ORSS and Applications to Inference

N. Balakrishnan

McMaster University
Hamilton, Ontario
Canada L8S 4K1

bala@univmail.cis.mcmaster.ca

Co-authored with

Tao Li

St. Francis Xavier University
Nova Scotia, Canada

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1. Preliminaries

$\boldsymbol{X} = \{X_1, \cdots, X_n\}$: independent identically distributed (IID) random variables with cdf $F(x)$ and pdf $f(x)$.

$\boldsymbol{X}_{\rm OS} = \{X_{1:n} \leq \cdots \leq X_{n:n}\}$: Order statistics obtained from $\boldsymbol{X}$.

• The pdf of $X_{r:n}$ $(1 \leq r \leq n)$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r} f(x). \qquad (1)$$

• The cdf of $X_{r:n}$ $(1 \leq r \leq n)$ is

$$F_{r:n}(x) = \sum_{j=r}^{n} \binom{n}{j} \{F(x)\}^{j} \{1 - F(x)\}^{n-j}. \qquad (2)$$

• The joint pdf of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is

$$\begin{aligned} f_{r,s:n}(x, y) \;=\; & \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \{F(x)\}^{r-1} f(x) \\ & \times \{F(y) - F(x)\}^{s-r-1} f(y) \{1 - F(y)\}^{n-s}, \\ & x < y; \end{aligned} \qquad (3)$$

see David and Nagaraja (2003) and Arnold *et al.* (1992).

$\boldsymbol{X} = \{X_1, \cdots, X_n\}$: independent nonidentically distributed (INID) random variables with cdf $F_i(x)$ and pdf $f_i(x)$, $i = 1, \cdots, n$.

$\boldsymbol{X}_{\text{OS}} = \{X_{1:n} \leq \cdots \leq X_{n:n}\}$: Order statistics from $\boldsymbol{X}$.

- The pdf of $X_{r:n}$ $(1 \leq r \leq n)$ is

$$f_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \sum_P \left[\prod_{k=1}^{r-1} F_{i_k}(x)\ f_{i_r}(x) \prod_{k=r+1}^{n} \{1 - F_{i_k}(x)\} \right]. \tag{4}$$

- The joint pdf of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is

$$\begin{aligned} f_{r,s:n}(x,y) &= \frac{1}{(r-1)!(s-r-1)!(n-s)!} \sum_P \left[\prod_{k=1}^{r-1} F_{i_k}(x)\ f_{i_r}(x) \right. \\ &\quad \left. \times \prod_{k=r+1}^{s-1} \{F_{i_k}(y) - F_{i_k}(x)\}\ f_{i_s}(y) \prod_{k=s+1}^{n} \{1 - F_{i_k}(y)\} \right], \\ &\qquad x < y, \end{aligned} \tag{5}$$

where $\sum_P$ denotes the summation over all $n!$ permutations of $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$.

• Another expression for $f_{r:n}(x)$ in the form of a permanent is as follows:

$$f_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \times$$

$$\begin{array}{c} + \\ \left| \begin{array}{ccc} F_1(x) & \cdots & F_n(x) \\ \vdots & & \vdots \\ F_1(x) & \cdots & F_n(x) \\ f_1(x) & \cdots & f_n(x) \\ 1-F_1(x) & \cdots & 1-F_n(x) \\ \vdots & & \vdots \\ 1-F_1(x) & \cdots & 1-F_n(x) \end{array} \right| \end{array}^{+} \quad \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} r-1 \text{ rows} \\ \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} n-r \text{ rows} \end{array} ,$$

where $^+|A|^+$ denotes the permanent of a matrix A (which is simply like the determinant, except that all the signs in its expansion are positive).

• Similarly, the joint pdf of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is

$$f_{r,s:n}(x,y) = \frac{1}{(r-1)!(s-r-1)!(n-s)!} \times$$

$$\begin{array}{c} + \qquad\qquad\qquad\qquad + \\ \left| \begin{array}{ccc} F_1(x) & \cdots & F_n(x) \\ \vdots & & \vdots \\ F_1(x) & \cdots & F_n(x) \\ f_1(x) & \cdots & f_n(x) \\ F_1(y)-F_1(x) & \cdots & F_n(y)-F_n(x) \\ \vdots & & \vdots \\ F_1(y)-F_1(x) & \cdots & F_n(y)-F_n(x) \\ f_1(y) & \cdots & f_n(y) \\ 1-F_1(y) & \cdots & 1-F_n(y) \\ \vdots & & \vdots \\ 1-F_1(y) & \cdots & 1-F_n(y) \end{array} \right| \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} r-1 \,\text{rows} \\ \\ \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} s-r-1 \,\text{rows}\,, \\ \\ \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} n-s \,\text{rows} \end{array}$$

$$x < y;$$

see Vaughan and Venables (1972) for details.

2. Ranked Set Sample (RSS)

The Procedure of Ranked Set Sample

$$
\begin{array}{cccccc}
 & & \underline{\text{Cycle 1}} & & & \\
\underline{X_{1:n}} & X_{2:n} & \cdots & X_{n:n} & \longrightarrow & X_{1(1)} \\
X_{1:n} & \underline{X_{2:n}} & \cdots & X_{n:n} & \longrightarrow & X_{1(2)} \\
\cdots & \cdots & \cdots & \cdots & \longrightarrow & \cdots \\
X_{1:n} & X_{2:n} & \cdots & \underline{X_{n:n}} & \longrightarrow & X_{1(n)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & \underline{\text{Cycle } m} & & & \\
\underline{X_{1:n}} & X_{2:n} & \cdots & X_{n:n} & \longrightarrow & X_{m(1)} \\
X_{1:n} & \underline{X_{2:n}} & \cdots & X_{n:n} & \longrightarrow & X_{m(2)} \\
\cdots & \cdots & \cdots & \cdots & \longrightarrow & \cdots \\
X_{1:n} & X_{2:n} & \cdots & \underline{X_{n:n}} & \longrightarrow & X_{m(n)}
\end{array}
$$

The RSS data so obtained are

$$\boldsymbol{X}_{\text{RSS}} = \Big\{X_{1(1)}, \cdot\cdot, X_{m(1)}, \cdot\cdot, X_{1(n)}, \cdot\cdot, X_{m(n)}\Big\}.$$

Observe that:

(a) $X_{i(r)}$'s are independent;

(b) For same r, $X_{i(r)}$'s are identically distributed;

(c) <u>If ranking is perfect</u>, the pdf of $X_{i(r)}$ is

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!} \, \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r} \, f(x).$$

Some Results and Developments

- (McIntyre, 1952):

$$\hat{\mu}_{\text{RSS}} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{r=1}^{n} X_{i(r)}.$$

- (Takahasi and Wakimoto, 1968; Dell and Clutter, 1972):

$$\mathrm{E}(\hat{\mu}_{\text{RSS}}) = \mu, \quad \mathrm{Var}\left(\hat{\mu}_{\text{RSS}}\right) < \mathrm{Var}(\bar{X});$$

$$\hat{\sigma}^2_{\text{RSS}} = \frac{1}{mn-1} \sum_{i=1}^{m} \sum_{r=1}^{n} (X_{i(r)} - \hat{\mu}_{\text{RSS}})^2$$

(biased for σ^2).

- (Stokes, 1980):

$$1 \le \frac{\mathrm{Var}(s^2)}{\mathrm{MSE}(\hat{\sigma}^2_{\mathrm{RSS}})}, \quad \text{for } mn \text{ sufficiently large.}$$

- (Chuiv and Sinha, 1998): Discussion of BLUE-RSS.
- (Stokes, 1995): Discussion of Fisher Information and MLE

$$I^{\mathrm{OS}}_{mn}(\mu) \le I^{\mathrm{RSS}}_{mn}(\mu), \quad I^{\mathrm{OS}}_{mn}(\sigma) \le I^{\mathrm{RSS}}_{mn}(\sigma).$$

- (Chen, 2000a): Estimation of quantiles using RSS.
- (Chen, Bai and Sinha, 2003): A book length account of RSS and related developments.

3. Ordered Ranked Set Sample

Let $N = mn$, and

$$\boldsymbol{X}_{\text{ORSS}} = \left\{X_{1:N}^{\text{ORSS}} \le X_{2:N}^{\text{ORSS}} \le \cdots \le X_{N:N}^{\text{ORSS}}\right\}$$

be the order statistics obtained from $\boldsymbol{X}_{\text{RSS}}$.

Theorem 1: The pdf of $X_{r:N}^{\text{ORSS}}$ is

$$f_{r:N}^{\text{ORSS}}(x) \;=\; \sum_{P} \sum_{k_1=j_1}^{n} \cdots \sum_{k_{r-1}=j_{r-1}}^{n} \sum_{k_{r+1}=0}^{j_{r+1}-1} \cdots \sum_{k_N=0}^{j_N-1} W_r^* f_{r':nN}(x),$$

where $\sum_P$ denotes the summation over all $N!$ permutations $(i_1, \cdots, i_N)$ of $(1, \cdots, N)$,

$$\begin{aligned}
j_a &= \begin{cases} [i_a/m] & \text{if } i_a/m = [i_a/m]\,, \\ [i_a/m]+1 & \text{if } i_a/m > [i_a/m]\,, \end{cases} \\
W_r^* &= \frac{(r'-1)!(nN-r')!}{(r-1)!(N-r)!(nN)!} \left[\prod_{\substack{a=1\\a\neq r}}^{N} \binom{n}{k_a}\right] \left[j_r \binom{n}{j_r}\right], \\
r' &= j_r + \sum_{\substack{a=1\\a\neq r}}^{N} k_a.
\end{aligned}$$

Theorem 2: The joint pdf of $X_{r:N}^{\text{ORSS}}$ and $X_{s:N}^{\text{ORSS}}$ $(1 \leq r < s \leq N)$ is

$$
\begin{aligned}
f_{r,s:N}^{\text{ORSS}}(x,y) = & \sum_{P} \sum_{k_1=j_1}^{n} \cdots \sum_{k_{s-1}=j_{s-1}}^{n} \sum_{k_s=0}^{j_s-1} \cdots \sum_{k_n=0}^{j_n-1} \sum_{l_1=0}^{n-k_1} \cdots \\
& \times \sum_{l_{r-1}=0}^{n-k_{r-1}} \sum_{l_{r+1}=k_{r+1}+1-j_{r+1}}^{k_{r+1}} \cdots \sum_{l_{s-1}=k_{s-1}+1-j_{s-1}}^{k_{s-1}-1} \sum_{l_{s+1}=0}^{k_{s+1}} \cdots \\
& \times \sum_{l_N=0}^{k_N} W_{r,s}^{*} f_{\tilde{r},\tilde{s}:nN}(x,y), \qquad\qquad x < y,
\end{aligned}
$$

where

$$
\begin{aligned}
W_{r,s}^{*} &= W_{j,k,l} \frac{(\tilde{r}-1)!(\tilde{s}-\tilde{r}-1)!(nN-\tilde{s})!}{(r-1)!(s-r-1)!(N-s)!(nN)!}, \\
W_{j,k,l} &= \left\{ \prod_{a=1}^{r-1} \binom{n}{k_a} \binom{n-k_a}{l_a} \right\} \left\{ j_r \binom{n}{j_r} \binom{n-j_r}{k_r-j_r} \right\} \\
& \times \left\{ \prod_{a=r+1}^{s-1} \binom{n}{k_a} \binom{k_a}{l_a} \right\} \left\{ j_s \binom{n}{j_s} \binom{j_s-1}{k_s} \right\} \left\{ \prod_{a=s+1}^{N} \binom{n}{k_a} \binom{k_a}{l_a} \right\}, \\
\tilde{r} &= \sum_{\substack{a=1 \\ a \neq r,s}}^{N} k_a + j_r + j_s - \sum_{\substack{a=r+1 \\ a \neq s}}^{N} l_a - k_s - 1, \\
\tilde{s} &= \sum_{\substack{a=1 \\ a \neq r,s}}^{N} k_a + j_s + \sum_{a=1}^{r-1} l_a.
\end{aligned}
$$

Moments of ORSS

The fact that

$$\sum_{r=1}^{N} X_{r:N}^{\text{ORSS}} = \sum_{i=1}^{m}\sum_{r=1}^{n} X_{i(r)}$$

readily yields the following identities:

$$\sum_{r=1}^{N} \mu_{r:N}^{\text{ORSS}} = m\sum_{r=1}^{n} \mu_{r:n}^{\text{OS}} = N\mu_{1:1},$$
$$\sum_{r=1}^{N} \mu_{r,r:N}^{\text{ORSS}} = m\sum_{r=1}^{n} \mathrm{E}(X_{i(r)}^2) = N\mu_{1,1:1},$$
$$\sum_{r=1}^{N-1}\sum_{s=r+1}^{N} \mu_{r,s:N}^{\text{ORSS}} = m^2\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mu_{r:n}^{\text{OS}}\mu_{s:n}^{\text{OS}} + \frac{m(m-1)}{2}\sum_{r=1}^{n}(\mu_{r:n}^{\text{OS}})^2.$$

Theorem 3: Suppose $\boldsymbol{X}_{\text{ORSS}} = \{X_{1:N}^{\text{ORSS}} \leq \cdots \leq X_{N:N}^{\text{ORSS}}\}$ is an ORSS from a symmetric distribution, say, about 0. Then,

$$\begin{aligned} X_{r:N}^{\text{ORSS}} &\overset{\mathrm{d}}{=} -X_{N+1-r:N}^{\text{ORSS}}, \qquad 1 \leq r \leq N, \\ \left(X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right) &\overset{\mathrm{d}}{=} \left(-X_{N+1-s:N}^{\text{ORSS}}, -X_{N+1-r:N}^{\text{ORSS}}\right), \\ &\qquad\qquad 1 \leq r < s \leq N. \end{aligned}$$

Theorem 3 immediately yields the following symmetry relations for moments of ORSS:

$$\begin{aligned}
\mu_{r:N}^{\mathrm{ORSS}} &= -\mu_{N+1-r:N}^{\mathrm{ORSS}}, && 1 \le r \le N,\\
\sigma_{r,r:N}^{\mathrm{ORSS}} &= \sigma_{N+1-r,N+1-r:N}^{\mathrm{ORSS}}, && 1 \le r \le N,\\
\mu_{r,s:N}^{\mathrm{ORSS}} &= \mu_{N+1-s,N+1-r:N}^{\mathrm{ORSS}}, && 1 \le r < s \le N,\\
\sigma_{r,s:N}^{\mathrm{ORSS}} &= \sigma_{N+1-s,N+1-r:N}^{\mathrm{ORSS}}, && 1 \le r < s \le N.
\end{aligned}$$

Remark 1: The mixture representations in Theorems 1 and 2 will enable us to compute the means, variances and covariances of ORSS as mixtures of the corresponding quantities of the usual OS. Theorem 3 will enable us to reduce the amount of computation considerably for symmetric distributions.

4. Best Linear Unbiased Estimator from ORSS (BLUE-ORSS)

Suppose $\boldsymbol{X}_{\text{ORSS}}$ is from a general location-scale distribution with location parameter θ_1 and scale parameter θ_2 (> 0).

Let $Z_{r:N}^{\text{ORSS}} = \left(X_{r:N}^{\text{ORSS}} - \theta_1\right)/\theta_2$ be the standardized ORSS. Let us denote

$$\mu_{r:N}^{\text{ORSS}} = \text{E}(Z_{r:N}^{\text{ORSS}}),$$
$$\sigma_{r,s:N}^{\text{ORSS}} = \text{Cov}(Z_{r:N}^{\text{ORSS}},\, Z_{s:N}^{\text{ORSS}}), 1 \le r \le s \le N.$$

Evidently,

$$\begin{aligned} \text{E}(X_{r:N}^{\text{ORSS}}) &= \theta_1 + \theta_2\, \mu_{r:N}^{\text{ORSS}}, \\ \text{Cov}(X_{r:N}^{\text{ORSS}},\, X_{s:N}^{\text{ORSS}}) &= \theta_2^2\, \sigma_{r,s:N}^{\text{ORSS}}. \end{aligned}$$

• It then easily follows that the BLUE-ORSS of $\boldsymbol{\theta} = [\theta_1, \theta_2]'$ is

$$\boldsymbol{\theta}^* = \left(\text{B}'\boldsymbol{\Sigma}^{-1}\text{B}\right)^{-1} \text{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_{\text{ORSS}},$$

and its variance-covariance matrix is

$$\text{Var}(\boldsymbol{\theta}^*) = \theta_2^2\left(\text{B}'\boldsymbol{\Sigma}^{-1}\text{B}\right)^{-1},$$

where

$$\mathrm{B} = [\mathbf{1}\ \ \boldsymbol{\mu}^{\mathrm{ORSS}}], \qquad \boldsymbol{\Sigma} = [\sigma_{r,s:N}^{\mathrm{ORSS}}]_{N\times N},$$

$$\mathbf{1} = [1, 1, \cdots, 1]', \ \boldsymbol{\mu}^{\mathrm{ORSS}} = \left[\mu_{1:N}^{\mathrm{ORSS}}, \cdots, \mu_{N:N}^{\mathrm{ORSS}}\right]'.$$

Theorem 4: Suppose $F\left(\frac{x-\theta_1}{\theta_2}\right)$ is symmetric about 0 and that $\left\{X_{1:N}^{\mathrm{ORSS}} \leq \cdots \leq X_{N:N}^{\mathrm{ORSS}}\right\}$ is the ORSS from $F\left(\frac{x-\theta_1}{\theta_2}\right)$. Then:

(1) The BLUE-ORSS of θ_1 is a symmetric function of ORSS, i.e. if it is

$$\theta_1^* = a_1 X_{1:N}^{\mathrm{ORSS}} + a_2 X_{2:N}^{\mathrm{ORSS}} + \cdots + a_N X_{N:N}^{\mathrm{ORSS}},$$

then $a_r = a_{N+1-r}$, $r = 1, 2, \cdots, N$;

(2) The BLUE-ORSS of θ_2 is a skew-symmetric function of ORSS, i.e. if it is

$$\theta_2^* = b_1 X_{1:N}^{\mathrm{ORSS}} + b_2 X_{2:N}^{\mathrm{ORSS}} + \cdots + b_N X_{N:N}^{\mathrm{ORSS}},$$

then $b_r = -b_{N+1-r}$, $r = 1, 2, \cdots, N$.

Comparisons between BLUE-ORSS, BLUE-RSS and BLUE-OS

Let (θ_1^*, θ_2^*), $(\hat{\theta}_1, \hat{\theta}_2)$ and $(\tilde{\theta}_1, \tilde{\theta}_2)$ denote the BLUE-ORSS, BLUE-RSS and BLUE-OS of (θ_1, θ_2), respectively. We then define:

$$
\begin{aligned}
RE^1 &= \frac{\mathrm{Var}(\hat{\theta}_1)}{\mathrm{Var}(\theta_1^*)}, \qquad RE^2 = \frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\theta_2^*)},\\
RE^3 &= \frac{\mathrm{Var}(\hat{\theta}_1)\mathrm{Var}(\hat{\theta}_2) - (\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2))^2}{\mathrm{Var}(\theta_1^*)\mathrm{Var}(\theta_2^*) - (\mathrm{Cov}(\theta_1^*, \theta_2^*))^2},\\
RE^4 &= \frac{\mathrm{Var}(\hat{\theta}_1) + \mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\theta_1^*) + \mathrm{Var}(\theta_2^*)},\\
RE^5 &= \frac{\mathrm{Var}(\tilde{\theta}_1)}{\mathrm{Var}(\theta_1^*)}, \qquad RE^6 = \frac{\mathrm{Var}(\tilde{\theta}_2)}{\mathrm{Var}(\theta_2^*)},\\
RE^7 &= \frac{\mathrm{Var}(\tilde{\theta}_1)\mathrm{Var}(\tilde{\theta}_2) - (\mathrm{Cov}(\tilde{\theta}_1, \tilde{\theta}_2))^2}{\mathrm{Var}(\theta_1^*)\mathrm{Var}(\theta_2^*) - (\mathrm{Cov}(\theta_1^*, \theta_2^*))^2},\\
RE^8 &= \frac{\mathrm{Var}(\tilde{\theta}_1) + \mathrm{Var}(\tilde{\theta}_2)}{\mathrm{Var}(\theta_1^*) + \mathrm{Var}(\theta_2^*)},\\
RE^9 &= \frac{\mathrm{Var}(\tilde{\theta}_1)}{\mathrm{Var}(\hat{\theta}_1)}, \qquad RE^{10} = \frac{\mathrm{Var}(\tilde{\theta}_2)}{\mathrm{Var}(\hat{\theta}_2)},\\
RE^{11} &= \frac{\mathrm{Var}(\tilde{\theta}_1)\mathrm{Var}(\tilde{\theta}_2) - (\mathrm{Cov}(\tilde{\theta}_1, \tilde{\theta}_2))^2}{\mathrm{Var}(\hat{\theta}_1))\mathrm{Var}(\hat{\theta}_2)) - (\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2))^2},\\
RE^{12} &= \frac{\mathrm{Var}(\tilde{\theta}_1) + \mathrm{Var}(\tilde{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1) + \mathrm{Var}(\hat{\theta}_2)}.
\end{aligned}
$$

Table 1. Variances, Covariances and Relative Efficiencies of BLUEs for the Two-parameter Exponential Distribution

$$f(x) = \frac{1}{\lambda} \exp\left(-\frac{x-a}{\lambda}\right),\ x > a,\ \lambda > 0$$

n	2	3	4	5	6	7	8	9	10
$\frac{\mathrm{Var}(a^*)}{\lambda^2}$	0.33078	0.08944	0.03943	0.02167	0.01352	0.00913	0.00656	0.00490	0.00380
$\frac{\mathrm{Var}(\lambda^*)}{\lambda^2}$	0.83673	0.34106	0.18977	0.12152	0.08457	0.06219	0.04759	0.03765	0.03053
$\frac{\mathrm{Cov}(a^*,\lambda^*)}{\lambda^2}$	-0.39626	-0.11685	-0.05407	-0.03064	-0.01951	-0.01339	-0.00970	-0.00732	-0.00572
$\frac{\mathrm{Var}(\hat{a})}{\lambda^2}$	0.87500	0.23301	0.09931	0.05274	0.03189	0.02101	0.01472	0.01079	0.00820
$\frac{\mathrm{Var}(\hat{\lambda})}{\lambda^2}$	1.50000	0.54029	0.28078	0.17202	0.11601	0.08340	0.06275	0.04888	0.03911
$\frac{\mathrm{Cov}(\hat{a},\hat{\lambda})}{\lambda^2}$	-1.00000	-0.28755	-0.12873	-0.07074	-0.04385	-0.02944	-0.02093	-0.01553	-0.01191
$\frac{\mathrm{Var}(\tilde{a})}{\lambda^2}$	0.50000	0.16667	0.08333	0.05000	0.03333	0.02381	0.01786	0.01389	0.01111
$\frac{\mathrm{Var}(\tilde{\lambda})}{\lambda^2}$	1.00000	0.50000	0.33333	0.25000	0.20000	0.16667	0.14286	0.12500	0.11111
$\frac{\mathrm{Cov}(\tilde{a},\tilde{\lambda})}{\lambda^2}$	-0.50000	-0.16667	-0.08333	-0.05000	-0.03333	-0.02381	-0.01786	-0.01389	-0.01111

Table 1 (Contd.)

n	2	3	4	5	6	7	8	9	10
RE^1	2.64524	2.60516	2.51829	2.43362	2.35968	2.30174	2.24498	2.20244	2.15978
RE^2	1.79268	1.58417	1.47957	1.41558	1.37185	1.34093	1.31863	1.29813	1.28082
RE^3	2.60947	2.56419	2.48074	2.40103	2.33177	2.27891	2.22884	2.18744	2.14677
RE^4	2.03423	1.79629	1.65827	1.56967	1.50797	1.46391	1.43080	1.40227	1.37802
RE^5	1.51157	1.86343	2.11324	2.30712	2.46633	2.60791	2.72354	2.83434	2.92695
RE^6	1.19512	1.46603	1.75647	2.05735	2.36498	2.67977	3.00179	3.32000	3.63924
RE^7	2.08757	3.29681	4.56902	5.90123	7.28909	8.75203	10.24474	11.79055	13.34928
RE^8	1.28478	1.54859	1.81785	2.09516	2.37895	2.67057	2.96810	3.26407	3.56047
RE^9	0.57143	0.71528	0.83916	0.94802	1.04520	1.13302	1.21317	1.28691	1.35521
RE^{10}	0.66667	0.92542	1.18715	1.45336	1.72393	1.99844	2.27644	2.55752	2.84133
RE^{11}	0.80000	1.28571	1.84180	2.45779	3.12598	3.84044	4.59644	5.39012	6.21829
RE^{12}	0.63158	0.86211	1.09623	1.33478	1.57758	1.82427	2.07443	2.32770	2.58376

Remark 2: ORSS is more efficient than RSS, and significantly more efficient than OS, while RSS is not as efficient as OS for small sample sizes ($n < 5$).

Table 2. Variances and Relative Efficiencies of BLUEs for the One-parameter Exponential Distribution $f(x) = \frac{1}{\lambda} \exp\left(-\frac{x}{\lambda}\right)$, $x > 0$, $\lambda > 0$

n	2	3	4	5	6	7	8	9	10
$\frac{\mathrm{Var}(\lambda^*)}{\lambda^2}$	0.36204	0.18841	0.11563	0.07819	0.05639	0.04257	0.03323	0.02671	0.02193
$\frac{\mathrm{Var}(\hat{\lambda})}{\lambda^2}$	0.35714	0.18544	0.11390	0.07714	0.05573	0.04215	0.03299	0.02653	0.02179
$\frac{\mathrm{Var}(\tilde{\lambda})}{\lambda^2}$	0.50000	0.33333	0.25000	0.20000	0.16667	0.14286	0.12500	0.11111	0.10000
$RE(\lambda^*, \hat{\lambda})$	0.98648	0.98427	0.98509	0.98661	0.98817	0.99008	0.99281	0.99319	0.99398
$RE(\lambda^*, \tilde{\lambda})$	1.38107	1.76922	2.16209	2.55783	2.95544	3.35595	3.76156	4.15990	4.56081
$RE(\hat{\lambda}, \tilde{\lambda})$	1.40000	1.79749	2.19482	2.59255	2.99080	3.38957	3.78880	4.18844	4.58844

Remark 3: ORSS is very slightly less efficient than RSS (with relative efficiency always more than 98%), but significantly more efficient than OS.

Table 3. Variances and Relative Efficiencies for the Normal Distribution $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$

n	2	3	4	5	6	7	8	9	10
$\frac{\mathrm{Var}(\mu^*)}{\sigma^2}$	0.34085	0.17198	0.10362	0.06924	0.04746	0.03719	0.02886	0.02311	0.01890
$\frac{\mathrm{Var}(\sigma^*)}{\sigma^2}$	0.49867	0.22085	0.13221	0.08996	0.05913	0.05046	0.04005	0.03259	0.02713
$\frac{\mathrm{Var}(\hat{\mu})}{\sigma^2}$	0.34085	0.17231	0.10399	0.06956	0.04979	0.03739	0.02910	0.02329	0.01906
$\frac{\mathrm{Var}(\hat{\sigma})}{\sigma^2}$	1.07080	0.39058	0.20836	0.13128	0.09093	0.06697	0.05151	0.04092	0.03333
$\frac{\mathrm{Var}(\tilde{\mu})}{\sigma^2}$	0.50000	0.33333	0.25000	0.20000	0.16667	0.14286	0.12500	0.11111	0.10000
$\frac{\mathrm{Var}(\tilde{\sigma})}{\sigma^2}$	0.57080	0.27548	0.18005	0.13332	0.10570	0.08750	0.07461	0.06502	0.05760
RE^1	1.00000	1.00187	1.00359	1.00470	1.04902	1.00537	1.00827	1.00787	1.00844
RE^2	2.14729	1.76851	1.57599	1.45926	1.53772	1.32726	1.28634	1.25538	1.22840
RE^3	2.14729	1.77182	1.58165	1.46612	1.61309	1.33439	1.29697	1.26527	1.23877
RE^4	1.68149	1.43287	1.32448	1.26157	1.32012	1.19069	1.16987	1.15270	1.13808
RE^5	1.46694	1.93816	2.41261	2.88861	3.51163	3.84158	4.33098	4.80816	5.29057
RE^6	1.14463	1.24735	1.36185	1.48196	1.78759	1.73404	1.86315	1.99465	2.12300
RE^7	1.67910	2.41757	3.28560	4.28081	6.27735	6.66146	8.06929	9.59061	11.23185
RE^8	1.27549	1.54979	1.82354	2.09373	2.55523	2.62824	2.89680	3.16185	3.42369
RE^9	1.46694	1.93454	2.40398	2.87509	3.34754	3.82108	4.29548	4.77059	5.24629
RE^{10}	0.53306	0.70531	0.86412	1.01556	1.16250	1.30648	1.44841	1.58888	1.72825
RE^{11}	0.78196	1.36446	2.07733	2.91982	3.89150	4.99215	6.22163	7.57990	9.06693
RE^{12}	0.75855	1.08159	1.37679	1.65962	1.93561	2.20733	2.47617	2.74299	3.00830

Remark 4: ORSS is more efficient than RSS, and significantly more efficient than OS.

Table 4. Variances and Relative Efficiencies of BLUEs for the Logistic Distribution $f(x) = \frac{1}{\beta}\frac{e^{-(x-\mu)/\beta}}{(1+e^{-(x-\mu)/\beta})^2}$, $-\infty < x < \infty$

n	2	3	4	5	6	7	8	9	10
$\frac{\mathrm{Var}(\mu^*)}{\beta^2}$	1.14493	0.54810	0.32108	0.21093	0.14918	0.11095	0.08600	0.06849	0.05586
$\frac{\mathrm{Var}(\beta^*)}{\beta^2}$	0.57587	0.27748	0.17078	0.11748	0.08633	0.06620	0.05259	0.04286	0.03561
$\frac{\mathrm{Var}(\hat{\mu})}{\beta^2}$	1.14493	0.56956	0.33781	0.22266	0.15747	0.11710	0.09042	0.07189	0.05850
$\frac{\mathrm{Var}(\hat{\beta})}{\beta^2}$	1.14493	0.45330	0.25214	0.16269	0.11433	0.08499	0.06577	0.05245	0.04283
$\frac{\mathrm{Var}(\tilde{\mu})}{\beta^2}$	1.64493	1.07247	0.79289	0.62824	0.51997	0.44343	0.38649	0.34247	0.30745
$\frac{\mathrm{Var}(\tilde{\beta})}{\beta^2}$	0.64493	0.33333	0.22539	0.17037	0.13697	0.11453	0.09841	0.08627	0.07679
RE^1	1.00000	1.03915	1.05209	1.05563	1.05554	1.05544	1.05142	1.04965	1.04729
RE^2	1.98818	1.63367	1.47638	1.38481	1.32429	1.28376	1.25047	1.22369	1.20260
RE^3	1.98818	1.69764	1.55329	1.46185	1.39784	1.35493	1.31477	1.28444	1.25947
RE^4	1.33070	1.23897	1.19941	1.17339	1.15406	1.14076	1.12696	1.11664	1.10776
RE^5	1.43671	1.95669	2.46942	2.97846	3.48547	3.99670	4.49412	5.00052	5.50378
RE^6	1.11993	1.20131	1.31977	1.45019	1.58650	1.72986	1.87108	2.01263	2.15619
RE^7	1.60901	2.35058	3.25907	4.31933	5.52970	6.91375	8.40886	10.06419	11.86719
RE^8	1.33070	1.70281	2.07025	2.43176	2.78937	3.14956	3.49872	3.85038	4.20044
RE^9	1.43671	1.88296	2.34715	2.82150	3.30208	3.78677	4.27433	4.76400	5.25525
RE^{10}	0.56329	0.73534	0.89392	1.04721	1.19800	1.34750	1.49630	1.64472	1.79294
RE^{11}	0.80929	1.38462	2.09817	2.95470	3.95589	5.10266	6.39568	7.83544	9.42233
RE^{12}	1.00000	1.37437	1.72605	2.07243	2.41701	2.76092	3.10457	3.44818	3.79183

Remark 5: ORSS is more efficient than RSS, and significantly more efficient than OS.

5. Nonparametric Confidence Intervals for Quantiles

Confidence Intervals for Quantiles

- The cdf $X_{r:N}^{\text{ORSS}}$ $(1 \le r \le N)$ is

$$\begin{aligned}
F_{r:N}^{\text{ORSS}}(x) &= \sum_{i=r}^{N} \sum_{S_i} \left\{ \prod_{l=1}^{i} F_{k_l:n}(x) \prod_{l=i+1}^{N} [1 - F_{k_l:n}(x)] \right\} \\
&= \sum_{i=r}^{N} \sum_{S_i} \left\{ \prod_{l=1}^{i} I_{F(x)}(k_l, n-k_l+1) \right. \\
&\qquad \left. \times \prod_{l=i+1}^{N} [1 - I_{F(x)}(k_l, n-k_l+1)] \right\},
\end{aligned}$$

where $\sum_{S_i}$ denotes the summation over all permutations $(j_1, \cdots, j_N)$ of $(1, \cdots, N)$ for which $j_1 < \cdots < j_i$ and $j_{i+1} < \cdots < j_N$, $I_p(a,b)$ is incomplete beta function, and

$$k_l = \begin{cases} [j_l/m] & \text{if } j_l/m = [j_l/m]\,, 1 \le l \le N, \\ [j_l/m] + 1 & \text{if } j_l/m > [j_l/m]\,, 1 \le l \le N. \end{cases}$$

• With $\xi_p = F^{-1}(p)$ as the p-th quantile $(0 < p < 1)$, we have

$$\begin{aligned}\Pr\left(X_{r:N}^{\text{ORSS}} \le \xi_p \le X_{s:N}^{\text{ORSS}}\right) &= \Pr\left\{U_{r:N}^{\text{ORSS}} \le p\right\} - \Pr\left\{U_{s:N}^{\text{ORSS}} \le p\right\} \\ &= \sum_{i=r}^{s-1} \sum_{S_i} \left\{ \prod_{l=1}^{i} I_p(k_l, n - k_l + 1) \right. \\ &\qquad \left. \times \prod_{l=i+1}^{N} \left[1 - I_p(k_l, n - k_l + 1)\right] \right\},\end{aligned}$$

where $U_{r:N}^{\text{ORSS}} = F\left(X_{r:N}^{\text{ORSS}}\right)$, $U_{s:N}^{\text{ORSS}} = F\left(X_{s:N}^{\text{ORSS}}\right)$.

• For constructing a confidence interval for ξ_p with confidence coefficient $\ge 1 - \alpha$:

(1) Choose r and s such that $s - r$ is as small as possible;

(2) For different (r, s) with same $s - r$, choose r and s such that $\mathrm{E}\left\{U_{s:N}^{\text{ORSS}} - U_{r:N}^{\text{ORSS}}\right\}$ is as small as possible.

• The upper confidence limit $X_{s_u}^{\text{ORSS}}$ for ξ_p:

$$s_u = \inf\left\{s : \Pr\left(\xi_p \le X_{s:N}^{\text{ORSS}}\right) \ge 1 - \alpha\right\}.$$

• The lower confidence limit $X_{s_l}^{\text{ORSS}}$ for ξ_p:

$$s_l = \sup\left\{s : \Pr\left(X_{s:N}^{\text{ORSS}} \le \xi_p\right) \ge 1 - \alpha\right\}.$$

Theorem 5: Suppose $0 < p < 1$, and ξ_p is the p-th quantile such that $F(\xi_p) = p$. Then:

(1) $\left[X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right]$ is a confidence interval for ξ_p with confidence coefficient $\geq 1 - \alpha$ if and only if $\left[X_{N-s+1:N}^{\text{ORSS}}, X_{N-r+1:N}^{\text{ORSS}}\right]$ is a confidence interval for ξ_{1-p} with confidence coefficient $\geq 1 - \alpha$, i.e.,

$$\begin{aligned} & \Pr\left(X_{r:N}^{\text{ORSS}} \leq \xi_p \leq X_{s:N}^{\text{ORSS}}\right) \geq 1 - \alpha \\ \Leftrightarrow \quad & \Pr\left(X_{N-s+1:N}^{\text{ORSS}} \leq \xi_{1-p} \leq X_{N-r+1:N}^{\text{ORSS}}\right) \geq 1 - \alpha; \end{aligned}$$

(2) $X_{s:N}^{\text{ORSS}}$ is an upper confidence limit for ξ_p with confidence coefficient $\geq 1 - \alpha$ if and only if $X_{N-s+1:N}^{\text{ORSS}}$ is a lower confidence limit for ξ_{1-p} with confidence coefficient $\geq 1 - \alpha$, i.e.,

$$\begin{aligned} & \Pr\left(\xi_p \leq X_{s:N}^{\text{ORSS}}\right) \geq 1 - \alpha \\ & \qquad \Leftrightarrow \Pr\left(X_{N-s+1:N}^{\text{ORSS}} \leq \xi_{1-p}\right) \geq 1 - \alpha. \end{aligned}$$

Table 5. 95% Confidence interval for the p-th quantile

n	$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
2									
3									
4				$[1,4]$	$[1,4]$	$[1,4]$			
5				$[1,4]$	$[1,5]$	$[2,5]$			
6			$[1,4]$	$[1,5]$	$[2,5]$	$[2,6]$	$[3,6]$		
7			$[1,4]$	$[1,5]^{**}$	$[2,6]$	$[3,7]^{**}$	$[4,7]$		
				$[2,6]$		$[2,6]$			
8			$[1,5]$	$[2,6]$	$[3,7]$	$[3,7]$	$[4,8]$		
9		$[1,4]$	$[1,5]$	$[2,6]$	$[3,7]$	$[4,8]$	$[5,9]$	$[6,9]$	
10		$[1,4]$	$[1,5]^{**}$	$[2,6]^{**}$	$[3,7]^{*}$	$[5,9]^{**}$	$[6,10]^{**}$	$[7,10]$	
			$[2,6]$	$[3,7]$	$[4,8]$	$[4,8]$	$[5,9]$		

$*$: the expected widths of $[3,7]$ and $[4,8]$ are the same.

$**$: that is chosen based on the minimum expected width.

Table 6. $100(1-\alpha)\%$ Upper confidence limit for the p-th quantile

	$p=0.1$		$p=0.2$		$p=0.3$		$p=0.4$		$p=0.5$		$p=0.6$		$p=0.7$		$p=0.8$		$p=0.9$	
n	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%
2	2	2	2	2	2	2	2											
3	2	2	2	2	3	3	3	3	3	3								
4	2	2	3	3	3	3	3	4	4	4	4	4						
5	2	2	3	3	3	4	4	4	4	5	5	5	5					
6	2	3	3	3	4	4	4	5	5	5	6	6	6	6				
7	2	3	3	4	4	4	5	5	6	6	6	6	7	7	7			
8	3	3	4	4	4	5	5	6	6	6	7	7	8	8	8			
9	3	3	4	4	5	5	6	6	7	7	7	8	8	9	9	9		
10	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10		

Table 7. $100(1-\alpha)\%$ Lower confidence limit for the p-th quantile

	$p=0.1$		$p=0.2$		$p=0.3$		$p=0.4$		$p=0.5$		$p=0.6$		$p=0.7$		$p=0.8$		$p=0.9$	
n	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%
2											1		1	1	1	1	1	1
3									1	1	1	1	1	1	2	2	2	2
4							1	1	1	1	2	1	2	2	2	2	3	3
5					1		1	1	2	1	2	2	3	2	3	3	4	4
6					1	1	1	1	2	2	3	2	3	3	4	4	5	4
7			1		1	1	2	2	2	2	3	3	4	4	5	4	6	5
8			1		1	1	2	2	3	3	4	3	5	4	5	5	6	6
9			1	1	2	1	3	2	3	3	4	4	5	5	6	6	7	7
10			1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

Comparison with Approximate Confidence Intervals for Quantiles

- Approximate confidence interval $[X_{l_1}^{\mathrm{ORSS}}, X_{l_2}^{\mathrm{ORSS}}]$ with confidence $1-\alpha$ and equal tail probabilities, is given by [see Chen (2000)]

$$l_1 \approx Np - Z_{1-\alpha/2}\sqrt{m\sum_{r=1}^{n} I_p(r, n-r+1)\,[1 - I_p(r, n-r+1)]},$$

$$l_2 \approx Np + Z_{1-\alpha/2}\sqrt{m\sum_{r=1}^{n} I_p(r, n-r+1)\,[1 - I_p(r, n-r+1)]},$$

and Z_a denotes the a-th quantile of $N(0,1)$.

- The approximate upper confidence limit $X_{L_u:N}^{\mathrm{ORSS}}$ and the approximate lower confidence limit $X_{L_l:N}^{\mathrm{ORSS}}$ can also be expressed as

$$L_l \approx Np - Z_{1-\alpha}\sqrt{m\sum_{r=1}^{n} I_p(r, n-r+1)\,[1 - I_p(r, n-r+1)]},$$

$$L_u \approx Np + Z_{1-\alpha}\sqrt{m\sum_{r=1}^{n} I_p(r, n-r+1)\,[1 - I_p(r, n-r+1)]}.$$

Table 8. 90% Approximate ORSS confidence interval for the p-th quantile, based on one and two cycles, with exact level of confidence

m	n	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
1	2	$[0,1]$	$[0,1]$	$[0,2]$	$[0,2]$	$[0,2]$	$[0,2]$	$[0,2]$	$[1,2]$	$[1,2]$
		80%	61%	95%	90%	81%	70%	55%	37%	20%
	3	$[0,1]$	$[0,2]$	$[0,2]$	$[0,2]$	$[0,3]$	$[1,3]$	$[1,3]$	$[1,3]$	$[2,3]$
		71%	95%	84%	69%	95%	85%	73%	54%	28%
	4	$[0,1]$	$[0,2]$	$[0,2]$	$[0,3]$	$[1,3]$	$[1,4]$	$[2,4]$	$[2,4]$	$[3,4]$
		62%	88%	68%	91%	75%	95%	83%	67%	36%
	5	$[0,1]$	$[0,2]$	$[0,3]$	$[1,3]$	$[1,4]$	$[2,4]$	$[2,5]$	$[3,5]$	$[4,5]$
		54%	79%	92%	74%	91%	73%	93%	76%	43%
	6	$[0,2]$	$[0,2]$	$[1,3]$	$[1,4]$	$[2,4]$	$[2,5]$	$[3,5]$	$[4,6]$	$[4,6]$
		94%	68%	80%	92%	71%	87%	64%	82%	54%
	7	$[0,2]$	$[0,3]$	$[1,3]$	$[1,4]$	$[2,5]$	$[3,6]$	$[4,6]$	$[4,7]$	$[5,7]$
		91%	94%	69%	81%	88%	93%	74%	90%	60%
	8	$[0,2]$	$[0,3]$	$[1,4]$	$[2,5]$	$[3,5]$	$[3,6]$	$[4,7]$	$[5,8]$	$[6,8]$
		87%	89%	90%	92%	69%	80%	86%	93%	66%
	9	$[0,2]$	$[0,3]$	$[1,4]$	$[2,5]$	$[3,6]$	$[4,7]$	$[5,8]$	$[6,9]$	$[7,9]$
		83%	82%	83%	85%	86%	89%	91%	95%	71%
	10	$[0,2]$	$[1,3]$	$[2,4]$	$[2,6]$	$[3,7]$	$[4,8]$	$[6,8]$	$[7,9]$	$[8,10]$
		78%	72%	69%	95%	95%	95%	68%	70%	75%
2	2		$[0,2]$	$[0,3]$	$[0,3]$	$[1,3]$	$[1,4]$	$[1,4]$	$[2,4]$	
			83%	95%	86%	70%	90%	80%	61%	
	3		$[0,3]$	$[0,3]$	$[1,4]$	$[1,5]$	$[2,5]$	$[3,6]$	$[3,6]$	
			94%	79%	87%	95%	83%	90%	79%	
	4	$[0,2]$	$[0,3]$	$[1,4]$	$[1,5]$	$[2,6]$	$[3,7]$	$[4,7]$	$[5,8]$	$[6,8]$
		83%	85%	85%	90%	93%	95%	81%	87%	60%
	5	$[0,2]$	$[0,4]$	$[1,5]$	$[2,6]$	$[3,7]$	$[4,8]$	$[5,9]$	$[6,10]$	$[6,10]$
		75%	94%	92%	91%	90%	91%	92%	94%	71%

Remark 6: The exact level of confidence, in most situations, for the approximate confidence interval based on one-cycle RSS, is considerably lower than 90% even for large n (particularly when p is away from 0.5). However, the confidence level becomes close to 90% when the number of cycles is more than one in the RSS.

Table 9. $100(1-\alpha)\%$ Approximate upper confidence limit for the p-th quantile, with exact level of confidence

	$p=0.1$		$p=0.2$		$p=0.3$		$p=0.4$		$p=0.5$		$p=0.6$		$p=0.7$		$p=0.8$		$p=0.9$	
n	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%	90%	95%
2	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2
	80%		**61%**		**45%**	**95%**	**90%**		**81%**		**70%**		**55%**		**39%**		**20%**	
3	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3
	71%		**46%**	**95%**	**84%**		**69%**		**50%**	**95%**	**87%**		**74%**		**54%**		**29%**	
4	1	1	2	2	2	2	3	3	3	3	3	4	4	4	4	4	4	4
	62%		**88%**		**68%**		**91%**		**76%**		**55%**	**95%**	**86%**		**67%**		**38%**	
5	1	1	2	2	2	3	3	3	4	4	4	4	4	5	5	5	5	5
	54%		**79%**		**51%**	**92%**	**75%**		**91%**		**75%**		**49%**	**93%**	**77%**		**46%**	
6	1	2	2	2	3	3	3	4	4	4	5	5	5	5	6	6	6	6
	46%	**94%**	**68%**		**83%**		**55%**	**92%**	**74%**		**88%**		**65%**		**85%**		**54%**	
7	2	2	2	3	3	3	4	4	5	5	5	6	6	6	7	7	7	7
	91%		**57%**	**94%**	**70%**		**81%**		**89%**		**64%**	**95%**	**78%**		**90%**		**60%**	
8	2	2	3	3	3	4	4	5	5	5	6	6	7	7	7	8	8	8
	87%		**89%**		**55%**	**91%**	**64%**	**94%**	**72%**		**80%**		**87%**		**54%**	**94%**	**67%**	
9	2	2	3	3	4	4	5	5	6	6	7	7	7	8	8	9	9	9
	83%		**82%**		**83%**		**85%**		**87%**		**90%**		**59%**	**93%**	**64%**	**96%**	**72%**	
10	2	2	3	3	4	4	5	6	6	7	7	8	8	8	9	9	10	10
	78%		**74%**		**72%**		**71%**	**95%**	**71%**	**95%**	**71%**	**96%**	**72%**		**73%**		**76%**	

Remark 7: The exact level of confidence, in most situations, for the approximate upper confidence limit, is considerably lower than 90% and 95%.

Table 10. $100(1-\alpha)\%$ Approximate lower confidence limit for the p-th quantile, with exact level of confidence

<table>
<tr><th></th><th colspan="2">$p = 0.1$</th><th colspan="2">$p = 0.2$</th><th colspan="2">$p = 0.3$</th><th colspan="2">$p = 0.4$</th><th colspan="2">$p = 0.5$</th><th colspan="2">$p = 0.6$</th><th colspan="2">$p = 0.7$</th><th colspan="2">$p = 0.8$</th><th colspan="2">$p = 0.9$</th></tr>
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Remark 8: The exact level of confidence, in most situations, for the approximate lower confidence limit, is considerably more than 90% and 95%.

Comparison with Intervals Based on the Usual Order Statistics

• Let $\tilde{I}_p$ and $\tilde{L}_p$, respectively, be the confidence interval for the p-th quantile based on OS and the expected length of this interval.

• Let I_p^* and L_p^*, respectively, be the confidence interval for the p-th quantile based on ORSS and the expected length of this interval.

• Then, the percentage reduction in L_p^* compared to $\tilde{L}_p$ can be defined as

$$PR = \frac{\tilde{L}_p - L_p^*}{\tilde{L}_p}.$$

Table 11. 90% OS confidence interval $\tilde{I}_p$ for the p-th quantile, its expected length $\tilde{L}_p$, the expected length L_p^* of the ORSS confidence interval, and the percentage reduction in L_p^* compared to $\tilde{L}_p$

n		$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
5	$\tilde{I}_p$				$[1, 5]$	$[1, 5]$	$[1, 5]$			
	$\tilde{L}_p$				0.6667	0.6667	0.6667			
	L_p^*				0.5574	0.5574	0.5574			
	PR				16.39%	16.39%	16.39%			
6	$\tilde{I}_p$				$[1, 5]$	$[1, 6]$	$[2, 6]$			
	$\tilde{L}_p$				0.5714	0.7143	0.5714			
	L_p^*				0.4697	0.4771	0.4697			
	PR				17.80%	33.21%	17.80%			
7	$\tilde{I}_p$			$[1, 6]$	$[1, 6]$	$[1, 6]/[2, 7]$	$[2, 7]$	$[2, 7]$		
	$\tilde{L}_p$			0.6250	0.6250	0.6250	0.6250	0.6250		
	L_p^*			0.4049	0.4129	0.5495	0.4129	0.4049		
	PR			35.21%	33.94%	12.08%	33.94%	35.21%		
8	$\tilde{I}_p$			$[1, 6]$	$[1, 6]$	$[2, 7]$	$[3, 8]$	$[3, 8]$		
	$\tilde{L}_p$			0.5556	0.5556	0.5556	0.5556	0.5556		
	L_p^*			0.3556	0.3631	0.3646	0.3631	0.3556		
	PR			36.00%	34.65%	34.38%	34.65%	36.00%		
9	$\tilde{I}_p$			$[1, 6]$	$[2, 7]$	$[1, 7]/[3, 9]$	$[3, 8]$	$[4, 9]$		
	$\tilde{L}_p$			0.5000	0.5000	0.6000	0.5000	0.5000		
	L_p^*			0.3229	0.4315	0.4339	0.4315	0.3229		
	PR			35.42%	13.70%	27.68%	13.70%	35.42%		
10	$\tilde{I}_p$			$[1, 6]$	$[2, 7]$	$[2, 8]/[3, 9]$	$[4, 9]$	$[5, 10]$		
	$\tilde{L}_p$			0.4545	0.4545	0.5455	0.4545	0.4545		
	L_p^*			0.2918	0.2938	0.2944	0.2938	0.2918		
	PR			35.80%	35.37%	46.03%	35.37%	35.80%		

Remark 9: The percentage reduction in L_p^* (the expected width of the confidence interval based on ORSS) compared to $\tilde{L}_p$ (the expected width of the confidence interval based on OS) is quite significant, and this amount of reduction increases as n increases in general.

6. Nonparametric Tolerance Intervals

• A tolerance interval that covers at least a proportion γ of the population with tolerance level β is defined as

$$\Pr\left\{\int_{X_{r:N}^{\mathrm{ORSS}}}^{X_{s:N}^{\mathrm{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta. \tag{6}$$

• The LHS of Eq. (6) can be rewritten as

$$\Pr\left\{U_{s:N}^{\mathrm{ORSS}} - U_{r:N}^{\mathrm{ORSS}} \geq \gamma\right\} = 1 - F_{W_{rs}^{\mathrm{ORSS}}}(\gamma),$$

where $U_{j:N}^{\mathrm{ORSS}} = F\left(X_{j:N}^{\mathrm{ORSS}}\right)$, and $W_{rs}^{\mathrm{ORSS}} = U_{s:N}^{\mathrm{ORSS}} - U_{r:N}^{\mathrm{ORSS}}$.

• The cdf of W_{rs}^{ORSS} can be derived as

$$F_{W_{rs}^{\mathrm{ORSS}}}(w) = \sum_{P}\sum_{k_1=j_1}^{n}\cdots\sum_{k_{s-1}=j_{s-1}}^{n}\sum_{k_s=0}^{j_s-1}\cdots\sum_{k_n=0}^{j_n-1}\sum_{l_{r+1}=0}^{k_{r+1}-1}\cdots\sum_{l_{s-1}=0}^{k_{s-1}-1}$$
$$\times\sum_{b_{r+1}=0}^{l_{r+1}}\cdots\sum_{b_{s-1}=0}^{l_{s-1}} D_{k,j,l,b}^{*}(r,s) I_w(s'-r', N'-s'-r'+1),$$

• Now, we need to choose r and s making $s-r$ as small as possible and

$$\Pr\left\{\int_{X_{r:N}^{\mathrm{ORSS}}}^{X_{s:N}^{\mathrm{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} \geq \beta. \tag{7}$$

Theorem 6: Suppose $0 < \gamma, \beta < 1$, then:

(1) $\left[X_{r:N}^{\text{ORSS}}, X_{s:N}^{\text{ORSS}}\right]$ is a tolerance interval that covers γ proportion of the population with tolerance level β if and only if $\left[X_{N-s+1:N}^{\text{ORSS}}, X_{N-r+1:N}^{\text{ORSS}}\right]$ is a tolerance interval that covers γ proportion of the population with tolerance level β, i.e.,

$$\Pr\left\{\int_{X_{r:N}^{\text{ORSS}}}^{X_{s:N}^{\text{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta$$
$$\Leftrightarrow \Pr\left\{\int_{X_{N-s+1:N}^{\text{ORSS}}}^{X_{N-r+1:N}^{\text{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta;$$

(2) $[X_{r:N}^{\text{ORSS}}, \infty)$ is a one-sided tolerance interval that covers γ proportion of the population with tolerance level β if and only if $(-\infty, X_{N-r+1:N}^{\text{ORSS}}]$ is a one-sided tolerance interval that covers γ proportion of the population with tolerance level β, i.e.

$$\Pr\left\{\int_{X_{r:N}^{\text{ORSS}}}^{\infty} f(x)\mathrm{d}x \geq \gamma\right\} = \beta$$
$$\Leftrightarrow \Pr\left\{\int_{-\infty}^{X_{N-r+1:N}^{\text{ORSS}}} f(x)\mathrm{d}x \geq \gamma\right\} = \beta.$$

Table 12. 90% Two-sided tolerance interval that covers γ proportion of the population

n	$\gamma = 0.1$	$\gamma = 0.2$	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
2									
3	$[1, 3]$	$[1, 3]$							
4	$[1, 3]$	$[1, 3]$	$[1, 4]$	$[1, 4]$					
	$[2, 4]$	$[2, 4]$							
5	$[1, 3]^*$	$[1, 4]$	$[1, 4]$	$[1, 5]$	$[1, 5]$				
	$[3, 5]^*$	$[2, 5]$	$[2, 5]$						
	$[2, 4]$								
6	$[1, 3]^*$	$[1, 4]^*$	$[1, 5]$	$[1, 5]$	$[1, 6]$	$[1, 6]$			
	$[4, 6]^*$	$[3, 6]^*$	$[2, 6]$	$[2, 6]$					
	$[2, 4]$	$[2, 5]$							
	$[3, 5]$								
7	$[1, 3]^*$	$[1, 4]^*$	$[1, 5]^*$	$[1, 6]$	$[1, 6]$	$[1, 7]$			
	$[5, 7]^*$	$[4, 7]^*$	$[3, 7]^*$	$[2, 7]$	$[2, 7]$				
	$[2, 4]$	$[2, 6]$	$[2, 5]$						
	$[4, 6]$	$[3, 6]$							
	$[3, 5]$								

$*$: intervals with the shortest expected width.

Table 12. (Contd.)

n	$\gamma = 0.1$	$\gamma = 0.2$	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
8	$[1,4]^*$	$[1,4]$	$[1,5]^*$	$[1,6]$	$[1,7]$	$[1,8]$	$[1,8]$		
	$[5,8]^*$	$[5,8]$	$[4,8]^*$	$[3,8]$	$[2,8]$				
	$[2,5]$		$[2,6]$	$[2,7]$					
	$[4,7]$		$[3,7]$						
	$[3,6]$								
9	$[1,4]^*$	$[1,5]^*$	$[1,6]^*$	$[1,7]^*$	$[1,7]^*$	$[1,8]$	$[1,9]$		
	$[6,9]^*$	$[5,9]^*$	$[4,9]^*$	$[3,9]^*$	$[3,9]^*$	$[2,9]$			
	$[2,5]$	$[2,6]$	$[2,7]$	$[2,8]$	$[2,8]$				
	$[5,8]$	$[4,8]$	$[3,8]$						
	$[3,6]$	$[3,7]$							
	$[4,7]$								
10	$[1,4]^*$	$[1,5]^*$	$[1,6]^*$	$[1,7]^*$	$[1,8]^*$	$[1,9]$	$[1,10]$		
	$[7,10]^*$	$[6,10]^*$	$[5,10]^*$	$[4,10]^*$	$[3,10]^*$	$[2,10]$			
	$[2,5]$	$[2,6]$	$[2,7]$	$[2,8]$	$[2,9]$				
	$[6,9]$	$[5,9]$	$[4,9]$	$[3,9]$					
	$[3,6]$	$[3,7]$	$[3,8]$						
	$[5,8]$	$[4,8]$							
	$[4,7]$								

$*$: intervals with the shortest expected width.

7. Fisher Information Comparison

Let the *one-cycle* RSS

$$\boldsymbol{X}_{\text{RSS}} = \left\{ X_{1(1)}, \cdots, X_{1(n)} \right\}$$

be from the population with pdf $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ and cdf $F\left(\frac{x-\mu}{\sigma}\right)$.

Fisher Information and MLE-RSS

- The score equations based on RSS are

$$\sum_{r=1}^{n} \frac{f'(z_{(r)})}{f(z_{(r)})} + \sum_{r=1}^{n} (r-1) \frac{f(z_{(r)})}{F(z_{(r)})} - \sum_{r=1}^{n} (m-r) \frac{f(z_{(r)})}{1-F(z_{(r)})} = 0,$$

$$n + \sum_{r=1}^{n} \frac{z_{(r)} f'(z_{(r)})}{f(z_{(r)})} + \sum_{r=1}^{n} (r-1) \frac{z_{(r)} f(z_{(r)})}{F(z_{(r)})} - \sum_{r=1}^{n} (m-r) \frac{z_{(r)} f(z_{(r)})}{1-F(z_{(r)})} = 0,$$

where $z_{(r)} = \frac{x_{(r)}-\mu}{\sigma}$.

- The Fisher information in RSS is

$$
\begin{aligned}
\hat{I}(\mu) &= \frac{n}{\sigma^2}\mathrm{E}\left[\frac{f'(Z)}{f(Z)}\right]^2 + \frac{n(n-1)}{\sigma^2}\mathrm{E}\left[\frac{f^2(Z)}{F(Z)[1-F(Z)]}\right], \\
\hat{I}(\mu,\sigma) &= \frac{n}{\sigma^2}\mathrm{E}\left[Z\left(\frac{f'(Z)}{f(Z)}\right)^2\right] + \frac{n(n-1)}{\sigma^2}\mathrm{E}\left[\frac{Zf^2(Z)}{F(Z)[1-F(Z)]}\right], \\
\hat{I}(\sigma) &= \frac{n}{\sigma^2}\mathrm{E}\left[\left(\frac{Zf'(Z)}{f(Z)}\right)^2 - 1\right] \\
&\qquad + \frac{n(n-1)}{\sigma^2}\mathrm{E}\left[\frac{[Zf(Z)]^2}{F(Z)[1-F(Z)]}\right],
\end{aligned}
$$

where Z is the corresponding standardized variable.

- For symmetric distribution, $\hat{I}(\mu,\sigma) = 0$.

- $\tilde{I}(\mu) \le \hat{I}(\mu), \qquad \tilde{I}(\sigma) \le \hat{I}(\sigma)$.

Fisher Information and MLE-ORSS

- The log-likelihood function is

$$
\begin{aligned}
l^* &= D - n \ln \sigma + \sum_{k=1}^{n} \ln f(z_k) \\
&\quad + \ln \left\{ \sum_{P} \prod_{k=1}^{n} \left[(F(z_k))^{i_k - 1} (1 - F(z_k))^{n - i_k} \right] \right\}.
\end{aligned}
$$

- The score equations based on ORSS are

$$
\begin{cases}
\sum_{k=1}^{n} \frac{f'(z_k)}{f(z_k)} + \frac{a_1}{b} = 0, \\
n + \sum_{k=1}^{n} \left(z_k \frac{f'(z_k)}{f(z_k)} \right) + \frac{a_2}{b} = 0,
\end{cases}
$$

where

$$
\begin{aligned}
a_1 &= \sum_{P} \left\{ \prod_{s=1}^{n} \left[(F(z_s))^{i_s - 1} (1 - F(z_s))^{n - i_s} \right] \right. \\
&\qquad \left. \times \sum_{k=1}^{n} \left[\left(\frac{i_k - 1}{F(z_k)} - \frac{n - i_k}{1 - F(z_k)} \right) \cdot f(z_k) \right] \right\}, \\
a_2 &= \sum_{P} \left\{ \prod_{s=1}^{n} \left[(F(z_s))^{i_s - 1} (1 - F(z_s))^{n - i_s} \right] \right. \\
&\qquad \left. \times \sum_{k=1}^{n} \left[\left(\frac{i_k - 1}{F(z_k)} - \frac{n - i_k}{1 - F(z_k)} \right) z_k f(z_k) \right] \right\}, \\
b &= \sum_{P} \prod_{s=1}^{n} \left[(F(z_s))^{i_s - 1} (1 - F(z_s))^{n - i_s} \right].
\end{aligned}
$$

- The Fisher information based on ORSS is

$$
\begin{aligned}
I^*(\mu) &= \frac{n}{\sigma^2}\mathrm{E}\left[\frac{f'(Z)}{f(Z)}\right]^2 + \frac{1}{\sigma^2}\mathrm{E}\left[\frac{a_3}{b}\right] + \frac{1}{\sigma^2}\mathrm{E}\left[\left(\frac{a_1}{b}\right)^2 - \frac{a_4}{b}\right], \\
I^*(\mu,\sigma) &= \frac{n}{\sigma^2}\mathrm{E}\left[Z\left(\frac{f'(Z)}{f(Z)}\right)^2\right] + \frac{1}{\sigma^2}\mathrm{E}\left[\frac{a_5}{b}\right] + \frac{1}{\sigma^2}\mathrm{E}\left[\frac{a_1 a_2}{b} - \frac{a_6}{b}\right], \\
I^*(\sigma) &= \frac{n}{\sigma^2}\mathrm{E}\left[\left(\frac{Zf'(Z)}{f(Z)}\right)^2 - 1\right] + \frac{1}{\sigma^2}\mathrm{E}\left[\frac{a_7}{b}\right] + \frac{1}{\sigma^2}\mathrm{E}\left[\left(\frac{a_2}{b}\right)^2 - \frac{a_8}{b}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
a_3 &= \sum_P \left\{ \prod_{s=1}^n \left[(F(z_s))^{i_s-1} (1-F(z_s))^{n-i_s} \right] \right. \\
&\quad \times \left[\sum_{k=1}^n \left(\left(\frac{i_k-1}{(F(z_k))^2} + \frac{n-i_k}{(1-F(z_k))^2} \right) f^2(z_k) \right) \right. \\
&\quad \left. \left. - \sum_{k=1}^n \left(\left(\frac{i_k-1}{F(z_k)} - \frac{n-i_k}{1-F(z_k)} \right) f'(z_k) \right) \right] \right\}, \\
a_4 &= \sum_P \left\{ \prod_{s=1}^n \left[(F(z_s))^{i_s-1} (1-F(z_s))^{n-i_s} \right] \right. \\
&\quad \left. \times \left[\sum_{k=1}^n \left(\frac{i_k-1}{F(z_k)} - \frac{n-i_k}{1-F(z_k)} \right) f(z_k) \right]^2 \right\},
\end{aligned}
$$

$$
\begin{aligned}
a_5 \;=\; & \sum_P \left\{ \prod_{s=1}^{n} \left[(F(z_s))^{i_s-1}(1-F(z_s))^{n-i_s} \right] \right. \\
& \times \left[\sum_{k=1}^{n} \left(\left(\frac{i_k-1}{(F(z_k))^2} + \frac{n-i_k}{(1-F(z_k))^2} \right) z_k f^2(z_k) \right) \right. \\
& \left. \left. - \sum_{k=1}^{n} \left(\left(\frac{i_k-1}{F(z_k)} - \frac{n-i_k}{1-F(z_k)} \right) z_k f'(z_k) \right) \right] \right\}, \\
a_6 \;=\; & \sum_P \left\{ \prod_{s=1}^{n} \left[(F(z_s))^{i_s-1}(1-F(z_s))^{n-i_s} \right] \right. \\
& \times \sum_{k=1}^{n} \left(\left(\frac{i_k-1}{F(z_k)} - \frac{n-i_k}{1-F(z_k)} \right) f(z_k) \right) \\
& \left. \times \sum_{k=1}^{n} \left(\left(\frac{i_k-1}{F(z_k)} - \frac{n-i_k}{1-F(z_k)} \right) z_k f(z_k) \right) \right\}, \\
a_7 \;=\; & \sum_P \left\{ \prod_{s=1}^{n} \left[(F(z_s))^{i_s-1}(1-F(z_s))^{n-i_s} \right] \right. \\
& \times \left[\sum_{k=1}^{n} \left(\left(\frac{i_k-1}{(F(z_k))^2} + \frac{n-i_k}{(1-F(z_k))^2} \right) z_k^2 f^2(z_k) \right) \right. \\
& \left. \left. - \sum_{k=1}^{n} \left(\left(\frac{i_k-1}{F(z_k)} - \frac{n-i_k}{1-F(z_k)} \right) z_k^2 f'(z_k) \right) \right] \right\}, \\
a_8 \;=\; & \sum_P \left\{ \prod_{s=1}^{n} \left[(F(z_s))^{i_s-1}(1-F(z_s))^{n-i_s} \right] \right. \\
& \left. \times \left[\sum_{k=1}^{n} \left(\left(\frac{i_k-1}{F(z_k)} - \frac{n-i_k}{1-F(z_k)} \right) f(z_k) \right) \right]^2 \right\}.
\end{aligned}
$$

Examples

$$RE(\mu_{\text{MLE}}) = \frac{\text{MSE}(\hat{\mu}_{\text{MLE}})}{\text{MSE}(\mu^*_{\text{MLE}})}, \quad RE(\sigma_{\text{MLE}}) = \frac{\text{MSE}(\hat{\sigma}_{\text{MLE}})}{\text{MSE}(\sigma^*_{\text{MLE}})}$$

• <u>Logistic Distribution</u>

$$\begin{aligned} I^*(\mu) &= \frac{n(n+1)}{6\sigma^2} = \hat{I}(\mu), \\ I^*(\mu,\sigma) &= 0 = \hat{I}(\mu,\sigma), \\ \hat{I}(\sigma) &= \frac{n}{\sigma^2}\left\{\text{E}[Z^2(1-2F(Z))]^2 - 1\right\} \\ &\qquad +\frac{n(n-1)}{\sigma^2}\text{E}[Z^2F(Z)(1-F(Z))], \\ I^*(\sigma) &= \hat{I}(\sigma) + \frac{1}{\sigma^2}\text{E}\left\{\left(\frac{a_2}{b}\right)^2 - \frac{a_8}{b}\right\} \le \hat{I}(\sigma). \end{aligned}$$

Table 13. Comparison of Fisher Information of σ between RSS and ORSS from Logistic(μ,σ^2) distribution

n	2	3	4	5	6	7	8	9	10
$\sigma^2\hat{I}(\sigma)$	3.29440	5.58191	8.29451	11.45899	15.03105	19.08059	23.50773	28.33388	33.67061
$\sigma^2 I^*(\sigma)$	3.04032	5.01068	7.36663	10.16772	13.35357	17.01754	21.04628	25.54500	30.92350

Table 14. Bias and MSE of MLEs based on RSS from Logistic(μ, σ^2) distribution

n	$\frac{\text{Bias}(\hat{\mu}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\hat{\mu}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{Bias}(\hat{\sigma}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\hat{\sigma}_{\text{MLE}})}{\sigma^2}$
2	-0.00123	1.14555	-2.12526	5.97207
3	0.00103	0.54891	-0.16112	0.19406
4	-0.00017	0.31869	-0.10695	0.12830
5	0.00110	0.20932	-0.07887	0.09209
6	0.00002	0.14808	-0.06070	0.06933
7	0.00019	0.11038	-0.04831	0.05477
8	0.00015	0.08548	-0.03993	0.04421
9	-0.00136	0.06875	-0.03254	0.03603
10	0.00169	0.05538	-0.02809	0.03043

Table 15. Bias and MSE of MLEs based on ORSS from Logistic(μ, σ^2) distribution

n	$\frac{\text{Bias}(\mu^*_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\mu^*_{\text{MLE}})}{\sigma^2}$	RE(μ_{MLE})	$\frac{\text{Bias}(\sigma^*_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\sigma^*_{\text{MLE}})}{\sigma^2}$	RE(σ_{MLE})
2	-0.00123	1.14555	1.00000	-1.67503	3.06856	1.94621
3	0.00108	0.54980	0.99838	-0.18265	0.21666	0.89570
4	-0.00018	0.31940	0.99780	-0.12261	0.14504	0.88461
5	0.00108	0.20972	0.99812	-0.09068	0.10460	0.88042
6	0.00000	0.14836	0.99809	-0.07018	0.07855	0.88265
7	0.00020	0.11055	0.99853	-0.05562	0.06178	0.88653
8	0.00010	0.08560	0.99866	-0.04604	0.04962	0.89091
9	-0.00137	0.06882	0.99904	-0.03726	0.04032	0.89374
10	0.00178	0.05545	0.99868	-0.03164	0.03368	0.90347

• Normal Distribution

Table 16. Comparison of Fisher Information between RSS and ORSS from Normal(μ, σ^2) distribution

n	$\sigma^2 \hat{I}(\mu)$	$\sigma^2 \hat{I}(\mu, \sigma)$	$\sigma^2 \hat{I}(\sigma)$	$\sigma^2 I^*(\mu)$	$\sigma^2 I^*(\mu, \sigma)$	$\sigma^2 I^*(\sigma)$
2	2.96123	0.00039	4.53074	2.95681	0.00044	4.20724
3	5.88288	-0.00561	7.62492	5.86996	-0.00493	6.88766
4	9.76617	-0.00149	11.23962	9.74324	-0.00198	10.04306
5	14.61027	-0.00437	15.40385	14.57601	-0.00446	13.71449
6	20.41955	0.00434	20.07481	20.37240	0.00335	17.86093
7	27.18308	0.00227	25.29647	27.12498	0.00216	22.56148
8	34.90793	-0.06191	31.15110	34.83470	-0.06409	27.85410
9	43.59402	-0.07638	37.63034	43.49940	-0.07851	33.69233
10	53.24776	-0.02337	44.40295	53.14073	-0.04098	39.92216

Table 17. Bias and MSE of MLEs based on RSS from Normal(μ, σ^2) distribution

n	$\frac{\text{Bias}(\hat{\mu}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\hat{\mu}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{Bias}(\hat{\sigma}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\hat{\sigma}_{\text{MLE}})}{\sigma^2}$
3	—	—	—	—
4	0.00141	0.10676	-0.12910	0.10160
5	-0.00396	0.06799	-0.09069	0.07030
6	-0.00262	0.04912	-0.07149	0.05349
7	-0.00073	0.03804	-0.05576	0.04285
8	-0.00170	0.02834	-0.04439	0.03381
9	-0.00012	0.02350	-0.04240	0.02838
10	-0.00424	0.01931	-0.03675	0.02501

Table 18. Bias and MSE of MLEs based on ORSS from Normal(μ, σ^2) distribution

n	$\frac{\text{Bias}(\mu^*_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\mu^*_{\text{MLE}})}{\sigma^2}$	RE(μ_{MLE})	$\frac{\text{Bias}(\sigma^*_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\sigma^*_{\text{MLE}})}{\sigma^2}$	RE(σ_{MLE})
3	-0.00054	0.17397	—	-0.22907	0.18255	—
4	0.00125	0.10706	0.99721	-0.14710	0.11498	0.88360
5	-0.00400	0.06814	0.99772	-0.10249	0.08032	0.87518
6	-0.00243	0.04935	0.99527	-0.08175	0.06104	0.87636
7	-0.00076	0.03815	0.99706	-0.06212	0.04820	0.88904
8	-0.00161	0.02840	0.99786	-0.04924	0.03838	0.88108
9	-0.00005	0.02355	0.99812	-0.04589	0.03152	0.90040
10	-0.00418	0.01936	0.99729	-0.04068	0.02745	0.91093

Remark 10: For logistic as well as normal distributions, the efficiency of the MLE-ORSS of μ is almost the same as the MLE-RSS of μ, while the relative efficiency of the MLE-ORSS of σ with respect to the MLE-RSS is around 90%.

Interestingly, for small n (say, 3), the Newton-Raphson method often does not converge based on RSS, but it does converge based on ORSS.

• One-parameter Exponential Distribution

Table 19. Comparison of Fisher Information between RSS and ORSS from Exponential(σ) distribution

n	2	3	4	5	6	7	8	9	10
$\sigma^2\hat{I}(\sigma)$	2.79955	5.41277	8.84513	13.08436	18.10901	23.95813	30.59065	38.15936	46.23967
$\sigma^2 I^*(\sigma)$	2.76277	5.32411	8.70176	12.88173	17.84192	23.61300	30.18296	37.64389	45.66347

Table 20. Comparison of bias and MSE of MLEs between RSS and ORSS from Exponential(σ) distribution

n	$\frac{\text{Bias}(\hat{\sigma}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{MSE}(\hat{\sigma}_{\text{MLE}})}{\sigma^2}$	$\frac{\text{Bias}(\sigma^*)}{\sigma^2}$	$\frac{\text{MSE}(\sigma^*_{\text{MLE}})}{\sigma^2}$	RE(σ)
2	0.01834	0.36682	0.02226	0.37486	0.97856
3	0.01415	0.19200	0.01685	0.19625	0.97837
4	0.01022	0.11638	0.01198	0.11881	0.97960
5	0.00731	0.07811	0.00851	0.07955	0.98190
6	0.00670	0.05614	0.00754	0.05712	0.98280
7	0.00540	0.04242	0.00604	0.04307	0.98494
8	0.00045	0.03290	0.00088	0.03335	0.98629
9	0.00428	0.02649	0.00447	0.02694	0.98345
10	0.00403	0.02133	0.00442	0.02158	0.98840

Remark 11: For the one-parameter exponential distribution, the efficiency of the MLE-ORSS of σ is almost the same as the MLE-RSS of σ (being more than 98%).

8. Tukey's Linear Sensitivity Measure Comparison

Let $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_n)'$ be a vector with distribution function $F_{\boldsymbol{\theta}}$, where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)'$.
Let $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)'$ and $\boldsymbol{\Sigma}$ denote, respectively, the mean vector and the variance-covariance matrix of $\boldsymbol{Y}$.
Assume that the partial derivatives

$$d_{ij} = \frac{\partial \mu_i}{\partial \theta_j} \quad i = 1, \cdots, n;\ j = 1, \cdots, k$$

all exist, and let $D = ((d_{ij}))$.

• The linear sensitivity measure contained in $\boldsymbol{Y}$ about $\boldsymbol{\theta}$ is

$$\boldsymbol{S}(\boldsymbol{Y}; \boldsymbol{\theta}) = \sup_A D'A'(A\Sigma A')^{-1}AD,$$

where *sup* is taken over those A for which $A\Sigma A'$ is non-singular.

• Then, the linear sensitivity measure becomes

$$\boldsymbol{S}(\boldsymbol{Y}; \boldsymbol{\theta}) = D'\Sigma D;$$

[see Nagaraja (1994) and Chandrasekar and Balakrishnan (2002) for details].

- Let us define the relative efficiencies as

$$RE(\mu_{\text{BLUE}}) = \frac{\boldsymbol{S}^*(\mu)}{\widehat{\boldsymbol{S}}(\mu)}, \quad RE(\sigma_{\text{BLUE}}) = \frac{\boldsymbol{S}^*(\sigma)}{\widehat{\boldsymbol{S}}(\sigma)},$$

where

$$(\boldsymbol{S}^*(\mu), \boldsymbol{S}^*(\sigma)) \quad \text{and} \quad \left(\widehat{\boldsymbol{S}}(\mu), \widehat{\boldsymbol{S}}(\sigma)\right)$$

are the sensitivity measures in ORSS and RSS, respectively, about μ and σ.

Table 21. Comparison of linear sensitivity between RSS and ORSS from Logistic(μ, σ^2) distribution

n	$\hat{\boldsymbol{S}}(\mu)$	$\hat{\boldsymbol{S}}(\sigma)$	$\boldsymbol{S}^*(\mu)$	$\boldsymbol{S}^*(\sigma)$	$RE(\mu_{\text{BLUE}})$	$RE(\sigma_{\text{BLUE}})$
2	0.87341	0.87341	0.87341	1.73650	1.00000	1.98818
3	1.75573	2.20602	1.82447	3.60392	1.03915	1.63367
4	2.96026	3.96609	3.11446	5.85548	1.05209	1.47638
5	4.49111	6.14675	4.74096	8.51207	1.05563	1.38481
6	6.35048	8.74662	6.70318	11.58308	1.05554	1.32429
7	8.53967	11.76594	9.01309	15.10464	1.05544	1.28376
8	11.05950	15.20538	11.62819	19.01391	1.05142	1.25047
9	13.91056	19.06575	14.60118	23.33060	1.04965	1.22369
10	17.09325	23.34781	17.90163	28.07809	1.04729	1.20260

Table 22. Comparison of linear sensitivity between RSS and ORSS from Normal(μ, σ^2) distribution

n	$\hat{\boldsymbol{S}}(\mu)$	$\hat{\boldsymbol{S}}(\sigma)$	$\boldsymbol{S}^*(\mu)$	$\boldsymbol{S}^*(\sigma)$	$RE(\mu_{\text{BLUE}})$	$RE(\sigma_{\text{BLUE}})$
2	2.93388	0.93388	2.93388	2.00532	1.00000	2.14729
3	5.80363	2.56028	5.81448	4.52789	1.00187	1.76851
4	9.61593	4.79934	9.65043	7.56371	1.00359	1.57599
5	14.37543	7.61739	14.44306	11.11572	1.00470	1.45926
6	20.08524	10.99780	21.06979	16.91149	1.04902	1.53772
7	26.74753	14.93140	26.89108	19.81793	1.00537	1.32726
8	34.36385	19.41280	34.64788	24.97151	1.00827	1.28634
9	42.93534	24.43864	43.27344	30.67988	1.00787	1.25538
10	52.46290	30.00675	52.90568	36.86042	1.00844	1.22840

Table 23. Comparison of linear sensitivity between RSS and ORSS from Exponential(μ, σ) distribution

n	$\hat{\boldsymbol{S}}(\mu)$	$\hat{\boldsymbol{S}}(\sigma)$	$\boldsymbol{S}^*(\mu)$	$\boldsymbol{S}^*(\sigma)$	$RE(\mu_{\text{BLUE}})$	$RE(\sigma_{\text{BLUE}})$
2	1.14286	0.66667	3.02314	1.19512	2.64524	1.79268
3	4.29170	1.85085	11.18056	2.93206	2.60516	1.58417
4	10.06989	3.56145	25.35888	5.26941	2.51829	1.47957
5	18.96039	5.81344	46.14243	8.22941	2.43362	1.41558
6	31.35595	8.61967	109.53242	11.82492	3.49319	1.37185
7	47.58677	11.99067	109.53242	16.07860	2.30174	1.34093
8	67.93731	15.93509	152.51807	21.01255	2.24498	1.31863
9	92.65721	20.46016	204.07229	26.55999	2.20244	1.29813
10	121.96878	25.57201	263.42567	32.75320	2.15978	1.28082

Table 24. Comparison of linear sensitivity between RSS and ORSS from Exponential(σ) distribution

n	2	3	4	5	6	7	8	9	10
RSS	2.80000	5.39246	8.77927	12.96275	17.94482	23.72697	30.31037	37.69593	45.88441
ORSS	2.76213	5.30766	8.64837	12.78917	17.73261	23.49164	30.09246	37.43908	45.60811
RE	0.98648	0.98427	0.98509	0.98661	0.98817	0.99008	0.99281	0.99319	0.99398

Remark 12: In the case of normal and logistic distributions, we observe that the REs are always more than 1. For the parameter μ, the RE is close to 1, but for σ, it is considerably larger than 1 (particularly for small n).

In the case of the two-parameter exponential distribution, the REs are considerably larger than 1 for both μ and σ.

These reveal that the BLUE-ORSS will be more efficient than the BLUE-RSS in these cases (and considerably so for small n).

In the case of the one-parameter exponential distribution, the RE is less than 1 but is around 99%. This means that the BLUE-ORSS has about the same efficiency as the BLUE-RSS.

9. Outliers and Robust Estimation

RSS when One Outlier is Involved

Suppose a single outlier, Y, is involved in the one-cycle RSS $\boldsymbol{X}_{\text{RSS}} = \{X_{(1)}, \cdots, X_{(n)}\}$.

- Then, the pdf of $X_{(r)}$ becomes

$$h_{r:n}^{\text{RSS}}(x) = \frac{1}{n} h_{r:n}^{\text{OS}}(x) + \frac{n-1}{n} f_{r:n}(x), \quad -\infty < x < \infty.$$

- Similarly, the joint pdf of $X_{(r)}$ and $X_{(s)}$ is

$$h_{r,s:n}^{\text{RSS}}(x,y) = \frac{1}{n} h_{r:n}^{\text{OS}}(x) f_{s:n}(y) + \frac{1}{n} f_{r:n}(x) h_{s:n}^{\text{OS}}(y) + \frac{n-2}{n} f_{r,s:n}(x,y),$$
$$-\infty < x < y < \infty,$$

where $f_{r:n}(x)$ and $h_{r:n}^{\text{OS}}(x)$ are, respectively, the pdf of the r-th order statistic in a sample of size n containing no outlier and one containing a single outlier with cdf $G(y)$ and pdf $g(y)$.

ORSS when One Outlier is Involved

Suppose one outlier Y, with cdf $G(y)$ and pdf $g(y)$, is contained in $\boldsymbol{X}_{\text{RSS}}$ from a population with cdf $F(x)$ and pdf $f(x)$. Let $\boldsymbol{X}_{\text{ORSS}} = \{X_{1:n}^{\text{ORSS}} \leq \cdots \leq X_{n:n}^{\text{ORSS}}\}$ be the ORSS obtained from $\boldsymbol{X}_{\text{RSS}}$.

Theorem 7: The pdf of $X_{r:n}^{\text{ORSS}}$ $(1 \leq r \leq n)$ is

$$h_{r:n}^{\text{ORSS}}(x) = \sum_P \sum_{j_1=i_1}^{n} \cdots \sum_{j_{r-1}=i_{r-1}}^{n} \sum_{j_{r+1}=0}^{i_{r+1}-1} \cdots \sum_{j_n=0}^{i_n-1} D_j^*(r) h_{\tilde{r}:n^2}^{\text{OS}}(x),$$

where $\sum_P$ and $i_a \, (a = 1, \cdots, n)$ are as defined earlier, and

$$\begin{aligned} D_j^*(r) &= \frac{(\tilde{r}-1)!(n^2-\tilde{r})!}{(r-1)!(n-r)!(n^2)!} \left[\prod_{\substack{a=1 \\ a \neq r}}^{n} \binom{n}{j_a} \right] \left[i_r \binom{n}{i_r} \right], \\ \tilde{r} &= i_r + \sum_{\substack{a=1 \\ a \neq r}}^{n} j_a. \end{aligned}$$

Theorem 8: The joint pdf of $X_{r:N}^{\text{ORSS}}$ and $X_{s:N}^{\text{ORSS}}$ $(1 \le r < s \le N)$ is

$$\begin{aligned} f_{r,s:N}^{\text{ORSS}}(x,y) = & \sum_{P} \sum_{j_1=i_1}^{n} \cdots \sum_{j_{s-1}=i_{s-1}}^{n} \sum_{j_s=0}^{i_s-1} \cdots \sum_{j_n=0}^{i_n-1} \sum_{k_1=0}^{n-j_1} \cdots \\ & \times \sum_{k_{r-1}=0}^{n-j_{r-1}} \sum_{k_{r+1}=j_{r+1}+1-i_{r+1}}^{j_{r+1}} \cdots \sum_{k_{s-1}=j_{s-1}+1-i_{s-1}}^{j_{s-1}-1} \sum_{k_{s+1}=0}^{j_{s+1}} \cdots \\ & \times \sum_{k_n=0}^{j_n} D_{i,j,k}^{*}(r,s) h_{r',s':n^2}(x,y), \qquad\qquad x < y, \end{aligned}$$

where

$$D_{i,j,k}^{*}(r,s) = D_{i,j,k} \frac{(\tilde{r}-1)!(\tilde{s}-\tilde{r}-1)!(n^2-\tilde{s})!}{(r-1)!(s-r-1)!(n-s)!(n^2)!},$$

$$\begin{aligned} D_{i,j,k} = & \left\{ \prod_{a=1}^{r-1} \binom{n}{j_a} \binom{n-j_a}{k_a} \right\} \cdot \left\{ i_r \binom{n}{i_r} \binom{n-i_r}{j_r-i_r} \right\} \\ & \times \left\{ \prod_{a=r+1}^{s-1} \binom{n}{j_a} \binom{j_a}{k_a} \right\} \left\{ i_s \binom{n}{i_s} \binom{i_s-1}{j_s} \right\} \left\{ \prod_{a=s+1}^{n} \binom{n}{j_a} \binom{j_a}{k_a} \right\}, \end{aligned}$$

$$\begin{aligned} r' &= \sum_{\substack{a=1 \\ a \ne r,s}}^{n} j_a + i_r + i_s - \sum_{\substack{a=r+1 \\ a \ne s}}^{n} k_a - j_s - 1, \\ s' &= \sum_{\substack{a=1 \\ a \ne r,s}}^{n} j_a + i_s + \sum_{a=1}^{r-1} k_a. \end{aligned}$$

Moments of ORSS when One Outlier is Involved

Theorem 9: For $n \geq 2$, $i = 1, 2, \cdots$, and under the assumption that a single outlier is involved in the RSS, we have

$$\sum_{r=1}^{n} v_{r:n}^{\mathrm{ORSS}(i)} = n\mu_{1:1}^{(i)} + \frac{1}{n}\left(v_{1:1}^{(i)} - \mu_{1:1}^{(i)}\right),$$

where

$$\begin{aligned} v_{r:n}^{\mathrm{ORSS}(i)} &= \int_{-\infty}^{+\infty} x^i h_{r:n}^{\mathrm{ORSS}}(x)\mathrm{d}x, \\ \mu_{1:1}^{(i)} &= \int_{-\infty}^{+\infty} x^i f(x)\mathrm{d}x, \\ v_{1:1}^{(i)} &= \int_{-\infty}^{+\infty} x^i g(x)\mathrm{d}x. \end{aligned}$$

Theorem 10: Suppose one outlier Y, with cdf $G(y)$ and pdf $g(y)$, is involved in $\boldsymbol{X}_{\text{RSS}}$ of size n from a population with cdf $F(x)$ and pdf $f(x)$. Suppose now that both $f(x)$ and $g(x)$ are both symmetric about 0.

With $\boldsymbol{X}_{\text{ORSS}}$ as the ORSS from $\boldsymbol{X}_{\text{RSS}}$, we have

$$\begin{aligned} h_{r:n}^{\text{ORSS}}(x) &= h_{n-r+1:n}^{\text{ORSS}}(x), \\ &\qquad -\infty < x < +\infty, \\ h_{r,s:n}^{\text{ORSS}}(x,y) &= h_{n-s+1,n-r+1:n}^{\text{ORSS}}(-y,-x), \\ &\qquad -\infty < x < y < +\infty. \end{aligned}$$

In this case, we have the following symmetry relations:

$$\begin{aligned} v_{r:n}^{\text{ORSS}} &= -v_{n+1-r:n}^{\text{ORSS}}, && 1 \le r \le n, \\ \omega_{r,r:n}^{\text{ORSS}} &= \omega_{n+1-r,n+1-r:n}^{\text{ORSS}}, && 1 \le r \le n, \\ v_{r,s:n}^{\text{ORSS}} &= -v_{n+1-s,n+1-r:n}^{\text{ORSS}}, && 1 \le r < s \le n, \\ \omega_{r,s:n}^{\text{ORSS}} &= \omega_{n+1-s,n+1-r:n}^{\text{ORSS}}, && 1 \le r < s \le n. \end{aligned}$$

Remark 13: The mixture representations in Theorems 8 and 9 will enable us to compute the means, variances and covariances of ORSS (when a single outlier is present in the RSS) as mixtures of the corresponding quantities of OS from a sample containing a single outlier.

Theorem 10 will enable us to reduce the amount of computation considerably for symmetric-outlier models.

Therefore, in the case of normal distribution, the tables of means, variances and covariances of OS from a single-outlier model [of David, Kennedy and Knight (1977)] can be used to discuss the robustness of estimators based on ORSS when a single outlier is present in the RSS.

Applications to Robust Estimation

• Normal Distribution

We now consider $\boldsymbol{X}_{\text{RSS}}$ from $N(\mu, \sigma^2)$ population while one outlier is involved in the RSS.

<u>For the estimation of μ, we consider:</u>

- μ^*_{BLUE}: BLUE-ORSS;
- $\hat{\mu}_{\text{BLUE}}$: BLUE-RSS;
- $\mu^*_{\text{BLUE}}(1)$: BLUE-ORSS with one observation censoed from each end;
- $T^*(1)$: ORSS trimmed mean with one observation censored from each end.

<u>For the estimation of σ, we consider:</u>

- σ^*_{BLUE}: BLUE-ORSS;
- $\hat{\sigma}_{\text{BLUE}}$: BLUE-RSS;
- $\sigma^*_{\text{BLUE}}(1)$: BLUE-ORSS with one observation censored from each end.

Table 25. Bias of estimators of μ, when a single outlier from $N(\lambda, 1)$ is involved in RSS from $N(0, 1)$

		λ						
n	Estimators	0	0.5	1.0	1.5	2.0	3.0	4.0
5	μ^*_{BLUE}	0	0.01985	0.03958	0.05626	0.07216	0.10115	0.12905
	$\hat{\mu}_{\text{BLUE}}$	0	0.01990	0.03925	0.05769	0.07517	0.10797	0.13941
	$\mu^*_{\text{BLUE}}(1)$	0	0.01953	0.03605	0.04825	0.05539	0.06089	0.06184
	$T^*(1)$	0	0.01952	0.03586	0.04805	0.05506	0.06039	0.06130
6	μ^*_{BLUE}	0	0.01386	0.02666	0.03950	0.05097	0.07176	0.09180
	$\hat{\mu}_{\text{BLUE}}$	0	0.01380	0.02712	0.03965	0.05133	0.07280	0.09306
	$\mu^*_{\text{BLUE}}(1)$	0	0.01364	0.02479	0.03390	0.03921	0.04311	0.04376
	$T^*(1)$	0	0.01364	0.02475	0.03382	0.03907	0.04289	0.04352
7	μ^*_{BLUE}	0	0.01011	0.01959	0.02797	0.03551	0.04820	0.06038
	$\hat{\mu}_{\text{BLUE}}$	0	0.01013	0.01984	0.02887	0.03717	0.05212	0.06603
	$\mu^*_{\text{BLUE}}(1)$	0	0.01000	0.01865	0.02496	0.02923	0.03243	0.03329
	$T^*(1)$	0	0.00999	0.01862	0.02486	0.02905	0.03217	0.03301
8	μ^*_{BLUE}	0	0.00773	0.01490	0.02114	0.02689	0.03622	0.04458
	$\hat{\mu}_{\text{BLUE}}$	0	0.00775	0.01514	0.02194	0.02820	0.03901	0.04902
	$\mu^*_{\text{BLUE}}(1)$	0	0.00764	0.01432	0.01919	0.02267	0.02536	0.02583
	$T^*(1)$	0	0.00764	0.01431	0.01916	0.02262	0.02528	0.02575
9	μ^*_{BLUE}	0	0.00608	0.01180	0.01665	0.02107	0.02793	0.03411
	$\hat{\mu}_{\text{BLUE}}$	0	0.00612	0.01193	0.01723	0.02197	0.03022	0.03768
	$\mu^*_{\text{BLUE}}(1)$	0	0.00606	0.01138	0.01528	0.01812	0.02023	0.02062
	$T^*(1)$	0	0.00606	0.01139	0.01530	0.01816	0.02029	0.02068
10	μ^*_{BLUE}	0	0.00500	0.00963	0.01351	0.01680	0.02224	0.02716
	$\hat{\mu}_{\text{BLUE}}$	0	0.00495	0.00963	0.01387	0.01763	0.02405	0.02978
	$\mu^*_{\text{BLUE}}(1)$	0	0.00496	0.00932	0.01253	0.01466	0.01655	0.01708
	$T^*(1)$	0	0.00496	0.00934	0.01259	0.01475	0.01670	0.01724

Table 26. MSE of estimators of μ, when a single outlier from $N(\lambda, 1)$ is involved in RSS from $N(0, 1)$

		λ						
n	Estimators	0	0.5	1.0	1.5	2.0	3.0	4.0
5	μ^*_{BLUE}	0.06924	0.06961	0.07220	0.07470	0.07942	0.09181	0.11045
	$\hat{\mu}_{\text{BLUE}}$	0.06956	0.06998	0.07128	0.07363	0.07735	0.09047	0.11288
	$\mu^*_{\text{BLUE}}(1)$	0.08266	0.08306	0.08535	0.08754	0.09012	0.09235	0.09272
	$T^*(1)$	0.08292	0.08333	0.08561	0.08777	0.09029	0.09241	0.09275
6	μ^*_{BLUE}	0.04746	0.04991	0.05110	0.05288	0.05514	0.06236	0.07337
	$\hat{\mu}_{\text{BLUE}}$	0.04979	0.04999	0.05061	0.05171	0.05343	0.05951	0.07010
	$\mu^*_{\text{BLUE}}(1)$	0.05712	0.05631	0.05750	0.05880	0.05977	0.06087	0.06103
	$T^*(1)$	0.05715	0.05640	0.05759	0.05889	0.05984	0.06090	0.06106
7	μ^*_{BLUE}	0.03719	0.03734	0.03784	0.03872	0.03983	0.04263	0.04714
	$\hat{\mu}_{\text{BLUE}}$	0.03739	0.03750	0.03783	0.03841	0.03929	0.04244	0.04798
	$\mu^*_{\text{BLUE}}(1)$	0.04087	0.04104	0.04151	0.04226	0.04289	0.04345	0.04368
	$T^*(1)$	0.04091	0.04108	0.04155	0.04228	0.04290	0.04343	0.04365
8	μ^*_{BLUE}	0.02886	0.02900	0.02933	0.02980	0.03045	0.03210	0.03451
	$\hat{\mu}_{\text{BLUE}}$	0.02910	0.02916	0.02936	0.02969	0.03019	0.03195	0.03509
	$\mu^*_{\text{BLUE}}(1)$	0.03108	0.03125	0.03155	0.03194	0.03232	0.03274	0.03282
	$T^*(1)$	0.03109	0.03126	0.03156	0.03195	0.03233	0.03274	0.03282
9	μ^*_{BLUE}	0.02311	0.02317	0.02342	0.02371	0.02409	0.02509	0.02652
	$\hat{\mu}_{\text{BLUE}}$	0.02329	0.02333	0.02345	0.02365	0.02395	0.02501	0.02690
	$\mu^*_{\text{BLUE}}(1)$	0.02452	0.02460	0.02483	0.02507	0.02531	0.02562	0.02566
	$T^*(1)$	0.02454	0.02461	0.02485	0.02509	0.02534	0.02564	0.02569
10	μ^*_{BLUE}	0.01890	0.01896	0.01912	0.01927	0.01954	0.02015	0.02110
	$\hat{\mu}_{\text{BLUE}}$	0.01906	0.01909	0.01917	0.01930	0.01949	0.02015	0.02134
	$\mu^*_{\text{BLUE}}(1)$	0.01986	0.01993	0.02008	0.02020	0.02038	0.02056	0.02062
	$T^*(1)$	0.01989	0.01997	0.02011	0.02024	0.02043	0.02061	0.02068

Table 27. Bias of estimators of σ, when a single outlier from $N(\lambda, 1)$ is involved in RSS from $N(0, 1)$

		λ						
n	Estimators	0	0.5	1.0	1.5	2.0	3.0	4.0
5	σ^*_{BLUE}	0	0.00488	0.01768	0.04017	0.06658	0.12577	0.18739
	$\hat{\sigma}_{\text{BLUE}}$	0	0.00494	0.01913	0.04090	0.06809	0.13125	0.19859
	$\sigma^*_{\text{BLUE}}(1)$	0	0.00452	0.01176	0.03063	0.04220	0.05268	0.05415
6	σ^*_{BLUE}	0	0.00332	0.01281	0.02810	0.04581	0.08602	0.12772
	$\hat{\sigma}_{\text{BLUE}}$	0	0.00343	0.01326	0.02826	0.04692	0.08994	0.13559
	$\sigma^*_{\text{BLUE}}(1)$	0	0.00331	0.01204	0.02187	0.03035	0.03768	0.03868
7	σ^*_{BLUE}	0	0.00237	0.00955	0.02040	0.03317	0.06200	0.09306
	$\hat{\sigma}_{\text{BLUE}}$	0	0.00252	0.00972	0.02066	0.03419	0.06517	0.09788
	$\sigma^*_{\text{BLUE}}(1)$	0	0.00184	0.00825	0.01595	0.02175	0.02734	0.02941
8	σ^*_{BLUE}	0	0.00209	0.00745	0.01555	0.02580	0.04741	0.07042
	$\hat{\sigma}_{\text{BLUE}}$	0	0.00193	0.00742	0.01574	0.02582	0.04922	0.07363
	$\sigma^*_{\text{BLUE}}(1)$	0	0.00217	0.00685	0.01257	0.01798	0.02229	0.02387
9	σ^*_{BLUE}	0	0.00155	0.00572	0.01207	0.01990	0.03694	0.05432
	$\hat{\sigma}_{\text{BLUE}}$	0	0.00152	0.00585	0.01237	0.02036	0.03838	0.05717
	$\sigma^*_{\text{BLUE}}(1)$	0	0.00144	0.00506	0.00973	0.01373	0.01813	0.01887
10	σ^*_{BLUE}	0	0.00107	0.00456	0.00997	0.01565	0.02948	0.04333
	$\hat{\sigma}_{\text{BLUE}}$	0	0.00123	0.00473	0.00998	0.01637	0.03069	0.04553
	$\sigma^*_{\text{BLUE}}(1)$	0	0.00107	0.00405	0.00816	0.01116	0.01505	0.01549

Table 28. MSE of estimators of σ, when a single outlier from $N(\lambda, 1)$ is involved in RSS from $N(0,1)$

		λ						
n	Estimators	0	0.5	1.0	1.5	2.0	3.0	4.0
5	σ^*_{BLUE}	0.08996	0.09036	0.09415	0.10145	0.11585	0.16776	0.25586
	$\hat{\sigma}_{\text{BLUE}}$	0.13128	0.13198	0.13487	0.14202	0.15623	0.21537	0.32230
	$\sigma^*_{\text{BLUE}}(1)$	0.24872	0.24898	0.25836	0.26393	0.27399	0.28332	0.28537
6	σ^*_{BLUE}	0.05913	0.06606	0.06784	0.07228	0.07937	0.10645	0.15337
	$\hat{\sigma}_{\text{BLUE}}$	0.09093	0.09125	0.09260	0.09602	0.10300	0.13308	0.18911
	$\sigma^*_{\text{BLUE}}(1)$	0.14930	0.14230	0.14636	0.14971	0.15307	0.15748	0.15859
	σ^*_{BLUE}	0.05046	0.05076	0.05178	0.05417	0.05820	0.07401	0.10191
7	$\hat{\sigma}_{\text{BLUE}}$	0.06697	0.06714	0.06785	0.06968	0.07348	0.09029	0.12239
	$\sigma^*_{\text{BLUE}}(1)$	0.09330	0.09386	0.09533	0.09726	0.09912	0.10138	0.10229
8	σ^*_{BLUE}	0.04005	0.04035	0.04095	0.04227	0.04488	0.05467	0.07198
	$\hat{\sigma}_{\text{BLUE}}$	0.05151	0.05161	0.05202	0.05308	0.05531	0.06538	0.08503
	$\sigma^*_{\text{BLUE}}(1)$	0.06659	0.06720	0.06784	0.06884	0.07014	0.07159	0.07219
9	σ^*_{BLUE}	0.03259	0.03283	0.03320	0.03406	0.03575	0.04206	0.05331
	$\hat{\sigma}_{\text{BLUE}}$	0.04092	0.04098	0.04123	0.04188	0.04327	0.04965	0.06231
	$\sigma^*_{\text{BLUE}}(1)$	0.05011	0.05050	0.05093	0.05161	0.05236	0.05337	0.05375
10	σ^*_{BLUE}	0.02713	0.02729	0.02751	0.02812	0.02920	0.03347	0.04111
	$\hat{\sigma}_{\text{BLUE}}$	0.03333	0.03336	0.03353	0.03395	0.03486	0.03907	0.04757
	$\sigma^*_{\text{BLUE}}(1)$	0.03935	0.03955	0.03979	0.04026	0.04075	0.04155	0.04171

Table 29. Variance of estimators of μ, when a single outlier from $N(0,\tau^2)$ is involved in RSS from $N(0,1)$

		τ				
n	Estimators	1.0	0.5	2.0	3.0	4.0
5	μ^*_{BLUE}	0.06924	0.06681	0.07706	0.09044	0.11121
	$\hat{\mu}_{\text{BLUE}}$	0.06956	0.06971	0.07049	0.07382	0.07995
	$\mu^*_{\text{BLUE}}(1)$	0.08266	0.07868	0.08699	0.08879	0.08951
	$T^*(1)$	0.08292	0.07880	0.08719	0.08895	0.08963
6	μ^*_{BLUE}	0.04746	0.04879	0.05404	0.06200	0.07453
	$\hat{\mu}_{\text{BLUE}}$	0.04979	0.04986	0.05022	0.05180	0.05473
	$\mu^*_{\text{BLUE}}(1)$	0.05712	0.05456	0.05835	0.05919	0.05965
	$T^*(1)$	0.05715	0.05462	0.05843	0.05925	0.05971
7	μ^*_{BLUE}	0.03719	0.03651	0.03909	0.04238	0.04739
	$\hat{\mu}_{\text{BLUE}}$	0.03739	0.03743	0.03761	0.03844	0.03999
	$\mu^*_{\text{BLUE}}(1)$	0.04087	0.03997	0.04195	0.04246	0.04287
	$T^*(1)$	0.04091	0.03998	0.04197	0.04247	0.04287
8	μ^*_{BLUE}	0.02886	0.02853	0.03006	0.03180	0.03470
	$\hat{\mu}_{\text{BLUE}}$	0.02910	0.02913	0.02923	0.02970	0.03058
	$\mu^*_{\text{BLUE}}(1)$	0.03108	0.03064	0.03185	0.03209	0.03234
	$T^*(1)$	0.03109	0.03065	0.03186	0.03209	0.03234
9	μ^*_{BLUE}	0.02311	0.02288	0.02384	0.02492	0.02655
	$\hat{\mu}_{\text{BLUE}}$	0.02329	0.02331	0.02337	0.02365	0.02419
	$\mu^*_{\text{BLUE}}(1)$	0.02452	0.02426	0.02500	0.02517	0.02528
	$T^*(1)$	0.02454	0.02428	0.02502	0.02520	0.02531
10	μ^*_{BLUE}	0.01890	0.01876	0.01940	0.02010	0.02113
	$\hat{\mu}_{\text{BLUE}}$	0.01906	0.01907	0.01911	0.01929	0.01963
	$\mu^*_{\text{BLUE}}(1)$	0.01986	0.01969	0.02019	0.02034	0.02039
	$T^*(1)$	0.01989	0.01973	0.02024	0.02039	0.02044

Table 30. Bias of estimators of σ, when a single outlier from $N(0, \tau^2)$ is involved in RSS from $N(0, 1)$

		τ				
n	Estimators	1.0	0.5	2.0	3.0	4.0
5	σ^*_{BLUE}	0	-0.01766	0.04450	0.09202	0.14019
	$\hat{\sigma}_{\text{BLUE}}$	0	-0.01700	0.04576	0.09664	0.14921
	$\sigma^*_{\text{BLUE}}(1)$	0	-0.02681	0.02414	0.03392	0.03844
6	σ^*_{BLUE}	0	-0.01175	0.03043	0.06277	0.09565
	$\hat{\sigma}_{\text{BLUE}}$	0	-0.01191	0.03145	0.06610	0.10180
	$\sigma^*_{\text{BLUE}}(1)$	0	-0.01708	0.01677	0.02424	0.02745
7	σ^*_{BLUE}	0	-0.00895	0.02190	0.04547	0.06947
	$\hat{\sigma}_{\text{BLUE}}$	0	-0.00883	0.02286	0.04780	0.07342
	$\sigma^*_{\text{BLUE}}(1)$	0	-0.01285	0.01220	0.01748	0.02054
8	σ^*_{BLUE}	0	-0.00668	0.01707	0.03453	0.05265
	$\hat{\sigma}_{\text{BLUE}}$	0	-0.00682	0.01732	0.03603	0.05518
	$\sigma^*_{\text{BLUE}}(1)$	0	-0.00864	0.01023	0.01438	0.01667
9	σ^*_{BLUE}	0	-0.00555	0.01315	0.02696	0.04036
	$\hat{\sigma}_{\text{BLUE}}$	0	-0.00543	0.01354	0.02803	0.04281
	$\sigma^*_{\text{BLUE}}(1)$	0	-0.00721	0.00791	0.01163	0.01298
10	σ^*_{BLUE}	0	-0.00456	0.01070	0.02120	0.03210
	$\hat{\sigma}_{\text{BLUE}}$	0	-0.00443	0.01086	0.02237	0.03406
	$\sigma^*_{\text{BLUE}}(1)$	0	-0.00591	0.00696	0.00938	0.01089

Table 31. MSE of estimators of σ, when a single outlier from $N(0, \tau^2)$ is involved in RSS from $N(0, 1)$

n	Estimators	τ				
		1.0	0.5	2.0	3.0	4.0
5	σ^*_{BLUE}	0.08996	0.08934	0.11386	0.17421	0.27259
	$\hat{\sigma}_{\text{BLUE}}$	0.13128	0.13171	0.13539	0.15102	0.18027
	$\sigma^*_{\text{BLUE}}(1)$	0.30523	0.30584	0.30613	0.30715	0.30788
6	σ^*_{BLUE}	0.05913	0.06587	0.07828	0.11065	0.16316
	$\hat{\sigma}_{\text{BLUE}}$	0.09093	0.09113	0.09301	0.10115	0.11665
	$\hat{\sigma}_{\text{BLUE}}(1)$	0.24872	0.23928	0.26278	0.26941	0.27267
7	σ^*_{BLUE}	0.05046	0.05032	0.05779	0.07698	0.10848
	$\hat{\sigma}_{\text{BLUE}}$	0.06697	0.06709	0.06813	0.07276	0.08171
	$\sigma^*_{\text{BLUE}}(1)$	0.09330	0.09249	0.09681	0.09835	0.09930
8	σ^*_{BLUE}	0.04005	0.04011	0.04459	0.05641	0.07650
	$\hat{\sigma}_{\text{BLUE}}$	0.05151	0.05158	0.05220	0.05502	0.06053
	$\sigma^*_{\text{BLUE}}(1)$	0.06659	0.06637	0.06872	0.06966	0.07047
9	σ^*_{BLUE}	0.03259	0.03270	0.03553	0.04341	0.05627
	$\hat{\sigma}_{\text{BLUE}}$	0.04092	0.04096	0.04135	0.04316	0.04673
	$\sigma^*_{\text{BLUE}}(1)$	0.05011	0.05014	0.05145	0.05222	0.05240
10	σ^*_{BLUE}	0.02713	0.02714	0.02913	0.03432	0.04312
	$\hat{\sigma}_{\text{BLUE}}$	0.03333	0.03335	0.03361	0.03482	0.03722
	$\sigma^*_{\text{BLUE}}(1)$	0.03935	0.03931	0.04025	0.04072	0.04094

Remark 14: From these computations, we can conclude that if the presence of an outlier is suspected, then the use of $\mu^*_{\text{BLUE}}(1)$ is highly recommended, as it has minimal loss of efficiency when there is no outlier, and also has very good robustness properties (in terms of both bias and MSE) when an outlier is indeed present.

In the case of $\sigma^*_{BLUE}(1)$, it has high loss of efficiency when there is no outlier, though develops good robustness properties (in terms of both bias and MSE) when an outlier is indeed present for large samples, but not for small samples.

♡ ♡ Hence, it will be of interest to propose some other estimator of σ that suffers minimal loss of efficiency when there is no outlier, and also possesses good robustness properties when an outlier is indeed present (even for small n).

• Exponential Distribution

We now consider $\boldsymbol{X}_{\text{RSS}}$ from Exp(θ) population while one outlier from Exp($\theta\delta$) is involved in the RSS.

For the estimation of θ, we consider:

- θ^*_{BLUE}: BLUE-ORSS;
- $\hat{\theta}_{\text{BLUE}}$: BLUE-RSS;
- $\theta^*_{\text{BLUE}}(1)$: BLUE-ORSS with one observation removed from right.

Table 32. Bias of estimators of θ, when a single outlier from Exp(δ) is involved in RSS from Exp(1)

n	Estimators	δ					
		1.0	2.0	4.0	5.0	8.0	10.0
5	θ^*_{BLUE}	0	0.03391	0.08770	0.11240	0.18371	0.23023
	$\hat{\theta}_{\text{BLUE}}$	0	0.03486	0.09183	0.11814	0.19409	0.24359
	$\theta^*_{\text{BLUE}}(1)$	0	0.02683	0.04793	0.05309	0.06156	0.06460
6	θ^*_{BLUE}	0	0.02307	0.05814	0.07393	0.11902	0.14826
	$\hat{\theta}_{\text{BLUE}}$	0	0.02376	0.06128	0.07831	0.12706	0.15866
	$\theta^*_{\text{BLUE}}(1)$	0	0.01885	0.03397	0.03770	0.04385	0.04607
7	θ^*_{BLUE}	0	0.01672	0.04116	0.05195	0.08247	0.10213
	$\hat{\theta}_{\text{BLUE}}$	0	0.01720	0.04353	0.05530	0.08870	0.11023
	$\theta^*_{\text{BLUE}}(1)$	0	0.01399	0.02536	0.02819	0.03287	0.03456
8	θ^*_{BLUE}	0	0.01258	0.03048	0.03838	0.05966	0.07388
	$\hat{\theta}_{\text{BLUE}}$	0	0.01300	0.03237	0.04091	0.06493	0.08034
	$\theta^*_{\text{BLUE}}(1)$	0	0.01078	0.01965	0.02198	0.02532	0.02684
9	θ^*_{BLUE}	0	0.00979	0.02344	0.02927	0.04545	0.05555
	$\hat{\theta}_{\text{BLUE}}$	0	0.01016	0.02494	0.03136	0.04930	0.06075
	$\theta^*_{\text{BLUE}}(1)$	0	0.00847	0.01562	0.01740	0.02043	0.02141
10	θ^*_{BLUE}	0	0.00792	0.01833	0.02310	0.03540	0.04301
	$\hat{\theta}_{\text{BLUE}}$	0	0.00815	0.01976	0.02474	0.03854	0.04731
	$\theta^*_{\text{BLUE}}(1)$	0	0.00698	0.01254	0.01433	0.01671	0.01754

Table 33. MSE of estimators of θ, when a single outlier from $\text{Exp}(\delta)$ is involved in RSS from Exp(1)

n	Estimators	δ					
		1.0	2.0	4.0	5.0	8.0	10.0
5	θ^*_{BLUE}	0.07819	0.08810	0.13338	0.17027	0.34085	0.50528
	$\hat{\theta}_{\text{BLUE}}$	0.07714	0.08757	0.13725	0.17804	0.36766	0.55105
	$\theta^*_{\text{BLUE}}(1)$	0.09648	0.10337	0.11096	0.11307	0.11675	0.11813
6	θ^*_{BLUE}	0.05639	0.06118	0.08196	0.09927	0.17771	0.25442
	$\hat{\theta}_{\text{BLUE}}$	0.05573	0.06079	0.08438	0.10376	0.19409	0.28171
	$\theta^*_{\text{BLUE}}(1)$	0.06615	0.06958	0.07347	0.07459	0.07624	0.07729
7	θ^*_{BLUE}	0.04257	0.04517	0.05587	0.06461	0.10564	0.14436
	$\hat{\theta}_{\text{BLUE}}$	0.04215	0.04488	0.05734	0.06757	0.11535	0.16181
	$\theta^*_{\text{BLUE}}(1)$	0.04827	0.05020	0.05224	0.05282	0.05409	0.05436
8	θ^*_{BLUE}	0.03323	0.03480	0.04105	0.04576	0.06831	0.09036
	$\hat{\theta}_{\text{BLUE}}$	0.03299	0.03459	0.04172	0.04756	0.07489	0.10152
	$\theta^*_{\text{BLUE}}(1)$	0.03684	0.03801	0.03929	0.03969	0.04024	0.04056
9	θ^*_{BLUE}	0.02671	0.02768	0.03139	0.03434	0.04793	0.06115
	$\hat{\theta}_{\text{BLUE}}$	0.02653	0.02752	0.03186	0.03540	0.05202	0.06824
	$\theta^*_{\text{BLUE}}(1)$	0.02911	0.02985	0.03065	0.03089	0.03135	0.03147
10	θ^*_{BLUE}	0.02193	0.02252	0.02486	0.02670	0.03529	0.04334
	$\hat{\theta}_{\text{BLUE}}$	0.02179	0.02244	0.02521	0.02747	0.03809	0.04845
	$\theta^*_{\text{BLUE}}(1)$	0.02358	0.02403	0.02462	0.02479	0.02508	0.02518

Remark 15: From these computations, in the case of the exponential distribution, we can conclude that if the presence of an outlier is suspected, then the use of $\theta^*_{\mathrm{BLUE}}(1)$ is highly recommended, as it has minimal loss of efficiency when there is no outlier, and also has very good robustness properties (in terms of both bias and MSE) when an outlier is indeed present.

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