Testing for General Fractional Integration in the Time Domain

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1 Background from the literature

General filter given by convolution of n fractional differences,

$$\Delta_{\gamma}(L;\delta) := \prod_{i=1}^{n} \xi_{\gamma_i}(L;\delta_i) \qquad (1)$$

with memory parameters

$$\delta = (\delta_1, ..., \delta_n)',$$

and frequencies

$$\gamma = (\gamma_1, ..., \gamma_n)', \quad 0 \leq \gamma_1 < \gamma_2 < ... < \gamma_n \leq \pi.$$

These frequencies characterize the long-run and/or the seasonal (cyclical) nature of the data.

The polynomials $\xi_{\gamma_i}(L;\delta_i)$ are defined as

$$\xi_{\gamma_1}(L;\delta_1) = (1-L)^{\delta_1}, \quad if \ \gamma_1 = 0;$$

$$\begin{aligned}(1-L)^{\delta_1} &= 1 - \delta_1 L - \frac{\delta_1(1-\delta_1)}{2}L^2 \\ &\quad - \frac{\delta_1(1-\delta_1)(2-\delta_1)L^3}{6} - \cdots\end{aligned}$$

where L denotes the conventional back-shift operator;

$$\xi_{\gamma_n}(L;\delta_n) = (1+L)^{\delta_n}, \quad if\ \gamma_n = \pi,$$

$$\begin{aligned}(1+L)^{\delta_n} &= 1+\delta_n L - \frac{\delta_n(1-\delta_n)}{2}L^2 \\ &+ \frac{\delta_n(1-\delta_n)(2-\delta_n)L^3}{6} \pm \cdots\end{aligned}$$

and

$$\xi_{\gamma_i}(L;\delta_i) = (1-2\cos\gamma_i L+L^2)^{\delta_i} \quad if\ \gamma_i \in (0,\pi),$$

where those frequencies are also known as Gegenbauer frequencies, (see, for instance, Gray, Zhang & Woodword, 1989),

$$(1-2\cos\gamma_i L + L^2)^{\delta_i} = \sum_{j=0}^{\infty} G_j(\gamma_i,\delta_i)L^j$$

where $G_j(\gamma_i,\delta_i)$ is called Gegenbauer function.

In particular, for $\gamma_i = \pi/2$:

$$\begin{aligned}\left(1+L^2\right)^{\delta_i} &= (1 - 2\cos\pi/2L + L^2)^{\delta_i} \\ &= \left[(1-e^{i\pi/2}L)(1-e^{-i\pi/2}L)\right]^{\delta_i} \\ &= 1+\delta_i L^2 - \frac{\delta_i\,(1-\delta_i)}{2}L^4 \\ &\quad + \frac{\delta_i\,(1-\delta_i)\,(2-\delta_i)\,L^6}{6} \pm \cdots\end{aligned}$$

Seasonal filters As an example for quarterly series ($S = 4$) with seasonal frequencies (on the unit circle),

$$\gamma_1 = 0\,,\ \gamma_2 = 2\pi/S = \pi/2\,,\ \gamma_{S/2+1} = 4\pi/S = \pi,$$

Hassler (1994) considers the flexible filter

$$(1-L)^{\delta_1}\left(1+L^2\right)^{\delta_2}(1+L)^{\delta_3}$$

while Porter-Hudak (1990) applies a rigid version ($\delta_1 = \delta_2 = \delta_3 = \delta$),

$$(1-L)^{\delta}\left(1+L^2\right)^{\delta}(1+L)^{\delta} = (1-L^4)^{\delta}\,.$$

The general filter in (1) is discussed in Woodward, Cheng & Gray (1998).

Stochastic process Consider

$$\Delta_\gamma(L;\delta)\, y_t = x_t\,, \quad t \in \mathbb{Z}\,, \tag{2}$$

and x_t is a stationary and invertible (ARMA) process. Then we know from Gray, Zhang & Woodward (1994), H (1994) and WCG (1998):

Proposition 1 *y_t is stationary if and only if all $\delta_i < 0.5$. If any $\delta_i > 0$, we have long memory in that autocorrelations are not absolutely summable. In this case the spectral density f_y has poles at γ_i of order $\lambda^{-2\delta_i}$:*

$$f_y(\gamma_i + \lambda) \;\sim\; G_i\, \lambda^{-2\delta_i}\,, \quad |\lambda| \to 0\,.$$

In case of $n = 1$ we obtain for autocorrelations at lag h

$$\rho_y(h) \;\sim\; C\, h^{2\delta_1 - 1}\,, \quad \gamma_1 = 0$$

$$\rho_y(h) \;\sim\; C\, \cos(h\gamma_1)\, h^{2\delta_1 - 1}\,, \quad \gamma_1 > 0$$

The effect of cycles of period $P = \frac{2\pi}{\gamma}$ can be seen from

$$Var(y_t) = 2\int_0^\pi f_y(\gamma)\, d\gamma\,.$$

Applications

- seasonal models have been applied e.g. by Porter-Hudak (1990) for monetary aggregates, and Gil-Alana & Robinson (2001) and Gil-Alana (2005) for studies on consumption and income data, and inflation,
- cyclical models have been used to explain interest rate dynamics (Ramachandran and Beaumont, 2001), industrial production (Dalla and Hidalgo, 2005), and nonimal exchange rates (Smallwood and Norrbin, 2006), atmospheric levels of CO_2 (Woodward, Cheng and Gray, 1998), wind speed (Bouette *et al.*, 2006), or power demand (Soares and Souza, 2006)

With n being specified, the null hypothesis in $\Delta_{\gamma}(L;\mathbf{d}+\theta)\, y_t = x_t$ is

$$\mathrm{H}_0 : \delta = \mathbf{d} \ \in \mathbb{R}^n \quad \text{or } H_0 : \theta = \mathbf{0},$$

against the alternative hypothesis

$$H_1 : \delta \neq \mathbf{d} \quad \text{or } H_1 : \theta \neq \mathbf{0}.$$

We extend Robinson (1991), Tanaka (1999), and accomplish Robinson (1994).

Assumptions:
$\mathcal{A}.1)$ *The observable data* $\{y_t\}$ *is generated from* $\Delta_{\gamma}(L;\mathbf{d})\, y_t = x_t,\ t = 1, ..., T,$ *with* $\Delta_{\gamma}(L;\mathbf{d})$ *defined in* (2), *and* $\mathbf{d}$ *being a possibly non-integer vector in* $\mathbb{R}^n$, $n \geq 1$. Further, $y_t = 0$ for $t \leq 0$.
$\mathcal{A}.2)$ *The innovation process* $\{x_t\}_{-\infty}^{\infty}$, *forms a martingale difference sequence (MDS) with* $E\left(\varepsilon_t^2\right) = \sigma^2 < \infty$, *with one of the following assumptions: 1)* x_t *is iid, or 2) is strictly stationary and ergodic with summable eight-order joint cumulants.*

Assuming a Gaussian log-likelihood function we obtain from (2)

$$\mathcal{L}(\delta, \sigma^2) = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(\Delta_\gamma\left(L;\delta\right)y_t\right)^2 \tag{3}$$

and the respective gradient of (3) evaluated under the null

$$\left.\frac{\partial\mathcal{L}(\delta,\sigma^2)}{\partial\theta}\right|_{\theta=0} = -\frac{1}{\sigma^2}\sum_{t=1}^{T}x_t\left.\left(\frac{\partial\Delta_\gamma\left(L;\delta\right)y_t}{\partial\theta}\right)\right|_{\theta=0}.$$

Note that if $\gamma_1 = 0$, *i.e.* the zero frequency is considered, then the partial derivative of $\Delta_\gamma\left(L;\delta\right)y_t$ on θ_1 is

$$\frac{\partial\Delta_\gamma\left(L;\delta\right)y_t}{\partial\theta_1} = \log\left(1-L\right)\left(1-L\right)^{\theta_1}\left(1-L\right)^{d_1}\left[\prod_{i=2}^{n}\xi_{\gamma_i}\left(L;\delta_i\right)\right]y_t$$

which reduces to

$$\log\left(1-L\right)\Delta_\gamma\left(L;\mathbf{d}\right)y_t = \log\left(1-L\right)x_t$$

when the score vector is evaluated at $\theta = \mathbf{0}$. Similarly, the partial derivatives with respect to θ_s,

$s = 2, ..., n-1$, for $\gamma_s \in (0, \pi)$ and θ_n for $\gamma_n = \pi$, when evaluated under the null hypothesis are given, respectively as,

$$\left.\frac{\partial \Delta_\gamma (L; \delta) y_t}{\partial \theta_s}\right|_{\mathrm{H}_0: \theta=0} = \log\left(1 - 2\cos\gamma_s L + L^2\right) x_t,$$

$$\left.\frac{\partial \Delta_\gamma (L; \delta) y_t}{\partial \theta_n}\right|_{\mathrm{H}_0: \theta=0} = \log(1+L) x_t.$$

The filters characterizing the score vector under the null hypothesis can be expanded as (Gradshteyn & Ryzhik, 2000):

$$\log\left(\mathcal{F}_{\gamma_k}\right) x_t = -\sum_{j=1}^{\infty} \omega_j (\gamma_k) x_{t-j},$$

where $\mathcal{F}_{\gamma_1} = 1 - L$, $\mathcal{F}_{\gamma_l} = \xi_{\gamma_l}(L; 1)\,,\, _l = 2, ..., n-1$, $\mathcal{F}_{\gamma_n} = 1 + L$ and $\omega_j(\gamma_k)$ are given in

Definition 2.1. *For all $j \geq 1$ and any $\gamma \in [0, \pi]$, define the non-stochastic weighting process $\omega_j(\gamma)$ as,*

$$\omega_j(\gamma) = \begin{cases} 1/j, & if\ \gamma = 0 \\ 2j^{-1}\cos(j\gamma), & if\ \gamma \in (0, \pi) \\ (-1)^j/j, & if\ \gamma = \pi \end{cases}.$$

For the whole vector $\omega_j = \left(\omega_j\left(\gamma_1\right), ..., \omega_j\left(\gamma_n\right)\right)'$.

Those weights are use to construct/define:

Definition 2.2.

$$x^*_{\gamma_s,t-1} = \sum_{j=1}^{t-1} \omega_j\left(\gamma_s\right) x_{t-j}$$

$$\mathbf{x}^*_{\gamma,t-1} = \left(x^*_{\gamma_1,t-1}, ..., x^*_{\gamma_n,t-1}\right)' = \sum_{j=1}^{t-1} \omega_j x_{t-j}.$$

It follows for the score function under the null

$$\left.\frac{\partial \mathcal{L}(\delta, \sigma^2)}{\partial \theta}\right|_{\mathrm{H}_0:\theta=0} = \frac{1}{\sigma^2}\sum_{t=2}^{T} x_t \left(\sum_{j=1}^{t-1} \omega_j x_{t-j}\right)$$
$$= \frac{1}{\sigma^2}\sum_{t=2}^{T} x_t \left(\mathbf{x}^*_{\gamma,t-1}\right).$$

With the Fisher information matrix estimated as outer product of gradients, the LM statistic is

(under the null)

$$LM_T = \left(\sum_{t=2}^{T} x_t \mathbf{x}^*_{\gamma,t-1}\right)' \left[\sum_{t=2}^{T} x_t^2 \mathbf{x}^*_{\gamma,t-1}\mathbf{x}'^*_{\gamma,t-1}\right]^{-1} \left(\sum_{t=2}^{T} x_t \mathbf{x}^*_{\gamma,t-1}\right).$$

Proposition 2 . *Under $\mathcal{A}.1$ and $\mathcal{A}.2$*

$$LM_T \;\Rightarrow\; \chi^2_{(n)},$$

as $T \to \infty$.

Remark 1 The asymptotic covariance matrix of the score vector is given by

$$\sigma^2 \sum_{j=1}^{\infty} \omega_j \omega_j' = \sigma^2 \Gamma_\gamma\,,$$

for which we have worked out explicit formulae.
Remark 2 We assume the frequencies of interest γ to be known. Different estimators (assuming n to be known!) have been proposed in the literature; Yajima (1996), Giriatis, Hidalgo,

and Robinson (2001), Hidalgo and Soulier (2004), Dalla and Hidalgo (2005), and Hidalgo (2007). Formal proofs of consistency are limited to the case $|d| < 1/2$.

REMARK 3 The results by Tanaka (1999) at the zero frequency only, are interesting and illustrative of the difficulties involved with the correction for autocorrelation in the differenced data.

3 Regression-based tests

To handle short-run autocorrelation we adopt the approach proposed by Breitung & Hassler (2002) and elaborated by Demetrescu, Kuzin & Hassler (2008).

$\mathcal{A}$.3) *The differences under the null satisfy* $a(L)\,x_t = v_t$, *where* $a(L) = 1 - \sum_j^p a_j L^j$, $p \geq 0$, *such that* $a(z)$ *has all its roots outside the unit circle and* $\{v_t\}$, *is strictly stationary and ergodic MDS satisfying the restrictions in Assumption* $\mathcal{A}$.2.

Consider the following lag-augmented least-squares regression $(t = p + 1, ..., T)$,

$$x_t = \sum_{\ell=1}^{n} \phi_\ell x^*_{\gamma_\ell,t-1} + \sum_{i=1}^{p} \zeta_i x_{t-i} + e_{t,p} \quad (4)$$

$$= \beta' X^*_{t,p} + e_{t,p} \quad (5)$$

where in practice, the unknown x_t is replaced by the differences und H_0: $\Delta_\gamma (L; \mathbf{d})\, y_t$.

Proposition 3 . *Under* $\mathcal{A}.1$ *and* $\mathcal{A}.3$ *with*

$$\beta_0 = (0, \ldots, 0, a_1, \ldots, a_p)'$$

and $\widehat{\beta}$ *from (5) it holds*

$$\sqrt{T}\,(\widehat{\beta} - \beta_0) \;\Rightarrow\; \mathcal{N}(0, V),$$

as $T \to \infty$, *where* V *can be estimated consistently.*

Let $\Upsilon^{(n)}_{Wp}$ denote the Wald-type test statistic testing from (4) for

$$\phi_1 = \cdots = \phi_n = 0$$

where we consider heteroskedasticity robust variance estimation like Eicker-White.

Proposition 4 . *Under* $\mathcal{A}.1$ *and* $\mathcal{A}.3$

$$\Upsilon_{Wp}^{(n)} \Rightarrow \chi^2_{(n)}$$

as $T \to \infty$.

REMARK 4 (COROLLARY) Similarly, the test could be performed in a "likelihood ratio" manner, by comparing sums of squares from (un)restricted versions of (4).

REMARK 5 (COROLLARY) Having a "rigid" model in mind,

$$\theta_1 = \cdots = \theta_n = \theta \quad \text{with } H_0 : \ \theta = 0 \, ,$$

we may perform the regression

$$x_t = \phi \left(\sum_{\ell=1}^{n} x^*_{\gamma_\ell, t-1} \right) + \sum_{i=1}^{p} \zeta_i x_{t-i} + e_{t,p} \quad (6)$$

and test $\phi = 0$. Then, the squared t-statistic follows $\chi^2_{(1)}$.

REMARK 6 If heteroskedasticity can be excluded a priori, we work under $\mathcal{A}.2.1$), and the statistics can be computed with "standard" variance estimators.

Remark 7 For $n = 1$ and $\gamma_1 = 0$, Breitung & Hassler (2002) stressed the analogy to Dickey-Fuller test, although all regressors are (asymptotically) stationary. Similarly, (4) is reminiscent of the HEGY test in that each frequency is covered by one regressor – only that here the regressors are not asymptotically orthogonal.

Remark 8 DKH (2008) for $n = 1$ and $\gamma_1 = 0$: They discuss the choice of p; they obtain the limiting distribution under local alternatives; they discuss the treatment of deterministic components – such that limiting distributions are not affected.

Remark 9 It was proposed to construct confidence sets by determining the region where H_0 is not rejected at a given level – one might even consider estimating parameters by searching for the null where the significance is weakest.

4 Finite sample performance

First experiment

$$(1 - 2\cos\gamma_s L + L^2)^{1+\theta} y_t = x_t$$

Second experiment

$$(1-2\cos(0.15)L+L^2)^{1+\theta_1}(1-2\cos(\frac{\pi}{2})L+L^2)^{1+\theta_2}y_t =$$

Table 1: Empirical rejection frequencies when the DGP is the simple GARMA model $(1-2\cos\gamma_s L+L^2)^{1+\theta}y_t = x_t, \quad x_t \sim iid\mathcal{N}(0,1)$.

	θ						
γ_s	-.3	-.2	-.1	0	.1	.2	.3
T=100							
$\frac{\pi}{10}$	.999	.984	.540	**.052**	.584	.981	.999
$\frac{2\pi}{10}$	.999	.933	.401	**.054**	.445	.927	.998
$\frac{3\pi}{10}$	.988	.810	.302	**.056**	.329	.832	.982
$\frac{4\pi}{10}$	.946	.689	.232	**.049**	.267	.721	.946
$\frac{5\pi}{10}$	.929	.630	.210	**.050**	.248	.686	.932
$\frac{6\pi}{10}$	.955	.683	.236	**.051**	.269	.730	.947
$\frac{7\pi}{10}$	.985	.826	.311	**.045**	.331	.836	.985
$\frac{8\pi}{10}$	.998	.929	.425	**.051**	.452	.933	.998
$\frac{9\pi}{10}$	.999	.982	.536	**.050**	.585	.984	.999
T=250							
$\frac{\pi}{10}$	.999	.999	.924	**.043**	.921	.999	.999
$\frac{2\pi}{10}$	.999	.999	.818	**.057**	.814	.999	.999
$\frac{3\pi}{10}$	.999	.997	.653	**.050**	.686	.995	.999
$\frac{4\pi}{10}$	.999	.979	.516	**.052**	.563	.980	.999
$\frac{5\pi}{10}$	.999	.971	.468	**.051**	.545	.968	.999
$\frac{6\pi}{10}$	.999	.980	.520	**.051**	.571	.978	.999
$\frac{7\pi}{10}$	.999	.998	.664	**.045**	.682	.994	.999
$\frac{8\pi}{10}$	.999	1.00	.811	**.050**	.816	.999	.999
$\frac{9\pi}{10}$	.999	.999	.918	**.045**	.913	.999	.999

Note: Empirical size is in bold.

Table 2: Empirical rejection frequencies when the DGP is the 2-factor GARMA model $(1-2\cos(0.15)L+L^2)^{1+\theta_1}(1-2\cos(\frac{\pi}{2})L+L^2)^{1+\theta_2}y_t = x_t,\ x_t \sim iid\mathcal{N}(0,1)$ and T=100

	Joint Restricted Test								Joint Unrestricted Test						
	θ_2								θ_2						
θ_1	-.3	-.2	-.1	0	.1	.2	.3	θ_1	-.3	-.2	-.1	0	.1	.2	.3
-.3	.999	.999	0.997	.959	.741	.362	.247	-.3	.999	.999	.999	.999	.999	.999	.999
-.2	.996	.992	.963	.834	.512	.220	.237	-.2	.994	.978	.974	.977	.990	.999	.999
-.1	.793	.731	.611	.398	.179	.098	.290	-.1	.892	.684	.502	.487	.693	.911	.985
.0	.126	.102	.082	**.047**	.067	.205	.480	.0	.857	.510	.161	**.049**	.205	.592	.893
.1	.631	.590	.583	.574	.625	.730	.853	.1	.988	.913	.741	.556	.535	.718	.898
.2	.987	.985	.982	.981	.982	.988	.993	.2	.999	.999	.992	.980	.974	.981	.991
.3	.999	.999	.999	.999	.999	.999	.999	.3	.999	.999	.999	.999	.999	.999	.999

Note: Empirical size is in bold.

Table 3: Empirical rejection frequencies when the DGP is the 2-factor GARMA model with ARMA errors:

$$(1-2\cos(0.15)L+L^2)^{1+\theta_1}(1-2\cos(\tfrac{\pi}{2})L+L^2)^{1+\theta_2}y_t = x_t,\ (1-0.5L)x_t = (1+0.5L)v_t,$$
$$v_t \sim iid\mathcal{N}(0,1)$$

T=100

Joint Restricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.300	.233	.148	.088	.067	.099	.142
-.2	.131	.120	.089	.059	.051	.072	.127
-.1	.063	.056	.055	.045	.041	.057	.096
.0	.047	.043	.046	**.043**	.049	.062	.080
.1	.065	.059	.063	.060	.061	.075	.086
.2	.093	.087	.092	.094	.092	.104	.113
.3	.126	.127	.123	.136	.127	.130	.139

Joint Unrestricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.204	.142	.122	.141	.202	.315	.381
-.2	.156	.097	.058	.069	.115	.160	.228
-.1	.137	.075	.046	.039	.058	.094	.138
.0	.121	.076	.046	**.037**	.044	.063	.090
.1	.113	.079	.058	.053	.053	.062	.075
.2	.103	.077	.073	.061	.068	.075	.085
.3	.105	.094	.085	.096	.091	.100	.105

T=500

Joint Restricted Test

θ_1 \ θ_2	-.3	-.2	-.1	.0	.1	.2	.3
-.3	.992	.955	.691	.225	.082	.316	.626
-.2	.897	.794	.525	.190	.071	.228	.534
-.1	.492	.389	.230	.093	.049	.179	.424
.0	.150	.113	.073	**.048**	.067	.175	.388
.1	.087	.090	.089	.115	.159	.258	.405
.2	.239	.255	.272	.294	.345	.401	.475
.3	.437	.448	.471	.493	.530	.543	.578

Joint Unrestricted Test

θ_1 \ θ_2	-.3	-.2	-.1	.0	.1	.2	.3
-.3	.981	.926	.834	.802	.862	.949	.979
-.2	.871	.680	.463	.386	.480	.653	.815
-.1	.570	.354	.170	.117	.177	.333	.518
.0	.264	.128	.064	**.053**	.092	.206	.360
.1	.126	.095	.075	.092	.134	.222	.338
.2	.192	.205	.215	.227	.272	.341	.405
.3	.371	.367	.394	.411	.446	.475	.511

Note: Empirical size is in bold. All tests are augmented using Schwert's rule.

Table 4: Empirical rejection frequencies when the DGP is the 2-factor GARMA model with AR errors: $(1 - 2\cos(0.15)L + L^2)^{1+\theta_1}(1 - 2\cos(\frac{\pi}{2})L + L^2)^{1+\theta_2}y_t = x_t,\ (1 - 0.5L)x_t = v_t,$ $v_t \sim iid\mathcal{N}(0, 1)$

T=100

Joint Restricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	0.1	.2	.3
-.3	.479	.336	.202	.108	.077	.104	.179
-.2	.250	.201	.139	.078	.061	.094	.160
-.1	.103	.087	.076	.049	.059	.093	.145
0	.056	.047	.050	**.041**	.050	.079	.132
.1	.062	.056	.054	.057	.062	.083	.122
.2	.104	.088	.083	.082	.088	.097	.135
.3	.139	.130	.123	.124	.126	.137	.151

Joint Unrestricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.334	.234	.191	.194	.255	.352	.467
-.2	.229	.138	.092	.089	.140	.232	.319
-.1	.224	.108	.055	.045	.082	.156	.249
0	.232	.101	.052	**.038**	.058	.127	.190
.1	.226	.118	.059	.046	.061	.115	.187
.2	.210	.124	.071	.057	.076	.128	.203
.3	.181	.118	.094	.083	.110	.164	.219

T=500

Joint Restricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.999	.966	.631	.153	.142	.527	.814
-.2	.967	.889	.564	.158	.108	.458	.766
-.1	.641	.538	.298	.093	.088	.409	.731
0	.205	.155	.088	**.044**	.118	.394	.706
.1	.094	.087	.090	.123	.225	.460	.693
.2	.232	.251	.269	.295	.398	.548	.710
.3	.438	.463	.481	.521	.574	.654	.750

Joint Unrestricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.995	.945	.833	.829	.934	.984	.998
-.2	.987	.812	.487	.397	.601	.853	.954
-.1	.947	.656	.236	.106	.268	.604	.834
0	.874	.513	.156	**.044**	.154	.464	.712
.1	.761	.437	.179	.101	.200	.453	.672
.2	.643	.431	.295	.247	.342	.527	.677
.3	.582	.496	.441	.441	.505	.600	.712

Note: Empirical size is in bold. All tests are augmented using Schwert's rule.

Table 5: Empirical rejection frequencies when the DGP is the 2-factor GARMA model with AR errors: $(1-2\cos(0.15)L+L^2)^{1+\theta_1}(1-2\cos(\frac{\pi}{2})L+L^2)^{1+\theta_2}y_t = x_t,\ (1-0.9L)x_t = v_t,$ $v_t \sim iid\mathcal{N}(0,1)$

T=100

Joint Restricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.085	.080	.062	.043	.040	.067	.114
-.2	.044	.042	.046	.037	.032	.056	.100
-.1	.037	.036	.040	.040	.041	.051	.078
0	.047	.042	.046	**.051**	.049	.065	.079
.1	.062	.068	.070	.072	.068	.082	.097
.2	.076	.075	.073	.084	.095	.087	.100
.3	.073	.077	.078	.078	.078	.086	.087

Joint Unrestricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.290	.143	.064	.034	.056	.105	.168
-.2	.297	.149	.070	.035	.047	.094	.167
-.1	.272	.125	.062	.039	.050	.107	.173
0	.230	.107	.056	**.043**	.065	.123	.208
.1	.157	.097	.058	.056	.081	.142	.226
.2	.120	.080	.055	.069	.094	.149	.219
.3	.081	.068	.056	.059	.078	.118	.170

T=500

Joint Restricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.913	.840	.667	.295	.083	.247	.587
-.2	.547	.474	.348	.184	.063	.173	.477
-.1	.172	.164	.118	.072	.0400	.111	.351
0	.063	.062	.055	**.051**	.053	.106	.248
.1	.088	.103	.080	.085	.100	.130	.197
.2	.133	.126	.123	.123	.119	.121	.161
.3	.105	.099	.092	.090	.081	.085	.088

Joint Unrestricted Test

θ_1 \ θ_2	-.3	-.2	-.1	0	.1	.2	.3
-.3	.995	.886	.549	.287	.391	.679	.866
-.2	.964	.738	.340	.148	.233	.524	.764
-.1	.843	.511	.185	.062	.138	.405	.656
0	.620	.319	.111	**.051**	.123	.320	.549
.1	.408	.217	.103	.066	.122	.258	.432
.2	.232	.150	.111	.090	.117	.184	.286
.3	.113	.091	.077	.076	.075	.102	.144

Note: Empirical size is in bold. All tests are augmented using Schwert's rule.