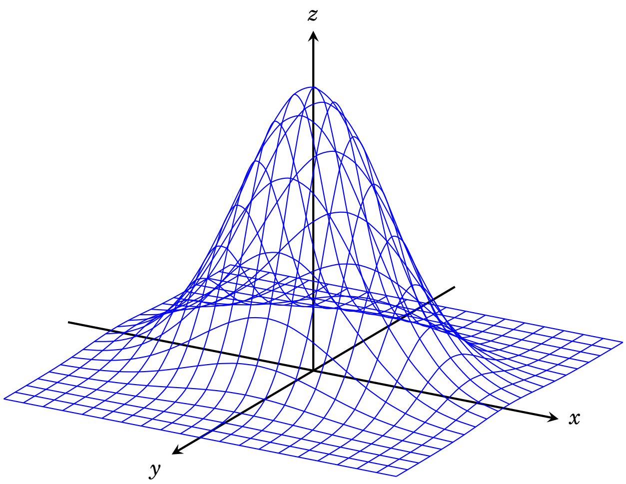
Bridging Course Mathematics

Solutions to Problems

Winter Semester 2023/24

Josef Leydold



August 4, 2023
Institute for Statistics and Mathematics · WU Wien

Solutions

1.1. (a) no subset;

(b) no subset, set equals $\{-11, 11\}$;

(c) subset;

(d) no subset.

1.2. (a) $\{1,3,6,7,9,10\}$;

(b) $\{6\}$;

(c) $\{1\}$;

(d) $\{2,4,5,7,8,10\}$;

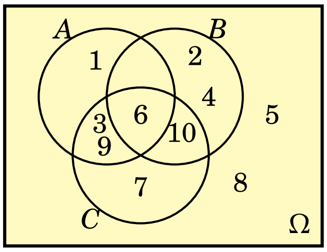
(e) $\{6,10\}$;

(f) $\{2,4,5,8\}$;

(g) $\{2,4\}$;

(h) $\{5,8\}$;

(i) $\{3,6,9\}$.



1.3. A.

1.4. A.

1.5. (a) $\overline{A} \cap \overline{B}$;

(b) A;

(c) $\emptyset$;

(d) C.

1.6. (a) the straight line segment between $(0,2)$ and $(1,2)$ in $\mathbb{R}^2$;

(b) the closed cube $[0,1]^3$ in $\mathbb{R}^3$;

(c) the half-open cube $[0,1] \times (0,1)^2$ in $\mathbb{R}^3$;

(d) cylinder in $\mathbb{R}^3$;

(e) unbounded stripe in $\mathbb{R}^2$;

(f) the closed cube $[0,1]^4$ in $\mathbb{R}^4$.

1.7. function: φ;

domain: $[0,\infty)$;

codomain: $\mathbb{R}$;

image (range): $[0,\infty)$;

function term: $\varphi(x) = x^\alpha$;

independent variable (argument): x;

dependent variable: y.

1.8. (a) no map;

(b) bijective map;

(c) map, neither one-to-one nor onto;

(d) one-to-one map, not onto.

1.9. (a) one-to-one;

(b) no map, as 0^{-2} is not defined;

(c) bijective;

(d) one-to-one, but not onto;

(e) bijective;

(f) no map, as $\{y \in [0,\infty) : x = y^2\}$ is a set and not an element of $\mathbb{R}$.

1.10. (a) map, neither one-to-one ($5' = 3' = 0$) nor onto;

(b) map, onto but not one-to-one;

(c) no map, as $(x^n)' = nx^{n-1} \notin \mathscr{P}_{n-2}$.

2.1. (b) and (d).

2.2. (a) $b^5 + ab^4 + a^2b^3 + a^3b^2 + a^4b + a^5$;

(b) $a_1 - a_6$;

(c) $a_1 - a_{n+1}$.

2.3. (a) $2\sum_{i=1}^n a_i b_i$;

(b) 0;

(c) $2\sum_{i=1}^n (x_i^2 + y_i^2)$;

(d) $x_0 - x_n$.

2.4. (a)

$$\begin{aligned}
&\big((x_1-\bar{x})^2 + (x_2-\bar{x})^2\big) - \big(\big(x_1^2 + x_2^2\big) - 2\bar{x}^2\big) \\
&= x_1^2 - 2x_1\bar{x} + \bar{x}^2 + x_2^2 - 2x_2\bar{x} + \bar{x}^2 - x_1^2 - x_2^2 + 2\bar{x}^2 \\
&= -2x_1\bar{x} + \bar{x}^2 - 2x_2\bar{x} + \bar{x}^2 + 2\bar{x}^2 \\
&= -2(x_1 + x_2)\bar{x} + 4\bar{x}^2 \\
&= -2(2\bar{x})\bar{x} + 4\bar{x}^2 = 0
\end{aligned}$$

(c)

$$\begin{aligned}
&\sum_{i=1}^{n}(x_i-\bar{x})^2-(\sum_{i=1}^{n}x_i^2-n\bar{x}^2)\\
&=\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i\bar{x}+n\bar{x}^2-\sum_{i=1}^{n}x_i^2+n\bar{x}^2\\
&=-2(n\bar{x})\bar{x}+2n\bar{x}^2=0
\end{aligned}$$

2.5.

$$\begin{aligned}
&\tfrac{1}{n}\textstyle\sum_{i=1}^{n}(x_i-\mu)^2-\left(\sigma^2+(\bar{x}-\mu)^2\right)\\
&=\tfrac{1}{n}\textstyle\sum_{i=1}^{n}((x_i-\bar{x})+(\bar{x}-\mu))^2-\sigma^2-(\bar{x}-\mu)^2\\
&=\tfrac{1}{n}\textstyle\sum_{i=1}^{n}(x_i-\bar{x})^2+\tfrac{1}{n}\sum_{i=1}^{n}(\bar{x}-\mu)^2\\
&\quad-\tfrac{2}{n}\textstyle\sum_{i=1}^{n}(x_i-\bar{x})(\bar{x}-\mu)-\sigma^2-(\bar{x}-\mu)^2\\
&=-\tfrac{2}{n}\textstyle\sum_{i=1}^{n}(x_i-\bar{x})(\bar{x}-\mu)\\
&=-\tfrac{2}{n}(\bar{x}-\mu)\textstyle\sum_{i=1}^{n}(x_i-\bar{x})\\
&=-2(\bar{x}-\mu)\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n}x_i-\tfrac{n}{n}\bar{x}\right)\\
&=-2(\bar{x}-\mu)(\bar{x}-\bar{x})\\
&=0
\end{aligned}$$

2.6. (a) x^2+1;

(b) x^4 if $x\geq 0$ and $-x^4$ if $x<0$;

(c) $x\,|x|$;

(d) $x\,|x|^{\alpha-1}$.

2.7. (a) $x^{\frac{1}{6}}y^{-\frac{1}{3}}$;

(b) $x^{\frac{3}{4}}$;

(c) $|x|$.

2.8. (a) monomial (polynomial) of degree 2;

(b) neither monomial nor polynomial;

(c) polynomial of degree 2;

(d) polynomial of degree 4;

(e) polynomial of degree 3;

(f) neither monomial nor polynomial;

(g) monomial (polynomial) of degree 0, i.e., a constant;

(h) polynomial of degree 1.

2.9. (a) $x^4+4x^3y+6x^2y^2+4xy^3+y^4$;

(b) $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

2.10. $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$

2.11. (a) $4xh$;

(b) $a(c-1)$;

(c) $A^3 - B^3$;

(d) $8xy(x^2+y^2)$.

2.12. All polynomials of degree 4 with these roots have the form $c(x+1)(x-2)(x-3)(x-4)$ with $c \neq 0$.
It cannot have more than 4 roots.

2.13. (a) $\frac{2x}{x^2-1}$;

(b) $\frac{1}{st}$;

(c) $\frac{1}{xyz}$;

(d) $\frac{(x^2+y^2)(x+y)}{xy(x-y)}$.

2.14. (a) $1+x$;

(b) $\frac{1+x^2}{1-x}$ (cannot be simplified);

(c) x^3;

(d) $x^3(1+x) = x^3 + x^4$;

(e) $x^2(1+x)(1+x^2) = x^2 + x^3 + x^4 + x^5$;

(f) $1 + x + x^2$.

2.15. (a) xy;

(b) $\frac{(x^2+y)(2x+y)}{(2x+1)2xy}$;

(c) $\frac{x(a-b)+a}{x(a+b)+b}$;

(d) x.

2.16. (a) $x^{\frac{1}{8}} - y^{\frac{1}{6}}$;

(b) $x^{\frac{1}{4}} + 2$;

(c) $2x^{\frac{51}{14}}$.

2.17. (a) $(1 + \frac{y}{\sqrt{x}})^{\frac{1}{3}}$;

(b) $\frac{1}{3x - \frac{1}{3}}$;

(c) $\frac{1}{(xy)^{\frac{1}{6}} + 3}$;

(d) $\sqrt{x} + \sqrt{y}$.

2.18. (a) 1;

(b) 2;

(c) 4;

(d) not defined;

(e) 0;

(f) -2;

(g) $\frac{1}{2}$;

(h) $-\frac{1}{2}$;

(i) not real.

2.19. (a) 2.47712;

(b) 4.7712.

2.20. (a) 10 000;

(b) -4;

(c) 27;

(d) $-\frac{\log_{10}(2)}{4} - \frac{1}{2}$;

(e) -3;

(f) -4.

2.21. (a) $y = \frac{1}{10}\, e^{x \ln 10}$;

(b) $y = 16\, e^{x \ln 4}$;

(c) $y = e^{x(\ln 3 + \ln 25)}$;

(d) $y = e^{x \ln \sqrt{1.08}}$;

(e) $y = 0.9\, e^{x \frac{1}{10} \ln 1.1}$;

(f) $y = \sqrt{q}\, e^{x \ln \sqrt{2}}$.

3.1. (a) $\{1-\sqrt{2}, 1, 1+\sqrt{2}\}$;
(b) $\{-\sqrt{2}, 0, \sqrt{2}\}$;
(c) $\{3\}$ (-1 not in domain).

3.2. (a) $x = \frac{\ln 3}{\ln 3 - \ln 2} \approx 2.710$;

(b) $x = \frac{\ln 9}{\ln 6} \approx 1.226$;

(c) $x = \frac{\ln 100}{\ln 5} \approx 2.861$;

(d) $x = \frac{\ln 50}{4 \ln 10} \approx 0.425$;

(e) $x = -\frac{\ln 2 + 4 \ln 10}{\ln 0{,}4} \approx 10.808$;

(f) $x = \frac{\ln 4}{\ln 9 - \ln 125} \approx -0.527$.

3.3. $x = \ln\left(a \pm \sqrt{a^2 - 1}\right)$

3.4. $x_1 = 0$,
$x_2 = \frac{27}{32} + \frac{\sqrt{217}}{32} \approx 1.304$,
$x_3 = \frac{27}{32} - \frac{\sqrt{217}}{32} \approx 0.383$.

3.5. (a) 1, (-2 does not satisfy the equation);
(b) 3.

3.6. (a) $(x+1)(x+3)$;
(b) $3\left(x-\frac{9+\sqrt{57}}{6}\right)\left(x-\frac{9-\sqrt{57}}{6}\right)$;
(c) $x(x+1)(x-1)$;
(d) $x(x-1)^2$;
(e) $(x+1)(x-1)^3$.

3.7. We get most of the solutions by means of the solution formula for quadratic equations or by factorizing the equation (displayed in square brackets [...]).
(a) $x=\frac{y}{y+1}$, $y=\frac{x}{1-x}$,
$[x(y+1)=y$ and $y(x-1)=-x$, resp.];
(b) $x=\frac{4y+1}{3y+2}$, $y=\frac{1-2x}{3x-4}$,
$[x(3y+2)=1+4y$ and $y(3x-4)=1-2x$, resp.];
(c) $x=y-1$ and $x=-y$; $y=x+1$ and $y=-x$,
$[(x-y+1)(x+y)=0]$;
(d) $x=-y$ and $x=\frac{1}{y}$; $y=-x$ and $y=\frac{1}{x}$,
$[(x+y)\cdot(xy-1)=0]$;
(e) $x=2-y$ and $x=-2-y$; $y=2-x$ and $y=-2-x$,
$[(x+y)^2=4]$;
(f) $x=-\frac{1}{3}y+\frac{5}{3}$ and $x=-\frac{1}{3}y-\frac{5}{3}$; $y=5-3x$ and $y=-5-3x$,
$[(3x+y)^2=25]$;
(g) $x=\pm\frac{3}{2}\sqrt{4-y^2}$, $y=\pm\frac{2}{3}\sqrt{9-x^2}$;
(h) $x=\pm\frac{3}{2}\sqrt{4+y^2}$, $y=\pm\frac{2}{3}\sqrt{x^2-9}$;
(i) $x=y-2\sqrt{y}+1$, $y=x-2\sqrt{x}+1$.

3.8. We get most of the solutions by means of the solution formula for quadratic equations or by factorizing the equation (displayed in square brackets [...]).
(a) $x=-\frac{1}{2}y\pm\frac{1}{2y}\sqrt{y^4+24y}$, $y=-\frac{1}{2}x\pm\frac{1}{2x}\sqrt{x^4+24x}$;
(b) $x=-y$ and $x=\frac{1}{y}$, $y=-x$ and $y=\frac{1}{x}$,
$[(x+y)\cdot(xy-1)=0]$;
(c) $x=(1\pm\sqrt{2})y$, $y=\frac{1}{1\pm\sqrt{2}}x$,
$[x^2-2xy-y^2=0]$;
(d) $x\in\mathbb{R}\setminus\{-y,\pm\sqrt{-y}\}$ arbitrary if $y=-1$ and $x=0$ else, $y\in\mathbb{R}\setminus\{-x,-x^2\}$ arbitrary if $x=0$ and $y=-1$ else,
$[x^2(1+y)=0]$;
(e) $x=\frac{1+3y-y^2}{y-1}$, $y=\frac{1}{2}(3-x)\pm\frac{1}{2}\sqrt{(x-3)^2+4(x+1)}$,
$[y^2-3y+xy-x-1=0]$;

(f) $x=\frac{y}{y^2-1}$, $y=\frac{1\pm\sqrt{1+4x^2}}{2x}$,
$[x(y^2-1)=y$ and $xy^2-y-x=0$, resp.];

(g) $x=-\frac{1}{8}(y+4)\pm\sqrt{\frac{1}{64}(y+4)^2-\frac{1}{4}\left(y+\frac{1}{y}\right)}$, $y=-2x\pm\sqrt{4x^2+\frac{1}{x+1}}$,
$[4x^2y+4xy+xy^2+y^2-1=0]$;

(h) $x=\frac{1}{2}y+\frac{1}{2}$ and $x=y-1$, $\quad y=2x-1$ and $y=y+1$,
$[(2x-y-1)\cdot(x-y+1)=0]$;

(i) $x=\frac{-3+\sqrt{17}}{4}y$ and $x=\frac{-3-\sqrt{17}}{4}y$, $\quad y=\frac{3+\sqrt{17}}{2}x$ and $y=\frac{3-\sqrt{17}}{2}x$,
$[2x^2+3xy-y^2=0]$.

3.9. (a) $a=1$, $b=0$, $c=1$;

(b) $a=3$, $b=1$, $c=3$.

3.10. (a) $L=[-1,0]\cup[3,\infty)$;

(b) $L=(-1,0)\cup(3,\infty)$;

(c) $L=\{1\}$;

(d) $L=\mathbb{R}$;

(e) $L=\emptyset$.

3.11. (a) $(-\infty,-\frac{2}{3}]\cup[\frac{1}{2},\infty)$;

(b) $(-\infty,-2)\cup(0,\infty)$;

(c) $(-\infty,\frac{22}{5}]\cup(5,\infty)$;

(d) $\left[\frac{3}{2}-\sqrt{\frac{25}{2}},-1\right)\cup\left(4,\frac{3}{2}+\sqrt{\frac{25}{2}}\right]$;

(e) $(-\frac{13}{2},-4]\cup[3,\frac{11}{2})$;

(f) $[-14,-9]\cup[1,6]$.

4.1. (a) increasing but neither alternating nor bounded;

(b) decreasing and bounded but not alternating;

(c) bounded but neither alternating nor monotone;

(d) increasing but neither alternating nor bounded;

(e) alternating and bounded but not monotone.

4.2. (a) $(2,6,12,20,30)$;

(b) $(0.333,0.583,0.783,0.95,1.093)$;

(c) $(1.072,2.220,3.452,4.771,6.185)$.

4.3. $a_n=2\cdot1.1^{n-1}$;

$a_7=3.543$.

4.4. (a) $(s_n)=\left(\frac{n(n-1)}{2}\right)=(0,1,3,\dots,45)$;

(b) $(s_n)=(n^2)=(1,4,9,\dots,100)$.

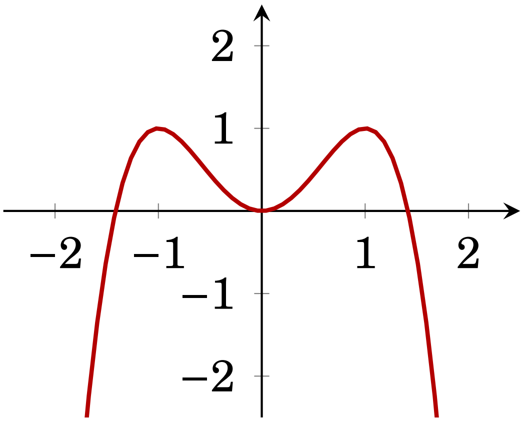
4.5. (a) $s_7 = \frac{1}{3} \cdot \frac{3^7-1}{3-1} = 364.33$;

(b) $s_7 = -\frac{1}{2} \cdot \frac{(-1/4)^7-1}{-1/4-1} = -0.400$.

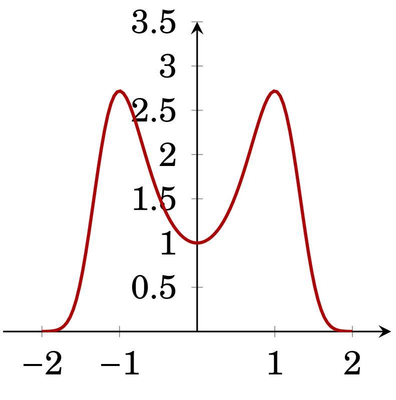
5.1. (a) $D_h = \mathbb{R} \setminus \{2\}$;
(b) $D_D = \mathbb{R} \setminus \{1\}$;
(c) $D_f = [2, \infty)$;
(d) $D_g = (\frac{3}{2}, \infty)$;
(e) $D_f = [-3, 3]$;
(f) $D_f = \mathbb{R}$;
(g) $D_f = \mathbb{R}$.

5.2. (a) $D_f = \mathbb{R} \setminus \{3\}$;
(b) $D_f = (-1, \infty)$;
(c) $D_f = \mathbb{R}$;
(d) $D_f = (-1, 1)$;
(e) $D_f = \mathbb{R}$;
(f) $D_f = \mathbb{R} \setminus \{0\}$.

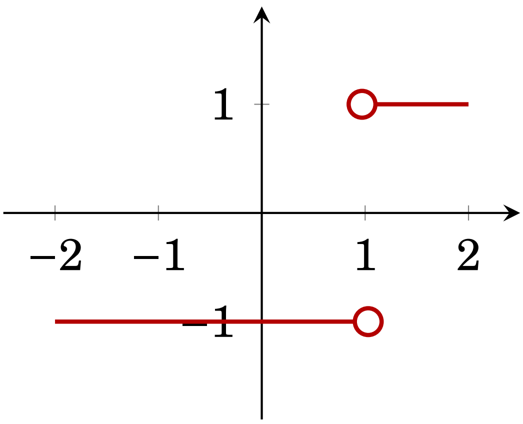
5.3.



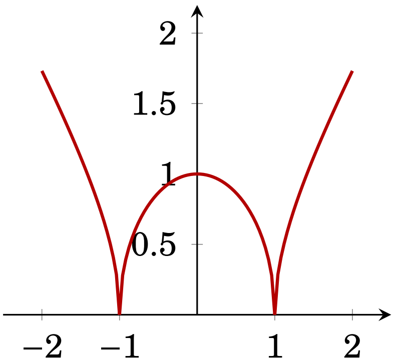
5.4.



5.5.



5.6.



5.7. (a) one-to-one, not onto;

(b) one-to-one, not onto;

(c) bijective;

(d) not one-to-one, not onto;

(e) not one-to-one, onto;

(f) bijective.

Beware! Domain and codomain are part of the function.

5.8. (a) $(f \circ g)(4) = f(g(4)) = f(9) = 98$;

(b) $(f \circ g)(-9) = f(g(-9)) = f(28) = 839$;

(c) $(g \circ f)(0) = g(f(0)) = g(-1) = 2$;

(d) $(g \circ f)(-1) = g(f(-1)) = g(-2) \approx 3.828$.

5.9. (a) $(f \circ g)(x) = (1+x)^2$,
$(g \circ f)(x) = 1 + x^2$,
$D_f = D_g = D_{f \circ g} = D_{g \circ f} = \mathbb{R}$;

(b) $(f \circ g)(x) = |x| + 1$,
$(g \circ f)(x) = (\sqrt{x} + 1)^2$,
$D_f = D_{g \circ f} = [0, \infty)$, $D_g = D_{f \circ g} = \mathbb{R}$;

(c) $(f \circ g)(x) = \frac{1}{\sqrt{x+2}}$,

$(g \circ f)(x) = \sqrt{\frac{1}{x+1}} + 1$,
$D_f = \mathbb{R} \setminus \{-1\}$, $D_g = D_{f \circ g} = [0, \infty)$, $D_{g \circ f} = (-1, \infty)$;

(d) $(f \circ g)(x) = 2 + |x - 2|$,
$(g \circ f)(x) = |x|$,
$D_f = D_{g \circ f} = [0, \infty)$, $D_g = D_{f \circ g} = \mathbb{R}$;

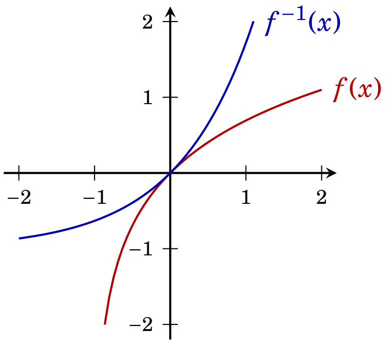
(e) $(f \circ g)(x) = (x-3)^2 + 2$,
$(g \circ f)(x) = x^2 - 1$,
$D_f = D_g = D_{f \circ g} = D_{g \circ f} = \mathbb{R}$;

(f) $(f \circ g)(x) = \frac{1}{1 + (\frac{1}{x})^2}$,
$(g \circ f)(x) = 1 + x^2$,
$D_f = D_{g \circ f} = \mathbb{R}$, $D_g = D_{f \circ g} = \mathbb{R} \setminus \{0\}$;

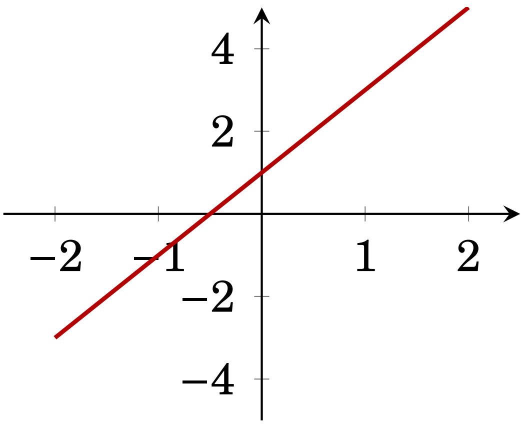
(g) $(f \circ g)(x) = x^2$,
$(g \circ f)(x) = \exp(\ln(x)^2) = x^{\ln x}$,
$D_f = D_{g \circ f} = (0, \infty)$, $D_g = D_{f \circ g} = \mathbb{R}$;

(h) $(f \circ g)(x) = \ln(x^3)$,
$(g \circ f)(x) = (\ln(x-1))^3 + 1$,
$D_f = D_{g \circ f} = (1, \infty)$, $D_g = \mathbb{R}$, $D_{f \circ g} = (0, \infty)$.

5.10. $f^{-1}(x) = e^x - 1$



5.11.



5.16. (a) $\boxed{18}$;

(b) $\boxed{11}$;

(c) $\boxed{1}$;

(d) $\boxed{12}$;

(e) $\boxed{9}$;

(f) $\boxed{4}$;

(g) $\boxed{6}$;

(h) $\boxed{16}$;

(i) $\boxed{10}$;

(j) $\boxed{3}$;

(k) $\boxed{8}$;

(l) $\boxed{13}$;

(m) $\boxed{15}$;

(n) $\boxed{5}$;

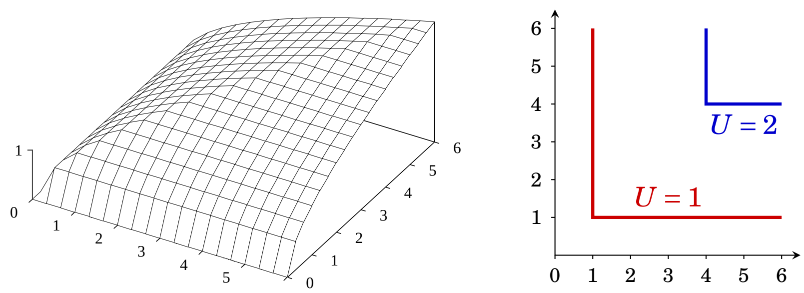
(o) $\boxed{14}$;

(p) $\boxed{17}$;

(q) $\boxed{2}$;

(r) $\boxed{7}$.

5.17.



6.1. (a) 7;

(b) $\frac{2}{7}$;

(c) divergent;

(d) divergent but tends to ∞;

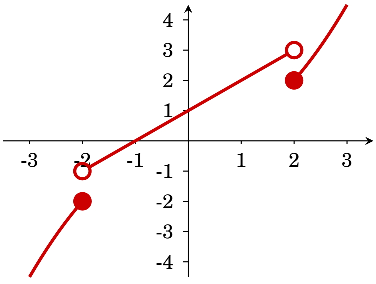
(e) 0.

6.2. (a) e^x;

(b) e^x;

(c) $e^{1/x}$.

6.3.



$\lim_{x\to-2^+} f(x) = -1$, $\lim_{x\to-2^-} f(x) = -2$, $\lim_{x\to-2} f(x)$ does not exist;

$\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) = \lim_{x\to0} f(x) = 1$;

$\lim_{x\to2^+} f(x) = 2$, $\lim_{x\to2^-} f(x) = 3$, $\lim_{x\to2} f(x)$ does not exist.

6.4. (a) $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = 1$;

(b) $\lim_{x\to0^-} \frac{1}{x} = -\infty$, $\lim_{x\to0^+} \frac{1}{x} = \infty$;

(c) $\lim_{x\to1^-} x = \lim_{x\to1^+} x = 1$.

6.5. (a) 0;

(b) 0;

(c) ∞;

(d) $-\infty$;

(e) 1.

6.6. (a) $\frac{1}{2}$;

(b) -4;

(c) -1;

(d) 0.

6.7. (a) 5;

(b) -5;

(c) 0;

(d) $-\infty$;

(e) $-\frac{1}{5}$.

6.8. (a) 1;

(b) $2x$;

(c) $3x^2$;

(d) nx^{n-1}.

6.9. (a) $\frac{2}{7}$;

(b) $-\frac{1}{4}$;

(c) $\frac{1}{3}$;

(d) $\frac{1}{2}$;

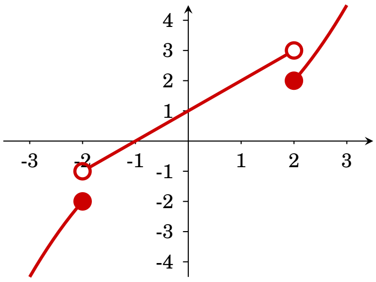
(e) $= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = 0$;

(f) ∞.

In (b) and (f) we cannot apply l'Hôpital's rule.

6.10. Look at numerator and denominator.

6.11.



$\lim_{x\to -2^+} f(x) = -1$, $\lim_{x\to -2^-} f(x) = -2$, $\lim_{x\to -2} f(x)$ does not exist;

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = 1$;

$\lim_{x\to 2^+} f(x) = 2$, $\lim_{x\to 2^-} f(x) = 3$, $\lim_{x\to 2} f(x)$ does not exist;

f is continuous at 0 but not in -2 and 2.

6.12. $\lim_{x\to 0^-} f(x) = -1$, $\lim_{x\to 0^+} f(x) = 1$.

Not continuous at 0 and thus not differentiable.

6.13. $\lim_{x\to 1} f(x) = 2$.
Continuous at 1, but not differentiable.

6.14. The respective functions are continuous in
(a) D;
(b) D;
(c) D;
(d) D;
(e) D;
(f) $\mathbb{R}\setminus\mathbb{Z}$;
(g) $\mathbb{R}\setminus\{2\}$.

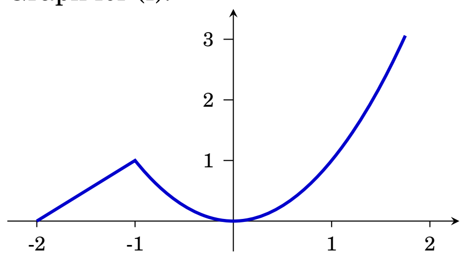
6.15. Yes.

6.16. $h = \frac{2}{5}$.

7.1. differentiable in

(a) $\mathbb{R}$, (b) $\mathbb{R}$, (c) $\mathbb{R}\setminus\{0\}$, (d) $\mathbb{R}\setminus\{-1,1\}$, (e) $\mathbb{R}\setminus\{-1,1\}$, (f) $\mathbb{R}\setminus\{-1\}$.

Graph for (f):



7.2. (a) $f'(x) = 16x^3 + 9x^2 - 4x, \quad f''(x) = 48x^2 + 19x - 4;$

(b) $f'(x) = -x\,e^{-\frac{x^2}{2}}, \quad f''(x) = (x^2 - 1)\,e^{-\frac{x^2}{2}};$

(c) = (b);

(d) $f'(x) = \frac{-2}{(x-1)^2}, \quad f''(x) = \frac{4}{(x-1)^3}.$

7.3. (a) $f'(x) = -\frac{2x}{(1+x^2)^2}, \quad f''(x) = \frac{6x^2-2}{(1+x^2)^3};$

(b) $f'(x) = -\frac{2}{(1+x)^3}, \quad f''(x) = \frac{6}{(1+x)^4};$

(c) $f'(x) = \ln(x), \quad f''(x) = \frac{1}{x};$

(d) $f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}.$

7.4. (a) $f'(x) = \frac{1}{cos(x)^2}, \quad f''(x) = \frac{2\sin(x)}{\cos(x)^3};$

(b) $f'(x) = \sinh(x), \quad f''(x) = \cosh(x);$

(c) $f'(x) = \cosh(x), \quad f''(x) = \sinh(x);$

(d) $f'(x) = -2x\sin(1+x^2), \quad f''(x) = -2\sin(1+x^2) - 4x^2\cos(1+x^2).$

7.5.

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \big(f(x)\cdot(g(x))^{-1}\big)' \\
&= f'(x)\cdot(g(x))^{-1} + f(x)\cdot\big((g(x))^{-1}\big)' \\
&= f'(x)\cdot(g(x))^{-1} + f(x)(-1)(g(x))^{-2}g'(x) \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\
&= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}
\end{aligned}$$

7.6. By means of the differential: $f(3.1) - f(3) \approx -0.001096$,

exact value: $f(3.1) - f(3) = -0.00124\ldots$

7.7. (a) $\varepsilon_g(x) = \frac{3x^3-4x^2}{x^3-2x^2}$,
1-elastic for $x = 1$ and $x = \frac{3}{2}$,
elastic for $x < 1$ and $x > \frac{3}{2}$,
inelastic for $1 < x < \frac{3}{2}$.

(b) $\varepsilon_h(x) = \beta$,

the elasticity of $h(x)$ only depends on parameter β and is constant on the domain of h.

7.8. (a) wrong;

(b) approximately;

(c) is correct;

(d) wrong.

7.9. Derivatives:

	(a)	(b)	(c)	(d)	(e)
f_x	1	y	$2x$	$2xy^2$	$\alpha x^{\alpha-1} y^{\beta}$
f_y	1	x	$2y$	$2x^2 y$	$\beta x^{\alpha} y^{\beta-1}$
f_{xx}	0	0	2	$2y^2$	$\alpha(\alpha-1)x^{\alpha-2} y^{\beta}$
$f_{xy} = f_{yx}$	0	1	0	$4xy$	$\alpha\beta x^{\alpha-1} y^{\beta-1}$
f_{yy}	0	0	2	$2x^2$	$\beta(\beta-1)x^{\alpha} y^{\beta-2}$

Derivatives at point $(1,1)$:

	(a)	(b)	(c)	(d)	(e)
f_x	1	1	2	2	α
f_y	1	1	2	2	β
f_{xx}	0	0	2	2	$\alpha(\alpha-1)$
$f_{xy} = f_{yx}$	0	1	0	4	$\alpha\beta$
f_{yy}	0	0	2	2	$\beta(\beta-1)$

7.10. Derivatives:
$f_x(x,y) = 2x\exp(x^2+y^2)$,
$f_y(x,y) = 2y\exp(x^2+y^2)$,
$f_{xx}(x,y) = (2+4x^2)\exp(x^2+y^2)$,
$f_{xy}(x,y) = f_{yx}(x,y) = 4xy\exp(x^2+y^2)$,
$f_{yy}(x,y) = (2+4y^2)\exp(x^2+y^2)$.

Derivatives at point $(0,0)$:
$f_x(0,0) = 0$,
$f_y(0,0) = 0$,
$f_{xx}(0,0) = 2$,
$f_{xy}(0,0) = f_{yx}(0,0) = 0$,
$f_{yy}0,0) = 2$.

7.11. Derivatives:

	(a)
f_x	$x(x^2+y^2)^{-1/2}$
f_y	$y(x^2+y^2)^{-1/2}$
f_{xx}	$(x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2}$
$f_{xy} = f_{yx}$	$-xy(x^2+y^2)^{-3/2}$
f_{yy}	$(x^2+y^2)^{-1/2} - y^2(x^2+y^2)^{-3/2}$

	(b)
f_x	$x^2(x^3+y^3)^{-2/3}$
f_y	$y^2(x^3+y^3)^{-2/3}$
f_{xx}	$2x(x^3+y^3)^{-2/3} - 2x^4(x^3+y^3)^{-5/3}$
$f_{xy} = f_{yx}$	$-2x^2y^2(x^3+y^3)^{-5/3}$
f_{yy}	$2y(x^3+y^3)^{-2/3} - 2y^4(x^3+y^3)^{-5/3}$

	(c)
f_x	$x^{p-1}(x^p+y^p)^{(1-p)/p}$
f_y	$y^{p-1}(x^p+y^p)^{(1-p)/p}$
f_{xx}	$(p-1)x^{p-2}(x^p+y^p)^{(1-p)/p}$ $-(p-1)x^{2(p-1)}(x^p+y^p)^{(1-2p)/p}$
$f_{xy} = f_{yx}$	$-(p-1)x^{p-1}y^{p-1}(x^p+y^p)^{(1-2p)/p}$
f_{yy}	$(p-1)y^{p-2}(x^p+y^p)^{(1-p)/p}$ $-(p-1)y^{2(p-1)}(x^p+y^p)^{(1-2p)/p}$

Derivatives at point $(1,1)$:

	(a)	(b)	(c)
f_x	$\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt[3]{4}}$	$2^{(1-p)/p}$
f_y	$\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt[3]{4}}$	$2^{(1-p)/p}$
f_{xx}	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt[3]{4}}$	$(p-1)2^{(1-2p)/p}$
$f_{xy} = f_{yx}$	$-\frac{1}{\sqrt{8}}$	$-\frac{1}{\sqrt[3]{4}}$	$-(p-1)2^{(1-2p)/p}$
f_{yy}	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt[3]{4}}$	$(p-1)2^{(1-2p)/p}$

7.12. Gradient:
(a) $\nabla f(x,y) = (1,1)$,
(b) $\nabla f(x,y) = (y,x)$,
(c) $\nabla f(x,y) = (2x,2y)$,
(d) $\nabla f(x,y) = (2xy^2, 2x^2y)$,
(e) $\nabla f(x,y) = (\alpha x^{\alpha-1} y^{\beta}, \beta x^{\alpha} y^{\beta-1})$,

Gradient at $(1,1)$:
(a) $\nabla f(1,1) = (1,1)$,
(b) $\nabla f(1,1) = (1,1)$,
(c) $\nabla f(1,1) = (2,2)$,
(d) $\nabla f(1,1) = (2,2)$,
(e) $\nabla f(1,1) = (\alpha,\beta)$.

7.13. Gradient:
(a) $\nabla f(x,y) = (x(x^2+y^2)^{-1/2},\ y(x^2+y^2)^{-1/2})$,
(b) $\nabla f(x,y) = (x^2(x^3+y^3)^{-2/3},\ y^2(x^3+y^3)^{-2/3})$,
(c) $\nabla f(x,y) = (x^{p-1}(x^p+y^p)^{(1-p)/p},\ y^{p-1}(x^p+y^p)^{(1-p)/p})$,

Gradient at $(1,1)$:
(a) $\nabla f(1,1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$,
(b) $\nabla f(1,1) = (\frac{1}{\sqrt[3]{4}}, \frac{1}{\sqrt[3]{4}})$,

(c) $\nabla f(1,1) = (2^{(1-p)/p}, 2^{(1-p)/p})$.

7.14. (a) $\mathbf{H}_f(1,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$;

(b) $\mathbf{H}_f(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

(c) $\mathbf{H}_f(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$;

(d) $\mathbf{H}_f(1,1) = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$;

(e) $\mathbf{H}_f(1,1) = \begin{pmatrix} \alpha(\alpha-1) & \alpha\beta \\ \alpha\beta & \beta(\beta-1) \end{pmatrix}$.

7.15. (a) $\mathbf{H}_f(1,1) = \begin{pmatrix} \frac{\sqrt{8}}{8} & -\frac{\sqrt{8}}{8} \\ -\frac{\sqrt{8}}{8} & \frac{\sqrt{8}}{8} \end{pmatrix}$;

(b) $\mathbf{H}_f(1,1) = \begin{pmatrix} \frac{\sqrt[3]{2}}{2} & -\frac{\sqrt[3]{2}}{2} \\ -\frac{\sqrt[3]{2}}{2} & \frac{\sqrt[3]{2}}{2} \end{pmatrix}$;

(c) $\mathbf{H}_f(1,1) = \begin{pmatrix} (p-1)2^{(1-2p)/p} & -(p-1)2^{(1-2p)/p} \\ -(p-1)2^{(1-2p)/p} & (p-1)2^{(1-2p)/p} \end{pmatrix}$.

7.16. (a) $h'(t) = f'(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = (2g_1(t), 2g_2(t)) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} = (2t, 2t^2) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} = 2t + 4t^3$.

(Composite function: $h(t) = f(\mathbf{g}(t)) = t^2 + t^4$.)

(b) $\mathbf{p}'(x,y) = \mathbf{g}'(f(x,y)) \cdot f'(x,y) = \begin{pmatrix} 1 \\ 2(x^2+y^2) \end{pmatrix} \cdot (2x, 2y) = \begin{pmatrix} 2x & 2y \\ 4x(x^2+y^2) & 4y(x^2+y^2) \end{pmatrix}$

(Composite function: $\mathbf{p}(x,y) = \mathbf{g}(f(x,y)) = \begin{pmatrix} x^2+y^2 \\ (x^2+y^2)^2 \end{pmatrix}$)

7.17. $\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 3x_1^2 & -1 \\ 1 & -3x_2^2 \end{pmatrix}$, $\mathbf{g}'(\mathbf{x}) = \begin{pmatrix} 0 & 2x_2 \\ 1 & 0 \end{pmatrix}$,

(a) $(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x}))\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 0 & 2(x_1 - x_2^3) \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3x_1^2 & -1 \\ 1 & -3x_2^2 \end{pmatrix}$
$= \begin{pmatrix} 2(x_1 - x_2^3) & 6(-x_1x_2^2 + x_2^5) \\ 3x_1^2 & -1 \end{pmatrix}$

(b) $(\mathbf{f} \circ \mathbf{g})'(\mathbf{x}) = \mathbf{f}'(\mathbf{g}(\mathbf{x}))\mathbf{g}'(\mathbf{x}) = \begin{pmatrix} 3x_2^4 & -1 \\ 1 & -3x_1^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2x_2 \\ 1 & 0 \end{pmatrix}$
$= \begin{pmatrix} -1 & 6x_2^5 \\ -3x_1^2 & 2x_2 \end{pmatrix}$.

7.18. Let $\mathbf{s}(t) = \begin{pmatrix} K(t) \\ L(t) \\ t \end{pmatrix}$. Then

$$\begin{aligned}\frac{dQ}{dt} &= \nabla Q(\mathbf{s}(t)) \cdot \mathbf{s}'(t) \\ &= (Q_K(\mathbf{s}(t)), Q_L(\mathbf{s}(t)), Q_t(\mathbf{s}(t)) \cdot \begin{pmatrix} K'(t) \\ L'(t) \\ 1 \end{pmatrix} \\ &= Q_K K'(t) + Q_L L'(t) + Q_t\,.\end{aligned}$$

8.1. (a) convex;

(b) concave;

(c) concave;

(d) convex if $\alpha \geq 1$ and $\alpha \leq 0$, concave if $0 \leq \alpha \leq 1$.

8.2. Monotonically decreasing in $[-1,3]$,
monotonically increasing in $(-\infty,-1]$ and $[3,\infty)$;

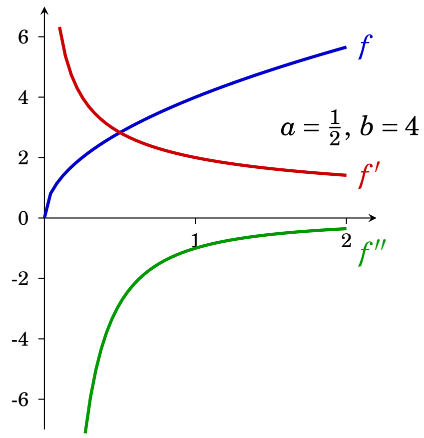
concave in $(-\infty,1]$,
convex in $[1,\infty)$.

8.3. (a) monotonically increasing (in $\mathbb{R}$),
concave in $(-\infty,0]$, convex in $[0,\infty)$;

(b) monotonically increasing $(-\infty,0]$, decreasing in $[0,\infty)$,
concave in $[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}]$, and convex in $(-\infty,-\frac{\sqrt{2}}{2}]$ and $[\frac{\sqrt{2}}{2},\infty)$;

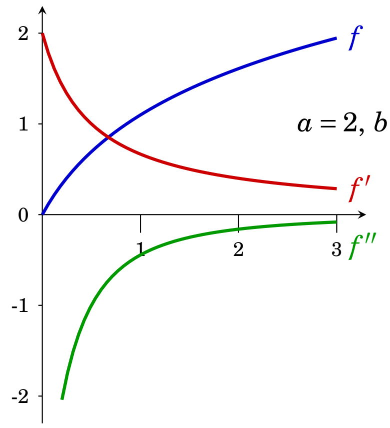
(c) monotonically increasing $(-\infty,0]$, decreasing in $[0,\infty)$,
concave in $[-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}]$, and convex in $(-\infty,-\frac{\sqrt{3}}{3}]$ and $[\frac{\sqrt{3}}{3},\infty)$.

8.4.



Compute all derivatives and verify properties (1)–(3).

8.5.



Compute all derivatives and verify properties (1)–(3).

8.6. $\left(\frac{1}{2}x+\frac{1}{2}y\right)^2-\left(\frac{1}{2}x^2+\frac{1}{2}y^2\right)=\frac{1}{4}x^2+\frac{1}{2}xy+\frac{1}{4}y^2-\frac{1}{2}x^2-\frac{1}{2}y^2=-\frac{1}{4}x^2+\frac{1}{2}xy-\frac{1}{4}y^2=-\left(\frac{1}{4}x^2-\frac{1}{2}xy+\frac{1}{4}y^2\right)=-\left(\frac{1}{2}x-\frac{1}{2}y\right)^2<0.$

8.7. As f is concave, $f''(x)\le 0$ for all x.
Hence $g''(x)=(-f(x))''=-f''(x)\ge 0$ for all x, i.e., g is convex.

8.8. As f is concave we find for all $x,y\in D_f$ and all $h\in[0,1]$,

$$f((1-h)x+h\,y)\ge(1-h)f(x)+h\,f(y)\,.$$

But then

$$\begin{aligned}g((1-h)x+h\,y)&=-f((1-h)x+h\,y)\\&\le-((1-h)f(x)+h\,f(y))\\&=(1-h)(-f(x))+h\,(-f(y))\\&=(1-h)g(x)+h\,g(y)\end{aligned}$$

i.e., g is convex by definition.

8.9. $h''(x)=\alpha f''(x)+\beta g''(x)\le 0$, i.e., h is concave. If $\beta<0$ then $\beta g''(x)$ is positive and the sign of $\alpha f''(x)+\beta g''(x)$ cannot be estimated any more.

8.10. Please find your own example.

8.11. Please find your own example.

8.12. (a) local maximum in $x=0$;

(b) local minimum in $x=1$, local maximum in $x=-1$;

(c) local minimum in $x=3$.

8.13. (a) global minimum in $x=1$, no global maximum;

(b) global maximum in $x=\frac{1}{4}$, no global minimum;

(c) global minimum in $x=0$, no global maximum;

(d) global minimum in $x = 1$, no global maximum;

(e) global maximum in $x = 1$, no global minimum.

8.14. (a) global maximum in $x = 12$, global minimum in $x = 8$;
(b) global maximum in $x = 6$, global minimum in $x = 3$;
(c) global maxima in $x = -2$ and $x = 2$,
global minima in $x = -1$ and $x = 1$.

9.1. (a) $2(x-1)e^x + c$;

(b) $-(x^2 + 2x + 2)e^{-x} + c$;

(c) $\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + c$;

(d) $\frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4 + c$;

(e) $\frac{1}{2}x^2(\ln(x))^2 - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + c$;

(f) $2x\sin(x) + (2 - x^2)\cos(x)$.

9.2. (a) $z = x^2$: $\frac{1}{2}e^{x^2} + c$;

(b) $z = x^2 + 6$: $\frac{2}{3}\left(x^2 + 6\right)^{\frac{3}{2}} + c$;

(c) $z = 3x^2 + 4$: $\frac{1}{6}\ln|3x^2 + 4| + c$;

(d) $z = x + 1$: $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$;

(e) $z = \ln(x)$: $\frac{1}{2}(\ln(x))^2 + c$.

9.3. (a) $z = \ln(x)$: $\ln|\ln(x)| + c$;

(b) $z = x^3 + 1$: $\frac{2}{9}(x^3 + 1)^{3/2} + c$;

(c) $z = 5 - x^2$: $-\sqrt{5 - x^2} + c$;

(d) $z = x - 3$: $7\ln|x - 3| + \frac{1}{2}x^2 + 2x + c$;

(e) $z = x - 8$: $\frac{2}{5}(x-8)^{5/2} + \frac{16}{3}(x-8)^{3/2} + c$.

9.4. (a) 39;

(b) $3e^2 - 3 \approx 19.16717$;

(c) 93;

(d) $-\frac{1}{6}$ (use radian!);

(e) $\frac{1}{2}\ln(8) \approx 1.0397$.

9.5. (a) $\frac{1}{2}$;

(b) $\frac{781}{10}$;

(c) $\frac{8}{3}$;

(d) $\frac{1}{2}\ln(5) - \frac{1}{2}\ln(2) \approx 0.4581$.

9.6. (a) $1 - e^{-2} \approx 0.8647$;

(b) $\frac{27}{4}$;

(c) 1;

(d) $2e^2 - 2 \approx 12.778$;

(e) $\frac{8}{3}\ln(2) - \frac{7}{9} \approx 1.07061$.

9.7. $\displaystyle\int_{-2}^{2} x^2 f(x)\,dx = \frac{1}{6}$.

9.8. $F(x) = \begin{cases} 0, & \text{for } x \le -1, \\ \frac{1}{2} + \frac{2x+x^2}{2}, & \text{for } -1 < x \le 0, \\ \frac{1}{2} + \frac{2x-x^2}{2}, & \text{for } 0 < x \le 1, \\ 1, & \text{for } x > 1. \end{cases}$