Measuring Change: Linear Logistic Models With Relaxed Assumptions (LLRA)

Models and Application

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What Are Linear Logistic Models With Relaxed Assumptions?

- They allow to analyse categorical repeated measurement data (categorical panel data)
- They are Item Response Models
- They are called “with Relaxed Assumptions” because they require neither unidimensionalty of items nor distributional assumptions about the latent trait
- They allow to contrast treatment and/or subject covariate effects
- They provide a framework for testing assumptions on (item) dimensionality, trend and covariate effects
- They measure change on a metric scale (actually a ratio scale)
- They are very rarely applied (especially in a business, economics, marketing etc. context)

About The Nature Of LLRA

- LLRA are Generalised Rasch Models
- Each item (or group of items) may - but need not - measure a single latent trait, hence it is a multi-dimensional model
- Change is modelled as a function of trend effects and covariate/treatment main and interaction effects
- Those effects can be specified by linear decomposition of virtual item parameters
- Due to conditional maximum likelihood estimation (CML), inference on effect parameters is completely independent of the trait parameters
- Item and person parameters are not of interest
- The same items are used at each occassion
- Treatment/covariate effects are assumed to be generalisable over subjects of a certain group

Questions The LLRA Can Answer

- Is there a treatment/covariate effect?
- Is there a trend effect?
- Are treatment/covariate/trend effects the same for a group of items?
- Are there any interaction effects between groups (e.g. gender and treatment)?
- Is there a specific functional form of trend and covariate/treatment effects over time (e.g. linear trend)?
- All combinations of the above.

Conditions And Assumptions

- All changes (positive or negative) must be independent of each other, including changes of the same person for other items
- Effect parameters take the same values for all subjects in a group
- There are no latent classes or unobserved heterogeneity due to subgroups
- Equal item discrimination (otherwise effect and discrimination parameters are confounded)

Its benefit is that - given the above holds - measurement of change is free of systematic influences of nuisance parameters, like item and person parameters.

Model Formulation - I

At T_1

$$P(X_{vih1} = 1|T_1) = \frac{\exp(h\theta_{vi1} + \omega_{ih})}{\sum_{l=0}^{m_i} \exp(l\theta_{vi1} + \omega_{il})},$$

At T_t

$$P(X_{viht} = 1|T_t) = \frac{\exp(h\theta_{vit} + \omega_{ih})}{\sum_{l=0}^{m_i} \exp(l\theta_{vit} + \omega_{il})} = \frac{\exp(h(\theta_{vi1} + \delta_{vit}) + \omega_{ih})}{\sum_{l=0}^{m_i} \exp(l(\theta_{vi1} + \delta_{vit}) + \omega_{il})},$$

θ_{vit} ... location of subject v for item i at T_t

$\delta_{vit} = \theta_{vit} - \theta_{vi1}$ ... amount of change of person v for trait i between time T_1 and T_t

h ... h-th response category $(h = 0, \ldots, m_i)$

ω_{ih} ... parameter for category h for item i

Model Formulation - II

The flexiblity of LLRA now arises from a (linear) reparametrisation of δ_{vit} to include different effects:

$$\delta_{vit} = \mathbf{w}_{it}^T \boldsymbol{\eta}$$

$\mathbf{w}_{it}^T$ ... row of design matrix W (covariate values, e.g. dosages or treatment groups) for trait i up to T_t.

$\boldsymbol{\eta}$ is a vector of parameters typically describing treatment or coavariate groups and trend and interactions etc.

$$\delta_{vit} = \sum_j q_{vjit}\lambda_{jit} + \tau_{it} + \sum_{j<l} q_{vjit} q_{vlit} \rho_{jlit}$$

q_{vjit} ... dosage of treatments j for trait i between T_1 and T_t.
λ_{jit} ... effect of the treatment j on trait i at T_t
τ_{it} ... trend effect on trait i for T_t.
ρ_{jlit} ... interaction effects of treatments j and l on trait i at T_t.

Estimation Of LLRA - I

Change effects in LLRA can be estimated via CML by using a trick:
Instead of assessing the change on the latent trait, change is estimated from differences in item difficulties

- In Rasch models, item and person parameters lie on the same latent trait
- A positive location change on the trait is equivalent to a negative shift of the item-(category) threshold
- Hence the same item at t different time points is viewed as t different "virtual" items whose locations differ by δ_{vit}
- CML can be applied to estimate those differences (conditional an difficulties and person parameters)

Detour: The LLTM For Measuring Change

If we have unidimensional, Rasch model conforming items, we can assess change by means of the LLTM with exactly the same trick of using “virtual items”

- Difficulties for each item are estimated at all time points
- Change is the difference between the difficulty at T_1 and the later time points
- In principle, different items can be used at each timepoint, since difficulties are known from the Rasch model
- The big difference to LLRA is that the baseline is estimated as well (the difficulties)

Estimation Of LLRA - II

Informal: The likelihood is the product of all positive and negative change patterns between two time points, e.g. for $(0, 1)$ and $(1, 0)$ for dichotomous items.

- Possibly a small part of information considered (all (1,1) and (0,0) paterns are discarded)
- Changes must occur in both directions
- Changes must not be too large
- Assumption of independence of changes

Data Structure - I

To fit LLRA with, e.g. eRm, the usual wide format has to be changed in to long format

Real Persons	T_1				T_2			
S_1	x_{111}	x_{121}	$\dots$	x_{1k1}	x_{112}	x_{122}	$\dots$	x_{1k2}
$\vdots$		$\vdots$				$\vdots$		
S_n	x_{n11}	x_{n21}	$\dots$	x_{nk1}	x_{n12}	x_{n22}	$\dots$	x_{nk2}

Virtual Persons	T_1	T_2
S^*_{11}	x_{111}	x_{112}
$\vdots$	$\vdots$	$\vdots$
S^*_{n1}	x_{n11}	x_{n12}
$\vdots$	$\vdots$	$\vdots$
S^*_{1k}	x_{1k1}	x_{1k2}
$\vdots$	$\vdots$	$\vdots$
S^*_{nk}	x_{nk1}	x_{nk2}

Data Structure - II

Group membership or covariate values must be specified so we can assign each persons item answers to the according group. This must be multiplied with the according number of items (assignment group), e.g. a treatment and a control, 3 items -> six groups

		Virtual Persons	T_1	T_2	Assignment Group
Item 1	TG	$S^*_{(TG)11}$	$x_{(TG)111}$	$x_{(TG)112}$	1
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$S^*_{(TG)n1}$	$x_{(TG)n11}$	$x_{(TG)n12}$	1
	CG	$S^*_{(CG)11}$	$x_{(CG)111}$	$x_{(CG)112}$	2
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$S^*_{(CG)n1}$	$x_{(CG)n11}$	$x_{(CG)n12}$	2
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
Item k	TG	$S^*_{(TG)1k}$	$x_{(TG)1k1}$	$x_{(TG)1k2}$	$2k-1$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$S^*_{(TG)nk}$	$x_{(TG)nk1}$	$x_{(TG)nk2}$	$2k-1$
	CG	$S^*_{(CG)1k}$	$x_{(CG)1k1}$	$x_{(CG)1k2}$	$2k$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$S^*_{(CG)nk}$	$x_{(CG)nk1}$	$x_{(CG)nk2}$	$2k$

Design Matrix

Since we are interested in estimation of $\boldsymbol{\eta}$, the crucial step is setting up the design matrix W.

In principle, W can be of any from, but we will use the following canonical structure:

- Each column corresponds to an effect
- The number of rows is the number of time points $\times$ the number of items $\times$ the number of covariate groups $\times$ the number of categories - 1
- The slowest index is the index of time points. Nested within time points are the item indices, and within items the group indices. The fastest index corresponds to response categories in case of polytomous items
- Non existing effects get a zero value
- For items with different numbers of catgories, the superfluous rows consist of zeros

Design Matrix - Generalisation Of Effects

After setting up a “quasi saturated” LLRA, the model can be simplified (e.g. effects can be generalised)

- Collapsing columns to equate effects (e.g. two treatment groups get the same effects)
- Specifiying certain functional forms (linear effects over time, etc.)
- Collapsing columns to equate effects for certain dimensions (i.e. items)
- Applying any kind of linear contrasts on the effects (e.g. treatment 1 and 2 together are 3 times as effective as treatment 3)
- “Classic” LLRA specifies a general trend over all items and groups and a treatment effect for each group over all items

Design Matrix - Example I

2 time points, 2 groups (control and treatment), k dichotomous items

		λ_1	λ_2	$\cdots$	λ_k	τ_1	τ_2	$\cdots$	τ_k
T_1	Item 1 $-$ TG								
	Item 1 $-$ CG								
	Item 2 $-$ TG								
	Item 2 $-$ CG								
	$\vdots$								
	Item k $-$ TG								
	Item k $-$ CG								
T_2	Item 1 $-$ TG	1				1			
	Item 1 $-$ CG					1			
	Item 2 $-$ TG		1				1		
	Item 2 $-$ CG						1		
	$\vdots$			$\vdots$				$\vdots$	
	Item k $-$ TG				1				1
	Item k $-$ CG								1

Design Matrix - Example II

2 time points, 2 groups (control and treatment), 3 items with 4 categories

				λ_1	λ_2	λ_3	τ_1	τ_2	τ_3	ω_{12}	ω_{13}	ω_{22}	ω_{23}	ω_{32}	ω_{33}
T_1	Item 1	TG	Cat 1												
			Cat 2							1					
			Cat 3								1				
		CG	Cat 1												
			Cat 2							1					
			Cat 3								1				
	Item 2	TG	Cat 1												
			Cat 2									1			
			Cat 3										1		
		CG	Cat 1												
			Cat 2									1			
			Cat 3										1		
	Item 3	TG	Cat 1												
			Cat 2											1	
			Cat 3												1
		CG	Cat 1												
			Cat 2											1	
			Cat 3												1
T_2	Item 1	TG	Cat 1	1			1								
			Cat 2	2			2			1					
			Cat 3	3			3				1				
		CG	Cat 1				1								
			Cat 2				2			1					
			Cat 3				3				1				
	Item 2	TG	Cat 1		1			1							
			Cat 2		2			2				1			
			Cat 3		3			3					1		
		CG	Cat 1					1							
			Cat 2					2				1			
			Cat 3					3					1		
	Item 3	TG	Cat 1			1			1						
			Cat 2			2			2					1	
			Cat 3			3			3						1
		CG	Cat 1						1						
			Cat 2						2					1	
			Cat 3						3						1

Design Matrix in R - I

Since these design matrices may become huge and are very sparse, setting them up manually might be cumbersome.

Hence we let R do it for us. What can help us is:

- The Kronecker product, `%x%`
- The Matrix package, `library(Matrix)`

For example: Setting up the former two design matrices.

Dichotomous

```
> des.tmp <- c(0, 1) %x% diag(3)
> design.2 <- cbind(des.tmp %x% c(1, 0), des.tmp %x% c(1, 1))
```

Design Matrix in R - II

For example: Setting up the former two design matrices.

Polytomous

```
> pseudodes <- matrix(c(1, 0, 1, 1), 2, 2)
> des0 <- diag(3) %x% pseudodes
> des0 <- des0[, c(1, 3, 5, 2, 4, 6)]
> des0 <- c(0, 1) %x% des0
> des1 <- des0 %x% c(1, 2, 3)
> c0 <- matrix(c(0, 1, 0, 0, 0, 1), 3, 2)
> c1 <- c(1, 1) %x% c0
> c2 <- diag(3) %x% c1
> des2 <- cbind(des1, rbind(c2, c2))
```

Collapsing is simply summing up the values in the according columns

```
> des2 <- cbind(des2[, 1], colSums(des2[, c(2, 3)]), des2[, c(4:12)])
```

Fitting with eRm - I

Example: Two groups, three items, two time points

```
> data <- matrix(unlist(dat), nc = 2)
> grps6 <- as.numeric(gl(6, 50, 300))
> res.llra <- LPCM(data, W = design.2, mpoints = 2, groupvec = grps6,
+     sum0 = F)
> res.llra
Results of LPCM estimation:

Call:  LPCM(X = data, W = design.2, mpoints = 2, groupvec = grps6, sum0 = F)

Conditional log-likelihood: -94.49457
Number of iterations: 16
Number of parameters: 6

Basic Parameters eta:
             eta 1      eta 2     eta 3     eta 4       eta 5      eta 6
Estimate 1.8918415 -0.7908925 1.9095409 0.4595322 -0.07410689 -0.9650805
Std.Err  0.8268503  0.5709425 0.6091365 0.3687384  0.38516366  0.4154649
```

Fitting with eRm - II

Do it yourself: Fit LLRA as in Hatzinger & Rusch (2009) for

- Example 1
- Example 2
- Example 4
- Example 5
- Example 6
- Example 7

See http://erm.r-forge.r-project.org/

LLRA functions for eRm

As you can see, this is a lot of work...

Functions in eRm will automatise this in the near future.

- `LLRA()`: A wrapper that does all of the above; restructure data, build design matrix, build assignment vector, fit LPCM
- `collapse_W()`: A function to conveniently collapse W
- `plotGR()`, `plotTR()`: Plot group and trend effects
- `anova()`: Model comparison with likelihood ratio test

References

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