Quadratic and Integer Programming

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1 Quadratic Programming

1.1 The Quadratic Programming Problem

Standard form:

$$\begin{aligned} \min_x \frac{1}{2} x^T Q x + c^T x & \\ Ax &= b \\ x &\geq 0 \end{aligned} \tag{1}$$

$A \in \mathbb{R}^{m \times n}$, $b, c \in \mathbb{R}^m$, $Q \in \mathbb{R}^{n \times n}$ is positive semidefinite $y^T Q y \geq 0$, $x \in \mathbb{R}^n$

e.g. OLS are QP without constraints, Markovitz mean-variance optimization problem

The dual problem:

$$\begin{aligned} \max_{x,y,s} b^T y - \frac{1}{2} x^T Q x & \\ A^T y - Qx + s &= c \\ x, s &\geq 0 \end{aligned} \tag{2}$$

Optimality conditions: *Karush-Kuhn-Tucker Theorem* (KKT conditions)
$\Rightarrow$ List of conditions which are necessarily satisfied at any local optimal solution

- *primal feasibility:* $Ax = b, x \geq 0$
- *dual feasibility:* $A^T y = Qx + s = c, s \geq 0$
- *complementary slackness:* $x_i, s_i = 0$ $(\forall i = 1, \ldots, n)$

In matrix notation

$$F(x,y,s) = \begin{bmatrix} A^T y - Qx + s - c \\ Ax - b \\ XSe \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (x,s) \geq 0$$

1.2 Interior Point Methods (IPM)

- To solve LPs in and QPs in polynomial time
- based on Newthon's method but modified to handle inequality constraints
- *Path-following algorithms, centered Newthon-direction, neighborhoods of the central path*

2 QP Models: Portfolio Optimization

2.1 Mean-Variance Optimization

Markowitz' theory of mean-variance optimization (MVO) is about the selection of portfolios of securities (or asset classes) in a manner that trades off the expected returns and the perceived risk of potential portfolios. Mathematically the formulation produces a quadratic programming problem:

$$\min_x \frac{1}{2} x^T Q x \tag{3}$$
$$\begin{aligned} \mu^T x &\geq R \\ Ax &= b \\ Cx &\geq d \end{aligned}$$

Where

- x is the proportion of the total funds invested in security i.
- $Q = (\sigma_{ij})$ is the $n \times n$ symmetric covariance matrix.
- μ and σ are the expected return and the standard deviation of the return.

- A is an $m \times n$ matrix e.g. the weights of the portfolio that should equal $b = 1$.
- C is a $p \times n$ matrix e.g. all the assets of the portfolio.
- d is a p-dimensional vector, if $p = 0$ short sales are not allowed.

2.2 Efficient Portfolio

Definition: Recall that a feasible portfolio x is called effcient if it has the maximal expected return among all portfolios with the same variance, or alternatively, if it has the minimum variance among all portfolios that have at least a certain expected return. The collection of effcient portfolios form the **effcient frontier** of the portfolio universe. The effcient frontier is often represented as a curve in a two-dimensional graph where the coordinates of a plotted point corresponds to the expected return and the standard deviation on the return of an effcient portfolio. Since we assume that Q is positive defnite, the variance is a strictly convex function of the portfolio variables and there exists a unique portfolio that has the minimum variance.

2.3 Markovitz Example

We apply Markowitz's MVO model to the problem of constructing a portfolio of US stocks, bonds and cash. We use historical data for the returns of these three asset classes: The S& P 500 index for the returns on stocks, the 10-year Treasury bond index for the returns on bonds, and we assume that the cash is invested in a money market account whose return is the 1-day federal fund rate. The times series for the Total Return are given for each asset between 1960 and 2003.

Let R_i denote the random rate of return of asset i. From the above historical data, we can compute the arithmetic mean rate of return for each asset:

$$r_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}$$

Or the geometric mean instead of the arithmetic mean:

$$\mu_i = (\prod_{t=1}^{T}(1 + r_{it}))^{\frac{1}{T}-1}$$

Now we set up the quadratic programm for portfolio optimization

$$
\begin{aligned}
\min 0.0284x_S^2 + 2\times 0.0039x_S x_B + 2\times 0.0002x_S x_M & & \\
+0.0114x_B^2 - 2\times -0.0012x_M^2 & & \\
0.1206x_S + 0.0785x_B + 0.0632x_M &\geq& R \\
x_S + x_B + x_M &=& 1 \\
x_S, x_B, X_M &\geq& 0
\end{aligned} \tag{4}
$$

and solving it for R = 6.5 to R = 10.5 % with increments of 0.5 % we get the optimal portfolios and can depict the optimal allocations on the effcient frontier.

2.4 MVO Examples

Exercise 39
Solve Markowitz's MVO model for constructing a portfolio of US stocks, bonds and cash using arithmetic means, instead of geometric means as above. Vary R from 6.5 % to 12 % with increments of 0.5 % . Compare with the results obtained above.

Exercise 40
In addition to the three securities given earlier (S & P 500 Index, 10-year Treasury Bond Index and Money Market), consider a 4th security (the NASDAQ Composite Index). Construct a portfolio consisting of the S & P 500 index, the NASDAQ index, the 10-year Treasury bond index and cash, using Markowitz's MVO model. Solve the model for different values of R.

2.5 Issues with Markovitz model

Diversification: model tends to produce portfolios with unreasonably large weights
Solutions:

- Additional constraints to ensure diversification (limits)
- Group securites to sectors and limit exposure to sector.

Transaction Costs: model does not account for transaction costs when reallocating
Solutions:

- Introduce a turnover constraint $\rightarrow$ the amount bought and sold is restricted.
- Include transaction costs (if known) directly in the model. E.g. proportional to the amount bought or sold.

Parameter Estimation: assumption of perfect information (on μ_i and σ_{ij} $\rightarrow$ small changes lead to large changes in "optimal" solution
Solutions:

- Sample the mean returns μ_i and the covariance coffcients σ_{ij} from a confdence interval around each parameter and then combine the portfolios obtained

- Generate a random value uniformly in an interval around the μ_i for each stock i repetetively and average portfolios obtained.

Exercise 42
Using historical returns of the stocks in the DJIA, estimate their mean μ_i and covariance matrix. Let R be the median of the μ_is.

- (i) Solve Markowitz's MVO model to construct a portfolio of stocks from the DJIA that has expected return at least R.
- (ii) Generate a random value uniformly in the interval $[0.95\mu_i; 1.05\mu_i]$, for each stock i.

Resolve Markowitz's MVO model with these mean returns, instead of μs as in (i). Compare the results obtained in (i) and (ii). (iii) Repeat three more times and average the five portfolios found in (i), (ii) and (iii). Compare this portfolio with the one found in (i).

2.6 Black-Litterman Model

Black and Litterman recommend to combine the investor's view with the market equilibrium. The expected return vector μ is assumed to have a probability distribution that is the product of two multivariate normal distributions. The frst distribution represents the returns at market equilibrium, the second distribution represents the investor's view about the μ_i 's.
The resulting distribution for μ is a multivariate normal distribution with mean

$$\bar{\mu} = [(\tau Q)^{-1} + P^T \Omega^{-1} P]^{-1} [(\tau Q)^{-1} \pi + P^T \Omega^{-1} q]$$

Black and Litterman use $\bar{\mu}$ as the vector of expected returns in the Markowitz model.

2.7 Black-Litterman Example

We use the expected returns on Stocks, Bonds and Money Market of the earlier examples for the vector π representing the market equilibrium. We need to choose the value of the small constant τ. We take $\tau = 0.1$. We have two views that we would like to incorporate into the model. First, we hold a strong view that the Money Market rate will be 2 % next year. Second, we also hold the view that S& P 500 will outperform 10-year Treasury Bonds by 5 % but we are not as confident about this view. These two views are expressed as follows

$$\begin{array}{lclll} \mu_M & = & 0.02 & \text{strong view:} & \omega_1 = 0.00001 \\ \mu_M & = & 0.02 & \text{weaker view:} & \omega_1 = 0.001 \end{array}$$

Then $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, q = \begin{pmatrix} 0.02 \\ 0.05 \end{pmatrix} \text{and} \Omega = \begin{pmatrix} 0.00001 & 0 \\ 0 & 0.001 \end{pmatrix}$

Solve it for $R = 4.0$ % to $R = 11.5$ % with increments of 0.5 %.

2.8 Mean-Absolute Deviation PO

Konno and Yamazaki propose a linear programming model instead of the classical quadratic model. Their approach is based on the observation that different measures of risk, such a volatility and L1-risk, are closely related, and that alternate measures of risk are also appropriate for portfolio optimization. This theorem implies that minimizing σ is equivalent to minimizing ω when $(R_1, \ldots, R_n)$ is multivariate normally distributed. With this assumption, the Markowitz model can be formulated as

$$\min E[| \sum_{i=1}^{n} (R_i - \mu_i) x_i |] \tag{5}$$

$$\text{subj. to} \sum_{i=1}^{n} \mu_i x_i \geq R$$

$$\sum_{i=1}^{n} x_i = 1$$

$$0 \leq x_i \leq m_i \quad \text{for} \quad i = 1, \ldots, n$$

the model can be rewritten as

$$\min \quad \sum_{i=1}^{T} y_t + z_t$$

subj. to $\quad y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \quad$ for $\quad t = 1, \ldots, T$
$\sum_{i=1}^{n} \mu_i x_i \geq R$
$\sum_{i=1}^{n} x_i = 1$
$0 \leq x_i \leq m_i \quad$ for $\quad i = 1, \ldots, n$
$y_t \geq 0, z_t \geq 0 \quad$ for $\quad t = 1, \ldots, T$

This is a linear program. Therefore this approach can be used to solve large scale portfolio optimization problems. (See page 141)

2.9 Other Examples for QP Models in Finance

Maximizing the Sharpe Ratio:

- When you look for a specific portfolio that lies on the efficient frontier you can define a risk you are willing to bear. Return/risk profiles of different combinations of a risky portfolio with the riskless asset can be represented as a straight line – the capital allocation line (CAL) – on the mean vs. standard deviation graph. This optimal CAL goes through a point on the effcient frontier and never goes above a point on the effcient frontier. It is called the reward-to-volatility ratio introduced by Sharpe.

- The standard strategy to find the portfolio maximizing the Sharpe ratio, often called the optimal risky portfolio, is the following: First, one traces out the effcient frontier on a two dimensional return vs. standard deviation graph. Then, the point on this graph corresponding to the optimal risky portfolio is found as the tangency point of the line going through the point representing the riskless asset and is tangent to the efcient frontier. Once this point is identifed, one can recover the composition of this portfolio from the information generated and recorded while constructing the effcient frontier.

Returns-Based Style Analysis:

- Constrained optimization techniques can also be used to determine the effective asset mix of a fund using only the return time series for the fund and a number of carefully chosen asset classes. The allocations in the portfolio can be interpreted as the fund's style and consequently, this approach has become to known as returns-based style analysis.
- It is used when only the returns of a fund is available but not the detailed holdings. A

generic linear factor model is used to explain the returns of passive investments with an error term ϵ_t that represents the contribution of active management.

- The objective is to determine a benchmark portfolio such that the difference between fund returns and the benchmark returns is as close to constant (i.e., variance 0) as possible.The fund return and benchmark return graphs should show two almost parallel lines.
- The problem is convex quadratic programming problem and is easily solvable using well-known optimization techniques such as interior-point methods.

3 Integer Programming

These are feasible programs in which some or all variables are restricted to be integers.

3.1 Combinatorial Auctions

Allows the bidder to submit bids on combinations of items

- $M = \{1, 2, \ldots, m\}$ … the set of items the auctioneer has to sell
- $B_j = (S_j, p_j)$ … a bid, $S_j \leq M$ is a nonempty set of items, p_j is the price offer for the set

Suppose the auctioneer has received

- n … the number of bids $B_1, \ldots, B_n$
- $x_j = \begin{cases} 1 & \text{if } B_j \text{ wins} \\ 0 & \text{otherwise} \end{cases}$

Solving the integer programm

$$\begin{aligned} \max \sum_{i=1}^{n} p_j x_j & & & & & \quad (6) \\ \text{subj. to} \sum_{j:i\in S_j} x_j & \leq & 1 & \text{for} & i = 1, \ldots, m \\ x_j & = & 0 \text{ or } 1 & \text{for} & j = 1, \ldots, n \end{aligned}$$

constraints impose that each item is sold at most once

R-Example

Multiple units of each item to sale:

- $B_j = (\lambda_1^j, \lambda_2^j, \ldots, \lambda_m^j; p_j)$
- λ_i^j … the desired number of units of item i
- p_j … the price offer
- u_i … number of units of item i for sale

$$
\begin{aligned}
\max \sum_{i=1}^{n} p_j x_j & & & & & \\
\text{subj. to} \sum_{j:i \in S_j} \lambda_i^j x_j & \leq & u_i & \text{for} & i = 1, \ldots, m & \\
x_j & = & 0 \text{ or } 1 & \text{for} & j = 1, \ldots, n &
\end{aligned} \tag{7}
$$

3.2 The Lockbox Problem

Lockbox: Often companies receive a large number of payments via checks in the mail have the bank set up a post office box for them, open their mail, and deposit any checks found.

From	L.A.	Cincinnati	Boston	Houston
West	2	4	6	6
Midwest	4	2	5	5
East	6	5	2	5
South	7	5	6	3

Tabelle 1: The average of days from mailing to clearing

Calculate the losses

From	L.A.	Cincinnati	Boston	Houston
West	60	120	180	180
Midwest	48	24	60	60
East	216	180	72	180
South	126	90	108	54

Tabelle 2: The average of days from mailing to clearing

Formulate the LP

$$y_j = \begin{cases} 1 & \text{if lockbox } j \text{ is opened} \\ 0 & \text{otherwise} \end{cases}$$

x_{ij} … 1 if region i sends to lockbox j

Minimize the total yearly costs (each region must assign to one lockbox)

$$\begin{aligned} \min 60x_{11} + 120x_{12} + 180x13 + \ldots + 90y_1 + 90y_2 + 90y_3 + 90y_4 \\ \text{subj. to} \sum_j x_{ij} &= 1 \forall i \end{aligned} \tag{8}$$

or a region can only be assigned to an open lockbox (8 constraints)

$$\text{subj. to (LA)} \quad x_{11} + x_{21} + x_{31} + x_{41} \leq 100y_1$$

R-Example

3.3 Constructing an Index Fund

3.3.1 A Large Scale Deterministic Model

Model that clusters the assets into groups of similar assets and selects one representative asset from each group to be included in the index fund portfolio

- ρ_{ij} … similarity between i and j (e.g. correlation between i and j)
- $y_j = \begin{cases} 1 & \text{if } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$
- $x_{ij} = \begin{cases} 1 & \text{if } j \text{ is the most similar stock in the index fund} \\ 0 & \text{otherwise} \end{cases}$

$$Z = \max \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} x_{ij} \tag{9}$$

$$\begin{aligned} \text{subj. to} \sum_{j=1}^{n} y_j &= q \\ \sum_{j=1}^{n} x_{ij} &= 1 \quad \text{for} \quad i = 1, \ldots, n \\ x_{ij} &\leq y_j \quad \text{for} \quad i, j = 1, \ldots, n \\ x_{ij}, y_j &= 0 \quad \text{or} \quad 1 \quad \text{for} \quad i, j = 1, \ldots, n \end{aligned}$$

If model has been solved and q stocks have been selected for the index fund, weight w_j is calculated for each j in the fund

$$w_j = \sum_{i=1}^{n} V_i x_{ij} \tag{10}$$

- w_j … total market value of the stocks represented by stock j
- V_i … market value of stock i

Also possible is an objective function that takes the weights w_j directly into account such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} V_i \rho_{ij} x_{ij} \tag{11}$$

but q stocks still need to be weighted here.

Solution strategy

Branch-and-Bound, very large formulation of the S&P 500:
250 000 variables, 250 000 constraints $x_{ij} \leq y_j$
$\Rightarrow$ Lagrangian relaxation defined for any vector $u = (u_1, \ldots, u_n)$

$$L(u) = \max \sum_{i=j}^{n} \sum_{j=1}^{n} \rho_{ij} x_{ij} + \sum_{i=1}^{n} u_i (1 - \sum_{j=1}^{n} x_{ij} \tag{12}$$

$$\begin{aligned} \text{subj. to} \sum_{j=1}^{n} y_j &= q \\ x_{ij} &\leq y_j \quad \text{for} \quad i,j = 1,\ldots,n \\ x_{ij}, y_j &= 0 \quad \text{or} \quad 1 \quad \text{for} \quad i,j = 1,\ldots,n \end{aligned}$$

Property 1:

$$L(u) \geq Z \tag{13}$$

where Z is the maximum for the model

Property 2:

$$L(u) = \max \sum_{j=1}^{n} C_j y_j + \sum_{i=1}^{n} u_i \tag{14}$$

$$\text{subj. to} \sum_{j=1}^{n} \sum_{j=1}^{n} y_j = q$$

$$y_j = 0 \quad \text{or} \quad 1 \quad \text{for} \quad j = 1, \dots, n$$

where $C_j = \sum_{i=1}^{n} (\rho_{ij} - u_i)^+$, if $\rho_{ij} - u_i > 0$
Property 3:

$$\text{if} \quad \rho_{ij} - u_i > 0 \quad \text{then} \quad x_{ij} = y_j \quad \text{otherwise} \quad x_{ij} = 0 \tag{15}$$

In an optimal solution of the Lagrangian relaxation y_j is equal to 1 for the largest values of C_j and the remaining y_j are equal to 0. Can also be used as heuristic solution for the model, yields to a feasible but not necessarily optimal solution but provides a lower bound on the optimum value Z.

3.4 A Linear Programming Model

Assumes that important characteristics of the market index to be tracked have already been identified (e.g. fraction f_i of index in each sector i, of companies with market capitalization in various range - small, medium, large)

- $a_{ij} = \begin{cases} 1 & \text{if company } j \text{ has characteristic } i \\ 0 & \text{otherwise} \end{cases}$

- x_{ij} ... the optimum weight of asset j in the portfolio (assume that initially the portfolio hast weights 0)

- y_j ... fraction of asset j bought

- z_j ... fraction sold

$$
\begin{aligned}
\min \sum_{j=1}^{n} (y_j + z_j) & & & & \\
\text{subj. to} \sum_{i=1} a_{ij} x_j &= f_i & \text{for} & \quad i = 1, \ldots, m \\
\sum_{j=1}^{n} x_j &= 1 & & \\
x_j - x_j^0 &\leq y_j & \text{for} & \quad j = 1, \ldots, n \\
x_j^0 - x_j &\leq z_j & \text{for} & \quad i = 1, \ldots, n \\
y_j, z_j, x_j &\geq 0 & \text{for} & \quad i = 1, \ldots, n
\end{aligned} \tag{16}
$$

R-Example

The End!