Paired Comparison Preference Models

The prefmod Package: Part I
Some Examples

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Paired Comparison Models

General Considerations:

the paired comparison models we consider in this talk are based on the Bradley-Terry Model (Biometrika, 1952)

- and will be used as statistical models describing and explaining data

- the models are similar to regression models (ANOVA) other types of dependent variables (multivariate, nominal)

Generalised Linear Models

- but will not be considered as measurement models in this case one should use another approach e.g. taken by Winkelmaier in the R package eba
Paired Comparisons

- method of data collection
- given a set of $J$ items

- individuals are asked to judge pairs of objects

  $j$ preferred to $k$

  $k$ preferred to $j$

- aim is to rank objects into a preference order
- obtain an overall ranking of the objects
Overview

• **LLBT models**: loglinear Bradley-Terry models

• Basic LLBT

• **Extended LLBT**
  - undecided response
  - subject covariates
  - object specific covariates

• **Pattern Models**
  - Paired comparison \(\rightarrow\) pattern models
  - Ranking \(\rightarrow\) pattern models
  - Rating \(\rightarrow\) pattern models
The Basic Bradley-Terry Model (BT)

for the each comparison \((jk)\) of object \(j\) to object \(k\) we observe:

- \(n(j \succ k)\) ... the number of times \(j\) is preferred to \(k\)
- \(n(k \succ j)\) ... the number of times \(k\) is preferred to \(j\)

\[
N(jk) = n(j \succ k) + n(k \succ j)
\]

total number of responses to comparison \((jk)\)

the probability that \(j\) is preferred to \(k\) in comparison \((jk)\)

\[
P(j \succ k) = \frac{\pi_j}{\pi_j + \pi_k}
\]

\(\pi\)'s are a called worth parameters
and are non-negative numbers

describing the location of the objects
The Basic Loglinear BT Model (LLBT)

the model can be formulated as a log-linear model following the usual Multinomial / Poisson - equivalence.

the expected value $m_{(j>k)}$ of $n_{(j>k)}$ is

$$m_{(j>k)} = N_{(jk)}p_{(j>k)}$$

$$P_{(j > k)} = \frac{\pi_{j}}{\pi_{j} + \pi_{k}} = c_{(jk)}\sqrt{\frac{\pi_{j}}{\pi_{k}}}$$

where $c_{(jk)}$ is constant for a given comparison

then our basic paired comparison model for one comparison is

$$\ln m_{(j>k)} = \mu_{(jk)} + \lambda_{j} - \lambda_{k}$$

$\lambda$’s are the object parameters
$\mu$’s are nuisance parameters

this model formulation is feasible for further extensions
terms and relations

- relation between $\pi$ and $\lambda$:

  $$\lambda_j = \ln \sqrt{\pi_j}$$

  $$\pi_j = \exp 2\lambda_j$$

- identifiability of $\pi$s is obtained by the restriction $\pi_J = 1$ via $\lambda_J = 0$

- the worth parameters are calculated by

  $$\pi_j = \frac{\exp(2\lambda_j)}{\sum_j \exp(2\lambda_j)}, j = 1, 2, \ldots, J$$

where $\sum_j \pi_j = 1$
LLBT Design Structure

- for 3 objects we have 3 comparisons

<table>
<thead>
<tr>
<th>PC</th>
<th>decision</th>
<th>counts</th>
<th>$\mu$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12)</td>
<td>$O_1$</td>
<td>$n_{(1\succ2)}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$O_2$</td>
<td>$n_{(2\succ1)}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(13)</td>
<td>$O_1$</td>
<td>$n_{(1\succ3)}$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>$O_3$</td>
<td>$n_{(3\succ1)}$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(23)</td>
<td>$O_2$</td>
<td>$n_{(2\succ3)}$</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>$O_3$</td>
<td>$n_{(3\succ2)}$</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

the design structure consists of counts (dependent variable) and the design matrix $X$ with:

$\mu$ which is a factor (dummies for $\mu_1, \mu_2, \mu_3$) and variates for the objects $O_1, O_2, O_3$
LLBT Model Structure

model formula for the first line is:

\[
\ln m_{(1 \rightarrow 2)} = \mu_1 + \lambda_1 - \lambda_2
\]

general matrix notation:

\[
\begin{pmatrix}
\ln m_{(1 \rightarrow 2)} \\
\ln m_{(2 \rightarrow 1)} \\
\ln m_{(1 \rightarrow 3)} \\
\ln m_{(3 \rightarrow 1)} \\
\ln m_{(2 \rightarrow 3)} \\
\ln m_{(3 \rightarrow 2)}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix}
\]
**WINE example (ficticious)**

4 sicilian red wines have been tasted pairwise by 1000 people (all of them prefer red to white wines)

aim of the study:

- preference order for different wines

data:

- PC-responses about the choices of 4 selected wines
- the wines are:
  - O1 = Nero D'Avola (also called 'Calabrese')
  - O2 = Gaglioppo (a red of Calabrian origin frequently grown in Sicily)
  - O3 = Perricone (Pignatello - esoteric, robust red)
  - O4 = Nerello (Mascalese - strong red)
Data preparation

- 4 objects (items) A,G,P,N
- number of comparisons is $4 \cdot \frac{3}{2} = 6$
- for data input the ordering of comparisons is important

<table>
<thead>
<tr>
<th>wines</th>
<th>wine first in comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_1$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>–</td>
</tr>
<tr>
<td>$G_2$</td>
<td>V1</td>
</tr>
<tr>
<td>$P_3$</td>
<td>V2</td>
</tr>
<tr>
<td>$N_4$</td>
<td>V4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>V1</th>
<th>V2</th>
<th>V3</th>
<th>V4</th>
<th>V5</th>
<th>V6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12)</td>
<td>(13)</td>
<td>(23)</td>
<td>(14)</td>
<td>(24)</td>
<td>(34)</td>
</tr>
</tbody>
</table>
# Coding

One possible coding for each comparison (V1, … V6) is

\[
(jk) = \begin{cases} 
1 & \text{if first object is preferred to second object } (j \succ k) \\
-1 & \text{if second object is preferred to first object } (k \succ j)
\end{cases}
\]

First respondent gave the following responses:

<table>
<thead>
<tr>
<th>V1</th>
<th>V2</th>
<th>V3</th>
<th>V4</th>
<th>V5</th>
<th>V6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12)</td>
<td>(13)</td>
<td>(23)</td>
<td>(14)</td>
<td>(24)</td>
<td>(34)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

**NOTE:**

- missing responses should be coded with NA
- if the coding is not (1,−1) but any other two numbers, the smaller number means the first object is preferred e.g. a coding with (0,1) ⇒ 0 means first object preferred
Wine data

• preparations

> library(prefmod)
> load("wine.Rdata")
> wnames <- c("A", "G", "P", "N")

produce barplots for all comparisons

> par(mfrow = c(2, 3))
> for (i in 1:6) barplot(table(wine[, i]))
> par(mfrow = c(1, 1))
Function: **llbtPC.fit()**

- preparations

```r
> library(prefmod)
> load("wine.Rdata")
> wnames <- c("A", "G", "P", "N")
```

- fit simple model

```r
> m1 <- llbtPC.fit(wine, nitems = 4, obj.names = wnames)
> m1
```
> m1
Call:

gnm(formula = formula, eliminate = elim, family = poisson, data = dfr)

Coefficients of interest:

A   G   P   N
1.118 1.073 0.794 NA

Deviance: 2.1059
Pearson chi-squared: 2.1101
Residual df: 3
• model fit was done by the package `gnm()`

• We see (parameter) estimates for $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4 = NA$ (is set to zero because of restriction for estimation)

• We do not see parameter estimates for $\mu_1, \mu_2, \mu_3, \mu_4$; these are nuisance parameters and they are eliminated by the procedure (fitted, but not displayed)

• Does the model fit?
  We can use the Deviance 2.106 degrees of freedom 3

  ```
  > dev1 <- round(m1$deviance, digits = 5)
  > df1 <- m1$df.residual
  > prob1 <- 1 - pchisq(dev1, df1)
  > print(prob1)
  [1] 0.55073
  ```

  Model fit is OK (probability is > 0.05)
Deviance: \[ \sum o_{ij} \ln \left( \frac{o_{ij}}{e_{ij}} \right) = \sum n_{ij} \ln \left( \frac{n_{ij}}{m_{ij}} \right) \]

under full model \( n_{ij} = m_{ij} \)

\( \chi^2 \) statistic: \[ \sum \frac{(o_{ij} - e_{ij})^2}{e_{ij}} = \sum \frac{(n_{ij} - m_{ij})^2}{m_{ij}} \]
Functions: `llbt.worth()`, `plotworth()`

- calculate worth parameters

```r
> worth1 <- llbt.worth(m1)
```

Remember:

\[
\pi_j = \frac{\exp(2\lambda_j)}{\sum_j \exp(2\lambda_j)}
\]

\[\lambda_A = 1.118, \lambda_G = 1.073, \lambda_P = 0.794, \lambda_N = 0\]

- or get estimates

```r
> est1 <- llbt.worth(m1, outmat = "lambda")
```

- plot worth or estimates

```r
> plotworth(worth1, ylab = "estimated worth")
> plotworth(est1)
```
extraction of parameter estimates

• How to get p.e. from \texttt{m1} (which is a \texttt{gnm} object)

all parameter estimates ($\mu_1, \mu_2, \ldots \lambda_1, \lambda_2, \ldots$)

\[
> \texttt{m1$coefficients} \\
\quad \begin{array}{cccc}
\text{A} & \text{G} & \text{P} & \text{N} \\
1.1183 & 1.0731 & 0.7935 & \text{NA} \\
\end{array} \\
\text{attr(,"eliminated")} \\
\]

• How to get parameter estimates of interest only? ($\lambda_1, \lambda_2, \ldots$)

\[
> \texttt{c1 <- coef(m1)} \\
> \texttt{c1} \\
\text{Coefficients of interest:} \\
\quad \begin{array}{cccc}
\text{A} & \text{G} & \text{P} & \text{N} \\
1.1183 & 1.0731 & 0.7935 & \text{NA} \\
\end{array}
\]
• Replace last parameter (NA) with 0 and give names

```r
> c1 <- c(c1[1:3], 0)
> names(c1) <- c("A", "G", "P", "N")
```

• How to get the \( \mu \)'s \((\mu_1, \mu_2, \ldots)\)

```r
> mu <- attr(m1$coefficients, "eliminated")
> mu[1:6]
> mu[4]
[1] 5.688
```
• How to get s.e. from m1

```r
> cov <- vcov(m1)
> se <- sqrt(diag(cov))
```

• How to make a confidence interval plot

```r
> up1 <- c1 + se * 1.96
> lo1 <- c1 - se * 1.96

> cbind(lo1, c1, up1)

<table>
<thead>
<tr>
<th>lo1</th>
<th>c1</th>
<th>up1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.05548</td>
<td>1.1183</td>
<td>1.18114</td>
</tr>
<tr>
<td>1.01080</td>
<td>1.0731</td>
<td>1.13542</td>
</tr>
<tr>
<td>0.73392</td>
<td>0.7935</td>
<td>0.85308</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.0000</td>
<td>0.00000</td>
</tr>
</tbody>
</table>
```
Plot estimates and confidence intervals

```r
> plot(1:4, c1, type = "b", ylim = c(0, 1.4))
> for (i in 1:4) {
+   lines(rep(i, 2), c(lo1[i], up1[i]), col = "red", lty = "dashed")
+   text(rep(i, 2), c1[i], names(c1)[i], pos = 2)
+ }
```
Parameter Interpretation

- **log odds** for preferring \( A \) to \( N \)

\[
\ln \left( \frac{P(A \succ N)}{P(N \succ A)} \right) = \ln \left( \frac{\frac{m(A \succ N)}{N(A,N)}}{\frac{m(N \succ B)}{N(A,N)}} \right) = \ln \left( \frac{m(A \succ N)}{m(N \succ B)} \right) = \ln m(A \succ N) - \ln m(N \succ A)
\]

insert equations

\[
\begin{align*}
\ln m(A \succ N) &= \mu(A,N) + \lambda_A - \lambda_N \\
- \ln m(N \succ A) &= -\mu(A,N) + \lambda_A - \lambda_N
\end{align*}
\]

- **log odds** preferring \( A \) to \( N \) are \( 2(\lambda_A - \lambda_N) \) is 2.237

parameter estimates are stored in \( c2[1] \) and \( c2[4] \)

\( \lambda_A \) is 1.118 and \( \lambda_N \) is 0 (reference object)

- the **odds** for preferring \( A \) to \( N \) are \( \exp(2.237) = 9.362 \)
• How would you calculate the expected preferences for $m(A \succ G)$ and for $m(G \succ A)$?

• the $\lambda$ parameter estimates are:

```r
> coef(m1)
Coefficients of interest:
     A     G     P     N
 1.1183 1.0731 0.7935 NA
```

• the estimates for the $\mu$s are:

```r
> attr(m1$coefficients, "eliminated")
```

• Hint 1: Use model formula to calculate the logarithm of the expected numbers with
$\ln m(A \succ G) = \mu_{(A,G)} + \lambda_A - \lambda_G$

• Hint 2: the expected numbers are $\exp(\ln m(A \succ G))$
• The observed preferences (counts) $n(i > j)$ are stored in

\[
> m1$y
 519 481 656 344 626 374 908 92 902 98 819 181
\]

• The expected preferences $m(i > j)$ are stored in

\[
> m1$fitted.values
522.59 477.41 656.92 343.08 636.27 363.73 903.49 96.51 895.32 104.68
1.V61 1.V62
830.19 169.81
\]

• The raw residuals are the difference between observed and expected preferences

\[
> m1$y - m1$fitted.values
-3.58549 3.58549 -0.92408 0.92408 -10.27047 10.27047 4.50957
-4.50957 6.68499 -6.68499 -11.19455 11.19455
\]
Example: CEMS exchange programme

students of the WU can study abroad visiting one of currently 17 CEMS universities

aim of the study:

- preference orderings of students for different locations
- identify reasons for these preferences

data:

- PC-responses about their choices of 6 selected CEMS universities for the semester abroad (London, Paris, Milan, Barcelona, St.Gall, Stockholm)
- answer: *can not decide* was allowed
- several covariates (e.g., gender, working status, language abilities, etc.)
LLBT Extensions Overview

- undecided responses (ties)
- subject covariates
- object specific covariates
Undecided Response

- undecided  3 possible responses

for each comparison between two objects $j$ and $k$ the response can be:

\[
\begin{align*}
j & \succ k \\
k & \succ j \\
j & = k
\end{align*}
\]

In our CEMS–example: comparing London (LO) to Paris (PA) the response can be:

\[
\begin{align*}
\text{LO} & \succ \text{PA} \\
\text{PA} & \succ \text{LO} \\
\text{LO} & = \text{PA}
\end{align*}
\]
LLBT with Undecided Response

Using the respecification of the probabilities suggested by Davidson and Beaver (1977):

the LLBT model formulas for the comparison \((jk)\) are now:
\[
\begin{align*}
\ln m_{(j \succ k)} &= \mu_{(jk)} + \lambda_j - \lambda_k \\
\ln m_{(k \succ j)} &= \mu_{(jk)} - \lambda_j + \lambda_k \\
\ln m_{(j = k)} &= \mu_{(jk)} + \gamma
\end{align*}
\]

where \(\gamma\) is the parameter for undecided response (could also be \(\gamma_{(jk)}\))

\(\lambda\)'s are the object parameters
\(\mu\)'s are nuisance parameters
Interpretation of parameter $\gamma$

- **log odds** for preferring $(j \succ k)$ to "no decision" $(j = k)$

\[
\ln \left( \frac{P(j \succ k)}{P(j = k)} \right) = \ln m(j \succ k) - \ln m(j = k)
\]

Insert equations

\[
\begin{align*}
\ln m(j \succ k) &= \mu(jk) + \lambda_j - \lambda_k \\
- \ln m(j = k) &= -\mu(jk) - \gamma
\end{align*}
\]

for $\lambda_j = \lambda_k$

- **log odds** preferring $(j \succ k)$ to "no decision" $(j = k)$ is $-\gamma$

if $\gamma < 0$ : there is an advantage in favour of a decision
## LLBT Design Structure with undecided counts

<table>
<thead>
<tr>
<th>PC</th>
<th>decision</th>
<th>counts</th>
<th>μ</th>
<th>γ</th>
<th>λ₁</th>
<th>λ₂</th>
<th>λ₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12)</td>
<td>O₁</td>
<td>(n(1\succ2))</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(12)</td>
<td>no</td>
<td>(n(1=2))</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(12)</td>
<td>O₂</td>
<td>(n(2\succ1))</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(13)</td>
<td>O₁</td>
<td>(n(1\succ3))</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>(13)</td>
<td>no</td>
<td>(n(1=3))</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(13)</td>
<td>O₃</td>
<td>(n(3\succ1))</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(23)</td>
<td>O₂</td>
<td>(n(2\succ3))</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>(23)</td>
<td>no</td>
<td>(n(2=3))</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(23)</td>
<td>O₃</td>
<td>(n(3\succ2))</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Design matrix consists of \(\mu, \lambda₁, \lambda₂, \lambda₃\) and additionally \(\gamma\) where \(\mu\) is a factor and the \(n\)'s are the **counts**
**Function:** `llbtPC.fit()`

**Preparations**

```r
> library(prefmod)
> load("cpc.Rdata")
> cities <- c("LO", "PA", "MI", "SG", "BA", "ST")
```

**Fit simple model**

```r
> m2 <- llbtPC.fit(cpc, nitems = 6, obj.names = cities)
> summary(m2)
```

**Fit model including effect for undecided**

```r
> m3 <- llbtPC.fit(cpc, nitems = 6, undec = TRUE, obj.names = cities)
> summary(m3)
```

**Compare models**

```r
> anova(m3, m2)
```
Functions:  `llbt.worth()`, `plotworth()`

calculate worth parameters

```r
> worth2 <- llbt.worth(m2)
> worth3 <- llbt.worth(m3)
> worthmat <- cbind(worth2, worth3)
```

plot them

```r
> plotworth(worthmat)
```
Subject Covariates

Are the preference orderings different for different groups of subjects?

For one subject covariate on $s$ levels we have now

$$\ln m_{(j \succ k)}|s = \mu_{(jk)s} + \lambda^S_s + (\lambda^O_j + \lambda^O_j^S) - (\lambda^O_k + \lambda^O_k^S)$$

where

- $\lambda^O$ object parameters (for subject baseline group)
- $\lambda^{OS}$ interaction parameter between objects and subject category
- $\lambda^S_s$ fixing the margin for category $s$ of covariate $S$ (nuisance)
- $\mu$'s nuisance parameters
### LLBT Design structure with 1 subject covariate

<table>
<thead>
<tr>
<th>PC</th>
<th>decision</th>
<th>counts</th>
<th>$\mu$</th>
<th>$\lambda^S$</th>
<th>$\gamma$</th>
<th>$\lambda^O_1$</th>
<th>$\lambda^O_2$</th>
<th>$\lambda^O_3$</th>
<th>$\lambda^{OS}_{12}$</th>
<th>$\lambda^{OS}_{22}$</th>
<th>$\lambda^{OS}_{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12)</td>
<td>$O_1$</td>
<td>$n(1\succ2)</td>
<td>1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(12)</td>
<td>no</td>
<td>$n(1=2)</td>
<td>1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(12)</td>
<td>$O_2$</td>
<td>$n(2\succ1)</td>
<td>1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<tr>
<td>(12)</td>
<td>$O_1$</td>
<td>$n(1\succ2)</td>
<td>2$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
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<td>-1</td>
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<tr>
<td>(12)</td>
<td>no</td>
<td>$n(1=2)</td>
<td>2$</td>
<td>4</td>
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<td>1</td>
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<tr>
<td>(12)</td>
<td>$O_2$</td>
<td>$n(2\succ1)</td>
<td>2$</td>
<td>4</td>
<td>1</td>
<td>0</td>
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<td>$\vdots$</td>
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</tr>
</tbody>
</table>

- $O$ and $OS$ are parameter of interest
- $MU \ast S$ not related to objects – nuisance parameters
**Options for `llbtPC.fit()`**: `formel`, `elim`

**fit null model (without subject covariates)**

```r
> mb0 <- llbtPC.fit(cpc, nitems = 6, undec = TRUE, formel = ~1, 
+     elim = ~SEX, obj.names = cities)
```

**fit model with different preference scales for SEX**

```r
> mbsex <- llbtPC.fit(cpc, nitems = 6, undec = TRUE, formel = ~SEX, 
+     elim = ~SEX, obj.names = cities)
```

**Plot for different preference scales for SEX**

female=1 and male =2

```r
> wbsex <- llbt.worth(mbsex)
> colnames(wbsex) <- c("female", "male")
> plotworth(wbsex, ylab = "estimated worth")
```
Interpretation of parameters of Interest

- $\lambda^O_j$ parameter estimate for $O_j$ for the reference group (all subjects covariates on level $= 0$ in dummy coding)

- $\lambda^O_{js}$ change of $\lambda^O_j$ for group $s$

<table>
<thead>
<tr>
<th>model: SEX estimates for object PA</th>
</tr>
</thead>
<tbody>
<tr>
<td>reference group</td>
</tr>
<tr>
<td>SEX1</td>
</tr>
<tr>
<td>SEX2</td>
</tr>
</tbody>
</table>
Interpretation of parameters of Interest

```r
> cmsw <- coef(mbsex)
> cmsw

Coefficients of interest:

<table>
<thead>
<tr>
<th></th>
<th>LO</th>
<th>PA</th>
<th>MI</th>
<th>SG</th>
<th>BA</th>
<th>ST</th>
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</thead>
<tbody>
<tr>
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<td>0.8444125</td>
<td>0.5177037</td>
<td>0.2604805</td>
<td>0.1505150</td>
<td>0.0788050</td>
<td>NA</td>
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<tr>
<td>u</td>
<td>L0:SEX2</td>
<td>PA:SEX2</td>
<td>MI:SEX2</td>
<td>SG:SEX2</td>
<td>BA:SEX2</td>
<td></td>
</tr>
<tr>
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<td>-0.2341781</td>
<td>-0.3117667</td>
<td>0.0647604</td>
<td>0.0042943</td>
</tr>
</tbody>
</table>

model: SEX estimates for object PA model in mbsex

\[
\lambda^{PA} + \lambda^{PA:SEX2}
\]

SEX1 0.518
SEX2 0.518 -0.234

\[
\lambda_{female}^{PA} = 0.518
\]
\[
\lambda_{male}^{PA} = 0.284
\]