) = \inf\{V_0(\lambda)\,;\ \lambda \in \mathbb{R}^N,\ V_1(\lambda) - X \in \mathcal{A}\}$$

where

$$\mathcal{A} = \begin{cases} \{X \in \mathcal{X}\,;\ \mathbb{P}(X \geq 0) = 1\} & (\text{superhedging}) \\ \{X \in \mathcal{X}\,;\ \mathbb{E}[u(X)] \geq k\} & (\text{utility } u) \\ \{X \in \mathcal{X}\,;\ \alpha(X) \geq k\} & (\text{acceptability index } \alpha) \end{cases}$$

can be interpreted as a price for X (from a seller's perspective).

Reference: Cochrane, Saa-Requejo (2000), Bernardo, Ledoit (2000), Carr, Geman, Madan (2001), Cherny, Madan (2009,2010), Arai (2011), Arai, Fukasawa (2014), ...

Motivating example (3)

Portfolio Management. Assume X represents a position at time 1. Then

$$\rho(X) = \inf\{r(X + V_1(\lambda) - V_0(\lambda))\,;\ \lambda \in \mathbb{R}^N\}$$

where

$$r(X) = \begin{cases} \mathrm{VaR}_\alpha(X) & \text{(Value at Risk)} \\ \mathrm{ES}_\alpha(X) & \text{(Expected Shortfall)} \end{cases}$$

can be interpreted as a market-based risk measure for X.

Reference: Föllmer, Schied (2002), Barrieu, El Karoui (2009), ...

Motivating example (4)

Capital Allocation/Systemic Risk. Assume that $X = (X_1, \ldots, X_d)$ represents the capital positions of d financial entities at time 1. Then

$$\rho(X) = \inf \left\{ \sum_{j=1}^{d} V_0(\lambda_j) \,;\; \lambda_1, \ldots, \lambda_d \in \mathbb{R}^N, \right.$$
$$\left. (X_1 + V_1(\lambda_1), \ldots, X_d + V_1(\lambda_d)) \in \mathcal{A} \right\}$$

where

$$\mathcal{A} = \begin{cases} \{X \in \mathcal{X}^d \,;\; X_j \in \mathcal{A}_j,\ \forall j = 1, \ldots, d\} \\ \{X \in \mathcal{X}^d \,;\; \mathbb{E}[u(X)] \geq k\} \quad (\text{multivariate utility } u) \end{cases}$$

can be interpreted as a systemic risk measure for X.

Reference: Burgert, Rüschendorf (2006), Ekeland, Schachermayer (2011), Armenti, Crépey, Drapeau, Papapantoleon (2017), Biagini, Fouque, Frittelli, Meyer-Brandis (2017), Feinstein, Rudloff, Weber (2017), ...

Objective of the presentation

Focus. We focus on the set-valued mapping $\mathcal{P} : \mathcal{X} \rightrightarrows \mathbb{R}^N$ defined by

$$\mathcal{P}(X) = \{\lambda \in \mathbb{R}^N ; \; X + V_1(\lambda) \in \mathcal{A}, \; V_0(\lambda) = \rho(X)\}$$

Every element of $\mathcal{P}(X)$ is called an optimal portfolio (of eligible assets).

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Goal. We address the following questions:

- existence of optimal portfolios?
- uniqueness of optimal portfolios?
- stability of optimal portfolios?

This requires studying the existence, uniqueness, and stability of the solutions of a nonlinear parametric optimization problem (featuring infinite-dimensional parameters).

Existence of optimal portfolios

Theorem. Define $\mathcal{R}_0 = \{V_1(\lambda)\,;\ \lambda \in \mathbb{R}^N,\ V_0(\lambda) = 0\}$. Then, the following are equivalent:

(a) $\mathcal{P}(X) \neq \emptyset$ for every $X \in \mathcal{X}$.

(b) $\mathcal{A} + \mathcal{R}_0$ is closed.

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(b) $\mathcal{A} + \mathcal{R}_0$ is closed.

Corollary. Assume that one of the following conditions holds:

(1) $\mathcal{A}$ is star-shaped (eg convex or conic) and $\mathcal{A} \cap \mathcal{R}_0 = \{0\}$.

(2) $\mathcal{A}$ is polyhedral (ie a finite intersection of halfspaces).

(3) $\mathcal{A}^\infty \cap \mathcal{R}_0 = \{0\}$ ($\mathcal{A}^\infty$ is the largest cone in $\mathcal{A}$).

Then, $\mathcal{P}(X) \neq \emptyset$ for every $X \in \mathcal{X}$.

The conditions in red stipulate the absence of (scalable) good deals.

Proposition. Assume that for every distinct $X, Y \in \partial\mathcal{A}$ we have

$$X - Y \in \mathcal{R}_0 \implies \lambda X + (1-\lambda)Y \in \mathrm{int}(\mathcal{A}) \text{ for some } \lambda \in (0,1).$$

Then, $|\mathcal{P}(X)| \leq 1$ for every $X \in \mathcal{X}$.

Uniqueness of optimal portfolios

Proposition. Assume that for every distinct $X, Y \in \partial\mathcal{A}$ we have

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Corollary. Assume that $\mathcal{A}$ is strictly convex. Then, $|\mathcal{P}(X)| \leq 1$ for every $X \in \mathcal{X}$.

Stability of optimal portfolios

Intuitively speaking, we want to ensure that

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Definition. (1) We say that $\mathcal{P}$ is upper semicontinuous at X if

$$\mathcal{U} \subset \mathbb{R}^N \text{ open} : \mathcal{P}(X) \subset \mathcal{U} \implies \exists \text{ neighborhood } \mathcal{U}_X : \mathcal{P}(\mathcal{U}_X) \subset \mathcal{U}.$$

(2) We say that $\mathcal{P}$ is lower semicontinuous at X if

$$\mathcal{U} \subset \mathbb{R}^N \text{ open} : \mathcal{P}(X) \cap \mathcal{U} \neq \emptyset \implies \begin{cases} \exists \text{ neighborhood } \mathcal{U}_X : \forall Y \in \mathcal{U}_X \\ \qquad \mathcal{P}(Y) \cap \mathcal{U} \neq \emptyset. \end{cases}$$

The above properties ensure that $\mathcal{P}$ does not shift away and, more specifically, does not explode (1) or shrink (2) as a result of a slight perturbation of X.

Upper semicontinuity

Theorem. The following statements are equivalent:

(*a*) $\mathcal{P}$ is upper semicontinuous on $\mathcal{X}$.

(*b*) $\mathcal{P}(\mathcal{K})$ is bounded for every compact $\mathcal{K} \subset \mathcal{X}$.

(*c*) For every $X \in \mathcal{X}$ we have

$$X_n \to X,\ \lambda_n \in \mathcal{P}(X_n) \implies \exists \lambda \in \mathcal{P}(X) : \lambda_{n_k} \to \lambda.$$

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Corollary. Assume that one of the following conditions holds:

(1) $\mathcal{A}$ is star-shaped and $\mathcal{P}(X)$ is bounded for all $X \in \mathcal{X}$.

(2) $\mathcal{A}^\infty \cap \mathcal{R}_0 = \{0\}$.

Then, $\mathcal{P}$ is upper semicontinuous on $\mathcal{X}$.

Lower semicontinuity

Theorem. The following statements are equivalent:

(a) $\mathcal{P}$ is lower semicontinuous on $\mathcal{X}$.

(b) For every $X \in \mathcal{X}$ we have

$$X_n \to X,\ \lambda \in \mathcal{P}(X) \implies \exists \lambda_n \in \mathcal{P}(X_n) : \lambda_n \to \lambda.$$

In other words, lower semicontinuity ensures that

$$Y \text{ is close to } X \text{ and } \lambda \in \mathcal{P}(X) \implies \exists \mu \in \mathcal{P}(Y) \text{ that is close to } \lambda.$$

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Theorem. If $\mathcal{A}$ is polyhedral, then $\mathcal{P}$ is lower semicontinuous on $\mathcal{X}$.

Corollary. We have lower semicontinuity if $\mathcal{A}$ is the positive cone or is based on Expected Shortfall provided that we work in finite dimension.

Failure of lower semicontinuity

Example. The map $\mathcal{P}$ fails to be lower semicontinuous on $\mathcal{X}$ in each of the following cases:

(1) $\mathcal{A}$ is based on Value at Risk (both in finite and infinite dimension).

Failure of lower semicontinuity

Example. The map $\mathcal{P}$ fails to be lower semicontinuous on $\mathcal{X}$ in each of the following cases:

(1) $\mathcal{A}$ is based on Value at Risk (both in finite and infinite dimension).

(2) $\mathcal{A}$ is a law-invariant convex cone in infinite dimension (with the exception of the acceptance set induced by the mean), eg:

- $\mathcal{A}$ is the positive cone
- $\mathcal{A}$ is based on Expected Shortfall
- $\mathcal{A}$ is based on a spectral risk measure
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- $\mathcal{A}$ is based on an expectile

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(3) $\mathcal{A}$ is convex, law-invariant, and is contained in some acceptance set based on Value at Risk in infinite dimension.

Definition. A continuous map $P : \mathcal{X} \to \mathbb{R}^N$ such that

$$P(X) \in \mathcal{P}(X) \quad \text{for every } X \in \mathcal{X}$$

is said to be a continuous portfolio selection.

Robust portfolio selections

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Michael's Selection Theorem. If $\mathcal{P}$ is lower semicontinuous on $\mathcal{X}$, then there exists a continuous portfolio selection.

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Michael's Selection Theorem. If $\mathcal{P}$ is lower semicontinuous on $\mathcal{X}$, then there exists a continuous portfolio selection.

In general, lower semicontinuity is only sufficient for the existence of continuous selections.

Goal. We address the following additional question:

- existence of continuous portfolio selections?

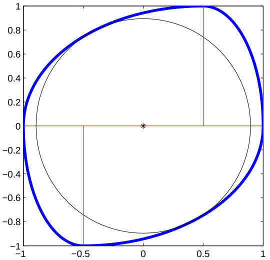
Failure of robust portfolio selections

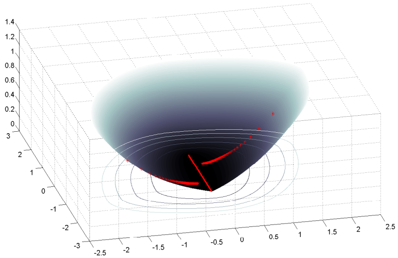
Example. The optimal portfolio map $\mathcal{P}$ always fails to admit robust portfolio selections if

(1) $\mathcal{A}$ is based on Value at Risk (both in finite and infinite dimension).

In addition, $\mathcal{P}$ may fail to admit robust portfolio selections if

(2) $\mathcal{A}$ is convex (both in finite and infinite dimension).





Stability of nearly-optimal portfolios

Focus. We focus on the set-valued mapping $\mathcal{P}_\varepsilon : \mathcal{X} \rightrightarrows \mathbb{R}^N$ defined by

$$\mathcal{P}_\varepsilon(X) = \{\lambda \in \mathbb{R}^N ;\ X + V_1(\lambda) \in \mathcal{A},\ V_0(\lambda) < \rho(X) + \varepsilon\}, \quad \varepsilon > 0$$

Every element of $\mathcal{P}_\varepsilon(X)$ is called a nearly-optimal portfolio.

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Theorem. Assume the following conditions are both satisfied:

(1) For every $X \in \mathcal{X}$ there exists $\lambda \in \mathbb{R}^N$ such that $X + V_1(\lambda) \in \mathrm{int}(\mathcal{A})$.

(2) $\mathrm{cl}(\mathrm{int}(\mathcal{A})) = \mathcal{A}$ (eg $\mathcal{A}$ is convex).

Then, $\mathcal{P}_\varepsilon$ is lower semicontinuous on $\mathcal{X}$.

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Then, $\mathcal{P}_\varepsilon$ is lower semicontinuous on $\mathcal{X}$.

Corollary. Assume that one of the following conditions holds:

(1) There exists $\lambda \in \mathbb{R}^N$ such that $V_1(\lambda) \in \mathrm{int}(\mathcal{X}_+)$.

(2) $\mathcal{A}$ is convex and there exists $\lambda \in \mathbb{R}^N$ such that $V_1(\lambda) \in \mathrm{int}(\mathcal{A}^\infty)$.

Then, $\mathcal{P}_\varepsilon$ is lower semicontinuous on $\mathcal{X}$.

Conclusions

- We discussed existence, uniqueness, and stability of optimal portfolios in a general one-period economy.
- Stability is understood in the sense of parametric optimization.
- We showed that stability breaks down in many important infinite-dimensional settings, eg:
  - superreplication
  - conic finance
  - pricing with acceptable risk, eg based on VaR and ES
  - (systemic) risk measurement, eg based on VaR and ES
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Thank you very much for your attention!