Dependence Structures of Financial Time Series

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Motivation

Financial Models often require iid. random variables

Models assume:

- Constant variance
- Absence of autocorrelation
- Normal Distribution

Do empirical data hold these assumptions?

Emprical Evidence of Constant Variance?

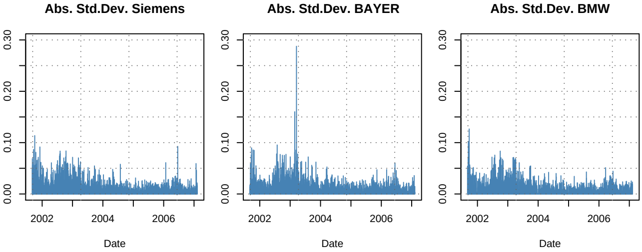


Figure: Absolute Returns

Emprical Evidence of Autocorrelation?

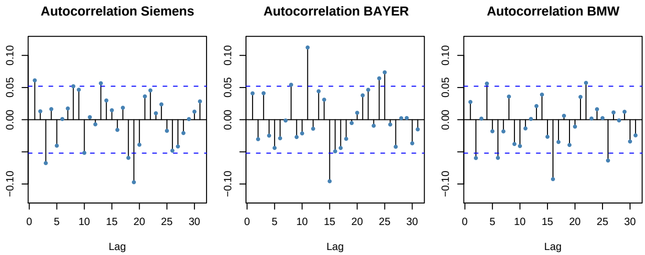


Figure: Autocorrelation of Return Series

Emprical Evidence of Normal Distribution?

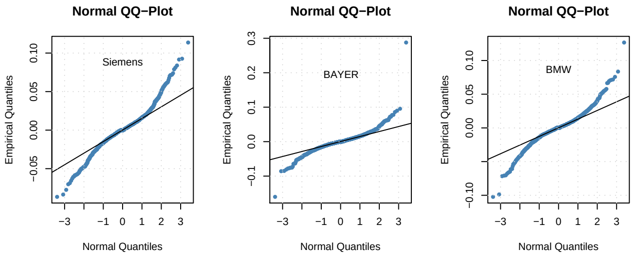


Figure: QQ-Plots of Return Series

Univariate Solution: Alternative Distributions?

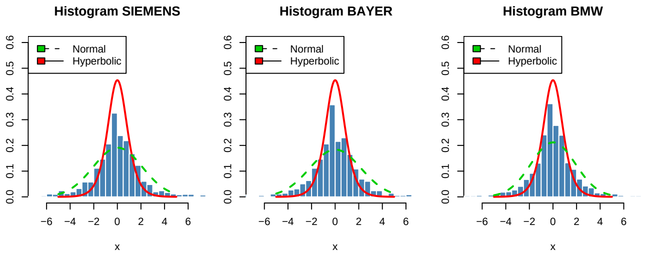


Figure: Histograms and GHP- vs. Normal Distribution

Consequences for Description of Dependence

- Multivariate Distribution need not to be elliptic!
- Pearson's Correlation is/might be wrong measure for dependence.
- Nevertheless, in Financial World Pearson's Correlation is most used measure to describe dependence structures.

Definition

Definition

Pearson's Correlation of the random variables X and Y is defined as

$$\rho(X, Y) = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}}, \tag{1}$$

when $\mathrm{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ holds and $\mathbb{V}[X]$ and $\mathbb{V}[Y]$ measures the variance of X and Y.

Limitations of Pearson's Correlation

Linear Transformation

$$\rho(X, Y) = \rho(\alpha + \beta X, \gamma + \delta Y) \tag{2}$$

for all $\alpha, \gamma \in \mathbb{R}$ and for $\beta, \delta > 0$.

Strictly monotone increasing Transformation

$$\rho(X, Y) \neq \rho(F(X), F(Y)). \tag{3}$$

for all stricltly monotone increasing functions $F : \mathbb{R} \mapsto \mathbb{R}$.

Limitations of Pearson's Correlation

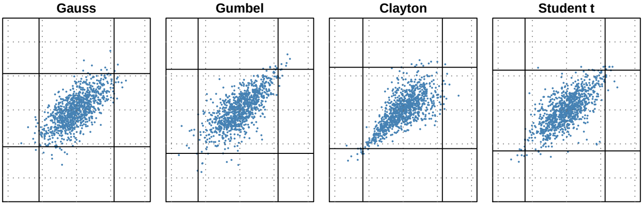


Figure: Simulation of random variates with correlation of 0.7

Dependence measures should be invariant!

Fallacies

- The marginal distributions and the pairwise linear correlation determines the joint distribution.
- Given the margins, one can attain for all the pairs a linear correaltion between $[-1, 1]$ through adjustment of the joint distribution.
- For a portfolio consisting of a linear combination of random variables the VaR is maximal when Pearson's Correlation is maximal.

Prerequisites

- Grounded function: $F(\mathbf{x}) = F(x_1, x_2, \ldots, x_{j-1}, s_j, x_{j+1}, \ldots, x_k) = 0$ if s_j is the lowest Element of S_j of the domain $S_1 \times S_2 \times \ldots \times S_n$.
- Volume of a function.
- k-increasing function: $F(s_1, s_2, \ldots, s_n) = 1$ if s_j for $i = 1, \ldots, n$ is greatest element of S_j of the domain $S_1 \times S_2 \times \ldots \times S_n$.

Definition of a Copula function

Definition

Let C be a function with domain I^k and let $\mathbf{U} = U_1, U_2, \ldots, U_n$ be standard uniform distributed random variables, then C is a Copula if

- C is grounded,
- C is k-increasing,
- $C(\mathbf{u}) = u_j$ for $u_1 = \cdots = u_{j-1} = u_{j+1} = \cdots = u_n = 1$ and $u_j \neq 1$.

Definition of a Copula function

Hence

$$V_C\left([0,u_1]\times[0,u_2]\times\ldots\times[0,u_k]\right)=C(u_1,u_2,\ldots,u_k)=C(\mathbf{u}) \tag{4}$$

holds, C fulfills all features of a distribution function and

$$C(\mathbf{u})=C(u_1,u_2,\ldots,u_k)=\mathbb{P}\left[U_1\leq u_1,U_2\leq u_2,\ldots,U_k\leq u_k\right] \tag{5}$$

holds for $U_i \sim \mathbb{U}[0,1]$.

Sklar's Theorem

Due to probability transformation of $U \sim \mathbb{U}[0,1]$ we have $F^{-1}(U) \sim F$ the following holds:

$$
\begin{aligned}
C\left(F_1(x_1), \ldots, F_n(x_n)\right) &= C(u_1, \ldots, u_n) \\
&= \mathbb{P}\left[U_1 \leq u_1, \ldots, U_n \leq u_n\right] \\
&= \mathbb{P}\left[U_1 \leq F_1(x_1), \ldots, U_n \leq F_n(x_n)\right] \\
&= \mathbb{P}\left[F_1^{-1}(U_1) \leq x_1, \ldots, F_n^{-1}(U_n) \leq x_n\right] \\
&= \mathbb{P}\left[X_1 \leq x_1, \ldots, X_n \leq x_n\right] \\
&= F(x_1, \ldots, x_n)
\end{aligned}
$$

Advantages of Copula functions

A copula **couples** the information about the the margins and information about the dependence to a multivariate distribution function!

Invariance of Copula functions

Let α_k be strictly monotone increasing functions and let $G_k(x)$ be the distribution function of $\alpha_k(X_k)$. Due to the probability transformation

$$
\begin{aligned}
G_k(x) &= \mathbb{P}\left[\alpha_k(X_k) \leq x\right] \\
&= \mathbb{P}\left[X_k \leq \alpha_k^{-1}(x)\right] \\
&= F_k\left(\alpha_k^{-1}(x)\right)
\end{aligned}
$$

holds.

Invariance of Copula functions

Using Sklar's Theorem we can derive

$$
\begin{aligned}
C_\alpha\left(G_1(x_1),\dots,G_n(x_n)\right) &= \mathbb{P}\left[\alpha_1(X_1)\le x_1,\dots,\alpha_n(X_n)\le x_n\right]\\
&= \mathbb{P}\left[X_1\le \alpha_1^{-1}(x_1),\dots,X_n\le \alpha_n^{-1}(x_n)\right]\\
&= C\left(F_1\left(\alpha_1^{-1}(x_1)\right),\dots,F_n\left(\alpha_n^{-1}(x_n)\right)\right)\\
&= C\left(G_1(x_1),\dots,G_n(x_n)\right).
\end{aligned}
$$

Hence, $C = C_\alpha$ and due to this Copula functions are invariant!

Fréchet-Höffding-Bounds

All Copula functions are bounded by the functions

$$C_l(u,v) = \max\{u+v-1,0\} \leq C(u,v) \leq \min\{u,v\} = C_u(u,v).$$

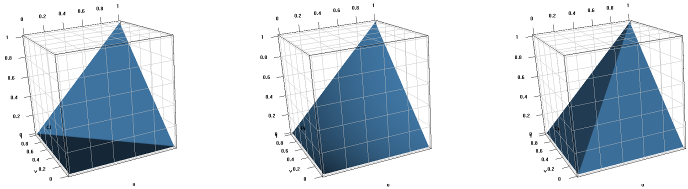


Figure: Graph of Fréchet-Bounds

Fréchet-Höffding-Bounds

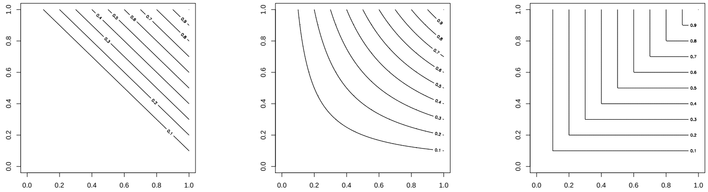


Figure: Graphs of Fréchet-Bounds

Describing Dependence

Independence

$$F(\mathbf{X}) = F_1(X_1)F_2(X_2)\cdots F_n(X_n) = C(F_1(X_1), F_2(X_2))$$

Comonotonicity - Perfect positive Dependence

Strictly monotone **increasing** transformation of 2 random variables leads to

$$F(x_1, x_2) = \min\{F_1(x_1), F_2(x_2)\} = C_u(F_1(X_1), F_2(X_2)),$$

which equals the upper Fréchet-Höffding-Bound.

Describing Dependence

Contermonotonicity - Perfect negative Dependence

Strictly monotone **increasing** transformation of one and strictly monotone **decreasing** transformation of the other random variable leads to

$$F(x_1, x_2) = \max\{F_1(x_1) + F_2(x_2) - 1, 0\} = C_l(F_1(X_1), F_2(X_2)),$$

which equals the lower Fréchet-Höffding-Bound.

Families of Copula functions

Elliptical Copulae

- Easy generation of random variates.

Archimedean Copulae

- Easy to construct Copulae through generator function.
- Archimidean Copulae can be adapted to many properties of empirical data.

Examples

Gauss Copula

$$C_G^{Ga}(\mathbf{u}) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \dots \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{(2\pi)^{\frac{n}{2}} |R|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}^T R^{-1}\mathbf{x}\right) dx_1 dx_2 \dots dx_n$$

Student-t Copula

$$C_{\nu,R}^t(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \dots \int_{-\infty}^{t_\nu^{-1}(u_n)} \frac{\Gamma\left(\frac{\nu+n}{2}\right) |R|^{-\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right) (\nu\pi)^{\frac{n}{2}}} \left(1 + \frac{1}{\nu}\mathbf{x}^T R^{-1}\mathbf{x}\right)^{-\frac{\nu+n}{2}} dx_1 dx_2 \dots dx_n$$

Gumbel Copula

$$C(u_1, u_2, \dots, u_n) = \exp\left(-\left(\sum_{i=1}^n (-\log u_i)^\alpha\right)^{\frac{1}{\alpha}}\right)$$

Clayton Copula

$$C(u_1, u_2, \dots, u_n) = \left(\sum_{i=1}^n u_i^{-\alpha} - n + 1\right)^{-\frac{1}{\alpha}}$$

Frank Copula

$$C(u_1, u_2, \dots, u_n) = -\frac{1}{\alpha} \log\left(1 + \frac{\prod_{i=1}^n (\mathrm{e}^{-\alpha u_i} - 1)}{(\mathrm{e}^{-\alpha} - 1)^{n-1}}\right)$$

Which properties should a dependence measure hold?

- Symmetry: $\delta(X,Y) = \delta(Y,X)$
- Normalistion: $-1 \;\leq\; \delta(X,Y) \;\leq\; 1$
- $\delta(X,Y) \;=\; 1$ if X,Y are comonotonic and
  $\delta(X,Y) \;=\; -1$ if X,Y are contermonotonic
- If $\alpha : \mathbb{R} \mapsto \mathbb{R}$ is a strictly monotone function, then

$$\delta\left(\alpha(X),Y\right) = \begin{cases} \delta(X,Y) & \text{if } \alpha \text{ is increasing,} \\ -\delta(X,Y) & \text{if } \alpha \text{ is decreasing.} \end{cases}$$

- X and Y independent $\Rightarrow \delta(X,Y) = 0$

Rank Correlations

Rank Correlations are based on the concept of concordance.

Definition

Let (x_i, y_i) and (x_j, y_j) be 2 samples of (X, Y).

Concordance

Concordant if $x_i < x_j$ and $y_i < y_j$ or alternatively $x_i > x_j$ and $y_i > y_j$ holds.

Disconcordance

Disconcordant if $x_i < x_j$ and $y_i > y_j$ or alternatively $x_i > x_j$ and $y_i < y_j$ holds.

Kendall's Tau

Measures concordance of 2 pairs.

Definition

$$\tau(X, Y) = \mathbb{P}\left[(X_1 - X_2)(Y_1 - Y_2) > 0\right] - \mathbb{P}\left[(X_1 - X_2)(Y_1 - Y_2) < 0\right]$$

Copula version

$$\tau = 1 - 4 \int_0^1 \int_0^1 \frac{\partial C(u, v)}{\partial u} \frac{\partial C(u, v)}{\partial v} du dv \tag{6}$$

Spearman's Rho

Measures concordance of 3 pairs.

Definition

$$\rho_S = 3\left(\mathbb{P}\left[(X_1 - X_2)(Y_1 - Y_3) > 0\right] - \mathbb{P}\left[(X_1 - X_2)(Y_1 - Y_3) < 0\right]\right)$$

Copula version

$$\rho_S = 12 \int_0^1 \int_0^1 C(u, v)\ dudv - 3 \qquad (7)$$

ρ_S equals Pearson's Correlation of the probability transformed random variables!

Gini's Gamma

Measure for monotone dependence.

Sample version

$$g = \frac{1}{\lfloor \frac{n^2}{2} \rfloor} \left(\sum_{i=1}^{n} |x_{(i)} + y_{(i)} - n - 1| - \sum_{i=1}^{n} |x_{(i)} - y_{(i)}| \right)$$

Continious version

$$\gamma = 2\mathbb{E}\left[|\frac{x_{(i)}}{n} + \frac{y_{(i)}}{n} - \frac{n+1}{n}| - |\frac{x_{(i)}}{n} - \frac{y_{(i)}}{n}| \right]$$

Copula version

$$\gamma = 4 \left(\int_0^1 C(u, 1-u) du - \int_0^1 (u - C(u, u)) \right) \quad (8)$$

Blomqvist's Beta

Similiar to Kendall's Tau, but based on median.

Definition

$$\beta = \mathbb{P}\left[(X-\bar{x})(Y-\bar{y})>0\right] - \mathbb{P}\left[(X-\bar{x})(Y-\bar{y})<0\right]$$

Copula version

$$\beta = 4\ C(0.5, 0.5) - 1 \quad (9)$$

Schweizer and Wolff's Simgma

Distance between C and the product Copula (independent).

Definition

$$\sigma_{SW} = 12 \int_0^1 \int_0^1 |C(u,v) - uv| \, dudv \tag{10}$$

Does not give information about sign of dependence, but can be usefull for interpretation of independence!

Tail Dependence

Do extreme realisations of 2 random variable occur togehter?
Let $X \sim F$ and $Y \sim G$.

Upper Tail Dependence

$$\lambda_U = \lim_{u \to 1^-} \mathbb{P}\left[Y > G^{-1}(u) | X > F^{-1}(u)\right]$$

Lower Tail Dependence

$$\lambda_L = \lim_{u \to 0^+} \mathbb{P}\left[Y \leq G^{-1}(u) | X \leq F^{-1}(u)\right]$$

Tail Dependence

Copula version

Upper Tail Dependence

$$\lambda_U = \lim_{u \to 1^-} \left(\frac{1 - 2u + C(u,u)}{1 - u} \right) \tag{11}$$

Lower Tail Dependence

$$\lambda_L = \lim_{u \to 0^+} \left(\frac{C(u,u)}{u} \right) \tag{12}$$

Tail Dependence

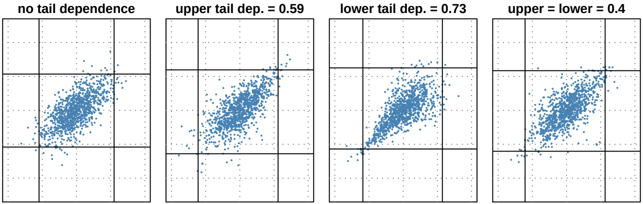


Figure: Simulation of random variates with different tail dependence

Which Copula functions fits to my data?

3 methods of Maximum-Likelihood-Estimation:

Exact Maximum Likelihood method (one stage method)

$$l(\theta) = \sum_{t=1}^{T} \ln c\left(F_1(x_{1,t}), F_2(x_{2,t}), \ldots, F_n(x_{n,t})\right) + \sum_{t=1}^{T} \sum_{j=1}^{n} \ln f_j(x_{j,t})$$

Inference Functions for Margins method (two stage method)

First step: Fit the margins

$$l_i(\alpha_i) = \sum_{t=1}^{T} \ln f_i(x_{i,t})$$

Which Copula functions fits to my data?

Second step: Fit the copula

$$l(\theta) = \sum_{t=1}^{T} \ln c\left(F_1(x_{1,t}|\hat{\alpha}_1), F_2(x_{2,t}|\hat{\alpha}_2), \ldots, F_n(x_{n,t}|\hat{\alpha}_n)\right)$$

Canonical Maximum Likelihood method

$$l(\theta) = \sum_{t=1}^{T} \ln c\left(\hat{F}_1(x_{1,t}), \hat{F}_2(x_{2,t}), \ldots, \hat{F}_n(x_{n,t})\right)$$

Criterias for model selection

Possible criterias:

- Objective function subject of Maximum-Likelihood-function
- Kolmogorov-Smirnov test
- Akaike's Information Criterion
- Bayesian Information Criterion
- Distance between fitted and empircal Copula

Replication of dependence structure

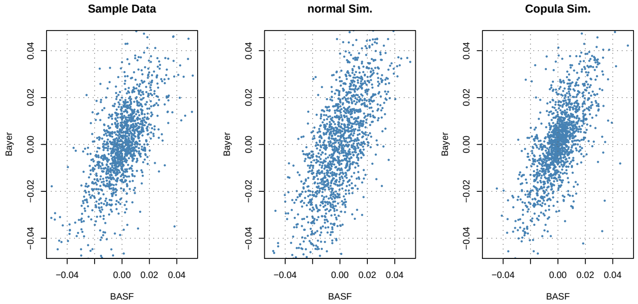


Figure: Sample data, normal simulation and copula simulation of BASF vs. BAYER

Replication of dependence structure

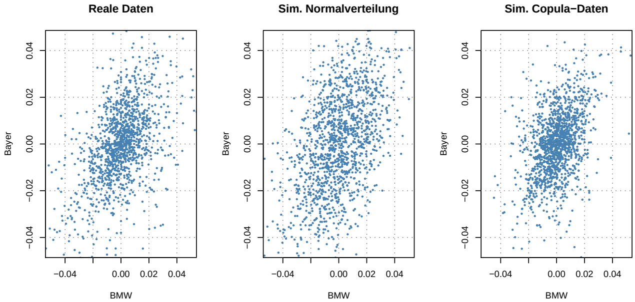
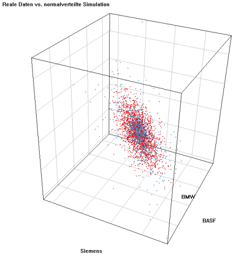


Figure: Sample data, normal simulation and copula simulation of BMW vs. BAYER

Replication of dependence structure



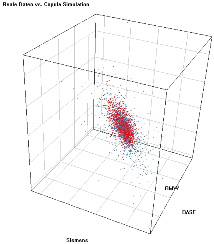


Figure: Copula simulation of BASF, Siemens and BMW

Comparison of various dependence measures

	BAS vs. BMW	SIE vs. BMW	BMW vs. BAY
ρ	0.636	0.614	0.520
τ	0.405	0.421	0.376
ρ_S	0.568	0.562	0.515
γ	0.458	0.464	0.418
β	0.405	0.418	0.380
σ_{SW}	0.567	0.663	0.504
λ_U	0.269	0.221	0.210
λ_L	0.264	0.221	0.210

Table: Comparison of various dependence measures

Re-Define Risk Matrix

Mean Variance Approach

Each element of the variance-covariance-matrix is defined as

$$\rho_{i,j}\sigma_i\sigma_j.$$

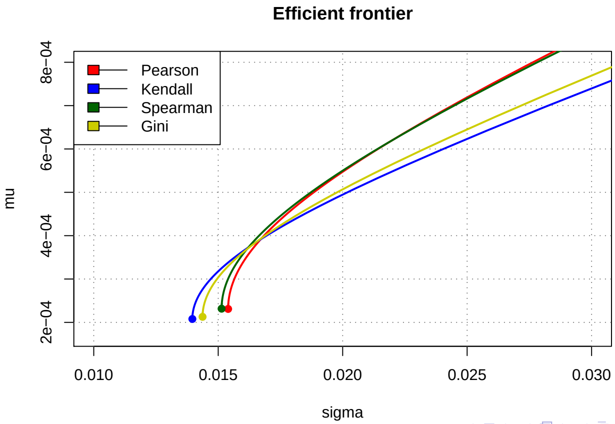
Alternative Approach

Replace ρ by other dependence measure:

$$\delta_{i,j}\sigma_i\sigma_j$$

What are the consequences for efficient frontier?

Effects on the efficient frontier



Research Outlook

Applications in Optimization

- Use alternative risk measures.
- (C)VaR optimization based on Copula calculations of VaR.
- Choose risk aversion factor based on position of the forecasts in empirical/estimated multivariate distribution (Copula) of the instruments.
- Alternatives for Mean-Variance-Optimization based on Copula Theory?

Research Outlook

Applications in Preprocessing

- Use alternative dependence measures for clustering of inputs.
- Try to reduce number of inputs with help of Copula functions.

Application in Model Building

- Use Copula functions for describing dependencies of inputs (especially for extreme events!).
- Try to calculate long-term forecasts which fit to empirical/fitted Copula of the instruments. Choose risk factor based on long-term forecasts.

Research Outlook

Applications in Alpha Selection

- Do selected forecasts fit to empirical/estimated Copula of the instruments?
- Do selected forecasts show same dependence structures as estimated/empirical Copula of the instruments? How can such forecasts be found?

Research Outlook

Methodological Part

- Limitations of Probability Theory (addition law).
- Copula are based on Probability Theory.
- What are consequences of limitations for measuring dependencies?
- Can Information Theory help?
- Information Theory and Dependence Structures.

Thank you for your attention!