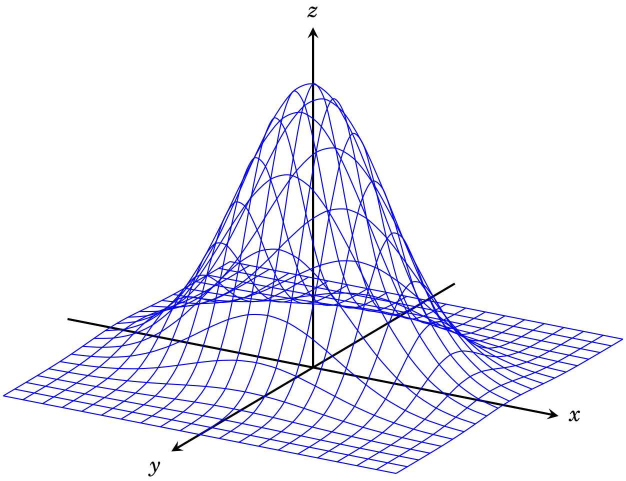
Bridging Course Mathematics

Winter Semester 2023/24

Josef Leydold



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Preface

We have made the experience that prior knowledge in mathematics (and statistics) of students of the master programme in *economics* differ heavily. We have students with courses in mathematics with a total of 25 ECTS points (or more) in their bachelor programme as well as students who did not attend any courses in mathematics at all. Their knowledge differ in basic *skills* (like computations with "symbols"), *tools* (like methods for optimization), and mathematical *reasoning* (i.e., proving one's claim).

In particular the following tasks cause issues for quite a few students:

- Drawing (or sketching) of graphs of functions.
- Transforming equations into equivalent ones.
- Handling inequalities.
- Correct handling of fractions.
- Calculations with exponents and logarithms.
- Obstructive multiplying of factors.
- Usage of mathematical notation.

Presented *"solutions"* of such calculation subtasks are surprisingly often *wrong*.

Thus this *bridging course* is intended to help participants to *close* possible knowledge gaps, and *rise* prior knowledge in *basic* mathematical *skills* to the same higher level.

Thus the chapters in this manuscript are organized as following:

- Mathematical notions and methods are repeated.
- Example problems are solved in details.
- A collection of problems with solutions allow to practice these methods. Details for finding these solution can be discussed in class and online sessions, resp.

It should be mentioned here that the subject matter may not be presented in a linear way as the intention of this course is to fill knowledge gaps rather than to teach new material.

There is another point of incomplete solutions. When students do homework or exam problems then they often submit mere calculations instead of an answer to the given question. These computations are of course necessary to give a correct answer. But it *is not* the answer to the problem. You also have to show that you can draw the right conclusions from your computations. Here is a very simple example.

Incomplete Solution. Example 0.1

Problem: Find all local minima of function $f(x) = (x-1)^2$.
"Solution": $f'(x) = 2(x-1) = 0 \Rightarrow x = 1 \quad f''(x) = 2 > 0$.
Of course all required computations are given. But the answer to the given question is missing[1]:
"Function f has a unique local minimum in $x = 1$."

I hope that this course and this manuscript will help to recall basic mathematical notion and methods and to gain confidence in computations. The presented matter is prerequisite for your courses in the master programme in "Economics".

Finally I want to recommend books from *Schaum's Outline Series* (McGraw Hill). They offer a lot of problems and exercise for training mathematical computations[2] (and related disciplines including microeconomics).

Josef Leydold

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[1] This may look pettifogging. However, one needs to know that "$f'(x_0) = 0$" is a necessary and "$f'(x_0) = 0$ and $f''(x_0) > 0$" is a sufficient condition for a local minimum x_0 of a two-times differentiable univariate function in order to give this answer.

[2] For example, Elliot Mendelson (2003), Beginning Calculus, 3rd ed., covers some of the chapters in this manuscript.

1

Sets and Maps

> Denke dir die Mengenlehre wäre als eine Art Parodie auf die Mathematik von einem Satiriker erfunden worden. – Später hätte man dann einen vernünftigen Sinn gesehen und sie in die Mathematik einbezogen.
>
> Ludwig Wittgenstein (1889–1951)

1.1 Sets

In this section we want to combine objects (e.g., natural numbers or all consumers in our economic model) in a single entity, called a *set*.

The notion of *set* is fundamental in modern mathematics. Its formal definition is surprisingly cumbersome. So we use a simple approach from naïve set theory.

Definition 1.1

Set. A **set** is a collection of *distinct* objects.

set

element

A member a of a set A is called an **element** and we write

$$a \in A$$

Sets are defined by *enumerating* or a *description* of their elements within *curly brackets* $\{\ldots\}$.

Example 1.2

The following two sets are equivalent:

$$A = \{2,4,6,8,10\}$$

$$\begin{aligned} B &= \{x \mid x \text{ is a positive even integer less than } 12\} \\ &= \{x \colon x \text{ is a positive even integer less than } 12\} \end{aligned}$$

In the latter set x is any object that fits the criteria provided.

⚠

Notice that the ordering in the enumeration of the elements is ignored. Moreover elements of a set must be discriminable.

Table 1.4

Some important sets

Symbol	Description
$\emptyset$	empty set (sometimes: $\{\}$)
$\mathbb{N}$	natural numbers $\{1,2,3,\ldots\}$
$\mathbb{Z}$	integers $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$
$\mathbb{Q}$	rational numbers $\{\frac{k}{n} \mid k,n \in \mathbb{Z}, n \neq 0\}$
$\mathbb{R}$	real numbers
$[a,b]$	closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
(a,b)	open interval $\{x \in \mathbb{R} \mid a < x < b\}$ (also: $]a,b[$)
$[a,b)$	half-open interval $\{x \in \mathbb{R} \mid a \leq x < b\}$
$\mathbb{C}$	complex numbers $\{a+bi \mid a,b \in \mathbb{R},\ i^2 = -1\}$

Example 1.3

Sets $\{1,2,3,4\}$ and $\{4,3,2,1\}$ are identical as there representations only differ in their ordering of the elements.
Collection[1] $\{1,1,2,3\}$ is not a set as elements "1" occurs twice.

We will make heavy use of sets. The most frequent ones are listed in Table 1.4.

Venn Diagram and Basic Set Operations

When we work with *sets* we assumes that these are subsets of a *universal superset* Ω called the **universe**.

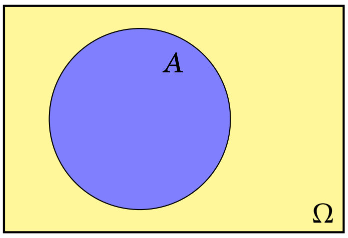
universe

Example 1.5

When we deal with intervals of real numbers like $[0,1)$, $[1,\infty)$, or $(2,4)$ we assume universe $\Omega = \mathbb{R}$.

Venn diagram

Sets can be represented (or visualized) by so called **Venn diagrams** . There universe Ω is drawn as a rectangle and sets are depicted as circles or ovals. Their interior represents the corresponding set.



subset

Set A is called a **subset** of B, $A \subseteq B$, if each element of A is also an element of B. That is, $A \subseteq B$ if $x \in A$ implies $x \in B$.

superset

Vice versa, B is then called a **superset** of A, $B \supseteq A$.

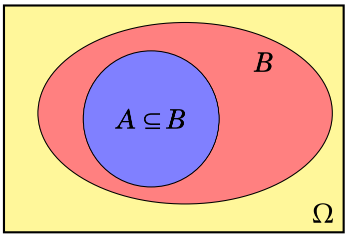
proper subset

Notice that $A \subseteq B$ includes the case where $A = B$. When this case is excluded then we call A a **proper subset**[2] of B, $A \subset B$. That is, $A \subset B$ if $A \subseteq B$ and $A \neq B$. In other words, A is a strictly smaller subset of B.

---

[1] Such a collection is sometimes called a *multiset*.

[2] Beware: There is an alternative notation in literature, where $A \subset B$ is used for subsets (where A is not necessarily distinct from B) and $A \subsetneq B$ for proper subsets.



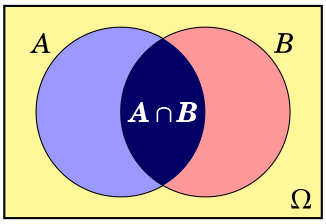
Example 1.6

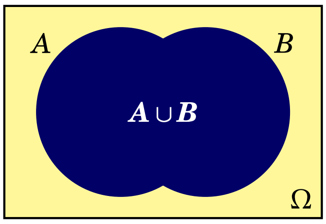
Let $\Omega = \mathbb{N}$, and $A = \{2,4,6,8\}$, $B = \{2,4\}$, $C = \{6,8\}$, $D = \{4,2\}$. Then (e.g.): $B \subset A$, $A \supset B$, $A \not\subseteq B$, $B = D$, $D \subseteq B$, $B \subseteq D$, $D \not\subset B$, $C \not\subseteq B$, and $B \not\subseteq C$.

When we have a collection of sets we can use basic set operations for creating new sets. Assume that A and B are two subsets in universe Ω.

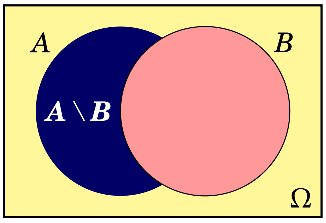
- **intersection** The **intersection** $A \cap B$ is the set all elements which are contained in both A *and* B.
- **disjoint** Two sets A and B are **disjoint** if $A \cap B = \emptyset$, i.e., if they share no elements.
- **union** The **union** $A \cup B$ is the set all elements which are contained in A *or* B (or both).

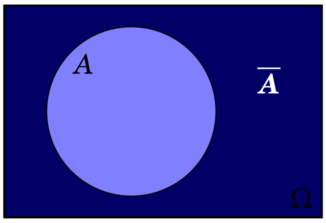
These operations can be well illustrates by means of Venn diagrams.





- **set difference** The **set difference** $A \setminus B$ is the set all elements in A which are *not* contained in B.
- **complements** The **complements** $\overline{A} = \Omega \setminus A$ is the set all elements in the universe that are not contained in A.





Example 1.8

Let $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$, and $A = \{2,4,6,8\}$, $B = \{1,2,3,4\}$, and $C = \{3,5\}$.
Then: $A \cap B = \{2,4\}$, $A \cup B = \{1,2,3,4,6,8\}$, $A \setminus B = \{6,8\}$, $B \setminus A = \{1,3\}$, $\overline{A} = \{1,3,5,7,9,10\}$.
Moreover, $A \cap C = \emptyset$, i.e., A and C are disjoint.

Table 1.7

Basic set operations

Symbol	Definition	Name
$A \cap B$	$\{x \mid x \in A \text{ and } x \in B\}$	*intersection*
$A \cup B$	$\{x \mid x \in A \text{ or } x \in B\}$	*union*
$A \setminus B$	$\{x \mid x \in A \text{ and } x \not\in B\}$	*set-theoretic difference*[a]
$\overline{A}$	$\Omega \setminus A$	*complement*

---

[a] also: $A - B$

Table 1.9

Rules for basic set operations

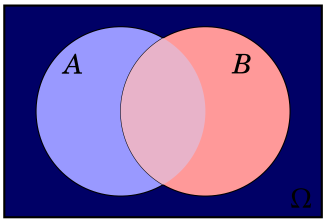
Rule	Name
$A \cup A = A \cap A = A$	*Idempotence*
$A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$	*Identity*
$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$	*Associativity*
$A \cup B = B \cup A$ and $A \cap B = B \cap A$	*Commutativity*
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	*Distributivity*
$\overline{A} \cup A = \Omega$ and $\overline{A} \cap A = \emptyset$ and $\overline{\overline{A}} = A$	
$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ and $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$	*de Morgan's laws*

Table 1.7 summarizes the basic set operations. Similarly to arithmetic computations with numbers there exist rules for these, see Table 1.9. For example, the intersection of any set A with its complement is the empty set $\emptyset$. These rules can be again visualized ("proved") by drawing Venn diagrams of either side of these equations.

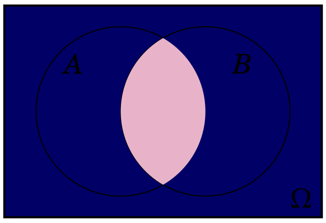
de Morgan's law

De Morgan's law is one such rule:

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$



$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$



Other Set Operations

In many applications we want to combine elements of two (or more) sets into ordered pairs (triples, n-tuples). The set of all these tuples is called the **Cartesian Product** $A \times B$ of the two sets A and B.

Cartesian Product

$$A \times B = \{(x,y) | x \in A, y \in B\}$$

In general we have $A \times B \neq B \times A$.

⚠

Example 1.10

The Cartesian product of $A = \{0,1\}$ and $B = \{2,3,4\}$ is

$$A \times B = \{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}\,.$$

We can visualize this Cartesion product by means of an array:

$A \times B$	2	3	4
0	$(0,2)$	$(0,3)$	$(0,4)$
1	$(1,2)$	$(1,3)$	$(1,4)$

Example 1.11

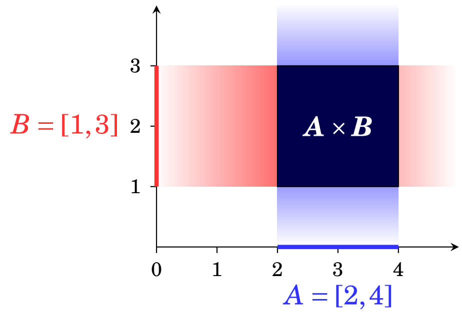
A well-known example are coordinates on a road map where the vertical side is partitioned into interval labeled by $\{A,B,C,D,E,F,G\}$ and the horizontal side is partitioned into intervals labeled by $\{1,2,3,4,5,6\}$. Then the map is partitioned into a grid where each rectangle correspond to the elements of the Cartesian product

$$\{A,B,C,D,E,F,G\} \times \{1,2,3,4,5,6\} = \{A1,A2,\ldots,A6,\ldots,\ldots,G1,\ldots,G6\}$$

Example 1.12

The Cartesian product of $A = [2,4]$ and $B = [1,3]$ is

$$A \times B = \{(x,y) \mid x \in [2,4] \text{ and } y \in [1,3]\}\,.$$



Example 1.13

The Cartesian product of two sets of reals numbers (which can be interpreted as one-dimensional number line) is (can be interpreted as) the two-dimension number plane.

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x_1,x_2) \mid x_1,x_2 \in \mathbb{R}\}\,.$$

For a given set A we can create the set of all subsets $\mathscr{P}(A)$ of A called the **powerset** of A,

powerset

$$\mathscr{P}(A) = \{S \mid S \subseteq A\}\,.$$

The elements of powerset $\mathscr{P}(A)$ are also sets.
Observe that we always have $\emptyset \in \mathscr{P}(A)$ and $A \in \mathscr{P}(A)$.

Example 1.14

Let $A = \{1,2,3\}$. Then $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A\}$.
Do not mix up: $1 \in A$, $\{1\} \subset A$, and $\{1\} \in \mathscr{P}(A)$.

⚠

Remark: Axiomatic Set Theory

Definition 1.1 of sets does not work in Mathematics as it causes contradictions. Take the collection of all sets (over all universes). As we can distinguish between two sets this collection is a *set* by our naïve definition. But then this set contains itself (as it is the set of all sets). So we can conclude that there is at least one set that contains itself. So let R be the class of all sets that are *not* contained in itself,

$$R = \{x \mid x \notin x\}\,.$$

Question: Is R contained in R, $R \in R$? We find

$$R \notin R \quad \Longleftrightarrow \quad R \in R$$

which obviously does not make sense[3]. So R cannot be a set albeit it is a subset of the "set" of all sets.
Conclusion: The collection of all sets cannot be a set.

This antinomy and in particular the idea of a set that contains itself may look weird. So let us look at a real world example. Assume that there is an old-fashioned library of books. This library contains a book that lists all books of the library. As it is also a book of this library it thus lists itself. Now we want to add a new book to this library that lists all books of the library that do not list themselves. Do we have to list this book in the book that do not list themselves?

In order to get rid of such antimonies sets are defined using so called **set of axioms** [4] that define basic properties of sets (like "the union of two sets is a set").

axiom

---

[3] This is called Russell's antinomy. It is related to the liars paradox.

[4] e.g., Zermelo-Fraenkel axioms, or ZF for short

1.2 Maps

Once we have sets we want to describe relations between the elements of two sets. The most important tool are functions that map every element in one set called *domain* to an element of another set called *range*.

Definition 1.15

Map. A **map** (or *mapping*) f is defined by

map

(i) a **domain** D,

domain

(ii) a **range** (*target set*, *co-domain*) W, and)

range

(iii) a *rule*, that maps each element of D to *exactly one* element of W.

$$f: D \to W, \quad x \mapsto y = f(x)$$

- x is called the **independent** variable, y the **dependent** variable.
- y is the **image** of x, x is the **preimage** of y.
- $f(x)$ is the **function term**, x is called the **argument** of f.
- $f(D) = \{y \in W : y = f(x) \text{ for some } x \in D\}$ is the **image** of f.

image

Maps are also called *functions* or *transformations*.

Injective · Surjective · Bijective

⚠

Note that each argument $x \in D_f$ of a function has *exactly one* image y. On the other hand, $y \in W_f$, may have any number of preimages. Thus we can characterize maps by their possible number of preimages.

- A map f is called **one-to-one** (or *injective*), if each element in the codomain has *at most one* preimage.

one-to-one

- It is called **onto** (or *surjective*), if each element in the codomain has *at least one* preimage.

onto

- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

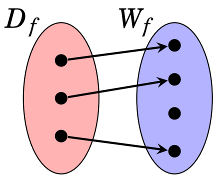
bijective

⚠

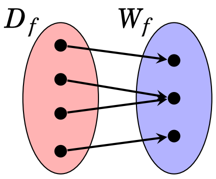
Injections have the important property

$$f(x) \neq f(y) \iff x \neq y$$

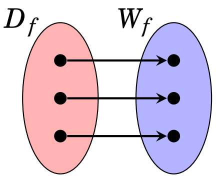
Maps can be visualized by means of arrows.



one-to-one
(not onto)



onto
(not one-to-one)



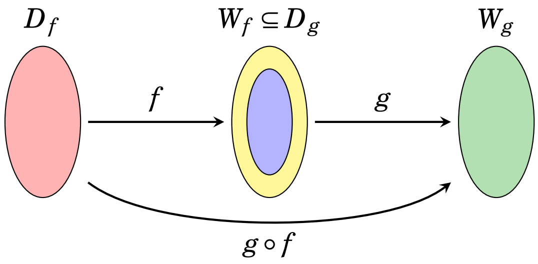
one-to-one and onto
(bijective)

Function Composition

Let $f : D_f \to W_f$ and $g : D_g \to W_g$ be functions with $W_f \subseteq D_g$. Then we can evaluate g at the image of f at some point x. Thus we get the **composite function**[5]

composite function

$$g \circ f : D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))$$

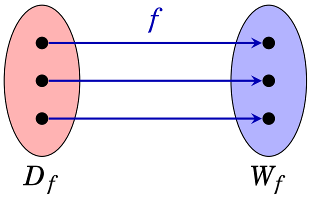


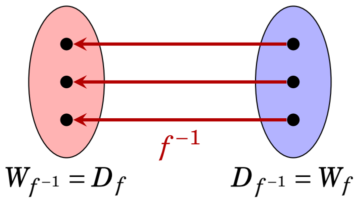
If $f : D_f \to W_f$ is a *bijection*, then every $y \in W_f$ can be uniquely mapped to its preimage $x \in D_f$. Thus we get a map

$$f^{-1} : W_f \to D_f,\ y \mapsto x = f^{-1}(y)$$

which is called the **inverse map** of f.

inverse map





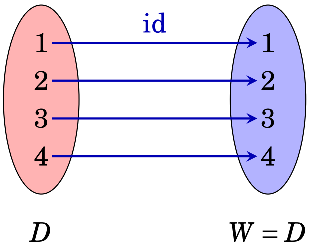
We obviously have for all $x \in D_f$ and $y \in W_f$,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y\,.$$

The most elementary function is the **identity map** id which maps its argument to itself, i.e.,

identity map

$$\mathrm{id} : D \to W = D,\ x \mapsto x$$



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[5] Read: "g composed with f", "g circle f", or "g after f".

The identity map has a similar role for compositions of functions as 1 has for multiplications of numbers:

$$f \circ \mathrm{id} = f \qquad \text{and} \qquad \mathrm{id} \circ f = f$$

Moreover,

$$f^{-1} \circ f = \mathrm{id}\colon D_f \to D_f \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}\colon W_f \to W_f$$

Real-valued Functions

Maps where domain and codomain are (subsets of) *real* numbers are called **real-valued functions** ,

real-valued function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto f(x)$$

and are the most important kind of functions.
The term **function** is often exclusive used for *real-valued* maps.
We will discuss such functions in more details later.

function

— Summary

- sets, subsets and supersets
- Venn diagram
- basic set operations
- de Morgan’s law
- Cartesian product
- maps
- one-to-one and onto
- inverse map and identity

— Exercises

1.1 Which of the the following sets is a subset of

$$A = \{x \mid x \in \mathbb{R} \text{ and } 10 < x < 200\}$$

(a) $\{x \mid x \in \mathbb{R} \text{ and } 10 < x \leq 200\}$

(b) $\{x \mid x \in \mathbb{R} \text{ and } x^2 = 121\}$

(c) $\{x \mid x \in \mathbb{R} \text{ and } 4\pi < x < \sqrt{181}\}$

(d) $\{x \mid x \in \mathbb{R} \text{ and } 20 < |x| < 100\}$

1.2 The set $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$ has subsets $A = \{1,3,6,9\}$, $B = \{2,4,6,10\}$ and $C = \{3,6,7,9,10\}$.

Draw the Venn diagram and give the following sets:

(a) $A \cup C$ (b) $A \cap B$ (c) $A \setminus C$

(d) $\overline{A}$ (e) $(A \cup C) \cap B$ (f) $(\overline{A} \cup B) \setminus C$

(g) $\overline{(A \cup C)} \cap B$ (h) $(\overline{A} \setminus B) \cap (\overline{A} \setminus C)$ (i) $(A \cap B) \cup (A \cap C)$

1.3 Mark the following set in the corresponding Venn diagram:

$$(A \cap \overline{B}) \cup (A \cap B)$$

1.4 Simplify the following set-theoretic expression:

$$(A \cap \overline{B}) \cup (A \cap B)$$

1.5 Simplify the following set-theoretic expressions:

(a) $\overline{(A \cup B)} \cap \overline{B}$ (b) $(A \cup \overline{B}) \cap (A \cup B)$

(c) $((\overline{A} \cup \overline{B}) \cap \overline{(A \cap \overline{B})}) \cap A$ (d) $(C \cup B) \cap \overline{(\overline{C} \cap \overline{B})} \cap (C \cup \overline{B})$

1.6 Describe the Cartesian products of

(a) $A = [0,1]$ and $P = \{2\}$.

(b) $A = [0,1]$ and $Q = \{(x,y)\colon 0 \leq x, y \leq 1\}$.

(c) $A = [0,1]$ and $O = \{(x,y)\colon 0 < x, y < 1\}$.

(d) $A = [0,1]$ and $C = \{(x,y)\colon x^2 + y^2 \leq 1\}$.

(e) $A = [0,1]$ and $\mathbb{R}$.

(f) $Q_1 = \{(x,y)\colon 0 \leq x, y \leq 1\}$ and $Q_2 = \{(x,y)\colon 0 \leq x, y \leq 1\}$.

1.7 We are given map

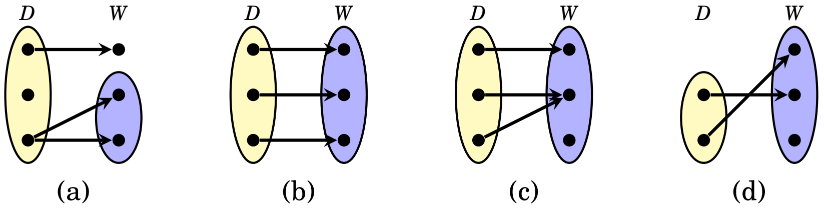
$$\varphi\colon [0,\infty) \to \mathbb{R},\ x \mapsto y = x^{\alpha} \qquad \text{for some } \alpha > 0$$

What are

- function name,
- domain,
- codomain,
- image (range),
- function term,
- argument,
- independent and dependent variable?

1.8 Which of these diagrams represent maps?

Which of these maps are one-to-one, onto, both or neither?



1.9

Which of the following are proper definitions of mappings?

Which of the maps are one-to-one, onto, both or neither?

(a) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^2$

(b) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^{-2}$

(c) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto x^2$

(d) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto \sqrt{x}$

(e) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \sqrt{x}$

(f) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \{y \in [0,\infty)\colon x = y^2\}$

1.10 Let $\mathscr{P}_n = \{\sum_{i=0}^{n} a_i x^i \colon a_i \in \mathbb{R}\}$ be the set of all polynomials in x of degree less than or equal to n.

Which of the following are proper definitions of mappings?

Which of the maps are one-to-one, onto, both or neither?

(a) $D : \mathscr{P}_n \to \mathscr{P}_n,\ p(x) \mapsto \frac{dp(x)}{dx}$ (derivative of p)

(b) $D : \mathscr{P}_n \to \mathscr{P}_{n-1},\ p(x) \mapsto \frac{dp(x)}{dx}$

(c) $D : \mathscr{P}_n \to \mathscr{P}_{n-2},\ p(x) \mapsto \frac{dp(x)}{dx}$

2

Terms

2.1 Terms

A mathematical expression like

$$B = R \cdot \frac{q^n - 1}{q^n(q-1)} \qquad \text{or} \qquad (x+1)(x-1) = x^2 - 1$$

contains symbols which denote mathematical objects. These symbols and compositions of symbols are called **terms** . Terms can be **term**

- **numbers** , **number**
- **constants** (symbols, which represent *fixed* values), **constant**
- **variables** (which are placeholders for *arbitrary* values), and **variable**
- *compositions* of terms.

We have to take care that a term may not be defined for some values of its variables.

Example 2.1

$\frac{1}{x-1}$ is only defined for $x \in \mathbb{R} \setminus \{1\}$.

$\sqrt{x+1}$ is only defined for $x \geq -1$.

The set of values for which a term is defined is called the **domain** of the term. **domain**

In this chapter we discuss various basic terms and rules for handling such expressions.

2.2 Sigma Notation

Sums with many terms that can be generated by some rule can be represented in a compact form called **summation** or **sigma notation**. **summation** **sigma notation**

$$\sum_{n=1}^{6} a_n = a_1 + a_2 + \cdots + a_6$$

Table 2.3

Rules for sigma notation

- $\sum_{i=1}^{n} a_i = \sum_{k=1}^{n} a_k$
- $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$
- $\sum_{i=1}^{n} c\,a_i = c \sum_{i=1}^{n} a_i$

where a_n … formula for the terms
n … index of summation
1 … first value of index n
6 … last value of n
The expression is read as the "sum of a_n as n goes from 1 to 6."
The first and last value can (and often are) given by symbols.

Example 2.2

The sum of the first 10 integers greater than 2 can be written in both notations as

$$\sum_{i=1}^{10} (2+i) = (2+1) + (2+2) + (2+3) + \cdots + (2+10)$$

The sigma notation can be seen as a convenient shortcut of the long expression on the r.h.s. It also avoids ambiguity caused by the ellipsis "…" that might look like an IQ test rather than an exact mathematical expression.

⚠

Keep in mind that all the *usual rules* for multiplication and addition (associativity, commutativity, distributivity) apply, see Table 2.3.

Example 2.4

Simplify

$$\sum_{i=2}^{n} (a_i + b_i)^2 - \sum_{j=2}^{n} a_j^2 - \sum_{k=2}^{n} b_k^2$$

Solution:

$$\begin{aligned}
\sum_{i=2}^{n} (a_i + b_i)^2 - \sum_{j=2}^{n} a_j^2 - \sum_{k=2}^{n} b_k^2 &= \sum_{i=2}^{n} (a_i + b_i)^2 - \sum_{i=2}^{n} a_i^2 - \sum_{i=2}^{n} b_i^2 \\
&= \sum_{i=2}^{n} \left((a_i + b_i)^2 - a_i^2 - b_i^2\right) \\
&= \sum_{i=2}^{n} \left(a_i^2 + 2a_i b_i + b_i^2) - a_i^2 - b_i^2\right) \\
&= \sum_{i=2}^{n} 2 a_i b_i \\
&= 2 \sum_{i=2}^{n} a_i b_i
\end{aligned}$$

2.3 Absolute Value

absolute value

modulus

The **absolute value** (or **modulus**) $|x|$ of a number x is its distance from origin 0 on the number line:

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

This formal definition may seem complicated. However, we will need it for computations. Observe that $|x|$ results in two (or more) cases which we have to handle separately.

Example 2.5

$|5| = 5$ and $|-3| = -(-3) = 3$.

We have the simple rules

$$|x| \cdot |y| = |x \cdot y| \qquad \text{and} \qquad |x| \geq x\,.$$

2.4 Powers and Roots

power

The n-th **power** of x (for $n \in \mathbb{N}$) is defined by[1]

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$$

where

- x is the **basis**, and
- n is the **exponent** of x^n.

basis

exponent

For *negative* exponents we define:

$$x^{-n} = \frac{1}{x^n}$$

root

A number y is called the **n-th root** $\sqrt[n]{x}$ of x, if $y^n = x$.
Computing the n-th root can be seen as the inverse operation of computing a power.

square root

⚠

We just write $\sqrt{x}$ for the **square root** $\sqrt[2]{x}$.
Symbol $\sqrt[n]{x}$ is used for the *positive* (real) root of x.
If we need the negative square root of 2 we have to write $-\sqrt{2}$.

---

[1] Read: “x raised to the n-th power”, “x raised to the power of n”, or “the n-th power of x”.

Table 2.6

Calculation Rule for Powers and Roots

$x^{-n} = \frac{1}{x^n}$	$x^0 = 1$	$(x \neq 0)$	$n, m \in \mathbb{N}$
$x^{n+m} = x^n \cdot x^m$	$x^{\frac{1}{m}} = \sqrt[m]{x}$	$(x \geq 0)$	
$x^{n-m} = \frac{x^n}{x^m}$	$x^{\frac{n}{m}} = \sqrt[m]{x^n}$	$(x \geq 0)$	
$(x \cdot y)^n = x^n \cdot y^n$	$x^{-\frac{n}{m}} = \frac{1}{\sqrt[m]{x^n}}$	$(x \geq 0)$	
$\left(x^n\right)^m = x^{n \cdot m}$			

Powers with *rational exponents* are defined by

$$x^{\frac{1}{m}} = \sqrt[m]{x} \qquad \text{for } m \in \mathbb{Z} \text{ and } x \geq 0;$$

and

$$x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad \text{for } m, n \in \mathbb{Z} \text{ and } x \geq 0.$$

⚠ **Important:**
For non-integer exponents the basis must be non-negative!

Rules for computations with powers and roots are listed in Table 2.6. The rule $x^0 = 1$ may seem weird. However, it becomes sensible if we do the following computation for an $x > 0$ and $n \in \mathbb{N}$ where we apply some of the other rules:

$$x^0 = x^{n-n} = x^n \cdot x^{-n} = \frac{x^n}{x^n} = 1\,.$$

⚠ **Important:** 0^0 is *not* defined!
It is not as there is no sensible way to do so. Observe that rules $x^0 = 1$ and $0^n = 0$ would imply two different values.

Powers x^α can also be generalized for $\alpha \in \mathbb{R}$. However, the definition is quite technical and out of scope for this course. We restrict ourselves to the fact that there is a button on the calculator for computing its value.

Example 2.7

$$\sqrt[3]{5^6} = (5^6)^{\frac{1}{3}} = 5^{(6 \cdot \frac{1}{3})} = 5^{\frac{6}{3}} = 5^2 = 25$$

$$(\sqrt[3]{5})^6 = (5^{\frac{1}{3}})^6 = 5^{(\frac{1}{3} \cdot 6)} = 5^{\frac{6}{3}} = 5^2 = 25$$

$$5^{4-3} = 5^1 = 5$$

$$5^{4-3} = \frac{5^4}{5^3} = \frac{625}{125} = 5$$

$$5^{2-2} = 5^0 = 1$$

$$5^{2-2} = \frac{5^2}{5^2} = \frac{25}{25} = 1$$

Example 2.8

$$\frac{(x \cdot y)^4}{x^{-2}y^3} = x^4 y^4 x^{-(-2)} y^{-3} = x^{4+2} y^{4-3} = x^6 y$$

$$\frac{(2x^2)^3(3y)^{-2}}{(4x^2y)^2(x^3y)} = \frac{2^3 x^{2\cdot 3} 3^{-2} y^{-2}}{4^2 x^{2\cdot 2} y^2 x^3 y} = \frac{\frac{8}{9} x^6 y^{-2}}{16x^7 y^3}$$

$$= \frac{1}{18} x^{6-7} y^{-2-3} = \frac{1}{18} x^{-1} y^{-5} = \frac{1}{18xy^5}$$

$$\left(3x^{\frac{1}{3}} y^{-\frac{4}{3}}\right)^3 = 3^3 x^{\frac{3}{3}} y^{-\frac{12}{3}} = 27xy^{-4} = \frac{27x}{y^4}$$

Sources of Errors: ⚠

- $-x^2$ is *not* equal to $(-x)^2$!
- $(x+y)^n$ is *not* equal to $x^n + y^n$! ⚠
- $x^n + y^n$ can*not* be simplified (in general)!

2.5 Polynomials

monomial

degree

A **monomial** is a real number, variable, or product of variables raised to positive integer powers. The **degree** of a monomial is the sum of all exponents of the variables (including a possible implicit exponent 1).

Example 2.9

$6x^2$ is a monomial of degree 2.

$3x^3y$ and xy^2z are monomials of degree 4.

$\sqrt{x}$ and $\frac{2}{3xy^2}$ are *not* monomials.

polynomial

A **polynomial** is a sum of (one or more) monomials. The **degree** of a polynomial is the maximum degree among its monomials.

Example 2.10

$4x^2y^3 - 2x^3y + 4x + 7y$ is a polynomial of degree 5.
Note that the degrees of the monomials are 5, 4, 1, 1, resp., with maximum degree 5.

Polynomials in *one variable* can be written in sigma notation as

$$P(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where x is the variable and $a_i \in \mathbb{R}$ are constants.

Binomial Theorem

Sometimes one has to compute the n-th power of the sum of two terms $x+y$. For the second power (i.e., squares) a simple straightforward computation yields

$$(x+y)^2 = x^2 + 2xy + y^2 \,.$$

For higher (large) values of n a summation formula can be derived.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

binomial coefficient

is called the **binomial coefficient**[2] and

$$n! = 1\cdot 2\cdot \ldots \cdot n$$

factorial

denotes the **factorial**[3] of n. The factorial function $n!$ just means to multiply a series of ascending natural numbers.

Example 2.11

$7! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7 = 5040$

These quantities originate from problems in combinatorics. The factorial operation counts the possible distinct sequences – the permutations – of n distinct objects. That is, $n!$ is the number of permutations of n objects. For convenience we set $0! = 1$. This corresponds to the observation that there is exactly one empty set (i.e., a set with 0 elements).

The binomial coefficient on the other hand counts the number of combinations of k elements that we draw from an urn with n distinct objects. In this case we do not care about the ordering of the objects. That is, $\binom{n}{k}$ is the number of subsets with k elements which we can draw from a set with n elements. For convenience we set $\binom{n}{0} = 1$. This corresponds to the observation that there is exactly one possibility to draw no object from the urn (and that there is exactly one empty subset).
We also have the following obvious relation:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

Observe that the factorial operation results in huge numbers even for moderate n. E.g., $15! = 1307674368000$. So there is some risk of

---

[2] Read: “n choose k”.

[3] Read: “n-factorial”.

overflow when using the above formula for computing binomial coefficients. Nevertheless, one can cancel factor $(n-k)!$ and obtain a formula for computing binomial coefficient even by simple calculators.

$$\binom{n}{k}=\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot\ldots\cdot\quad 1}$$

Example 2.12

The number of possible combinations in Austrian lottery is given by

$$\binom{45}{6}=\frac{45\cdot 44\cdot 43\cdot 42\cdot 41\cdot 40}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}=8\,145\,060\,.$$

Observe that $45!\approx 1.196\cdot 10^{56}$.

Example 2.13

By means of the binomial theorem we obtain:

$$(x+y)^2=\binom{2}{0}x^2+\binom{2}{1}xy+\binom{2}{2}y^2=x^2+2xy+y^2$$

$$(x+y)^3=\binom{3}{0}x^3+\binom{3}{1}x^2y+\binom{3}{2}xy^2+\binom{3}{3}y^3=x^3+3x^2y+3xy^2+y^3$$

Multiplication

The product of two polynomials of degree n and m, resp., is a polynomial of degree $n+m$.

Example 2.14

The product of polynomials of degree 2 and 3, resp, has degree 5.

$$\begin{aligned}
&(2x^2+3x-5)\cdot(x^3-2x+1)=\\
&\qquad=2x^2\cdot x^3+2x^2\cdot(-2x)+2x^2\cdot 1\\
&\qquad\quad+3x\cdot x^3+3x\cdot(-2x)+3x\cdot 1\\
&\qquad\quad+(-5)\cdot x^3+(-5)\cdot(-2x)+(-5)\cdot 1\\
&\qquad=2x^5+3x^4-9x^3-4x^2+13x-5
\end{aligned}$$

Polynomials can be *divided* analogously to the division of integers.

Example 2.15

$$\begin{array}{rl}
 & (x^3+x^2\quad+0x-2):(x-1)=\boldsymbol{x^2+2x+2}\\
\boldsymbol{x^2}\cdot(x-1)\longrightarrow & \underline{x^3-x^2}\\
 & \quad 2x^2+0x\\
\boldsymbol{2x}\cdot(x-1)\longrightarrow & \quad\underline{2x^2-2x}\\
 & \qquad\quad 2x-2\\
\boldsymbol{2}\cdot(x-1)\longrightarrow & \qquad\quad\underline{2x-2}\\
 & \qquad\qquad\quad 0
\end{array}$$

We thus yield $\quad x^3+x^2-2=(x-1)\cdot(x^2+2x+2)$.

If the *divisor* is not a factor of the *dividend*, then we obtain a *remainder*.

The process of expressing a polynomial as the product of polynomials of smaller degree (**factor**) is called **factorization**.

factorization

Example 2.16

$$2x^2+4xy+8xy^3=2x\cdot(x+2y+4y^3)$$

$$x^2-y^2=(x+y)\cdot(x-y)$$

$$x^2-1=(x+1)\cdot(x-1)$$

$$x^2+2xy+y^2=(x+y)\cdot(x+y)=(x+y)^2$$

$$x^3+y^3=(x+y)\cdot(x^2-xy+y^2)$$

These products can be easily verified by multiplying their factors.

⚠

The factorizations

$$(x^2-y^2)=(x+y)\cdot(x-y)$$

and

$$(x-y)=(\sqrt{x}+\sqrt{y})\cdot(\sqrt{x}-\sqrt{y})$$

can be very useful and should be memorized.

The “Ausmultiplizierreflex”

Factorizing a polynomial is often *very hard* while multiplying its factors is fast and easy[4].

⚠

A factorized expression contains more information than their expanded counterpart.

⚠

In my experience many students have an *acquired “Ausmultiplizierreflex”*: *Instantaneously* (and *without thinking*) they *multiply* all factors which often turns a simple problem into a difficult one.

⚠

Suppress your “Ausmultiplizierreflex”!

Think first and multiply factors only when it seems to be useful!

Linear Factors

A polynomial of *degree* 1 is called a **linear term**.

linear term

Example 2.17

$a+bx+y+ac$ is a linear term in x and y, if a, b and c are constants.

$xy+x+y$ is not linear, as xy has degree 2.

linear factor

A factor (polynomial) of degree 1 is called a **linear factor**.
A polynomial in one variable x with root x_1 has linear factor $(x-x_1)$.

If a polynomial in x of degree n,

$$P(x)=\sum_{i=0}^{n} a_i x^i$$

has n real roots $x_1,x_2,\dots,x_n$, then it can be written[5] as the product of the n linear factors $(x-x_i)$:

$$P(x)=a_n \prod_{i=1}^{n}(x-x_i)$$

2.6 Rational Terms

rational term

A **rational term** is one of the form

$$\frac{P(x)}{Q(x)}$$

numerator

denominator

where $P(x)$ and $Q(x)$ are polynomials called **numerator** and **denominator**, resp.
Alternatively one can write $P(x)/Q(x)$.

The domain of a rational term is $\mathbb{R}$ without the roots of the denominator.

Example 2.18

$\dfrac{x^2+x-4}{x^3+5}$ is a rational term with domain $\mathbb{R}\setminus\{-\sqrt[3]{5}\}$.

⚠

Beware! Expression $\dfrac{0}{0}$ is not defined.

⚠

Many years of teaching experiences show that calculations with rational terms is an obscure thing for at least a few students and many errors show up in there solutions. Table 2.19 shows the complete list of calculation rules. There are no other general rules. Really. In particular addition of two rational terms is a frequent source of errors.

⚠

Very important! *Really!*
You have to expand fractions such that they have a **common denominator** *before* you add them!

Example 2.20

$$\frac{x^2-1}{x+1}=\frac{(x+1)(x-1)}{x+1}=x-1$$

$$\frac{4x^3+2x^2}{2xy}=\frac{2x^2(2x+1)}{2xy}=\frac{x(2x+1)}{y}$$

---

[4] The RSA public key encryption is based on this idea.

[5] This is a special case of the Fundamental Theorem of Algebra.

Table 2.19

Rule for fractions and rational terms

Let $b, c, e \neq 0$.

$\frac{c \cdot a}{c \cdot b} = \frac{a}{b}$	*Reduce*
$\frac{a}{b} = \frac{c \cdot a}{c \cdot b}$	*Expand*
$\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$	*Multiplying*
$\frac{a}{b} : \frac{e}{c} = \frac{a}{b} \cdot \frac{c}{e}$	*Dividing*
$\frac{\frac{a}{b}}{\frac{e}{c}} = \frac{a \cdot c}{b \cdot e}$	*Compound fraction*
$\frac{a}{b} + \frac{d}{b} = \frac{a+d}{b}$	*Addition with common denominator*
$\frac{a}{b} + \frac{d}{c} = \frac{a \cdot c + d \cdot b}{b \cdot c}$	*Addition*

$$\frac{x+1}{x-1} + \frac{x-1}{x+1} = \frac{(x+1)^2 + (x-1)^2}{(x-1)(x+1)}$$
$$= \frac{x^2 + 2x + 1 + x^2 - 2x + 1}{(x-1)(x+1)} = 2\,\frac{x^2+1}{x^2-1}$$

⚠ Again I want to stress at this point that there happen *a lot of mistakes* in calculations that involve rational terms. The following examples of such fallacies are collected from students' exams.

Sources of Errors: ⚠

$\frac{a+c}{b+c}$	is *not* equal to	$\frac{a}{b}$
$\frac{x}{a} + \frac{y}{b}$	is *not* equal to	$\frac{x+y}{a+b}$
$\frac{a}{b+c}$	is *not* equal to	$\frac{a}{b} + \frac{a}{c}$

Example 2.21

$$\frac{x+2}{y+2} \neq \frac{x}{y}$$

$$\frac{1}{2} + \frac{1}{3} \neq \frac{1}{5}$$

$$\frac{1}{x^2+y^2} \neq \frac{1}{x^2} + \frac{1}{y^2}$$ ⚠

2.7 Exponent and Logarithm

We already have discussed powers of a number x. We can see this as **power function** of x with a fixed exponent α:

power function

$$(0,\infty) \to (0,\infty),\ x \mapsto x^\alpha \quad \text{for some } \textit{fixed exponent } \alpha \in \mathbb{R}$$

We can, however, also keep the basis fixed and get a function of its exponent, the **exponential function**:

exponential function

$$\mathbb{R} \to (0,\infty),\ x \mapsto a^x \quad \text{for some } \textit{fixed basis } a \in (0,\infty)$$

The inverse of power function x^α is root $\sqrt[\alpha]{x} = x^{1/\alpha}$ which is a power function again. The inverse of the exponential function is called the logarithm.

Definition 2.22

logarithm

Logarithm. A number y is called the **logarithm** to basis a, if $a^y = x$. The logarithm is the *exponent of a number to basis a*. We write

$$y = \log_a(x) \quad \Longleftrightarrow \quad x = a^y$$

Albeit any positive number can be used as basis of an exponential function, the most important ones are:

- the **natural logarithm** $\ln(x)$ with basis $e = 2.7182818\ldots$ (called *Euler's number*).
- the **common logarithm** $\lg(x)$ with basis 10 (also called *decadic* or *decimal logarithm*).

natural logarithm

common logarithm

Example 2.23

Compute the following logarithms (without using a calculator).

$$\log_{10}(100) = 2, \quad \text{as } 10^2 = 100$$

$$\log_{10}\left(\tfrac{1}{1000}\right) = -3, \quad \text{as } 10^{-3} = \tfrac{1}{1000}$$

$$\log_2(8) = 3, \quad \text{as } 2^3 = 8$$

$$\log_{\sqrt{2}}(16) = 8, \quad \text{as } \sqrt{2}^8 = 2^{8/2} = 2^4 = 16$$

Pocket calculators usually have a button for computing exponential function and logarithm to respective basis e and 10. Logarithms to other basis can however computed from these values by means of the following conversion formula[6]:

$$a^x = e^{x\ln(a)} \qquad \text{and} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Table 2.26

Rules for exponent and logarithm

$a^{x+y} = a^x \cdot a^y$	$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$
$a^{x-y} = \frac{a^x}{a^y}$	$\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
$(a^x)^y = a^{x \cdot y}$	$\log_a(x^\beta) = \beta \cdot \log_a(x)$
$(a \cdot b)^x = a^x \cdot b^x$	
$a^{\log_a(x)} = x$	$\log_a(a^x) = x$
$a^0 = 1$	$\log_a(1) = 0$
	$\log_a(x)$ has domain $x > 0$!

Example 2.24

$$\log_2(123) = \frac{\ln(123)}{\ln(2)} \approx \frac{4.812184}{0.6931472} = 6.942515$$

$$3^7 = e^{7\ln(3)}$$

⚠

Important: Often one can see $\log(x)$ without an explicit basis. In this case the basis is (should be) implicitly given by the context of the book or article.

- In **mathematics**: *natural* logarithm
  (financial mathematics, programs like **R**, *Mathematica*, ...)
- In **applied sciences**: *common* logarithm
  (economics, pocket calculator, Excel, ...)

We conclude this section by listing the rules for computations with exponential function and logarithm in Table 2.26. Observe that the rules for the exponential function analogous to the rules for powers in Table 2.6 on page 16.

Example 2.25

We derive our conversion formulas above:

$$a^x = (e^{\ln(a)})^x = e^{x\ln(a)}\ .$$

$$\ln(x) = \ln\left(a^{\log_a(x)}\right) \quad \text{which implies} \quad \log_a(x) = \ln(x)/\ln(a)\ .$$

---

[6] The formula holds analogously for the common logarithm $\lg(x)$ instead of the natural logarithm $\ln(x)$.

— Summary

- sigma notation
- absolute value
- powers and roots
- monomials and polynomials
- binomial theorem
- multiplication, factorization, and linear factors
- trap door “Ausmultiplizierreflex”
- fractions, rational terms and many fallacies
- exponent and logarithm

— Exercises

2.1 Which of the following expressions is equal to the summation

$$\sum_{i=2}^{10} 5(i+3)\,?$$

(a) $5(2+3+4+\ldots+9+10+3)$

(b) $5(2+3+3+3+4+3+5+3+6+3+\ldots+10+3)$

(c) $5(2+3+4+\ldots+9+10)+5\cdot 3$

(d) $5(2+3+4+\ldots+9+10)+9\cdot 5\cdot 3$

2.2 Compute and simplify:

(a) $\displaystyle\sum_{i=0}^{5} a^i b^{5-i}$

(b) $\displaystyle\sum_{i=1}^{5} (a_i - a_{i+1})$

(c) $\displaystyle\sum_{i=1}^{n} (a_i - a_{i+1})$

Remark: The sum in (c) is a so called *telescoping sum*.

2.3 Simplify the following summations:

(a) $\displaystyle\sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 - \sum_{k=1}^{n} (a_k - b_k)^2$

(b) $\displaystyle\sum_{i=1}^{n} (a_i b_{n-i+1} - a_{n-i+1} b_i)$

(c) $\displaystyle\sum_{i=1}^{n} (x_i + y_i)^2 + \sum_{j=1}^{n} (x_j - y_j)^2$

(d) $\displaystyle\sum_{j=0}^{n-1} x_j - \sum_{i=1}^{n} x_i$

2.4 *Arithmetic mean* (“average”)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and *variance*

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

are important location and dispersion parameters in statistics.

Variance σ^2 can be computed by means of formula

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

which requires to read data (x_i) only once.
Verify

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

for

(a) $n = 2$, (b) $n = 3$,

(c) $n \geq 2$ arbitrary.

Hint: Show that

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 - \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) = 0.$$

2.5 Let μ be the "true" value of a metric variate and $\{x_i\}$ the results of some measurement with stochastic errors. Then

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

is called the *mean square error* of sample $\{x_i\}$.

Verify:

$$MSE = \sigma^2 + (\bar{x} - \mu)^2$$

i.e., the MSE is the sum of

- the variance of the measurement (*dispersion*), and
- the squared deviation of the sample mean from μ (*bias*).

2.6 Find simple equivalent formulas for the following expressions without using absolute values:

(a) $|x^2 + 1|$ (b) $|x| \cdot x^3$

Find simpler expressions by means of absolute values:

(a) $\begin{cases} x^2, & \text{for } x \geq 0, \\ -x^2, & \text{for } x < 0. \end{cases}$

(b) $\begin{cases} x^\alpha, & \text{for } x > 0, \\ -(-x)^\alpha & \text{for } x < 0, \end{cases}$
for some fixed $\alpha \in \mathbb{R}$.

2.7 Simplify the following expressions:

(a) $\dfrac{(xy)^{\frac{1}{3}}}{x^{\frac{1}{6}} y^{\frac{2}{3}}}$ (b) $\dfrac{1}{(\sqrt{x})^{-\frac{3}{2}}}$ (c) $\left(\dfrac{|x|^{\frac{1}{3}}}{|x|^{\frac{1}{6}}}\right)^6$

2.8 Which of the following expressions are monomials or polynomials?

What is their degree?

Assume that x, y, and z are variables and all other symbols represent constants.

(a) x^2 (b) $x^{2/3}$

(c) $2x^2 + 3xy + 4y^2$ (d) $(2x^2 + 3xy + 4y^2)(x^2 - z^2)$

(e) $(x - a)(y - b)(z + 1)$ (f) $x\sqrt{y} - \sqrt{x}y$

(g) $ab + c$ (h) $x\sqrt{a} - \sqrt{b}y$

2.9 Compute

(a) $(x + y)^4$ (b) $(x + y)^5$

by means of the binomial theorem.

2.10 Show by means of the binomial theorem that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Hint: Use $1 + 1 = 2$.

2.11 Simplify the following expressions:

(a) $(x + h)^2 - (x - h)^2$ (b) $(a + b)c - (a + bc)$

(c) $(A - B)(A^2 + AB + B^2)$ (d) $(x + y)^4 - (x - y)^4$

2.12

(a) Give a polynomial in x of degree 4 with roots -1, 2, 3, and 4.

(b) What is the set of all such polynomials?

(c) Can such a polynomial have other roots?

2.13 Simplify the following expressions:

(a) $\frac{1}{1+x}+\frac{1}{x-1}$ (b) $\frac{s}{st^2-t^3}-\frac{1}{s^2-st}-\frac{1}{t^2}$

(c) $\frac{\frac{1}{x}+\frac{1}{y}}{xy+xz+y(z-x)}$ (d) $\frac{\frac{x+y}{y}}{\frac{x-y}{x}}+\frac{\frac{x+y}{x}}{\frac{x-y}{y}}$

2.14 Factorize and reduce the fractions:

(a) $\frac{1-x^2}{1-x}$ (b) $\frac{1+x^2}{1-x}$

(c) $\frac{x^3-x^4}{1-x}$ (d) $\frac{x^3-x^5}{1-x}$

(e) $\frac{x^2-x^6}{1-x}$ (f) $\frac{1-x^3}{1-x}$

2.15 Simplify the following expressions:

(a) $y(xy+x+1)-\frac{x^2y^2-1}{x-\frac{1}{y}}$ (b) $\frac{\frac{x^2+y}{2x+1}}{\frac{2xy}{2x+y}}$

(c) $\frac{\frac{a}{x}-\frac{b}{x+1}}{\frac{a}{x+1}+\frac{b}{x}}$ (d) $\frac{2x^2y-4xy^2}{x^2-4y^2}+\frac{x^2}{x+2y}$

2.16 Simplify the following expressions:

(a) $\frac{x^{\frac{1}{4}}-y^{\frac{1}{3}}}{x^{\frac{1}{8}}+y^{\frac{1}{6}}}$ (b) $\frac{\sqrt{x}-4}{x^{\frac{1}{4}}-2}$

(c) $\frac{\frac{2}{x^{-\frac{1}{7}}}}{x^{-\frac{7}{2}}}$

2.17 Simplify the following expressions:

(a) $\dfrac{(\sqrt{x}+y)^{\frac{1}{3}}}{x^{\frac{1}{6}}}$ (b) $\dfrac{1}{3\sqrt{x}-1}\cdot\dfrac{1}{1+\frac{1}{3\sqrt{x}}}\cdot\dfrac{1}{\sqrt{x}}$

(c) $\dfrac{(xy)^{\frac{1}{6}}-3}{(xy)^{\frac{1}{3}}-9}$ (d) $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

2.18 Compute without a calculator:

(a) $\log_2(2)$ (b) $\log_2(4)$

(c) $\log_2(16)$ (d) $\log_2(0)$

(e) $\log_2(1)$

(a) $\log_2\left(\frac{1}{4}\right)$ (b) $\log_2\left(\sqrt{2}\right)$

(c) $\log_2\left(\frac{1}{\sqrt{2}}\right)$ (d) $\log_2(-4)$

2.19 Compute without a calculator:

(a) $\log_{10}(300)$ (b) $\log_{10}(3^{10})$

Use $\log_{10}(3) = 0.47712$.

2.20 Compute (simplify) without a calculator:

(a) $0.01^{-\log_{10}(100)}$ (b) $\log_{\sqrt{5}}\left(\frac{1}{25}\right)$ (c) $10^{3\log_{10}(3)}$

(d) $\frac{\log_{10}(200)}{\log_{\frac{1}{\sqrt{7}}}(49)}$ (e) $\log_8\left(\frac{1}{512}\right)$ (f) $\log_{\frac{1}{3}}(81)$

2.21 Represent the following expression in the form $y = A\,e^{cx}$ (i.e., determine A and c):

(a) $y = 10^{x-1}$ (b) $y = 4^{x+2}$ (c) $y = 3^x 5^{2x}$

(d) $y = 1.08^{x-\frac{x}{2}}$ (e) $y = 0.9\cdot 1.1^{\frac{x}{10}}$ (f) $y = \sqrt{q}\,2^{x/2}$

3

Equations and Inequalities

3.1 Equations

In mathematics, an **equation** is a statement that asserts the equality of two expressions.

equation

l.h.s. = r.h.s.

Similar to terms there are sets associated with equations.

- The **domain** is the intersection of the domains of all involved terms restricted to a feasible region (e.g., non-negative numbers).
- The **solution set** is the set of objects from the domain that solve the equation(s).

domain

solution set

Finding (all) solutions of an equation can be quite tedious and a source of many errors. It may happen that a solution cannot be expressed in closed form. It may even happen that a solution does not exist at all.

In this section we discuss methods and strategies to find such solutions.

Transformation into an Equivalent Equation

The first step often will be to transform the given equation into an *equivalent simpler* equation. That is, the new equation has the *same solution*. The following operations may work:

- **Add** or **subtract** a number or term on both sides of the equation.
- **Multiply** or **divide** by a *non-zero* number or term on both sides of the equation.
- Take the **logarithm** or **antilogarithm** on both sides.

A useful strategy is to *isolate* the unknown entity on one side of the equation.

Sources of Errors

⚠ **Beware!** These operations may change the *domain* of the equation. This may or may not alter the solution set.

In particular this happens if a rational term is reduced or expanded by a factor that contains the unknown.

⚠ **Important!** Verify that both sides are *strictly positive* before taking the *logarithm*.

⚠ **Beware!** Any term that contains the unknown may *vanish* (becomes 0). Then

- multiplication may result in an *additional* but *invalid* "solution".
- division may *eliminate* a *valid* solution.

The next three examples demonstrate these issues. Of course they are very simple but similar errors might occur in more complex expressions. Then the error may not be so obvious.

Example 3.1

Non-equivalent Domains. Equation

$$\frac{(x-1)(x+1)}{x-1} = 1$$

can be transformed into the seemingly equivalent equation

$$x+1 = 1$$

by reducing the rational term by factor $(x-1)$. However, the latter has domain $\mathbb{R}$ while the given equation has domain $\mathbb{R} \setminus \{1\}$.

Fortunately, the solution set $L = \{0\}$ remains unchanged by this transformation.

Example 3.2

Multiplication Changes Solution Set. Multiplication of

$$\frac{x^2+x-2}{x-1} = 1$$

by $(x-1)$ yields

$$x^2+x-2 = x-1$$

with solution set $L = \{-1, 1\}$.

However, $x = 1$ is not in the domain of $\frac{x^2+x-2}{x-1}$ and thus not a valid solution of our equation.

Example 3.3

Division Discards Solution. If we divide both sides of equation

$$(x-1)(x-2)=0 \qquad \text{(with solution set } L=\{1,2\})$$

by $(x-1)$ we obtain equation

$$x-2=0 \qquad \text{(with solution set } L=\{2\})$$

Thus solution $x=1$ has been discarded by this division.

We can detect false solutions easily by verfication[1]. However, we cannot recover solutions that are "lost" by an erroneous transformation. In particular **division** by an unknown expression is quite dangerous.

division

⚠

Important! We need a *case-by-case* analysis when we divide by some term that contains an unknown:

Case 1: Division is *allowed* (the divisor is *non-zero*).

Case 2: Division is *forbidden* (the divisor is *zero*).

Example 3.4

Find all solutions of $(x-1)(x-2)=0$:

Case $x-1\neq 0$:
Division by $(x-1)$ yields equation $x-2=0$ with solution $x_1=2$.

Case $x-1=0$:
This implies solution $x_2=1$.

So we have solution set $L=\{1,2\}$.

Remark: We shortly will discuss a better method for finding roots.

Example 3.5

Find the solutions of the following system of two equations in two unknowns:

$$\begin{cases} xy-x=0 \\ x^2+y^2=2 \end{cases}$$

Addition and division in the first equation yields:

$$xy=x \rightsquigarrow y=\frac{x}{x}=1$$

Substituting into the second equation then gives $x=\pm 1$.
Seemingly solution set: $L=\{(-1,1),(1,1)\}$.

However: Division is only allowed if $x\neq 0$.

The value $x=0$ also satisfies the first equation (for every y).
So we have the (now correct) solution set[2]:
$L=\{(-1,1),(1,1),(0,\sqrt{2}),(0,-\sqrt{2})\}$.

---

[1] Substitute the solutions from your computations into the given equation and verify that equality holds.

[2] As we have two unknowns we get pairs (x,y) of solutions.

Factorization

Factorizing a term can be a suitable method for finding roots[3].

Example 3.6

Find all roots of the following equation:

$$\begin{cases} x\,y - x = 0 \\ x^2 + y^2 = 2 \end{cases}$$

The first equation $x\,y - x = x \cdot (y-1) = 0$ implies

$$x = 0 \quad or \quad y - 1 = 0 \quad \text{(or both).}$$

Case $x = 0$:
Then the second equation $x^2 + y^2 = 0^2 + y^2 = 2$ implies $y = \pm\sqrt{2}$.

Case $y - 1 = 0$:
Then $y = 1$ and $x^2 + y^2 = x^2 + 1^2 = 2$ implies $x = \pm 1$.

So the solution set is given by $L = \{(-1,1),(1,1),(0,\sqrt{2}),(0,-\sqrt{2})\}$.

Verification

A (seemingly correct) solution can be easily *verified* by *substituting* it into the given equation.

If unsure, verify the correctness of your solution.

Hint for your exams:
If a (homework or exam) problem asks for verification of a given solution, then simply substitute into the equation.
There is no need to solve the equation from scratch.

Linear Equations

linear equation

Linear equations only contain *linear terms* and can (almost) always be solved.

Example 3.7

Express annuity R from the formula for the present value

$$B_n = R \cdot \frac{q^n - 1}{q^n(q-1)}\ .$$

As R appears in degree 1 we have a linear equation which can be solved by dividing by (non-zero) constant $\frac{q^n-1}{q^n(q-1)}$:

$$R = B_n \cdot \frac{q^n(q-1)}{q^n - 1}$$

Remark: Systems of linear equations can be solved by methods from Linear Algebra, in particular by Gaussian eliminination.

Equations with Absolute Value

absolute value

An equation with **absolute value** can be seen as an abbreviation for a *system* of two (or more) equations:

$$|x| = 1 \iff x = 1 \text{ or } -x = 1$$

The solution set consists of the solutions of all of these equations.

Example 3.8

Find all solutions of $|2x-3| = |x+1|$.

The solution set is the union of the respective solutions of the two equations

$$\begin{aligned} (2x-3) &= (x+1) && \text{with solution } x_1 = 4 \\ -(2x-3) &= (x+1) && \text{with solution } x_2 = \tfrac{2}{3} \end{aligned}$$

The remaining two equations $-(2x-3) = -(x+1)$ and $(2x-3) = -(x+1)$ are equivalent to the above ones.

We thus find solution set $L = \left\{\frac{2}{3}, 4\right\}$.

Equations with Exponents or Logarithms

Equations where the unknown is an exponent can (sometimes) be solved by taking the logarithm:

- Isolate the term with the unknown on one side of the equation.
- Take the *logarithm* on both sides.

Example 3.9

Solve equation $2^x = 32$.

By taking the logarithm we obtain

$$\begin{aligned} & 2^x = 32 \\ \Leftrightarrow\quad & \ln(2^x) = \ln(32) \\ \Leftrightarrow\quad & x\ln(2) = \ln(32) \\ \Leftrightarrow\quad & x = \tfrac{\ln(32)}{\ln(2)} = 5 \end{aligned}$$

Solution set: $L = \{5\}$.

Example 3.10

Compute the term n of a loan over K monetary units and accumulation factor q from formula

$$X = K \cdot q^n \frac{q-1}{q^n - 1}$$

for installment X.

---

[3] That is, points where a term vanishes.

First we isolate the unknown n on the l.h.s. of this equation:

$$\begin{aligned}
X &= K\cdot q^n \tfrac{q-1}{q^n-1} && \mid\ \cdot(q^n-1)\\
X(q^n-1) &= K q^n (q-1) && \mid\ -K q^n (q-1)\\
q^n(X-K(q-1))-X &= 0 && \mid\ +X\\
q^n(X-K(q-1)) &= X && \mid\ :(X-K(q-1))\\
q^n &= \tfrac{X}{X-K(q-1)}
\end{aligned}$$

Then we can use the logarithm for computing n:

$$\begin{aligned}
q^n &= \tfrac{X}{X-K(q-1)} && \mid\ \ln\\
n\ln(q) &= \ln(X)-\ln(X-K(q-1)) && \mid\ :\ln(q)\\
n &= \tfrac{\ln(X)-\ln(X-K(q-1))}{\ln(q)}
\end{aligned}$$

Thus we have

$$n = \frac{\ln(X)-\ln(X-K(q-1))}{\ln(q)}$$

Equations which contain (just) the logarithm of the unknown can (sometimes) be solved by taking the antilogarithm.

Find the solution of $\ln(x+1)=0$. Example 3.11

Using the antilogarithm we find

$$\begin{aligned}
& \ln(x+1)=0\\
\Leftrightarrow\quad & e^{\ln(x+1)}=e^0\\
\Leftrightarrow\quad & x+1=1\\
\Leftrightarrow\quad & x=0
\end{aligned}$$

Solution set: $L=\{0\}$.

Equations with Powers or Roots

An Equation that contains *only one* power of the unknown which in addition has *integer* degree can be solved by *calculating roots*.

Important! ⚠

- Take care that the equation may not have a (unique) solution (in $\mathbb{R}$) if the power has *even* degree.
- If its degree is *odd*, then the solution always exists and is unique (in $\mathbb{R}$).

The solution set of $x^2=4$ is $L=\{-2,2\}$. Example 3.12

Equation $x^2=-4$ does not have a (real) solution, $L=\emptyset$.

The solution set of $x^3=-8$ is $L=\{-2\}$.

equation with roots

We can solve an **equation with roots** by squaring or taking the power of both sides.

Example 3.13

We get the solution of $\sqrt[3]{x-1}=2$ by taking the third power:

$$\sqrt[3]{x-1}=2 \quad\Longleftrightarrow\quad x-1=2^3 \quad\Longleftrightarrow\quad x=9$$

⚠

Beware!
Squaring an equation with square roots may create *additional* but invalid solutions (cf. multiplying with possibly negative terms on page 32).

Example 3.14

Squaring “non-equality” $-3\neq 3$ yields equality $(-3)^2=3^2$.

⚠

Beware!
The domain of an equation with roots often is just a subset of $\mathbb{R}$.
For roots with even root degree the *radicand* must not be negative.

⚠

Important!

***Always** verify solutions of equations with roots!*

Example 3.15

Solve equation $\sqrt{x-1}=1-\sqrt{x-4}$.

Domain is $D=\{x|x\geq 4\}$.
Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1-\sqrt{x-4} && |\ ^2\\
x-1 &= 1-2\cdot\sqrt{x-4}+(x-4) && |\ -x+3 \quad |:2\\
1 &= -\sqrt{x-4} && |\ ^2\\
1 &= x-4\\
x &= 5
\end{aligned}$$

However, substitution gives $\sqrt{5-1}=1-\sqrt{5-4}$ or equivalently $2=0$ which is a *false* statement.
Thus we get the empty solution set $L=\emptyset$.

Example 3.16

Solve equation $\sqrt{x-1}=1+\sqrt{x-4}$.

Domain is $D=\{x|x\geq 4\}$.
Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1+\sqrt{x-4} && |\ ^2\\
x-1 &= 1+2\cdot\sqrt{x-4}+(x-4) && |\ -x+3 \quad |:2\\
1 &= \sqrt{x-4} && |\ ^2\\
1 &= x-4\\
x &= 5
\end{aligned}$$

Now substitution gives $\sqrt{5-1}=1+\sqrt{5-4}$ or equivalently $2=2$ which is a *true* statement.
Thus we get the non-empty solution $L=\emptyset$.

Algebraic Equations

quadratic equation

A **quadratic equation** is one of the form

$$a x^2 + b x + c = 0 \qquad \text{Solution:} \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or in standard form

$$x^2 + p x + q = 0 \qquad \text{Solution:} \quad x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

algebraic equation

Quadratic equations are a special case of **algebraic equations** (*polynomial equations*)

$$P_n(x) = 0$$

where $P_n(x)$ is a polynomial of degree n.

There exist closed form solutions for algebraic equations of degree 3 (*cubic equation*) and 4 (*quartic equation*), resp. However, these are rather tedious[4].

For polynomials of degree 5 or higher no general formula does exist.

An algebraic equation can be solved by reducing its degree recursively.

1. Search for a root x_1 of $P_n(x)$
   (e.g. by trial and error, by means of Vieta's formulas, or by means of Newton's method)
2. We obtain a linear factor $(x - x_1)$ of $P_n(X)$.
3. By division $P_n(x) : (x - x_1)$ we get a polynomial $P_{n-1}(x)$ of degree $n - 1$.
4. If $n - 1 = 2$, solve the resulting quadratic equation.
   Otherwise goto Step 1.

Example 3.17

Find all roots of the cubic equation

$$x^3 - 6x^2 + 11x - 6 = 0.$$

By educated guess we find solution $x_1 = 1$.

Division by the linear factor $(x - x_1) = (x - 1)$ yields

$$(x^3 - 6x^2 + 11x - 6) : (x - 1) = x^2 - 5x + 6$$

Quadratic equation $x^2 - 5x + 6 = 0$ has solutions $x_2 = 2$ and $x_3 = 3$.
The solution set is thus $L = \{1, 2, 3\}$.

---

[4] See for example Cardano's formula.

Roots of Products

A *product* of two (or more) terms $f(x)\cdot g(x)$ is zero if and only if *at least one* factor is zero:

$$f(x)=0 \quad \textit{or} \quad g(x)=0 \quad \text{(or both).}$$

Example 3.18

Equation $x^2\cdot(x-1)\cdot e^x=0$ is satisfied if (and only if)

- $x^2=0$ (and thus $x=0$), or
- $x-1=0$ (and thus $x=1$), or
- $e^x=0$ (no solution).

Thus we have solution set $L=\{0,1\}$.

⚠

Important!
If a polynomial is already factorized one should resist to expand this expression[5].

Example 3.19

The roots of polynomial

$$(x-1)\cdot(x+2)\cdot(x-3)=0$$

are obviously 1, −2 and 3. But the roots of the expanded expression

$$x^3-2x^2-5x+6=0$$

are hard to find.

3.2 Inequalities

inequality

We get an **inequality** by comparing two terms by means of one of the "inequality" symbols

$\leq$	(less than or equal to),
$<$	(less than),
$>$	(greater than),
$\geq$	(greater than or equal to):

$$\text{l.h.s.} \leq \text{r.h.s.}$$

strict

solution set

The inequality is called **strict** if equality does not hold.

The **solution set** of an inequality is the set of all numbers in its domain that satisfy the inequality (or all inqualities in a system of inequality).
Usually this is an (open or closed) interval or union of intervals.

---

[5] See the comments on the "Ausmultiplizierreflex" in Section 2.5 on page 20.

Transformation into Equivalent Inequalities

The first step often will be to transform the given inequalities into *equivalent simpler* ones.
Ideally we try to isolate the unknown on one side of the inequality.

⚠

Beware!
If we *multiply* an inequality by some *negative* number, then the *direction* of the inequality symbol is **reverted**.

We have to take care about three cases:

- Case: term is *greater* than zero.
  Direction of inequality symbol is *not revert*.
- Case: term is *less* than zero.
  Direction of inequality symbol *is revert*.
- Case: term is *equal* to zero.
  Multiplication or division is *forbidden*!

Example 3.20

Find all solutions of $\frac{2x-1}{x-2} \leq 1$.

We multiply inequality by $(x-2)$.

- Case $x-2>0 \Leftrightarrow x>2$:
  We find $2x-1 \leq x-2 \Leftrightarrow x \leq -1$,
  a contradiction to our assumption $x>2$.
- Case $x-2<0 \Leftrightarrow x<2$:
  The inequality symbol is reverted, and
  we find $2x-1 \geq x-2 \Leftrightarrow x \geq -1$.
  Hence $x<2$ and $x \geq -1$.
- Case $x-2=0$:
  The $x=2$ which is not in the domain of this inequality.

The solution set is the interval $L=[-1,2)$.

⚠

Sources of Errors:
Inequalities with polynomials *cannot* be directly solved by transformations.

⚠

Important!
One *must not* simply replace the equality sign "=" in the solution formula for quadratic equations by an inequality sign.

Example 3.21

We want to find all solutions of

$$x^2-3x+2 \leq 0$$

Invalid approach: $x_{1,2} \leq \frac{3}{2} \pm \sqrt{\frac{9}{4}-2} = \frac{3}{2} \pm \frac{1}{2}$
and thus $x \leq 1$ (and $x \leq 2$) which would imply "solution" set $L=(-\infty,1]$.

However, $0 \in L$ but violates the inequality as $0^2-3\cdot 0+2=2 \not\leq 0$.

Inequalites with polynomials can be solved by the following procedure:

inequalites with polynomial

1. Move all terms on the l.h.s. and obtain an expression of the form $T(x) \leq 0$ (and $T(x) < 0$, resp.).

2. Compute all roots $x_1 < \ldots < x_k$ of $T(x)$,
   i.e., solve equation $T(x) = 0$ as we would with any polynomial as described above.

3. These roots decompose the *domain* into intervals I_j.

   These are open if the inequality is *strict* (with $<$ or $>$), and closed otherwise.

   In each of these intervals the inequality now holds
   either in *all or* in *none* of its points.

4. Select some point $z_j \in I_j$ which is not on the boundary.

   If z_j satisfies the corresponding *strict* inequality,
   then I_j belongs to the solution set, else none of its points.

(Example 3.21 cont). Find all solutions of

Example 3.22

$$x^2 - 3x + 2 \leq 0 .$$

The solutions of $x^2 - 3x + 2 = 0$ are $x_1 = 1$ and $x_2 = 2$.

We obtain the three intervals $(-\infty, 1]$, $[1,2]$, and $[2,\infty)$ and check by means of respective points 0, $\frac{3}{2}$, and 3 whether the inequality is satisfied in each of these. These points are choosen at random in the interior of the respective intervals. We find

$$\begin{array}{rrl} (-\infty,1] & \text{not satisfied:} & 0^2 - 3\cdot 0 + 2 = 2 \not\leq 0 \\ [1,2] & \text{satisfied:} & \left(\frac{3}{2}\right)^2 - 3\cdot\frac{3}{2} + 2 = -\frac{1}{4} < 0 \\ [2,\infty) & \text{not satisfied:} & 3^2 - 3\cdot 3 + 2 = 2 \not\leq 0 \end{array}$$

Thus the solution set of this inequality is $L = [1,2]$.

The same recipe can works inequalities where all terms are *continuous*.

It even can be generalized in the following way: If there is any point where $T(x)$ is *not continuous*, then we also have to use this point for decomposing the domain into intervals.

Furthermore, we have to take care when the domain of the inequality is a union of two or more disjoint intervals.

Rational terms. Find all solutions of inequality

Example 3.23

$$\frac{x^2 + x - 3}{x - 2} \geq 1 .$$

Its domain is the union of two intervals: $(-\infty,2)\cup(2,\infty)$ because the numerator has a root in 2.

We find for the solutions of the equation $\frac{x^2+x-3}{x-2}=1$ by

$$\frac{x^2+x-3}{x-2}=1 \iff x^2+x-3=x-2 \iff x^2-1=0$$

and thus $x_1=-1, x_2=1$.

So he get four intervals:

$(-\infty,-1]$, $[-1,1]$, $[1,2)$ and $(2,\infty)$.

We check by means of four points whether the inequality holds in these intervals:

$(-\infty,-1]$	not satisfied:	$\frac{(-2)^2+(-2)-3}{(-2)-2}=\frac{1}{4}\not>1$
$[-1,1]$	satisfied:	$\frac{0^2-0-3}{0-2}=\frac{3}{2}>1$
$[1,2)$	not satisfied:	$\frac{1.5^2+1.5-3}{1.5-2}=-\frac{3}{2}\not>1$
$(2,\infty)$	satisfied:	$\frac{3^2+3-3}{3-2}=9>1$

The solution set is $L=[-1,1]\cup(2,\infty)$.

Inequalities with absolute values can also be solved by the above procedure.

inequality with absolute value

However, we also can see such an inequality as a system of two (or more) inequalities:

$$|x|<1 \iff x<1 \text{ and } x>-1$$

$$|x|>1 \iff x>1 \text{ or } x<-1$$

— Summary

- equations and inequalities
- domain and solution set
- transformation into equivalent problem
- possible errors with multiplication and division
- equations with powers and roots
- equations with polynomials and absolute values
- roots of polynomials
- equations with exponents and logarithms
- method for solving inequalities

— Exercises

3.1 Solve the following equations:

(a) $|x(x-2)| = 1$

(b) $|x+1| = \dfrac{1}{|x-1|}$

(c) $\left|\dfrac{x^2-1}{x+1}\right| = 2$

3.2 Solve the following equations:

(a) $2^x = 3^{x-1}$

(b) $3^{2-x} = 4^{\frac{x}{2}}$

(c) $2^x 5^{2x} = 10^{x+2}$

(d) $2 \cdot 10^{x-2} = 0.1^{3x}$

(e) $\frac{1}{2^{x+1}} = 0.2^x 10^4$

(f) $(3^x)^2 = 4 \cdot 5^{3x}$

3.3 Function

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

is called the *hyperbolic cosine*.

Find all solutions of

$$\cosh(x) = a$$

Hint: Use auxiliary variable $y = e^x$. Then the equation simplifies to $(y + \frac{1}{y})/2 = a$.

3.4 Solve the following equation:

$$\ln\left(x^2\left(x - \frac{7}{4}\right) + \left(\frac{x}{4} + 1\right)^2\right) = 0$$

3.5 Solve the following equations:

(a) $\sqrt{x+3} = x+1$

(b) $\sqrt{x-2} = \sqrt{x+1} - 1$

3.6 Compute all roots and decompose into linear factors:

(a) $f(x) = x^2 + 4x + 3$

(b) $f(x) = 3x^2 - 9x + 2$

(c) $f(x) = x^3 - x$

(d) $f(x) = x^3 - 2x^2 + x$

(e) $f(x) = (x^2-1)(x-1)^2$

3.7 Solve with respect to x and with respect to y:

(a) $xy + x - y = 0$ (b) $3xy + 2x - 4y = 1$

(c) $x^2 - y^2 + x + y = 0$ (d) $x^2y + xy^2 - x - y = 0$

(e) $x^2 + y^2 + 2xy = 4$ (f) $9x^2 + y^2 + 6xy = 25$

(g) $4x^2 + 9y^2 = 36$ (h) $4x^2 - 9y^2 = 36$

(i) $\sqrt{x} + \sqrt{y} = 1$

3.8 Solve with respect to x and with respect to y:

(a) $xy^2 + yx^2 = 6$ (b) $xy^2 + (x^2 - 1)y - x = 0$

(c) $\frac{x}{x+y} = \frac{y}{x-y}$ (d) $\frac{y}{y+x} = \frac{y-x}{y+x^2}$

(e) $\frac{1}{y-1} = \frac{y+x}{2y+1}$ (f) $\frac{yx}{y+x} = \frac{1}{y}$

(g) $(y + 2x)^2 = \frac{1}{1+x} + 4x^2$ (h) $y^2 - 3xy + (2x^2 + x - 1) = 0$

(i) $\frac{y}{x+2y} = \frac{2x}{x+y}$

3.9 Find constants a, b and c such that the following equations hold for all x in the corresponding domains:

(a) $$\frac{x}{1+x} - \frac{2}{2-x} = -\frac{2a + bx + cx^2}{2 + x - x^2}$$

(b) $$\frac{x^2 + 2x}{x+2} - \frac{x^2 + 3}{x+3} = \frac{a(x-b)}{x+c}$$

3.10 Solve the following inequalities:

(a) $x^3 - 2x^2 - 3x \geq 0$ (b) $x^3 - 2x^2 - 3x > 0$

(c) $x^2 - 2x + 1 \leq 0$ (d) $x^2 - 2x + 1 \geq 0$

(e) $x^2 - 2x + 6 \leq 1$

3.11 Solve the following inequalities:

(a) $7 \leq |12x + 1|$ (b) $\dfrac{x+4}{x+2} < 2$

(c) $\dfrac{3(4-x)}{x-5} \leq 2$ (d) $25 < (-2x + 3)^2 \leq 50$

(e) $42 \leq |12x + 6| < 72$ (f) $5 \leq \dfrac{(x+4)^2}{|x+4|} \leq 10$

4

Sequences and Series

> "Can you do Addition?" the White Queen asked.
> "What's one and one and one and one and one and one and one and one and one and one?"
> "I don't know," said Alice. "I lost count."
>
> Lewis Carroll (1832–1898)

4.1 Sequences

Sequences by an important role in mathematics. The are used for discrete models for dynamics systems. But they used for defining concepts like continuity of derivatives of real functions.

Definition 4.1

Sequence. A **sequence** is an enumerated collection of objects in which repetitions are allowed. These objects are called **members** or **terms** of the sequence.

sequence

members

terms

In this chapter we are interested in *sequences of numbers*.

Formally a sequence is a special case of a *map*:

$$a\colon \mathbb{N} \to \mathbb{R}, n \mapsto a_n$$

Sequences are denoted by $(a_n)_{n=1}^{\infty}$ or just (a_n) for short[1].

Sequences can be defined

- by *enumerating*[2] of its terms,
- by a *formula*, or
- by **recursion** (Each term is determined by its predecessor(s)).

recursion

---

[1] An alternative notation used in literature is $\langle a_n\rangle_{n=1}^{\infty}$.

[2] Enumeration looks more like an IQ test rather than a mathematical object and should be avoided.

Table 4.3

Properties of a sequence

Property	Definition	
monotonically increasing	$a_{n+1} \geq a_n$	for all $n \in \mathbb{N}$
monotonically decreasing	$a_{n+1} \leq a_n$	
alternating	$a_{n+1} \cdot a_n < 0$	i.e. the sign changes
bounded	$\lvert a_n \rvert \leq M$	for some $M \in \mathbb{R}$

Example 4.2

The following sequences are equivalent:

Enumeration: $(a_n) = (1, 3, 5, 7, 9, \dots)$

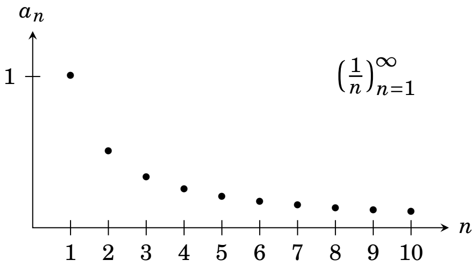
Formula: $(a_n) = (2n-1)$

Recursion: $a_1 = 1,\ a_{n+1} = a_n + 2$

graphically representation

A sequence (a_n) can be **graphically represented** in the following way.

(1) By drawing tuples (n, a_n) in the plane, or



(2) By drawing points on the number line.

We will discuss sequences later in the following chapters. Here we only list the most basic characterizations, see Table 4.3.

Example 4.4

Sequence $\left(\frac{1}{n}\right)$ is

- monotonically decreasing,
- bounded, as for all $n \in \mathbb{N}$, $|a_n| = |1/n| \leq M = 1$ (we could also choose $M = 1000$),
- but *not* alternating.

Arithmetic Sequence:

Formula:

$$a_n = a_1 + (n-1)\cdot d$$

Recursion:

$$a_{n+1} = a_n + d$$

Differences of consecutive terms are constant:

$$a_{n+1} - a_n = d$$

Each term is the *arithmetic mean* of its neighboring terms:

$$a_n = \tfrac{1}{2}(a_{n+1} + a_{n-1})$$

Arithmetic series:

$$s_n = \tfrac{n}{2}(a_1 + a_n)$$

Table 4.6

Arithmetic sequence

4.2 Series

The sum of the first n terms of sequence $(a_i)_{i=1}^{\infty}$,

$$s_n = \sum_{i=1}^{n} a_i$$

is called the n-th **partial sum** of the *sequence*.

partial sum

The sequence (s_n) of all partial sums is called the **series** of the sequence.

series

Example 4.5

The series of sequence $(a_i) = (2i-1)$ is

$$(s_n) = \left(\sum_{i=1}^{n}(2i-1)\right) = (1,4,9,16,25,\dots) = (n^2)\,.$$

Series play an important role in the approximation of real functions. A well-known example are Taylor series which will be discussed in the next course of this programme.

Two sequences and their corresponding series are important in finance, in particular for calculation of annuities: the arithmetic and the geometric series. We summarize their properties in Tables 4.6 and 4.7, resp.

Table 4.7

Geometric sequence

Geometric Sequence:

Formula:

$$a_n = a_1 \cdot q^{n-1}$$

Recursion:

$$a_{n+1} = a_n \cdot q$$

Ratios of consecutive terms are constant:

$$\frac{a_{n+1}}{a_n} = q$$

Each term is the *geometric mean* of its neighboring terms:

$$a_n = \sqrt{a_{n+1} \cdot a_{n-1}}$$

Geometric series:

$$s_n = a_1 \cdot \frac{q^n - 1}{q - 1} \qquad \text{for } q \neq 1$$

⚠

Sources of errors:
Indices of sequences may also start with 0 (instead of 1). Formulæ for arithmetic and geometric sequences and series are then slightly changed.

Arithmetic sequence:

$$a_n = a_0 + n \cdot d \qquad \text{and} \qquad s_n = \tfrac{n+1}{2}(a_0 + a_n)$$

Geometric sequence:

$$a_n = a_0 \cdot q^n \qquad \text{and} \qquad s_n = a_0 \cdot \frac{q^{n+1} - 1}{q - 1} \quad (\text{for } q \neq 1)$$

— Summary

- sequence
- formula and recursion
- series and partial sums
- arithmetic and geometric sequence

— Exercises

4.1 Draw the first 10 elements of the following sequences.

Which of these sequences are monotone, alternating, or bounded?

(a) $\left(n^2\right)_{n=1}^{\infty}$ (b) $\left(n^{-2}\right)_{n=1}^{\infty}$

(c) $\left(\sin(\pi/n)\right)_{n=1}^{\infty}$ (d) $a_1 = 1,\ a_{n+1} = 2a_n$

(e) $a_1 = 1,\ a_{n+1} = -\frac{1}{2}a_n$

4.2 Compute the first 5 partial sums of the following sequences:

(a) $2n$ (b) $\dfrac{1}{2+n}$ (c) $2^{n/10}$

4.3 We are given a geometric sequence $\left(a_n\right)$ with $a_1 = 2$ and relative change 0.1, i.e., each term of the sequence is increased by 10% compared to its predecessor.

Give formula and term a_7.

4.4 Compute the first 10 partial sums of the arithmetic series for

(a) $a_1 = 0$ and $d = 1$, (b) $a_1 = 1$ and $d = 2$.

4.5 Compute $\sum_{n=1}^{N} a_n$ for

(a) $N = 7$ and $a_n = 3^{n-2}$ (b) $N = 7$ and $a_n = 2(-1/4)^n$

5

Real Functions

> Jede Entdeckung wird gleich in die Gesamtheit der Wissenschaften geleitet und hört damit gewissermaßen auf, Entdeckung zu sein, sie geht im Ganzen auf und verschwindet, man muss schon einen wissenschaftlich geschulten Blick haben, um sie dann noch zu erkennen.
>
> Franz Kafka (1883–1924)

5.1 Real Functions

real function

Real functions are maps where both *domain* and *codomain* are (unions of) intervals in $\mathbb{R}$.

Often only function terms are given but neither domain nor codomain. Then domain and codomain are implicitly given as following:

- *Domain* D_f of the function is the largest *sensible* subset of the domain of the function terms (i.e., where the terms are defined).
- *Codomain* is the image set of the function

  $$f(D) = \{y \mid y = f(x) \text{ for a } x \in D_f\}\,.$$

Example 5.1

Production function $f(x) = \sqrt{x}$ is an abbreviation for

$$f\colon [0,\infty) \to [0,\infty),\ x \mapsto f(x) = \sqrt{x}$$

There are no negative amounts of goods.
Moreover, $\sqrt{x}$ is not real for $x < 0$.

Its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ is an abbreviation for

$$f'\colon (0,\infty) \to (0,\infty),\ x \mapsto f'(x) = \frac{1}{2\sqrt{x}}$$

Note the open interval $(0,\infty)$; $\frac{1}{2\sqrt{x}}$ is not defined for $x = 0$.

5.2 Graph of a Function

Each pair $(x, f(x))$ corresponds to a point in the xy-plane.
The set of all these points forms a curve called the **graph** of function f.

graph

$$\mathcal{G}_f = \{(x,y) \mid x \in D_f,\ y = f(x)\}$$

Graphs are a quite powerful tool to visualize functions. They allow to detect many properties of the given function.

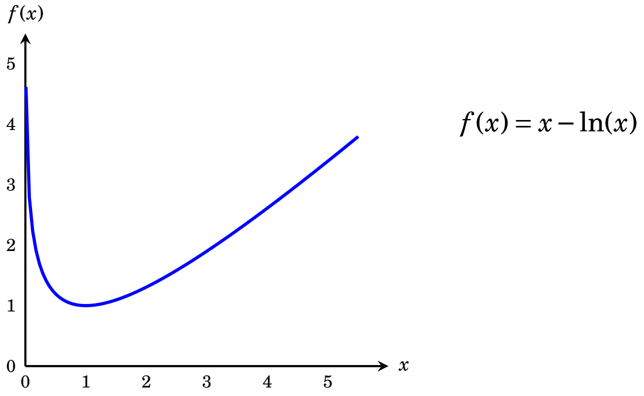


Figure 5.2

Graph of function $f(x) = x - \ln(x)$

How to Draw a Graph

Drawing the graph of a function is a quite challenging tasks. Many mistakes happen and the resulting picture often does not provide useful information about the underlying function. Even worse, the drawing can be quite misleading.

Unfortunately, drawing the graph requires some expectation about the shape of this graph. So this may look like an chicken-and-egg problem. However, this just means that we either need some experiences or we run some drawing-inspect-improve cycles.

Here is a simple recipe for drawing graphs of a function:

1. Get an idea about the possible shape of the graph. One should be able to sketch graphs of elementary functions by heart.

2. Find an appropriate range for the x-axis.
   (It should show a characteristic detail of the graph.)

3. Create a table of function values and draw the corresponding points into the xy-plane.

   If known, use characteristic points like local extrema or inflection points.

4. Check if the curve can be constructed from the drawn points.
   If not *add* adapted points to your table of function values.

5. Fit the curve of the graph through given points in a proper way.

Example 5.3

Draw the graph of function $f(x) = x - \ln x$.

By educated guess we use domain $[0, 5]$ for drawing the graph. We evaluate the function at some points, plot these, and join them with a smooth line (because we know that all terms in f are differentiable). We obtain the following first draft:

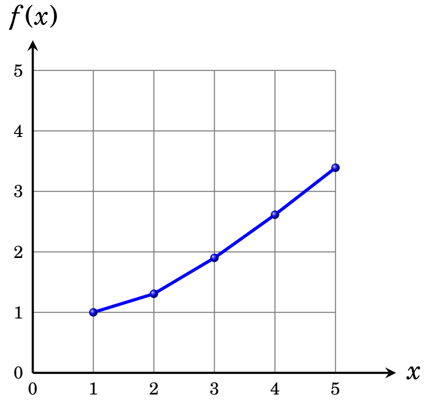


Table of function values:

x	$f(x)$
0	`ERROR`
1	1
2	1.307
3	1.901
4	2.614
5	3.391

Figure 5.4

At this point we realize that our calculator returns an error message when we try to evaluate $f(0)$. So it is not clear what happens close to 0.

⚠ When our calculator returns an error code like `ERROR` then[1]:

- The function is *not* defined at 0. Really.
- The calculator does *not* mean $f(0) = 0$.
- The calculator does *not* mean $f(0) = f(1) = 1$ just because 1 is the argument nearby 0 where we got a result.
- It does *not* mean, that f is not defined in subinterval $(0, 1)$. So the above graph is *not* complete.

⚠ However, when unsure there is a way out of his problem:
Add more evaluations to your table!

So if we some points in subinterval $(0, 1)$ the situation becomes much clearer.

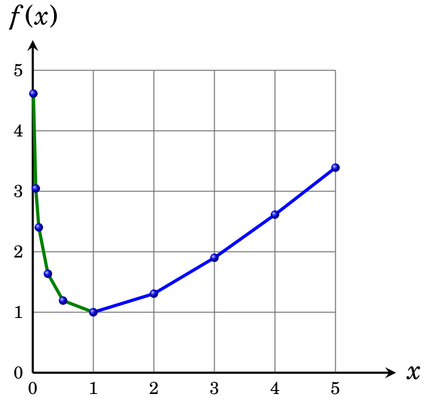


Table of function values:

x	$f(x)$
0	`ERROR`
1	1
2	1.307
3	1.901
4	2.614
5	3.391
0.5	1.193
0.25	1.636
0.1	2.403
0.05	3.046
0.01	4.615

Figure 5.5

---

[1] These are wrong assumptions from students' exams.

Finally when draw the curve of this graph smoothly (and remove the grid) we arrive at Figure 5.2.

Remark: As already written above one should already have some idea about the graph of the given function. In Example 5.3 one should know that the logarithm log tends to $-\infty$ when argument x converges to 0. So one should expect[2] that f tends to ∞ as seen in Figure 5.2.

Extrema and Inflection Points

Additional information about the function can be quite helpful for drawing its graph. In particular the location of extrema or inflection points often is an great help. Here is an example.

Example 5.6

Graph of function $f(x) = \frac{1}{15}(3x^5 - 20x^3)$. It has a local maximum in -2, a local minimum in 2 and three inflection points in $-\sqrt{2}$, 0, and $\sqrt{2}$, resp.

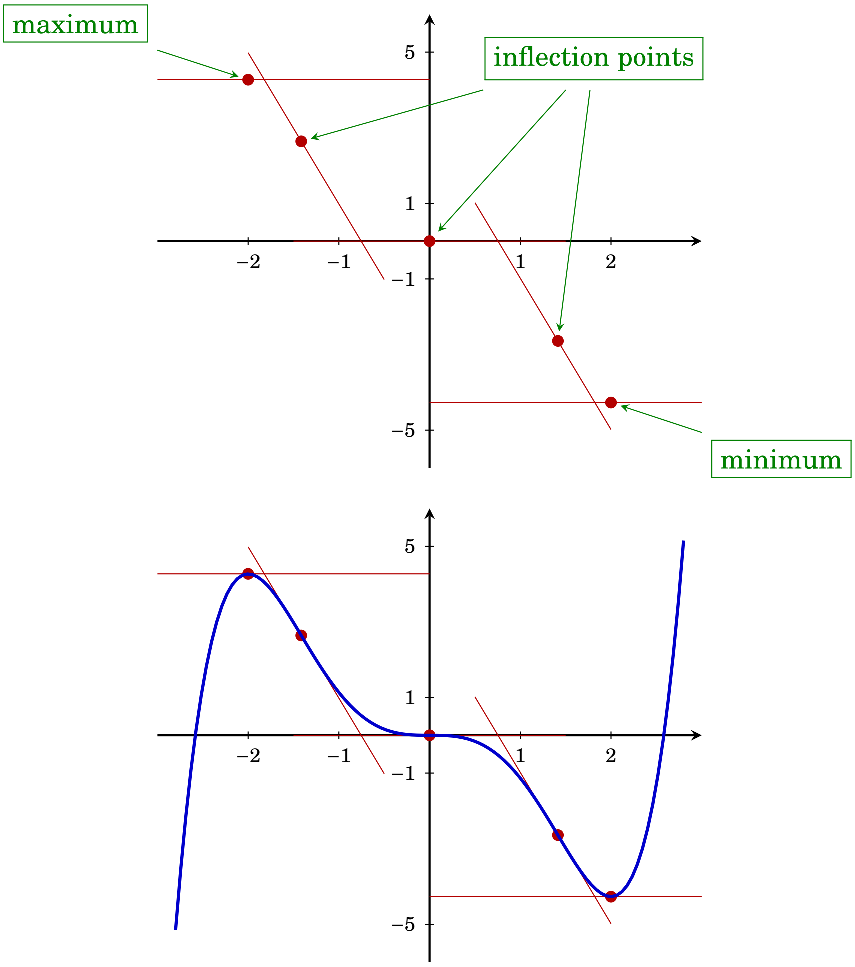


Figure 5.7

Function with extrema and inflection points

---

[2] Some more powerful calculators return `INF` instead of `ERROR` and remind us about this behavior of f close to 0.

Sketch of a Function Graph

Often a *sketch* of the graph is sufficient. Then the exact function values are not so important. Axes may not have scales.

However, it is important that the sketch clearly shows all characteristic details of the graph (like extrema or important function values).

⚠ A sketch of a graph must not be as a sloppily drawn curve. If important properties of the function are not shown like in l.h.s. of Figure 5.11 below then even the sketch is plain wrong.

Sketches can also be drawn like a *caricature*:
They stress prominent parts and properties of the function.

Sources of Errors When Drawing Graphs

Graphs in students' solution show quite a few mistakes. Here is list of the most frequent errors when drawing function graphs that should be avoided:

- ⚠ *Table of values is too small:*

  It is not possible to construct the curve from the computed function values.

- ⚠ *Important points are ignored:*

  Ideally extrema and inflection points should be known and used.

- ⚠ *Range for x and y-axes not suitable:*

  The graph is tiny or important details vanish in the "noise" of handwritten lines (or pixel size in case of a computer program).

Here is a gallery of typical examples of such errors. The l.h.s. shows graphs which show some defects. The graph of the r.h.s. is much better.

Example 5.8

Function $f(x) = \frac{1}{3}x^3 - x$ is smooth, that is, its graph does not have any sharp kink as shown in the picture on the l.h.s. When in doubt use a larger table of function values.

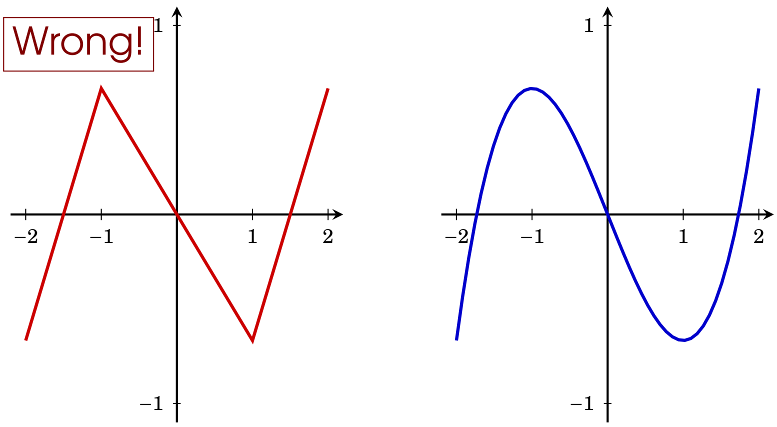


Figure 5.9

Function $f(x) = \frac{1}{3}x^3 - x$

Example 5.10

The graph of function $f(x) = x^3$ has slope 0 in $x = 0$. The graph on the l.h.s. does not show this important property of. So it is completely wrong. It shows the graph of a different function.

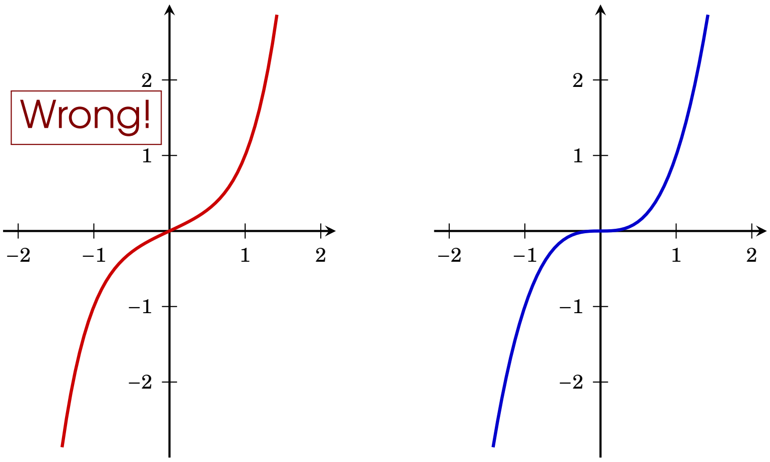


Figure 5.11

Function $f(x) = x^3$

Example 5.12

Function $f(x) = \exp(\frac{1}{3}x^3 + \frac{1}{2}x^2)$ has a local maximum in $x = -1$. However, due to the scaling of the y-axis it is hardly visible.

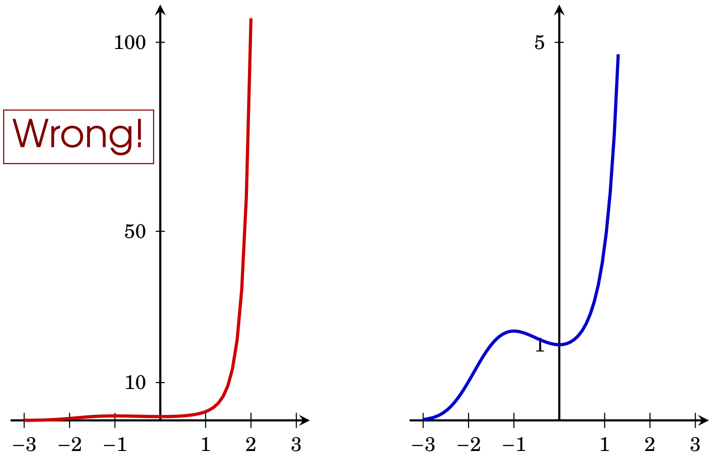


Figure 5.13

Function

$f(x) = \exp(\frac{1}{3}x^3 + \frac{1}{2}x^2)$

Example 5.14

Function $f(x) = \frac{1}{3}x^3 - x$ is defined in interval $[-2, 2]$. This is correctly given in the figure on the l.h.s. However, the scaling of the x- and y-axes is chosen not appropriately so that the graph is tiny and thus too small.

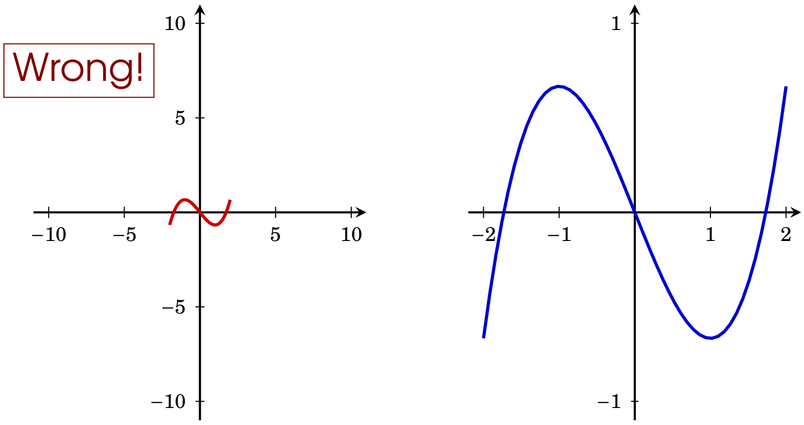


Figure 5.15

Function $f(x) = \frac{1}{3}x^3 - x$

Example 5.16

In this example the domain of function $f(x) = \frac{1}{3}x^3 - x$ is restricted to the interval $[0, 2]$. However, the figure on the l.h.s. shows the graph for a function with different domain $[-2, 2]$. Albeit the function term is the same the entire function is not equal to the given one.

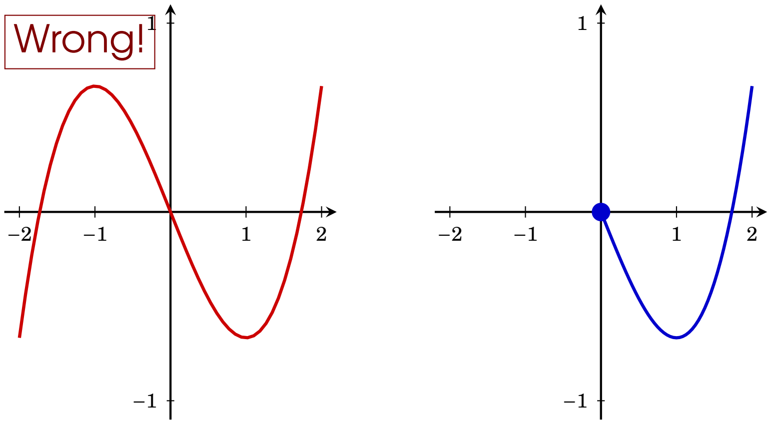


Figure 5.17

Function $f(x) = \frac{1}{3}x^3 - x$ on interval $[0, 2]$

⚠

Important: Domain and codomain are an essential part of a function. When the domain is changed then we get a different function.

It is *important* that one already has an *idea of the shape* of the function graph *before* drawing the curve.

⚠

Even a graph drawn by means of a computer program can differ significantly from the correct curve.

Example 5.18

Consider function $f(x) = \sin\left(\frac{1}{x}\right)$. It is the composition $f = g \circ h$ of functions $h(x) = \frac{1}{x}$ and $g(x) = \sin(x)$. Observe that $h(x) = \frac{1}{x}$ maps the unit interval $(0,1]$ onto the unbounded interval $[1,\infty)$. Function $g(x) = \sin(x)$ then is a periodic function that oscillates infinitely often with range $[-1,1]$ in domain $[1,\infty)$, see Figure 5.41 on page 66 below. So we expect infinitely many maxima and minima for $f(x) = \sin\left(\frac{1}{x}\right)$ which accumulate at 0.

However, if we start a program for plotting function graphs then we obtain the following picture:

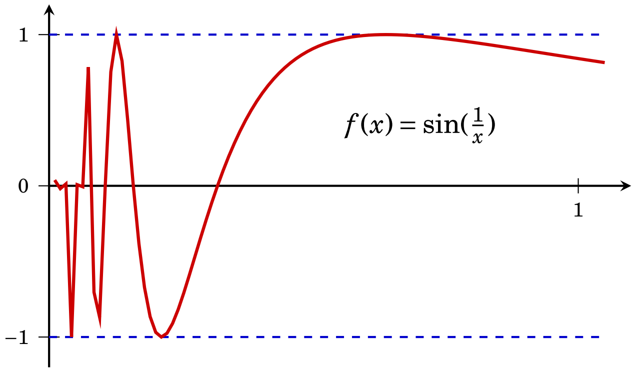


Figure 5.19

Function $f(x) = \sin\left(\frac{1}{x}\right)$

Obviously the region close to 0 is not drawn correctly[3]. In order to get more information, we zoom into this region and obtain a quite artificial plot that visualizes numerical round-off errors of the used plotting software rather than a realistic picture of the function graph[4]:

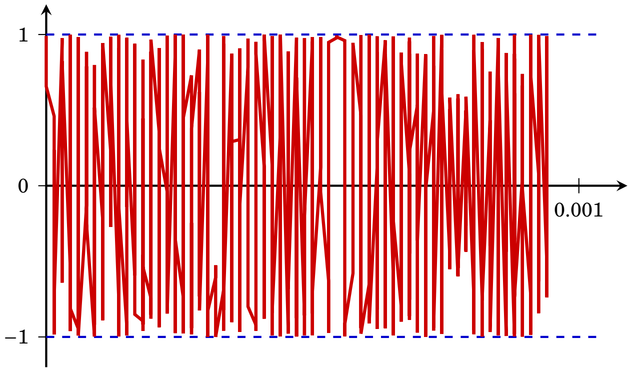


Figure 5.20

Function $f(x) = \sin\left(\frac{1}{x}\right)$ close to 0

---

[3] The picture can be improved by using better parameters for the plot. However, the main problem does not disappear: The plotting program cannot show infinitely many oscillations.

[4] It must be noted here, that this plotting program is designed for illustrating books and manuscripts rather than showing very accurate plots. However, the main problem does not disappear with better software: The plotting program cannot show infinitely many oscillations. They still show numerical artifacts.

Piece-wise Defined Functions

The function term can be defined differently in subintervals of the domain. The graph of such a function can be drawn in the following way:

1. Mark the given subintervals.
2. Draw the function graphs in each subdomain separately.
   Take care about the boundaries of these subintervals.
3. At the boundary points of these subintervals we have to mark which points belong to the graph and which do not:

   • (belongs) and ∘ (does not belong).

Example 5.21

Consider function

$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1, \\ x, & \text{for } x \ge 1. \end{cases}$$

Its domain is partitioned[5] into three subintervals. On each of these function terms are given. We have to draw the graphs of the respective functions

- $f_1(x) = 1$ on subinterval $(-\infty, 0]$,
- $f_2(x) = 1 - \frac{x^2}{2}$ on subinterval $[0,1]$, and
- $f_3(x) = x$ on subinterval $[1, \infty)$.

Finally we mark point $(1,1)$ by • (as $x = 1$ belongs to the third subdomain and $f_3(1) = 1$) and point $(1, \frac{1}{2})$ by ∘ (as $f_2(1) = \frac{1}{2}$ but $x = 1$ does not belong the second subdomain). Since f is continuous[6] at $x = 0$ there is no need to mark point $(0,1)$.

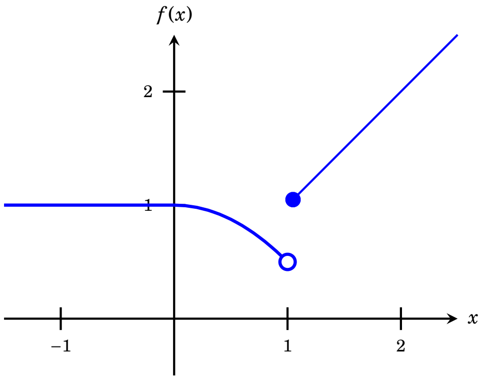


Figure 5.22

Piece-wise defined function

---

[5] That is, these subsets are disjoint and their union is equal to the domain.

[6] See Section 6.4 on page 89.

5.3 Bijectivity

Recall that each argument of a function has exactly one image and that the number of preimages of an element in the codomain can vary. Thus we can characterize maps by their possible number of preimages.

- A map f is called **one-to-one** (or *injective*), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or *surjective*), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

one-to-one

onto

bijective

Recall that a function has an *inverse* if and only if it is *bijective* (i.e., one-to-one and onto).

How can we determine whether a real function is one-to-one or onto? That is, how many preimage may a $y \in W_f$ have? Here is a simple **horizontal test** to solve this problem (for functions that behave “nice”).

horizontal test

(1) Draw the graph of the given function.

(2) Mark some $y \in W$ on the y-axis and draw a line parallel to the x-axis (*horizontal*) through this point.

(3) The number of intersection points of horizontal line and graph coincides with the number of preimages of y.

(4) Repeat Steps (2) and (3) for a *representative*

(5) Interpretation: If all horizontal lines intersect the graph in

   (a) *at most one* point, then f is *one-to-one*;

   (b) *at least one* point, then f is *onto*;

   (c) *exactly one* point, then f is *bijective*.

Example 5.23

Consider function $f_1\colon [-1,2] \to \mathbb{R}, x \mapsto x^2$.
We find a horizontal line at $y = 0.7 \in \mathbb{R}$ which intersects the graph twice and hence f_1 is not one-to-one. On the other hand horizontal line at $y = 4.7 \in \mathbb{R}$ does not intersects the graph at all and hence f_1 is not onto.

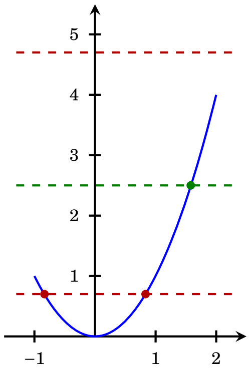


Figure 5.24

Neither one-to-one nor onto

Example 5.25

Consider function $f_2\colon [0,2] \to \mathbb{R}, x \mapsto x^2$.
It differs from function f_1 from Example 5.23 by its domain. Now we cannot find a horizontal line that intersects the graph more than once. Thus f_2 is one-to-one. However, horizontal line at $y = 4.7 \in \mathbb{R}$ does not intersects the graph at all and hence f_1 is not onto.

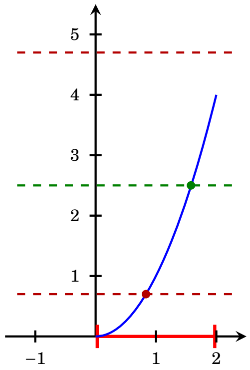


Figure 5.26

One-to-one but not onto

Example 5.27

Now restrict the range of f_2 in Example 5.25 to its image. Then we obtain function $f_3\colon [0,2] \to [0,4], x \mapsto x^2$.
Now all horizontal lines that might not intersect the graph of the function are excluded by construction. The codomain only contains points that occur as images of f_3. Thus f_3 is one-to-one and onto.

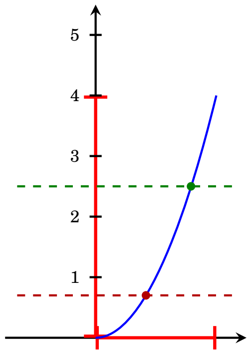


Figure 5.28

One-to-one and onto

Beware! *Domain* and *codomain* are part of the function! ⚠

5.4 Special Functions

Function Composition

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$. Then

$$g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))$$

is called **composite function**[7].

composite function

---

[7] Read: "g composed with f", "g circle f", or "g after f".

Example 5.29

Let $g\colon \mathbb{R} \to [0,\infty)$, $x \mapsto g(x) = x^2$ and $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = 3x - 2$. Then

$$(g \circ f)\colon \mathbb{R} \to [0,\infty),\ x \mapsto (g \circ f)(x) = g(f(x)) = g(3x-2) = (3x-2)^2$$

$$(f \circ g)\colon \mathbb{R} \to \mathbb{R},\ x \mapsto (f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2 - 2$$

Inverse Function

inverse function

If $f : D_f \to W_f$ is a *bijection*, then there exists a so called **inverse function**

$$f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)$$

with the property

$$f^{-1} \circ f = \mathrm{id} \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}$$

That is,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y$$

for all $x \in D_f$ and $y \in W_f$.

⚠

Beware: Do not mix up symbol $f^{-1}(x)$ (or sometimes just f^{-1}) for the inverse function of f with symbol y^{-1} for the -1st power of y which is equal to the reciprocal $\frac{1}{y}$ of y.

We get the function term of the inverse by *interchanging* the roles of *argument* x and *image* y.
That is, we get the term for the inverse function by expressing x as function of y.

Example 5.30

Compute the inverse function of

$$y = f(x) = 2x - 1\,.$$

By rearranging we obtain

$$y = 2x - 1 \quad \Longleftrightarrow \quad y + 1 = 2x \quad \Longleftrightarrow \quad \frac{1}{2}(y+1) = x$$

Thus the term of the inverse function is $f^{-1}(y) = \frac{1}{2}(y+1)$.

As arguments are usually denoted by x we write

$$f^{-1}(x) = \frac{1}{2}(x+1)\,.$$

Example 5.31

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.

We get the graph of the inverse of a function by the following geometric operation. Interchanging of x and y corresponds to a reflection across the median between x- and y-axis. Figure 5.32 shows the graph of $f(x) = x^3$ (in blue) and of its inverse $f^{-1}(x) = \sqrt[3]{x}$ (in red). A point (x_0, y_0) on the blue graph of f is reflected to point (y_0, x_0) on the red graph of inverse f^{-1}.

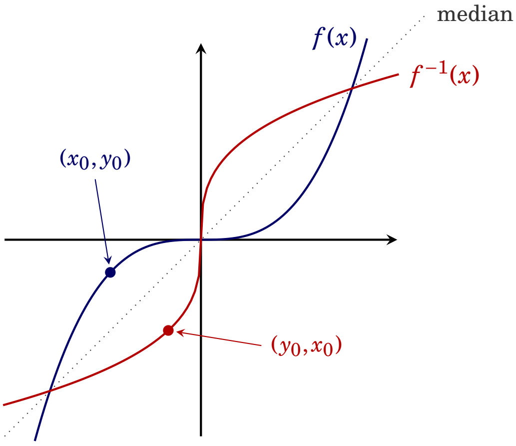


Figure 5.32

Geometric interpretation of inverse function

5.5 Elementary Functions

In this section we shortly discuss the most basic functions. Their properties must be known and it should be possible to sketch the graphs of these functions without a table of values.

Linear Function

linear function

slope

intercept

Linear functions are the simplest class of real functions. They are described by the two parameter **slope** k and **intercept** d that gives the function value at 0.

$$f(x) = k\,x + d$$

The graph of a linear function is a straight line.

⚠

Use a ruler to draw this straight line!

If you draw this line freehand you might get an arc that looks like a piece of a parabola. It is sufficient to compute only two function values.

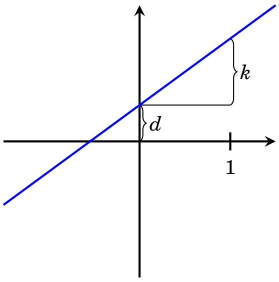


Figure 5.33

Linear function

Absolute Value

absolute value

We have already discussed the **absolute value** in Sect. 2.3 on page 15.

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

Its graph has a sharp kink at 0.

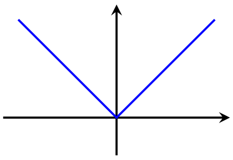


Figure 5.34

Absolute value

Power Function

power function

The class of **power functions** with integer coefficients are still quite simple (as they can compute by multiplication and division).

$$f : x \mapsto x^n, \quad n \in \mathbb{Z} \qquad D = \begin{cases} \mathbb{R} & \text{for } n \geq 0 \\ \mathbb{R} \setminus \{0\} & \text{for } n < 0 \end{cases}$$

The shape of the function graphs depend on the parameter n.

- If n is *even*, then the power function is symmetric, i.e., $f(-x) = f(x)$ for all $x \in D$, see Figure 5.35.
- If n is *odd*, then the power function is skew-symmetric, i.e., $f(-x) = -f(x)$ for all $x \in D$, see Figure 5.36.
- If $n \geq 0$, then $f(0) = 0$.
- If $n < 0$, then f tends to $\pm\infty$ for $x \to 0$.

Rules for computations with powers are listed in Sect. 2.4 on page 15.

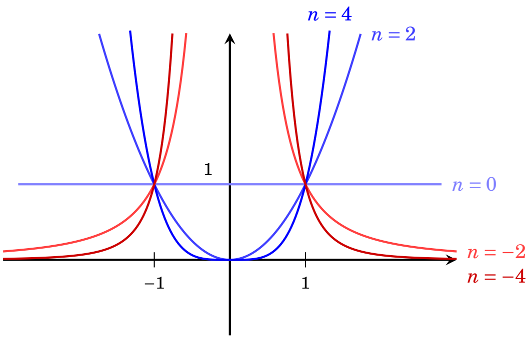


Figure 5.35

Powers with even exponents

Power function with *real* (non-integer) exponents have non-negative domain. Recall that these also include square roots $\sqrt{x} = x^{\frac{1}{2}}$ and cubic

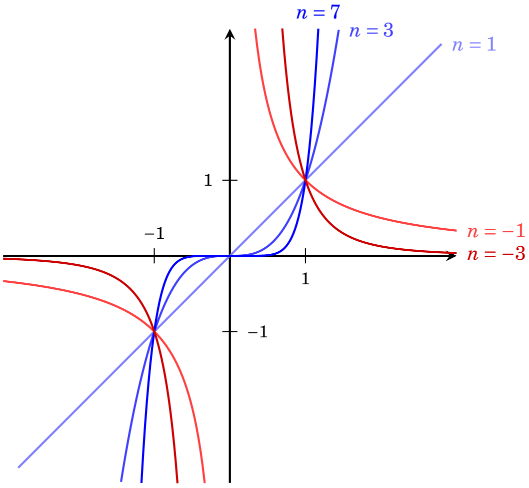


Figure 5.36

Powers with odd exponents

roots $\sqrt[3]{x} = x^{\frac{1}{3}}$.

$$f : x \mapsto x^{\alpha} \qquad \alpha \in \mathbb{R} \qquad D = \begin{cases} [0,\infty) & \text{for } \alpha \geq 0 \\ (0,\infty) & \text{for } \alpha < 0 \end{cases}$$

The shape of the function graphs depend on the parameter α, see Figure 5.37.

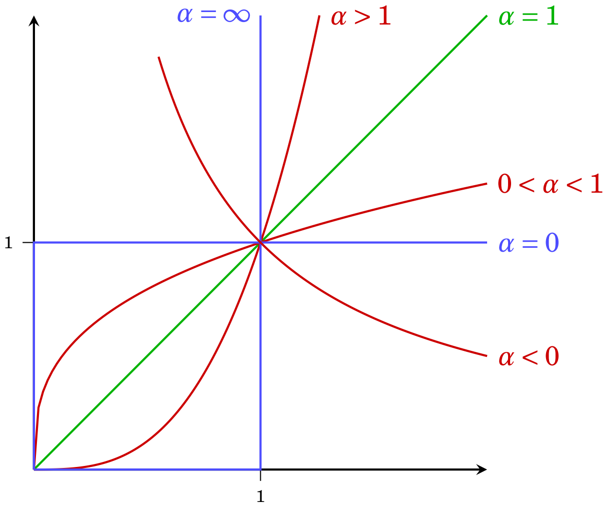


Figure 5.37

Powers with non-integer exponents

Polynomials

polynomial

Polynomials of degree n are sums of power with natural exponents:

$$f(x) = \sum_{k=0}^{n} a_k x^k , \qquad \text{where } a_i \in \mathbb{R}, \text{ for } i = 1, \dots, n, \text{ and } a_n \neq 0.$$

Rational Functions

The function term of a **rational function** is a rational expression, i.e., the ratio of polynomials, see Sect. 2.6 on page 21.

rational function

$$D \to \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and domain $D = \mathbb{R} \setminus \{\text{roots of } q\}$.

Exponential Function and Logarithm

The argument of a power functions is the basis of power while the exponent is kept fixed. In an **exponential function** the argument is the basis is kept fixed, usually Euler's number $e = 2,7182818\ldots$.

exponential function

$$\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto \exp(x) = e^x$$

However, any positive basis a can be used.

$$\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto a^x\,, \qquad a > 0$$

The graph of an exponential function is given in Figure 5.38.
Rules for computations with exponent are listed in Sect. 2.7 on page 23.

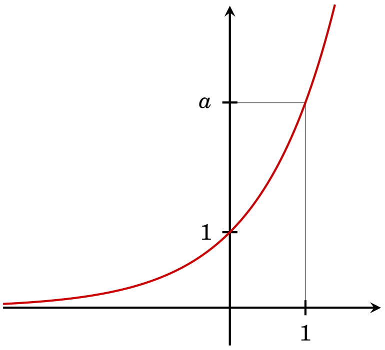


Figure 5.38

Exponential function

The **logarithm** is the inverse of the exponential function.

logarithm

$$\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log(x) = \ln(x)$$

Or if we use an arbitrary basis $a > 0$:

$$\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log_a(x)$$

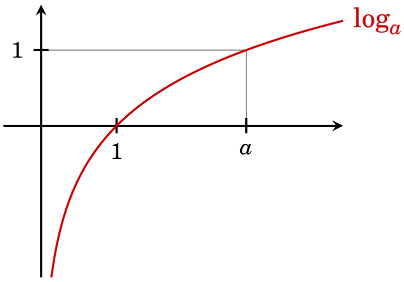


Figure 5.39

Logarithm function

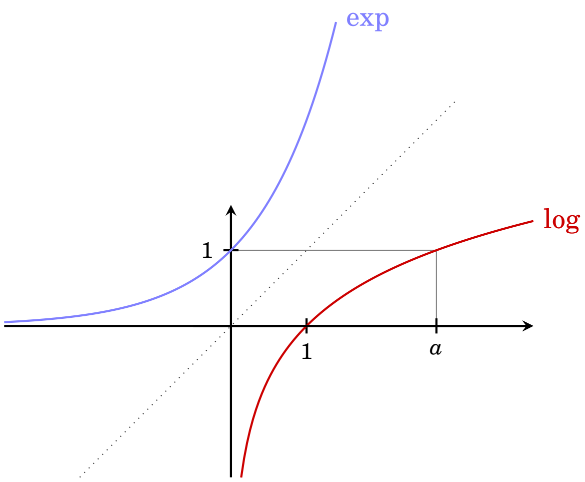


Figure 5.40

Logarithm function – inverse of exponential function

Trigonometric Functions

Sine and cosine are used to model oscillating and periodic behavior.

Sine :

sine

$$\mathbb{R} \to [-1,1],\ x \mapsto \sin(x)$$

Cosine :

cosine

$$\mathbb{R} \to [-1,1],\ x \mapsto \cos(x)$$

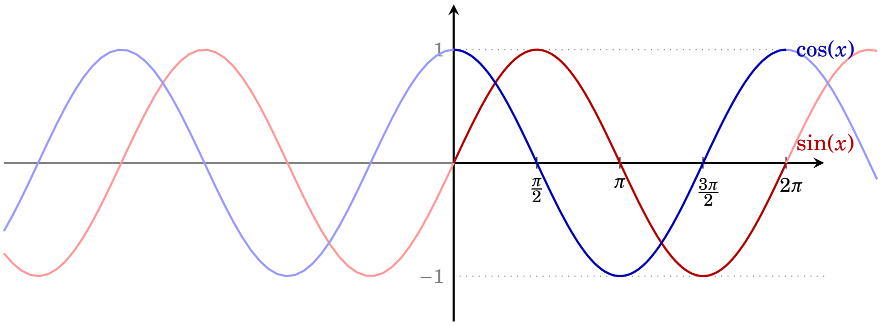


Figure 5.41

Sine and cosine function

Beware!
Trigonometric functions use **radian** for their arguments, i.e., angles are measured by means of the length of arcs on the unit circle and not by degrees. A right angle then corresponds to $x = \pi/2$.

⚠

radian

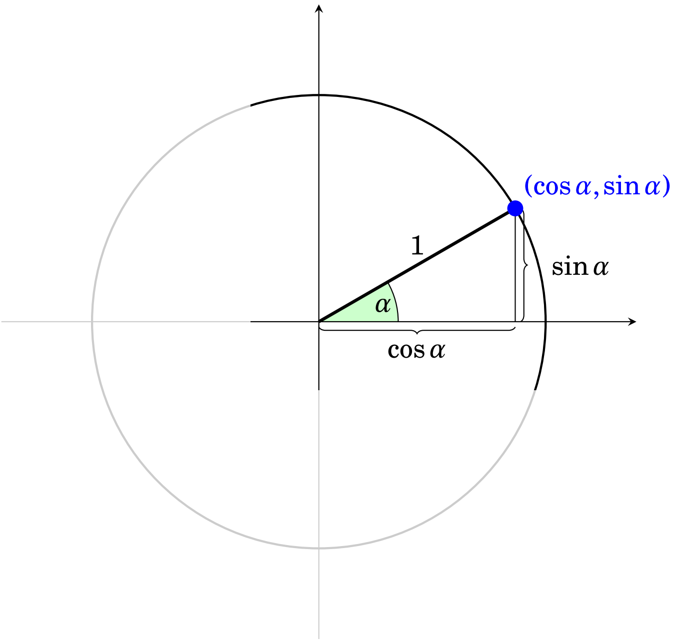


Figure 5.42

Definition of sine and cosine

Observe that Sine and Cosine are **periodic** functions, that is, we find for all $k \in \mathbb{Z}$,

periodic

$$\sin(x + 2k\pi) = \sin(x)$$
$$\cos(x + 2k\pi) = \cos(x)$$

From Figure 5.42 we also can derive the following relation between sin and cos:

$$\sin^2(x) + \cos^2(x) = 1$$

5.6 Multivariate Functions

A **function of several variables** (or **multivariate function**) is a function with more than one argument which evaluates to a real number.

function of several variables

multivariate function

$$f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$$

Arguments x_i are the **variables** of function f.

variable

Example 5.43

$$f(x, y) = \exp(-x^2 - 2y^2)$$

is a bivariate function in variables x and y.

$$p(x_1, x_2, x_3) = x_1^2 + x_1x_2 - x_2^2 + 5x_1x_3 - 2x_2x_3$$

is a function in the three variables x_1, x_2, and x_3.

Graphs of Bivariate Functions

Bivariate functions (i.e., of *two* variables) can be visualized by its graph:

$$\mathcal{G}_f = \{(x,y,z) \mid z = f(x,y) \text{ for } x,y \in \mathbb{R}\}$$

It can be seen as the two-dimensional *surface* of a three-dimensional landscape.

Remark:

The notion of **graph** exists analogously for functions of three or more variables.

graph

$$\mathcal{G}_f = \{(\mathbf{x},y) \mid y = f(\mathbf{x}) \text{ for an } \mathbf{x} \in \mathbb{R}^n\}$$

However, it can hardly be used to visualize such functions.

In practice one will use a computer program for drawing graphs of bivariate functions. These create a two-dimensional mesh in as three-dimensional space which is then projected onto the two-dimension screen or sheet of paper. These programs also allow to rotate the point of view so that we can inspect details of graph.

Nevertheless, one should have an idea, what these computer programs do and which artifacts can happen. In dimension two there are much more points required for drawing the graph than in dimension one (see Sect. 5.2) and there is a higher risk that one gets fooled by the resulting picture.

Example 5.44

We want to draw the graph of bivariate function $f(x,y) = \exp(-x^2 - 2y^2)$.

1. We draw a grid in the xy-plane. The z-axis corresponds to function values.

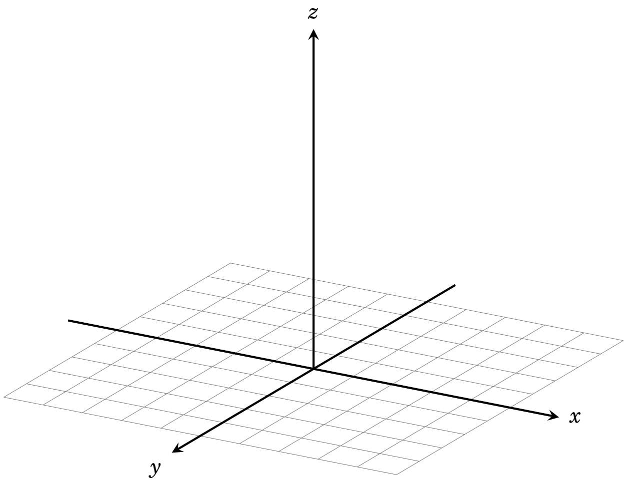


Figure 5.45

Graph of a bivariate function – the grid

2. We compute the functions values $z = f(x,y)$ for all grid points (similarly to the table of function values for univariate functions). Thus we get triples (x,y,z) which we mark in our picture. We do this by drawing a straight line parallel to the z-axes which starts in grid point (x,y) and has length $z = f(x,y)$.

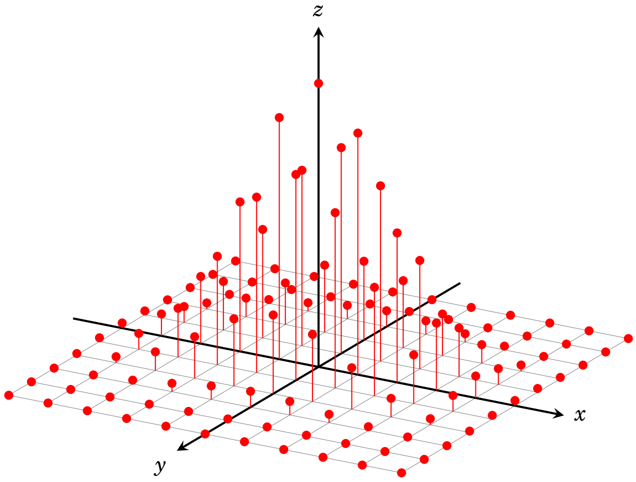


Figure 5.46

Graph of a bivariate function – function values

3. We connect the points that correspond to function values with lines. We join (x_1,y_1,z_1) with (x_2,y_2,z_2) whenever points (x_1,y_1) and (x_2,y_2) are joined in the two-dimensional grid in the xy-plane. Thus we get a mesh that approximates the graph of the function.

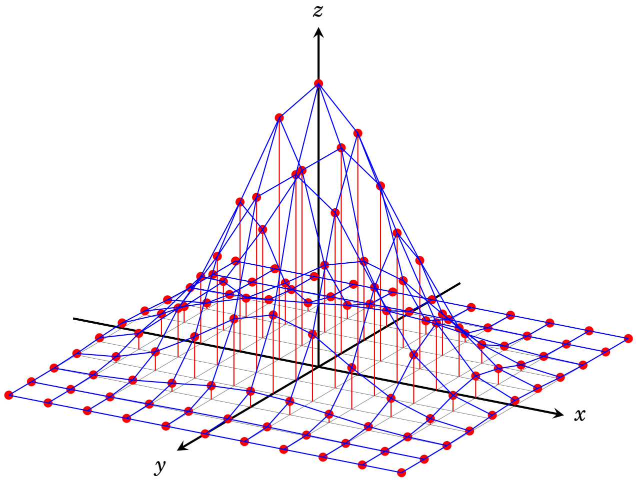


Figure 5.47

Graph of a bivariate function – mesh

4. We may replace the piece-wise straight lines by smooth curves. However, this assumes that the given function is continuous and smooth.

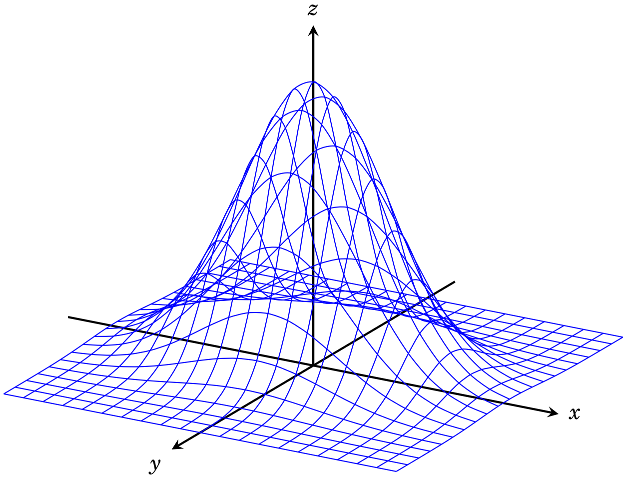


Figure 5.48

Graph of a bivariate function – smoothed mesh

5. Finally we may assume that the graph is opaque and remove the lines that are below the surface and thus invisible by our point of view. Sophisticated computer programs often add a source of light.

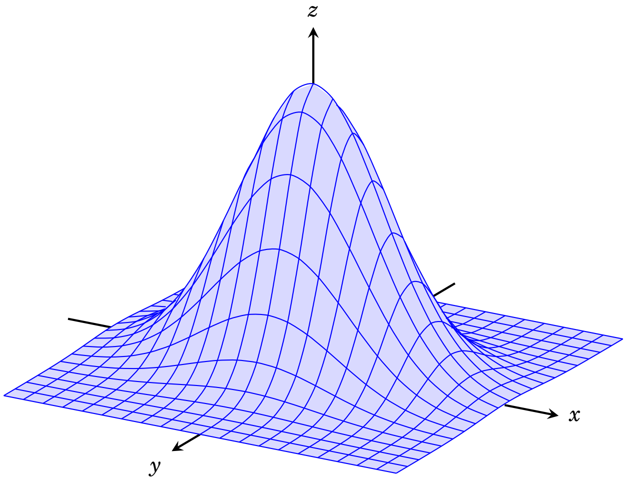


Figure 5.49

Graph of a bivariate function

Contour Lines of Bivariate Functions

Let $c \in \mathbb{R}$ be fixed. Then the set of all points (x, y) in the real plane with $f(x, y) = c$ is called **contour line** of function f. Obviously function f is *constant* on each of its contour lines.

contour line

There exist other names which are used in different contexts:

- **indifference curve**
- **isoquant**
- **level set** (is a generalization of a contour line for functions of any number of variables.)

indifference curve

isoquant

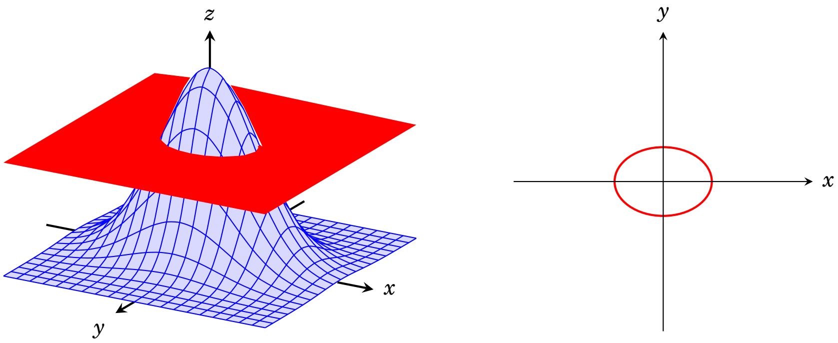
level set

A collection of contour lines can be seen as a kind of “hiking map” for the “landscape” (graph) of the function.

Example 5.50

Here is an illustration for the bivariate function $f(x,y) = \exp(-x^2 - 2y^2)$ from Example 5.44.

1. We choose a level c. This corresponds to plane parallel to the xy-plane through point $(0,0,c)$. It intersects[8] the graph in some curve (Figure 5.51, l.h.s.) which we then project into the xy-plane (Figure 5.51, r.h.s.) We obtain this projected line by solving equation $f(x,y) = c$ w.r.t. x and y. That is, we have to find all points where this equation holds. In our example this reduces to equation $x^2 + 2y^2 = -c$ that describes an ellipsis[9].



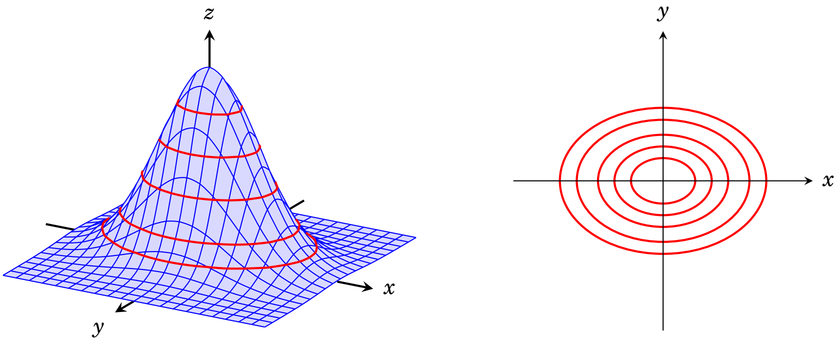
graph

contour lines

Figure 5.51

A single contour line of a bivariate function

2. We now repeat this procedure for a couple of suitable levels c.



graph

contour lines

Figure 5.52

Contour lines of a bivariate function

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[8] This intersection may be the empty set. Then there is no contour line that corresponds to this level.

[9] Please take a look into your mathematics book from high school.

3. Finally we may colorize the regions depending on the values of the levels. We thus get a “hiking map” of the “landscape” of our function.

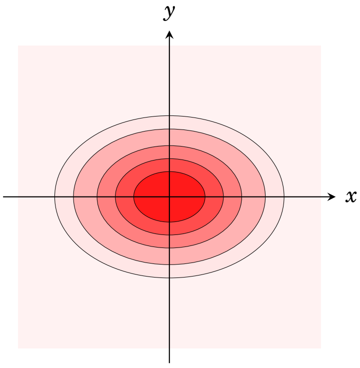


Figure 5.53

“Hiking map” of a bivariate function

5.7 Indifference Curves

indifference curve

Indifference curves are determined by an equation

$$F(x,y) = 0$$

Thus they can be seen as a special case of contour lines with level $c = 0$.

implicit function

Such indifference curves correspond to **implicit functions** . We can (try to) draw such curves by expressing one of the variables as function of the other one (i.e., solve the equation w.r.t. one of the two variables).

Thus we may get an univariate function. The graph of this function coincides with the indifference curve. We then draw the graph of this univariate function by the method described in Section 5.2 above.

Example 5.54

Cobb-Douglas Function. We want to draw indifference curve

$$x^{\frac{1}{3}} y^{\frac{2}{3}} = 1 , \qquad x, y > 0.$$

Expressing x by y yields:

$$x = \frac{1}{y^2}$$

Alternatively we can express y by x:

$$y = \frac{1}{\sqrt{x}}$$

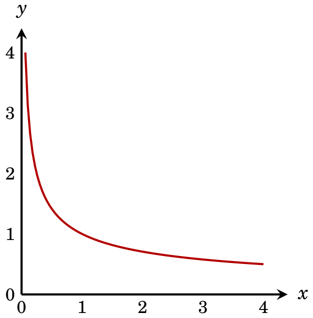


Figure 5.55

Indifference curve of a Cobb-Douglas function

Example 5.56

CES Function. We want to draw indifference curve

$$\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)^2=4\,,\qquad x,y>0.$$

Expressing x by y yields:

$$y=\left(2-x^{\frac{1}{2}}\right)^2$$

(Take care about the domain of this curve!)

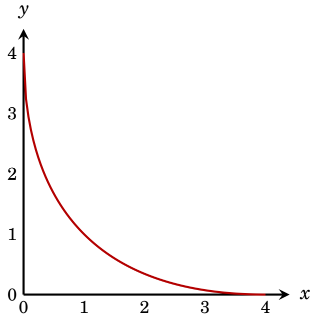


Figure 5.57

Indifference curve of a CES function

5.8 Paths

A function

$$s\colon\mathbb{R}\to\mathbb{R}^n,\ t\mapsto s(t)=\begin{pmatrix}s_1(t)\\ \vdots\\ s_n(t)\end{pmatrix}$$

is called a **path** in $\mathbb{R}^n$. Variable t is often interpreted as *time*.

path

The graph of a function $\mathbb{R}\to\mathbb{R}^2$ is a curve in the plane. In mathematical software these are often called **parametric curve**.

parametric curve

Example 5.58

Consider path

$$[0,\infty)\to\mathbb{R}^2,\ t\mapsto\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$$

The image set of this path is the unit circle (see Figure 5.42 on page 67).

Figure 5.59 shows some snapshots of this path. It starts in $(1,0)$ for $t=0$. When $t=\frac{\pi}{3}$ is has moved along the unit circle counter-clock wise to position that corresponds to an angle of 60°. It continues along its way on the unit circle. When $t=2\pi$ it again arrives at its starting point and continues with a second round.

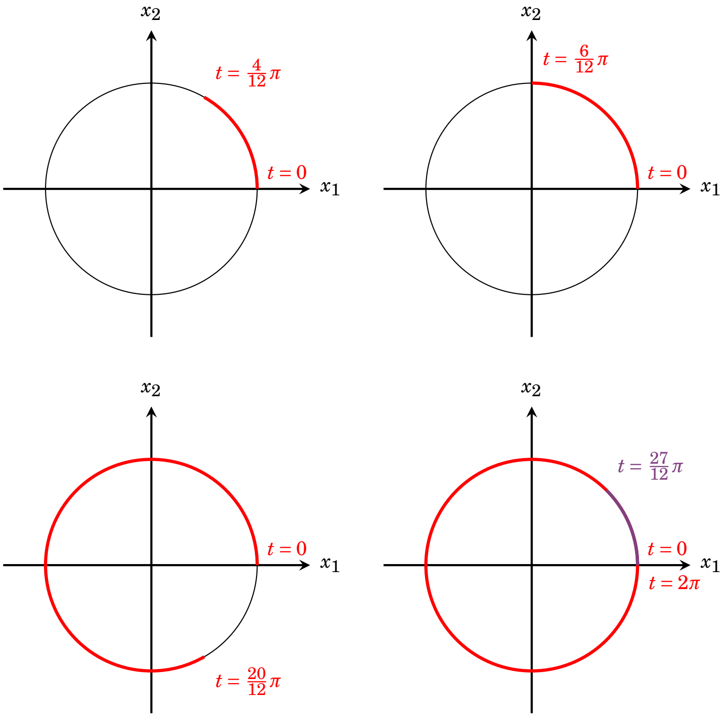


Figure 5.59

Snapshots of a path in $\mathbb{R}^2$

5.9 Generalized Real Functions

The idea of multivariate functions and pathes can generalized to **vector-valued functions** where both the domain and the target set are higher dimensional spaces.

vector-valued function

$$\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}$$

Example 5.60

All the functions discussed in this chapter are special cases.

- Univariate functions:

$$\mathbb{R} \to \mathbb{R},\ x \mapsto y = x^2$$

- Multivariate functions:

$$\mathbb{R}^2 \to \mathbb{R},\ \mathbf{x} \mapsto y = x_1^2 + x_2^2$$

- Paths:

$$[0,1) \to \mathbb{R}^n,\ s \mapsto (s, s^2)^t$$

- Linear maps:

$$\mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{A} \text{ is an } m \times n\text{-Matrix}$$

— Summary

- real functions
- implicit domain
- graph of a function
- sources of errors
- piece-wise defined functions
- one-to-one and onto
- function composition
- inverse function
- elementary functions
- multivariate functions
- paths
- vector-valued functions

— Exercises

5.1 Give the largest possible domain of the following functions?

(a) $h(x) = \frac{x-1}{x-2}$
(b) $D(p) = \frac{2p+3}{p-1}$
(c) $f(x) = \sqrt{x-2}$
(d) $g(t) = \frac{1}{\sqrt{2t-3}}$
(e) $f(x) = 2 - \sqrt{9-x^2}$
(f) $f(x) = 1 - x^3$
(g) $f(x) = 2 - |x|$

5.2 Give the largest possible domain of the following functions?

(a) $f(x) = \frac{|x-3|}{x-3}$
(b) $f(x) = \ln(1+x)$
(c) $f(x) = \ln(1+x^2)$
(d) $f(x) = \ln(1-x^2)$
(e) $f(x) = \exp(-x^2)$
(f) $f(x) = (e^x - 1)/x$

5.3 Draw the graph of function

$$f(x) = -x^4 + 2x^2$$

in interval $[-2,2]$.

5.4 Draw the graph of function

$$f(x) = e^{-x^4+2x^2}$$

in interval $[-2,2]$.

5.5 Draw the graph of function

$$f(x) = \frac{x-1}{|x-1|}$$

in interval $[-2,2]$.

5.6 Draw the graph of function

$$f(x) = \sqrt{|1-x^2|}$$

in interval $[-2,2]$.

5.7 Draw the graphs of the following functions and determine whether these functions are one-to-one or onto (or both).

(a) $f: [-2,2] \to \mathbb{R}, x \mapsto 2x+1$
(b) $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}$
(c) $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^3$
(d) $f: [2,6] \to \mathbb{R}, x \mapsto (x-4)^2 - 1$
(e) $f: [2,6] \to [-1,3], x \mapsto (x-4)^2 - 1$
(f) $f: [4,6] \to [-1,3], x \mapsto (x-4)^2 - 1$

5.8 Let $f(x) = x^2 + 2x - 1$ and $g(x) = 1 + |x|^{\frac{3}{2}}$.

Compute

(a) $(f \circ g)(4)$

(b) $(f \circ g)(-9)$

(c) $(g \circ f)(0)$

(d) $(g \circ f)(-1)$

5.9 Determine $f \circ g$ and $g \circ f$.

What are the domains of f, g, $f \circ g$ and $g \circ f$?

(a) $f(x) = x^2, \quad g(x) = 1 + x$

(b) $f(x) = \sqrt{x} + 1, \quad g(x) = x^2$

(c) $f(x) = \frac{1}{x+1}, \quad g(x) = \sqrt{x} + 1$

(d) $f(x) = 2 + \sqrt{x}, \quad g(x) = (x-2)^2$

(e) $f(x) = x^2 + 2, \quad g(x) = x - 3$

(f) $f(x) = \frac{1}{1+x^2}, \quad g(x) = \frac{1}{x}$

(g) $f(x) = \ln(x), \quad g(x) = \exp(x^2)$

(h) $f(x) = \ln(x-1), \quad g(x) = x^3 + 1$

5.10 Find the inverse function of

$$f(x) = \ln(1 + x)$$

Draw the graphs of f and f^{-1}.

5.11 Draw the graph of function

$$f(x) = 2x + 1$$

in interval $[-2, 2]$.

Hint: Two points and a ruler are sufficient.

5.12 Draw (sketch) the graph of power function

$$f(x) = x^n$$

in interval $[0, 2]$ for

$$n = -4, -2, -1, -\tfrac{1}{2}, 0, \tfrac{1}{4}, \tfrac{1}{2}, 1, 2, 3, 4\,.$$

5.13 Draw (sketch) the graphs of the following functions in interval $[-2, 2]$:

(a) $f(x) = \dfrac{x}{x^2 + 1}$

(b) $f(x) = \dfrac{x}{x^2 - 1}$

(c) $f(x) = \dfrac{x^2}{x^2 + 1}$

(d) $f(x) = \dfrac{x^2}{x^2 - 1}$

5.14 Draw (sketch) the graph of the following functions:

(a) $f(x) = e^x$

(b) $f(x) = 3^x$

(c) $f(x) = e^{-x}$

(d) $f(x) = e^{x^2}$

(e) $f(x) = e^{-x^2}$

(f) $f(x) = e^{-1/x^2}$

(g) $\cosh(x) = (e^x + e^{-x})/2$

(h) $\sinh(x) = (e^x - e^{-x})/2$

5.15 Draw (sketch) the graph of the following functions:

(a) $f(x) = \ln(x)$

(b) $f(x) = \ln(x+1)$

(c) $f(x) = \ln\left(\frac{1}{x}\right)$

(d) $f(x) = \log_{10}(x)$

(e) $f(x) = \log_{10}(10x)$

(f) $f(x) = (\ln(x))^2$

5.16 Assign the following functions to the graphs $\boxed{1}$ – $\boxed{18}$:

(a) $f(x) = x^2$

(b) $f(x) = \frac{x}{x+1}$

(c) $f(x) = \frac{1}{x+1}$

(d) $f(x) = \sqrt{x}$

(e) $f(x) = x^3 - 3x^2 + 2x$

(f) $f(x) = \sqrt{|2x - x^2|}$

(g) $f(x) = -x^2 - 2x$

(h) $f(x) = (x^3 - 3x^2 + 2x)\operatorname{sgn}(1 - x) + 1$
(sgn(x) = 1 if $x \geq 0$ and -1 otherwise.)

(i) $f(x) = e^x$

(j) $f(x) = e^{x/2}$

(k) $f(x) = e^{2x}$

(l) $f(x) = 2^x$

(m) $f(x) = \ln(x)$

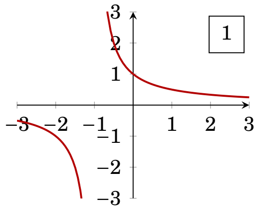
(n) $f(x) = \log_{10}(x)$

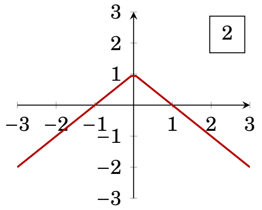
(o) $f(x) = \log_2(x)$

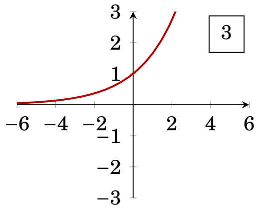
(p) $f(x) = \sqrt{4 - x^2}$

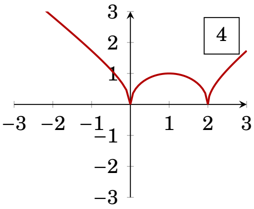
(q) $f(x) = 1 - |x|$

(r) $f(x) = \prod_{k=-1}^{2}(x + k)$





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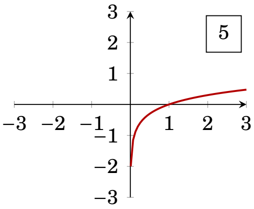
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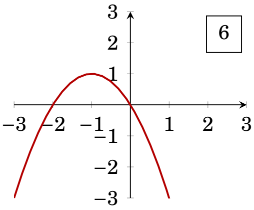
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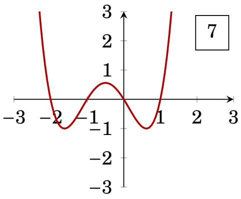
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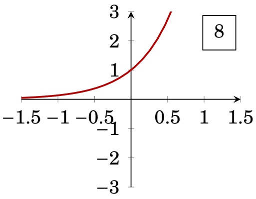
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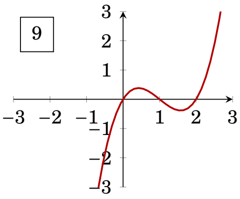
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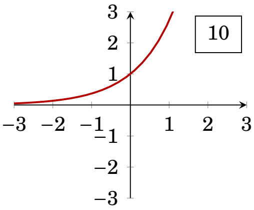
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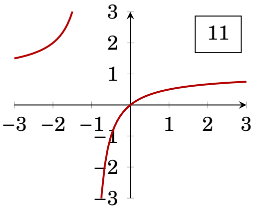
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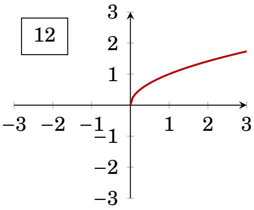
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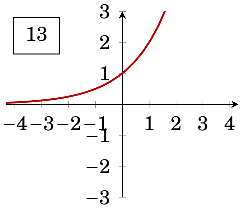
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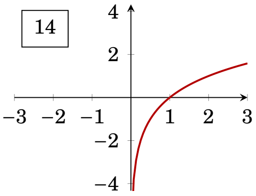
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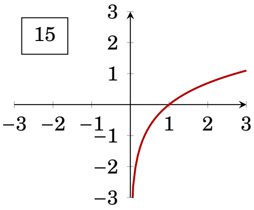
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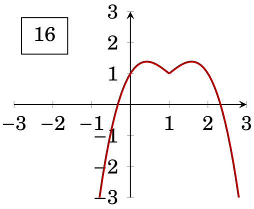
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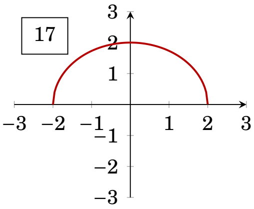
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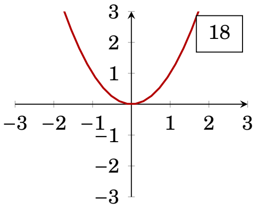


5.17 In a simplistic model we are given utility function U of a household w.r.t. two complementary goods (e.g. left and right shoes):

$$U(x_1, x_2) = \sqrt{\min\{x_1, x_2\}}, \quad x_1, x_2 \geq 0.$$

(a) Sketch the graph of U.

(b) Sketch the contour lines for $U = U_0 = 1$ and $U = U_1 = 2$.

5.18 Draw the following indifference curves:

(a) $x + y^2 - 1 = 0$

(b) $x^2 + y^2 - 1 = 0$

(c) $x^2 - y^2 - 1 = 0$

5.19 Sketch the graphs of the following paths:

(a) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

(b) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$

(c) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t\,\cos(2\pi t) \\ t\,\sin(2\pi t) \end{pmatrix}$

6

Limits

Many issues in analysis result from inspecting function values at some point. Then it is important that we approach this point without actually touching it. A possible idea is to get closer and closer to this point stepwise. Such steps can be implemented by a sequence of arguments. For describing the idea behind "approaching" we need the notion of a limit.

This we first define limits of sequences and use this concept to define limits of functions. Finally we see how this idea can be used to define such concepts of continuity and derivatives (in Chapter 7).

6.1 Limit of a Sequence

Consider the following sequence of numbers

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \tfrac{1}{n}\right)_{n=1}^{\infty} = (-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \ldots)$$

The terms of this sequence *tend* to 0 with increasing n. At some point we cannot distinguish any more between the tick marks for the elements of this sequence and the red tick mark for 0.

Figure 6.1

We could try to use magnification glass.

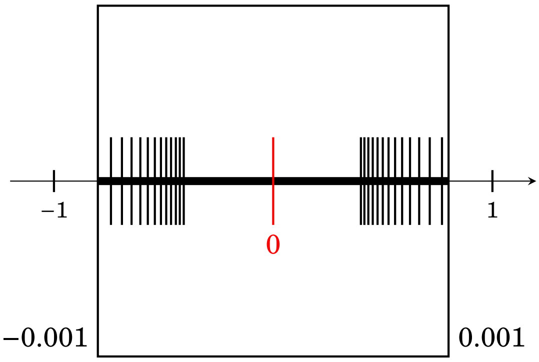
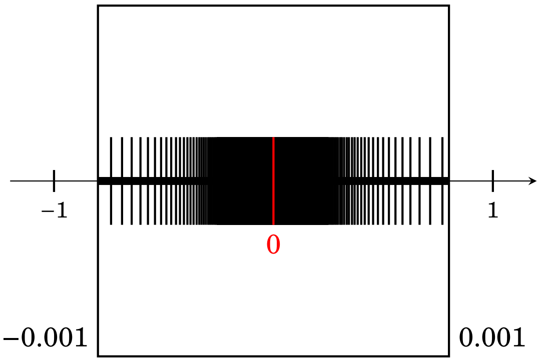


Figure 6.2

However, if we add more terms of the sequence we again arrive at some point where we again cannot distinguish between the tick marks for the elements of this sequence and the red tick mark for 0.

Figure 6.3



Of course we could use a glass with stronger magnification. But for the above sequence we always will reach a point where we cannot distinguish between the tick marks for such an element of this sequence (and all elements following this one) and the red tick mark for 0.

We say that sequence (a_n) *converges* to 0. Here is a formal definition.

Definition 6.4

limit

Limit of a Sequence. A number $a \in \mathbb{R}$ is a **limit** of sequence $(a_n)_{n=1}^{\infty}$, if there *exists an N for every interval*[1] $(a-\varepsilon, a+\varepsilon)$ such that $a_n \in (a-\varepsilon, a+\varepsilon)$ for all $n \geq N$; i.e., all terms following a_N are contained in this interval[2]. We write[3]

$$(a_n) \to 0 \quad \text{or} \quad \lim_{n\to\infty} a_n = 0$$

convergent

divergent

A sequence that has a *limit* is called **convergent**. It **converges** to its limit. A sequence *without* a limit is called **divergent**.

It can be shown that a limit of a sequence is *uniquely* defined (if it exists).

Example 6.5

Sequence

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \tfrac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \dots\right)$$

has limit $a = 0$.

For example, if we set $\varepsilon = 0.3$, then all terms following a_4 are contained in interval $(a-\varepsilon, a+\varepsilon)$.

If we set $\varepsilon = \frac{1}{1000000}$, then all terms starting with the 1 000 001-st term are contained in the interval.

Thus

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0\,.$$

---

[1] Mathematicians like to use ε for a very small positive number.

[2] This can be equivalently written as: $|a_n - a| < \varepsilon$ for all $n \geq N$.

[3] Read: "limit of a_n for n tends to ∞".

Table 6.7

Limits of important sequences

$$\lim_{n\to\infty} c \quad = c \text{ for all } c \in \mathbb{R}$$

$$\lim_{n\to\infty} n^\alpha = \begin{cases} \infty, & \text{for } \alpha > 0, \\ 1, & \text{for } \alpha = 0, \\ 0, & \text{for } \alpha < 0. \end{cases}$$

$$\lim_{n\to\infty} q^n = \begin{cases} \infty, & \text{for } q > 1, \\ 1, & \text{for } q = 1, \\ 0, & \text{for } -1 < q < 1, \\ \nexists, & \text{for } q \leq -1. \end{cases}$$

$$\lim_{n\to\infty} \frac{n^a}{q^n} = \begin{cases} 0, & \text{for } |q| > 1, \\ \infty, & \text{for } 0 < q < 1, \\ \nexists, & \text{for } -1 < q < 0, \end{cases} \qquad \text{for } |q| \notin \{0,1\}.$$

$$\lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n = e = 2.7182818\ldots$$

Example 6.6

Sequence $\left(a_n\right)_{n=1}^\infty = \left(\frac{1}{2^n}\right)_{n=1}^\infty = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right)$ converges to 0:

$$\lim_{n\to\infty} a_n = 0$$

Sequence $\left(b_n\right)_{n=1}^\infty = \left(\frac{n-1}{n+1}\right)_{n=1}^\infty = \left(0, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots\right)$ is convergent:

$$\lim_{n\to\infty} b_n = 1$$

Sequence $\left(c_n\right)_{n=1}^\infty = \left((-1)^n\right)_{n=1}^\infty = \left(-1, 1, -1, 1, -1, 1, \ldots\right)$ is divergent.

Sequence $\left(d_n\right)_{n=1}^\infty = \left(2^n\right)_{n=1}^\infty = \left(2, 4, 8, 16, 32, \ldots\right)$ is divergent, but tends to ∞. By abuse of notation we write[4]:

$$\lim_{n\to\infty} d_n = \infty$$

Computing limits can be a very challenging task. Thus we only look at a few examples. Table 6.7 lists limits of some important sequences. Notice that the limit of $\lim_{n\to\infty} \frac{n^a}{q^n}$ just says that in a product of a power sequence with an exponential sequence the latter dominates the limits.

Table 6.8 list rules for computations with limits that often allow to reduce a given problem to the limits listed in Table 6.7.

Example 6.9

$$\lim_{n\to\infty} \left(2 + \frac{3}{n^2}\right) = 2 + 3 \underbrace{\lim_{n\to\infty} n^{-2}}_{=0} = 2 + 3 \cdot 0 = 2$$

---

[4] "In mathematics, *abuse of notation* occurs when an author uses a mathematical notation in a way that is not entirely formally correct, but which might suggest the correct intuition (while possibly minimizing errors and confusion at the same time)." (`https://en.wikipedia.org/wiki/Abuse_of_notation`)

Table 6.8

Rules for limits of sequences

Let (a_n) and (b_n) be two sequences with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Let (c_n) be a bounded sequence. Then

(1)	$\lim_{n\to\infty}(\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n$	for all $\alpha, \beta \in \mathbb{R}$
(2)	$\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$	
(3)	$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$	for $b \neq 0$
(4)	$\lim_{n\to\infty}(a_n \cdot c_n) = 0$	if $a = 0$
(5)	$\lim_{n\to\infty} a_n^k = a^k$	for $k \in \mathbb{N}$

$$\lim_{n\to\infty}(2^{-n} \cdot n^{-1}) = \lim_{n\to\infty} \frac{n^{-1}}{2^n} = 0$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{3}{n^2}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(2-\frac{3}{n^2}\right)} = \frac{1}{2}$$

$$\lim_{n\to\infty} \underbrace{\sin(n)}_{\text{bounded}} \cdot \underbrace{\frac{1}{n^2}}_{\to 0} = 0$$

⚠

Important!
When we apply these rules (to rational terms) we have to take care that we never obtain expressions of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $0 \cdot \infty$.

These expressions are **not defined**!

Example 6.10

Here we cannot directly apply the rules for rational terms:

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \frac{\lim_{n\to\infty} 3n^2+1}{\lim_{n\to\infty} n^2-1} = \frac{\infty}{\infty} \quad \text{(not defined!)}$$

⚠

Trick: *Reduce* the fraction by the *largest power* in its *denominator*.

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \lim_{n\to\infty} \frac{\cancel{n}^2}{\cancel{n}^2} \cdot \frac{3+n^{-2}}{1-n^{-2}} = \frac{\lim_{n\to\infty} 3+n^{-2}}{\lim_{n\to\infty} 1-n^{-2}} = \frac{3}{1} = 3$$

Euler's number

Euler's number is a very important number which occurs quite naturally in mathematics. One of these points is the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7182818284590\ldots$$

This limit has many applications including finance (e.g., continuous compounding).

We get exponential function e^x by means of the following limit:

Example 6.11

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(1+\frac{1}{n/x}\right)^n\\ &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} \qquad \left(m=\frac{n}{x}\right)\\ &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x = e^x\end{aligned}$$

6.2 Limit of a Function

What happens with the value of a function f, if the argument x tends to some value x_0 (which need not belong to the domain of f)?

Function

Example 6.12

$$f(x)=\frac{x^2-1}{x-1}$$

is not defined in $x=1$.

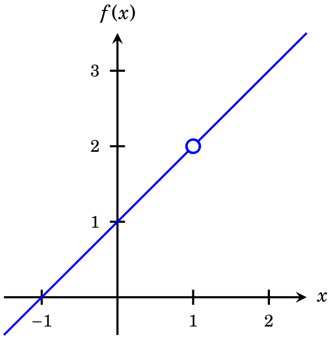


Figure 6.13

By factorizing and reducing we get function[5]

$$g(x)=x+1=\begin{cases}f(x), & \text{if } x\neq 1\\ 2, & \text{if } x=1\end{cases}$$

Suppose we approach argument $x_0=1$.
Then the value of function $f(x)=\dfrac{x^2-1}{x-1}$ tends to 2.
We say that $f(x)$ *converges* to 2 when x tends to 1 and write

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2\,.$$

While the limit it is quite “obvious” from Figure 6.13, we nevertheless are in need of formal definition of convergence. The trick from Example 6.12 – cancel out common factors from rational terms – only works

---

[5] As $\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}=x+1$.

Let $\lim_{x \to x_0} f(x) = y$ and $\lim_{x \to x_0} g(x) = z$. Then

(1) $\lim_{x \to x_0} (c \cdot f(x) + d) = c \cdot y + d$ for all $c, d \in \mathbb{R}$

(2) $\lim_{x \to x_0} (f(x) + g(x)) = y + z$

(3) $\lim_{x \to x_0} (f(x) \cdot g(x)) = y \cdot z$

(4) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{y}{z}$ for $z \neq 0$

(5) $\lim_{x \to x_0} (f(x))^k = a^k$ for $k \in \mathbb{N}$

Table 6.15

Rules for limits of functions

in rare cases. So one approach is to use sequences[6] that converge to argument $x_0 = 1$ and look at the sequence of the corresponding function values. In other words, we approach x_0 step-wise where the step size becomes smaller and smaller when we approach x_0. We do not want to step on x_0 directly as it may not be part of the domain of f.

Definition 6.14

limit

Limit of a Function. If sequence $\left(f(x_n)\right)_{n=1}^{\infty}$ of function values converges to number y_0 for *every* convergent sequence $\left(x_n\right)_{n=1}^{\infty} \to x_0$ of arguments, then y_0 is called the **limit** of f as x approaches x_0.
We write

$$\lim_{x \to x_0} f(x) = a \quad \text{or} \quad f(x) \to a \text{ for } x \to x_0$$

⚠

Point x_0 need not belong to the domain of f. Limit y_0 need not belong to the codomain of f.

From our definition of limits of functions by means of convergent sequences we immediately conclude that rules analogous to Table 6.8 also for limits of functions, see Table 6.15.

How Can We Find Limits?

As for sequences finding limits of functions can be quite challenging. The following recipe is suitable for “simple” functions:

1. Draw the graph of the function.
2. Mark x_0 on the x-axis.
3. Follow the graph with your pencil until we reach x_0 starting from *right* of x_0.

---

[6] In literature you will find other equivalent definitions as well.

4. The y-coordinate of your pencil in this point is then the so called **right-handed limit** of f as x approaches x_0 (from above)[7]:

right-handed limit

$$\lim_{x\to x_0^+} f(x)$$

5. Analogously we get the **left-handed limit** of f as x approaches x_0 (from below)[8]:

left-handed limit

$$\lim_{x\to x_0^-} f(x)$$

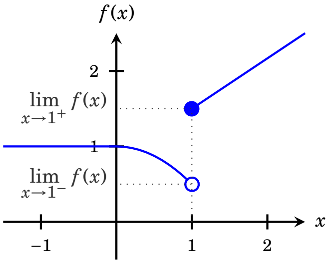
6. If both limits *coincide*, then the limit exists and we have

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$$

Otherwise the limit does not exist.

Example 6.16

Consider the following function:



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

Figure 6.17

Limit does not exist

The limit of f at $x_0 = 1$ does not exist as

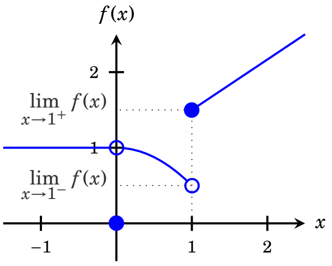
$$0.5 = \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) = 1.5\,.$$

The limits at other points, however, do exist.
For example, for $x_0 = 0$ we find

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 1\,.$$

Example 6.18

The following example is similar to Figure 6.17 above.



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

Figure 6.19

Limit does not exist

---

[7] Other notations: $\lim_{x\downarrow x_0} f(x)$ or $\lim_{x\searrow x_0} f(x)$).

[8] Other notations: $\lim_{x\uparrow x_0} f(x)$ or $\lim_{x\nearrow x_0} f(x)$).

The only difference is the function value at $x_0 = 0$. Nevertheless, we find

$$\lim_{x\to 0^-} f(x) = 1 = \lim_{x\to 0^+} f(x)\,.$$

Consequently the limit does exist and

$$\lim_{x\to 0} f(x) = 1\,.$$

Unbounded Function

It may happen that $f(x)$ tends to ∞ (or $-\infty$) if x tends to x_0.
We then write (by abuse of notation):

$$\lim_{x\to x_0} f(x) = \infty$$

Example 6.20

Function $f(x) = \frac{1}{x^2}$ is not bounded.

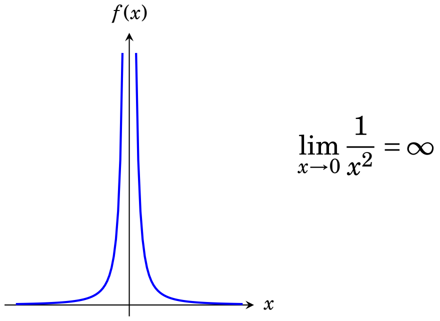


Figure 6.21

Unbounded function

Note that $f(x) = \frac{1}{x^2}$ is not defined for $x = 0$.

Limit as $x \to \infty$

By abuse of language we can define the *limit* analogously for $x_0 = \infty$ and $x_0 = -\infty$, resp.

Limit

$$\lim_{x\to \infty} f(x)$$

exists, if $f(x)$ converges whenever x tends to infinity.

Example 6.22

Consider function $f(x) = \frac{1}{x^2}$ (see Figure 6.21). Then

$$\lim_{x\to \infty} \frac{1}{x^2} = 0 \quad \text{and} \quad \lim_{x\to -\infty} \frac{1}{x^2} = 0$$

6.3 L'Hôpital's Rule

Suppose we want to compute

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}$$

and find

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0 \qquad (\text{or } = \pm\infty)$$

However, expressions like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are not defined.

Recall: You must not reduce the fraction by 0 or ∞! ⚠

In such a case **l'Hôpital's rule** rule is can be quite useful. **l'Hôpital's rule**

Assume that f and g are differentiable in x_0 and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0 \qquad (\text{or } = \infty \text{ or } = -\infty).$$

Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Example 6.23

$$\lim_{x\to 2}\frac{x^3-7x+6}{x^2-x-2} = \lim_{x\to 2}\frac{3x^2-7}{2x-1} = \frac{5}{3}$$

$$\lim_{x\to\infty}\frac{\ln x}{x^2} = \lim_{x\to\infty}\frac{\frac{1}{x}}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0$$

L'Hôpital's rule can be applied iteratively: Example 6.24

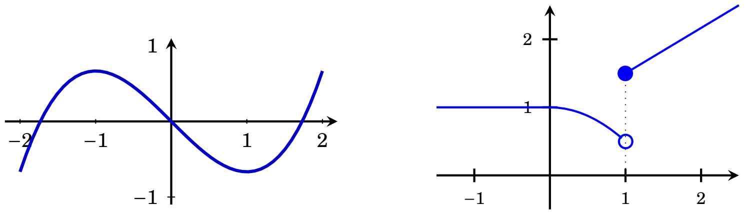
$$\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2} = \lim_{x\to 0}\frac{1-(1+x)^{-1}}{2x} = \lim_{x\to 0}\frac{(1+x)^{-2}}{2} = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{e^x-x-1}{x^2} = \lim_{x\to 0}\frac{e^x-1}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$$

6.4 Continuity

We may observe that we can draw the graph of a function *without removing the pencil from paper*. We call such functions **continuous**. **continuous**

For some other functions we have to *raise* the pencil and jump to a different point when we proceed with our drawing. At such points the function has a **jump discontinuity**. **jump discontinuity**



continuous

jump discontinuity at $x = 1$

Figure 6.25

Continuous and discontinuous function

This picture of drawing graphs is helpful for understanding the idea of continuity. Nevertheless, we of course need a formal definition.

Definition 6.26

Continuity. Function $f : D \to \mathbb{R}$ is called **continuous** *at* $x_0 \in D$, if

continuous

(i) $\lim_{x \to x_0} f(x)$ exists, and

(ii) $\lim_{x \to x_0} f(x) = f(x_0)$.

The function is called **continuous** if it is continuous *at all* points of its domain.

continuous

⚠

Continuous functions have the following important property.

Memorize it!

$$\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right)$$

⚠

Note that continuity is a *local* property of a function.
Moreover, observe that x_0 *must be* an element of domain D of the function.

Example 6.27

Function f from Figure 6.17 is not continuous in $x = 1$ as $\lim_{x \to 1} f(x)$ does not exist. So f is not a continuous function.

However, it is still continuous in all $x \in \mathbb{R} \setminus \{1\}$. For example at $x = 0$, $\lim_{x \to 0} f(x)$ does exist and $\lim_{x \to 0} f(x) = 1 = f(0)$.

Example 6.28

Function f from Figure 6.19 is similar to that from Figure 6.17. Again it is not continuous in $x = 1$. Moreover, it is also not continuous at $x = 0$, either. Then $\lim_{x \to 0} f(x) = 1$ does exist but is distinct from $f(0) = 0$, $\lim_{x \to 0} f(x) \neq f(0)$.

How Can We Check for Continuity?

Observe that our definition of *continuity* is based on the concept of function limits. So verifying that a given function is continuous or has one or more jump discontinuities can be challenging. The following recipe is suitable for "nice" functions:

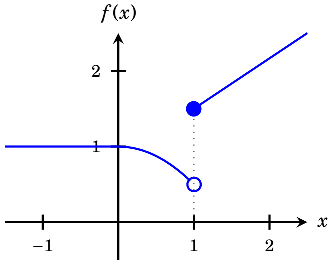
1. Draw the graph of the given function.
2. At all points of the *domain*, where we *have to raise* the pencil from paper the function is *not continuous*.
3. At all other points of the domain (where we need not raise the pencil) the function is *continuous*.

Example 6.29

Consider the following function:

Figure 6.30

Jump discontinuity



$$f(x)=\begin{cases}1, & \text{for } x<0\\ 1-\frac{x^2}{2}, & \text{for } 0\le x<1\\ \frac{x}{2}+1, & \text{for } x\ge 1\end{cases}$$

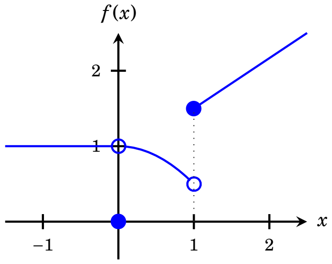
When we draw its graph we have to raise the pencil when (and only when) we arrive at $x_0 = 1$. Thus the function has a jump discontinuity in $x_0 = 1$ but is continuous in all other points, i.e., $\mathbb{R}\setminus\{1\}$. We say that f is not continuous.

Example 6.31

Consider the following slightly modified function:

Figure 6.32

Jump discontinuity



$$f(x)=\begin{cases}1, & \text{for } x<0\\ 0, & \text{for } x=0\\ 1-\frac{x^2}{2}, & \text{for } 0<x<1\\ \frac{x}{2}+1, & \text{for } x\ge 1\end{cases}$$

When we draw its graph we have to raise the pencil when (and only when) we arrive at $x = 0$ or at $x = 1$. Thus the function has jump discontinuities in 0 and 1 but is continuous in all other points, i.e., $\mathbb{R}\setminus\{0,1\}$. We say that f is not continuous.

Limits of Continuous Functions

If function f is known to be *continuous*, then its limit $\lim_{x\to x_0} f(x)$ exists for all $x_0 \in D_f$ and we obviously find

$$\lim_{x\to x_0} f(x) = f(x_0)\,.$$

Example 6.33

Polynomials are always continuous. Hence

$$\lim_{x\to 2}(3x^2-4x+5) = 3\cdot 2^2 - 4\cdot 2 + 5 = 9\,.$$

— Summary

- limit of a sequence
- limit of a function
- convergent and divergent
- Euler's number
- rules for limits
- l'Hôpital's rule
- continuous functions

— Exercises

6.1 Compute the following limits:

(a) $\lim_{n\to\infty}\left(7+\left(\frac{1}{2}\right)^n\right)$ (b) $\lim_{n\to\infty}\left(\frac{2n^3-6n^2+3n-1}{7n^3-16}\right)$

(c) $\lim_{n\to\infty}\left(n^2-(-1)^n n^3\right)$ (d) $\lim_{n\to\infty}\left(\frac{n^2+1}{n+1}\right)$

(e) $\lim_{n\to\infty}\left(\frac{n \bmod 10}{(-2)^n}\right)$

$a \bmod b$ is the remainder after integer division, e.g., $17 \bmod 5 = 2$ and $12 \bmod 4 = 0$.

6.2 Compute the following limits:

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$ (b) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$ (c) $\lim_{n\to\infty}\left(1+\frac{1}{nx}\right)^{n}$

6.3 Draw the graph of function

$$f(x)=\begin{cases}-\frac{x^2}{2}, & \text{for } x\le -2,\\ x+1, & \text{for } -2<x<2,\\ \frac{x^2}{2}, & \text{for } x\ge 2.\end{cases}$$

and determine $\lim_{x\to x_0^+} f(x)$, $\lim_{x\to x_0^-} f(x)$, and $\lim_{x\to x_0} f(x)$ for $x_0=-2$, 0 and 2:

$$\begin{array}{lll}\lim_{x\to-2^+} f(x) & \lim_{x\to-2^-} f(x) & \lim_{x\to-2} f(x)\\ \lim_{x\to0^+} f(x) & \lim_{x\to0^-} f(x) & \lim_{x\to0} f(x)\\ \lim_{x\to2^+} f(x) & \lim_{x\to2^-} f(x) & \lim_{x\to2} f(x)\end{array}$$

6.4 Determine the following left-handed and right-handed limits:

(a) $\lim_{x\to0^-} f(x)$

$\lim_{x\to0^+} f(x)$

for $f(x)=\begin{cases}1, & \text{for } x\ne 0,\\ 0, & \text{for } x=0.\end{cases}$

(b) $\lim_{x\to0^-}\frac{1}{x}$

$\lim_{x\to0^+}\frac{1}{x}$

(c) $\lim_{x\to1^-} x$

$\lim_{x\to1^+} x$

6.5 Determine the following limits:

(a) $\lim_{x\to\infty} \frac{1}{x+1}$

(b) $\lim_{x\to 0} x^2$

(c) $\lim_{x\to\infty} \ln(x)$

(d) $\lim_{x\to 0} \ln|x|$

(e) $\lim_{x\to\infty} \frac{x+1}{x-1}$

6.6 Determine

(a) $\lim_{x\to 1^+} \frac{x^{3/2}-1}{x^3-1}$

(b) $\lim_{x\to -2^-} \frac{\sqrt{|x^2-4|^2}}{x+2}$

(c) $\lim_{x\to 0^-} \lfloor x \rfloor$

(d) $\lim_{x\to 1^+} \frac{x-1}{\sqrt{x-1}}$

Remark: $\lfloor x \rfloor$ is the largest integer less than or equal to x.

6.7 Determine

(a) $\lim_{x\to 2^+} \frac{2x^2-3x-2}{|x-2|}$

(b) $\lim_{x\to 2^-} \frac{2x^2-3x-2}{|x-2|}$

(c) $\lim_{x\to -2^+} \frac{|x+2|^{3/2}}{2+x}$

(d) $\lim_{x\to 1^-} \frac{x+1}{x^2-1}$

(e) $\lim_{x\to -7^+} \frac{2|x+7|}{x^2+4x-21}$

6.8 Compute

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

for

(a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = x^3$

(d) $f(x) = x^n$, for $n \in \mathbb{N}$.

6.9 Compute the following limits:

(a) $\lim_{x\to 4} \frac{x^2-2x-8}{x^3-2x^2-11x+12}$

(b) $\lim_{x\to -1} \frac{x^2-2x-8}{x^3-2x^2-11x+12}$

(c) $\lim_{x\to 2} \frac{x^3-5x^2+8x-4}{x^3-3x^2+4}$

(d) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

(e) $\lim_{x\to 0^+} x\ln(x)$

(f) $\lim_{x\to\infty} x\ln(x)$

6.10 If we apply l'Hôpital's rule on the following limit we obtain

$$\lim_{x\to 1} \frac{x^3+x^2-x-1}{x^2-1} = \lim_{x\to 1} \frac{3x^2+2x-1}{2x} = \lim_{x\to 1} \frac{6x+2}{2} = 4\,.$$

However, the correct value for the limit is 2.

Why does l'Hôpital's rule not work for this problem?

How do you get the correct value?

6.11 Draw the graph of function

$$f(x) = \begin{cases} -\frac{x^2}{2}, & \text{for } x \le -2, \\ x+1, & \text{for } -2 < x < 2, \\ \frac{x^2}{2}, & \text{for } x \ge 2. \end{cases}$$

and compute $\lim_{x\to x_0^+} f(x)$, $\lim_{x\to x_0^-} f(x)$, and $\lim_{x\to x_0} f(x)$ for $x_0 = -2$, 0, and 2.

Is function f continuous at these points?

6.12 Determine the left and right-handed limits of function

$$f(x) = \begin{cases} x^2+1, & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -x^2-1, & \text{for } x < 0. \end{cases}$$

at $x_0 = 0$.

Is function f continuous at this point?

Is function f differentiable at this point?

6.13 Is function

$$f(x) = \begin{cases} x+1, & \text{for } x \le 1, \\ \frac{x}{2} + \frac{3}{2}, & \text{for } x > 1, \end{cases}$$

continuous at $x_0 = 1$?

Is it differentiable at $x_0 = 1$?

Compute the limit of f at $x_0 = 1$.

6.14 Sketch the graphs of the following functions.

Which of these are continuous (on its domain)?

(a) $D = \mathbb{R}, f(x) = x$

(b) $D = \mathbb{R}, f(x) = 3x + 1$

(c) $D = \mathbb{R}, f(x) = e^{-x} - 1$

(d) $D = \mathbb{R}, f(x) = |x|$

(e) $D = \mathbb{R}^+, f(x) = \ln(x)$

(f) $D = \mathbb{R}, f(x) = \lfloor x \rfloor$

(g) $D = \mathbb{R}, f(x) = \begin{cases} 1, & \text{for } x \leq 0, \\ x+1, & \text{for } 0 < x \leq 2, \\ x^2, & \text{for } x > 2. \end{cases}$

Remark: $\lfloor x \rfloor$ is the largest integer less than or equal to x.

6.15 Sketch the graph of

$$f(x) = \frac{1}{x}.$$

Is it continuous?

6.16 Determine a value for h, such that function

$$f(x) = \begin{cases} x^2 + 2hx, & \text{for } x \leq 2, \\ 3x - h, & \text{for } x > 2, \end{cases}$$

is continuous.

7

Derivatives

> Ein neuer Zweig der Mathematik, der bis zu der Kunst vorgedrungen ist, mit unendlich kleinen Größen zu rechnen, gibt jetzt auch in anderen komplizierten Fällen der Bewegung Antwort auf die Fragen, die bisher unlösbar schienen.
>
> Лев Николаевич Толстой (1817–1875)

7.1 Differential Quotient

Let $f: \mathbb{R} \to \mathbb{R}$ be some function. Then the ratio

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

is called **difference quotient** of f at x. It can be geometrically interpreted as the slope of the secant between x_0 and x.

difference quotient

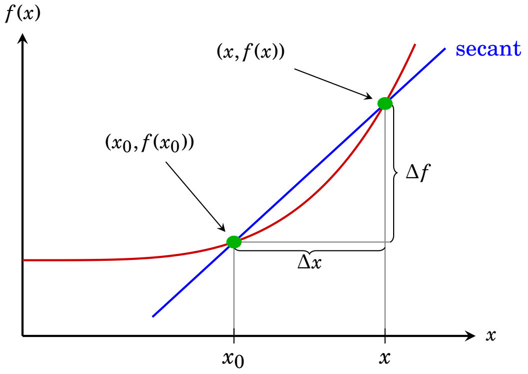


Figure 7.1

Difference quotient

It also can be seen as the rate of change of function f between arguments x_0 and x. So we naturally may ask if the change $\Delta x = x - x_0$ of the argument tends to 0. What is the instantaneous change of f at x_0?

If the *limit* of the difference quotient

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

exists, then function f is called **differentiable** at x_0. This limit is then called **differential quotient** or **(first) derivative** of function f at x_0. We write

differentiable

differential quotient

derivative

$$f'(x_0) \qquad \text{or} \qquad \left.\frac{df}{dx}\right|_{x=x_0}$$

Function f is called *differentiable*, if it is differentiable at each point of its domain.

⚠

Observe that differentiability is a local property of a function. Hence we say that f is differentiable in point x_0.

The differential quotient gives the *slope of the tangent* to the graph of function $f(x)$ at x_0.

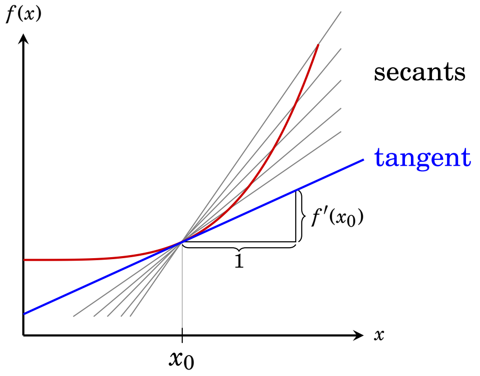


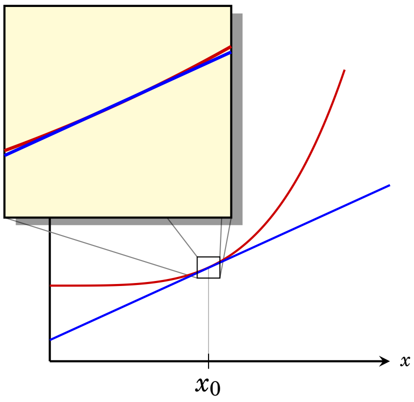
Figure 7.2

Differential quotient

The differential quotient can also be seen as the slope of a marginal function (as in “marginal utility” or “marginal revenue”).

Figure 7.3

Marginal function



Existence of Differential Quotient

Function f is differentiable at all points, where we can draw the tangent (with finite slope) uniquely to the graph.

In each points where this is *not* possible f is not differentiable. Then the limit of the difference quotient does not exist We can find such points when we inspect the graph of f.
In particular these points are

- jump discontinuities

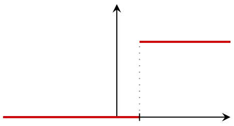


Figure 7.4

Jump discontinuity

- "kinks" in the graph of the function

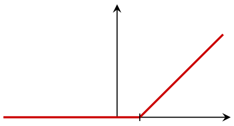


Figure 7.5

Kinks

- vertical tangents (infinite slope)

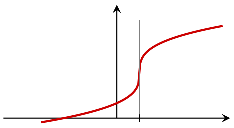


Figure 7.6

Vertical tangent

Computation of the Differential Quotient

We can compute a differential quotient by determining the limit of difference quotients.

Example 7.7

Let $f(x) = x^2$. The we find for the first derivative

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{(x_0+h)^2 - {x_0}^2}{h} \\ &= \lim_{h\to 0} \frac{{x_0}^2 + 2x_0h + h^2 - {x_0}^2}{h} \\ &= \lim_{h\to 0} \frac{2x_0h + h^2}{h} = \lim_{h\to 0}(2x_0 + h) \\ &= 2x_0 \end{aligned}$$

7.2 Derivative

Recall that differential quotient $\left.\frac{df}{dx}\right|_{x_0}$ is a local property of function f at some point x_0.

$f(x)$	$f'(x)$
c	0
x^α	$\alpha \cdot x^{\alpha-1}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

Table 7.8

Derivatives of elementary functions

▸ $(c \cdot f(x))' = c \cdot f'(x)$	
▸ $(f(x)+g(x))' = f'(x)+g'(x)$	Summation rule
▸ $(f(x)\cdot g(x))' = f'(x)\cdot g(x)+f(x)\cdot g'(x)$	Product rule
▸ $(f(g(x)))' = f'(g(x))\cdot g'(x)$	Chain rule
▸ $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{(g(x))^2}$	Quotient rule

Table 7.9

Computation rules for derivatives

We can, however, create a new function f' that maps every x to the differential quotient of f at x.

$$f' : D \to \mathbb{R},\ x \mapsto f'(x) = \left.\frac{df}{dx}\right|_x$$

Function f' is called the **first derivative** of function f. Its domain D is the set of all points where f is differentiable (i.e., where the limit of the difference quotient) exists.

first derivative

Computing derivatives of functions can be quite challenging. Fortunately this already has been done for important functions by mathematicians in the 18th century. Table 7.8 lists derivatives of elementary functions.

Of course in practice one needs derivatives of compositions of functions. These can be derived from elementary functions by means of the rules from Table 7.9.

Example 7.10

Applying the rules from Table 7.9 yields:

$$\left(3x^3+2x-4\right)' = 3\cdot 3\cdot x^2+2\cdot 1-0 = 9x^2+2$$

$$\left(e^x\cdot x^2\right)' = \left(e^x\right)'\cdot x^2+e^x\cdot\left(x^2\right)' = e^x\cdot x^2+e^x\cdot 2x$$

$$\left((3x^2+1)^2\right)' = 2(3x^2+1)\cdot 6x$$

$$\left(\sqrt{x}\right)' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\left(a^x\right)' = \left(e^{\ln(a)\cdot x}\right)' = e^{\ln(a)\cdot x} \cdot \ln(a) = a^x \ln(a)$$

$$\left(\frac{1+x^2}{1-x^3}\right)' = \frac{2x\cdot(1-x^3)-(1+x^2)\cdot 3x^2}{(1-x^3)^2}$$

Higher Order Derivatives

Observe that the derivative f' of some function f is again a function. So we can try to compute the derivative of the first derivative; provided that f' is differentiable.

Thus we obtain the

- **second derivative** $f''(x)$ of function f,
- **third derivative** $f'''(x)$, etc.,
- n**-th derivative** $f^{(n)}(x)$.

second derivative $f''(x)$

third derivative $f'''(x)$

n-th derivative $f^{(n)}(x)$

Other notations of higher derivatives in literature are:

- $f''(x) = \frac{d^2f}{dx^2}(x) = \left(\frac{d}{dx}\right)^2 f(x)$
- $f^{(n)}(x) = \frac{d^nf}{dx^n}(x) = \left(\frac{d}{dx}\right)^n f(x)$

Example 7.11

The first six derivatives of function

$$f(x) = x^4 + 2x^2 + 5x - 3$$

are

$$\begin{aligned} f'(x) &= (x^4 + 2x^2 + 5x - 3)' = 4x^3 + 4x + 5 \\ f''(x) &= (4x^3 + 4x + 5)' = 12x^2 + 4 \\ f'''(x) &= (12x^2 + 4)' = 24x \\ f^{(4)}(x) &= (24x)' = 24 \\ f^{(5)}(x) &= 0 \\ f^{(6)}(x) &= 0 \end{aligned}$$

7.3 Differential

We can estimate the derivative $f'(x_0)$ approximately by means of the difference quotient with *small* change Δx:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \approx \frac{\Delta f}{\Delta x}$$

Vice verse we can estimate the change Δf of f for *small* changes Δx approximately by the first derivative of f:

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x$$

This approximation has the following properties:

- $f'(x_0) \cdot \Delta x$ is a *linear function* in Δx (when x_0 is fixed).
- The graph of this linear function corresponds to the tangent of f at x_0.
- It is the *best possible* approximation of f by a linear function *around* x_0.
- This approximation is useful only for *"small"* values of Δx.

This approximation becomes exact if Δx (and thus Δf) becomes *infinitesimally small*. We then write dx and df instead of Δx and Δf, resp.

$$df(x_0) = f'(x_0)\,dx$$

differential

Symbols df and dx are called the **differentials** of function f and the independent variable x, resp.

Differential df can be seen as a linear function in dx. We can use it to compute f approximately around x_0.

$$f(x_0 + h) \approx f(x_0) + df(x_0)(h) \qquad \text{where } df(x_0)(h) = f'(x_0)h$$

Example 7.12

Let $f(x) = e^x$.

Differential of f at point $x_0 = 1$:

$$df(x_0) = f'(1)\,dx = e^1\,dx$$

Approximation of $f(1.1)$ by means of this differential gives

$$h = \Delta x = (x_0 + h) - x_0 = 1.1 - 1 = 0.1$$

$$f(1.1) \approx f(1) + df(1)h = e + e \cdot 0.1 \approx 2.99$$

For comparison, the exact value is $f(1.1) = 3.004166\ldots$

⚠

The approximation does not work for large values[1] of h:
$f(2) \approx f(1) + df(1) \cdot 1 = e + e \cdot 1 \approx 5.44$ but $f(2) = 2e = 7.389056\ldots$.

The differential is the mathematical concept behind the idea of "marginal function". The marginal revenue is just a linear approximation of the (non-linear) revenue function.

---

[1] The magnitude of "large" depends on function f and location x_0.

7.4 Elasticity

The first derivative of a function gives *absolute* rate of change of f at x_0. Hence it depends on the scales used for argument and function values.

However, often *relative* rates of change are more appropriate. In particular relative changes ignore scaling effects.

We obtain *scale invariance* and *relative* rate of changes by

$$\frac{\text{change of function value relative to value of function}}{\text{change of argument relative to value of argument}} = \frac{\frac{f(x+\Delta x)-f(x)}{f(x)}}{\frac{\Delta x}{x}}$$

and thus

$$\lim_{\Delta x\to 0}\frac{\frac{f(x+\Delta x)-f(x)}{f(x)}}{\frac{\Delta x}{x}} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\cdot\frac{x}{f(x)} = f'(x)\cdot\frac{x}{f(x)}$$

Expression

$$\varepsilon_f(x) = x\cdot\frac{f'(x)}{f(x)}$$

is called the **elasticity** of f at point x.

elasticity

Example 7.13

Let $f(x) = 3\,e^{2x}$. Then

$$\varepsilon_f(x) = x\cdot\frac{f'(x)}{f(x)} = x\cdot\frac{6\,e^{2x}}{3\,e^{2x}} = 2x$$

Let $f(x) = \beta x^{\alpha}$. Then

$$\varepsilon_f(x) = x\cdot\frac{f'(x)}{f(x)} = x\cdot\frac{\beta\,\alpha\,x^{\alpha-1}}{\beta x^{\alpha}} = \alpha$$

An alternative Approach

The relative rate of change of f w.r.t. absolute change of argument x can be expressed as

$$\ln(f(x))' = \frac{f'(x)}{f(x)}$$

What happens if we compute the derivative of $\ln(f(x))$ w.r.t. $\ln(x)$?

Let $v = \ln(x)$ (or equivalently $x = e^v$). Then derivation by means of the chain rule yields

$$\frac{d(\ln(f(x)))}{d(\ln(x))} = \frac{d(\ln(f(e^v)))}{dv} = \frac{f'(e^v)}{f(e^v)}e^v = \frac{f'(x)}{f(x)}x = \varepsilon_f(x)$$

That is,

$$\varepsilon_f(x) = \frac{d(\ln(f(x)))}{d(\ln(x))}$$

A Shortcut

In microeconomic calculations often shortcuts like following computations are used. The underlying mathematical principle are substitution and chain rule.

We can use the chain rule *formally* in the following way to derive the above formula for the elasticity. Let

- $u = \ln(y)$,
- $y = f(x)$,
- $x = e^v \quad \Leftrightarrow \quad v = \ln(x)$

Then we find

$$\frac{d(\ln f)}{d(\ln x)} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv} = \frac{1}{y} \cdot f'(x) \cdot e^v = \frac{f'(x)}{f(x)} x$$

⚠ Notice that $\frac{du}{dv}$ is a mere symbol (for the limit of the difference quotient) and not a ratio of numbers. So the above expansion to $\frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv}$ is not a exact transformation but rather a shortcut for a longer mathematical derivation.

Elastic Functions

A Function f is called
- ▸ **elastic** in x, if $\left|\varepsilon_f(x)\right| > 1$
- ▸ **1-elastic** in x, if $\left|\varepsilon_f(x)\right| = 1$
- ▸ **inelastic** in x, if $\left|\varepsilon_f(x)\right| < 1$

For elastic functions we then have:
The value of the function changes *relatively* faster than the value of the argument.

Example 7.14

Function $f(x) = 3e^{2x}$ has elasticity $\varepsilon_f(x) = 2x$. Thus[2]

- 1-elastic, for $x = -\frac{1}{2}$ and $x = \frac{1}{2}$;
- inelastic, for $-\frac{1}{2} < x < \frac{1}{2}$;
- elastic, for $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.

⚠ **Beware:**
Function f is elastic if the *absolute value* of the *elasticity* is greater than 1. So it may happen that the elasticity of an elastic function is negative (to be precise: less than -1).

Example 7.15

Elastic Demand. Let $q(p)$ be an *elastic* demand function, where p is the price. We have: $p > 0$, $q > 0$, and $q' < 0$ (q is decreasing). Hence

$$\varepsilon_q(p) = p \cdot \frac{q'(p)}{q(p)} < -1\,.$$

What happens to the revenue (= price × selling)?

$$\begin{aligned} u'(p) &= (p \cdot q(p))' = 1 \cdot q(p) + p \cdot q'(p) \\ &= q(p) \cdot (1 + \underbrace{p \cdot \frac{q'(p)}{q(p)}}_{=\varepsilon_q < -1}) \\ &< 0 \end{aligned}$$

In other words, the revenue decreases if we raise prices.

Analogously we obtain, that $u'(p) > 0$, if q is inelastic, and $u'(p) = 0$, if q is 1-elastic.

7.5 Partial Derivatives

Derivatives of univariate functions are a very powerful tool, see Chapter 8 below.

So we wish to have such a tool for multivariate functions as well. In a first step towards such a tool is to keep all but one variable fixed and thus obtain a univariate function. Thus we investigate the rate of change of function $f(x_1,\ldots,x_n)$, when variable x_i changes and the other variables remain fixed. Limit

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(\ldots,x_i+\Delta x_i,\ldots) - f(\ldots,x_i,\ldots)}{\Delta x_i}$$

is called the (first) **partial derivative** of f w.r.t. x_i.

partial derivative

Other notations for partial derivative $\frac{\partial f}{\partial x_i}$ in literature:

- $f_{x_i}(\mathbf{x})$ (derivative w.r.t. variable x_i)
- $f_i(\mathbf{x})$ (derivative w.r.t. the i-th variable)
- $f_i'(\mathbf{x})$ (i-th component of the gradient)

Albeit notation f_i is quite common in microeconomics there is a high risk of confusion with the i-th component of a vector values function. I thus recommend not to use this particular notation.

Computation of Partial Derivatives

We obtain partial derivatives $\frac{\partial f}{\partial x_i}$ by applying the rules for *univariate* functions for variable x_i while we treat *all other* variables *as constants*.

Example 7.16

The first partial derivatives of

$$f(x_1,x_2)=\sin(2x_1)\cdot\cos(x_2)$$

are

$$f_{x_1}=2\cdot\cos(2x_1)\cdot\underbrace{\cos(x_2)}_{\text{treated as constant}}$$

and

$$f_{x_2}=\underbrace{\sin(2x_1)}_{\text{treated as constant}}\cdot(-\sin(x_2))$$

Higher Order Partial Derivatives

higher order partial derivatives

We can compute partial derivatives of partial derivatives analogously to their univariate counterparts and obtain **higher order partial derivatives**:

$$\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x}) \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x})$$

Other notations for partial derivative $\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x})$ used in literature:

- $f_{x_i x_k}(\mathbf{x})$ (derivative w.r.t. variables x_i and x_k)
- $f_{ik}(\mathbf{x})$ (derivative w.r.t. the i-th and k-th variable)
- $f''_{ik}(\mathbf{x})$ (component of the Hessian matrix with indices i and k)

If all second order partial derivatives exists and are *continuous*, then the order of differentiation does not matter

$$\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x})=\frac{\partial^2 f}{\partial x_i \partial x_k}(\mathbf{x})$$

Remark: Practically all differentiable functions in economic models have this property.

Example 7.17

Compute the first and second order partial derivatives of

$$f(x,y)=x^2+3xy$$

First order partial derivatives:

$$f_x=2x+3y \qquad f_y=0+3x$$

Second order partial derivatives:

$$\begin{array}{ll} f_{xx}=\ 2 & f_{yx}=\ 3 \\ f_{xy}=\ 3 & f_{yy}=\ 0 \end{array}$$

As can be easily verified we have $f_{xy}=f_{yx}$.

---

[2] See Section 3.2 on page 39 for solving inequalities with absolute values.

7.6 Gradient

We collect all *first order partial derivatives* into a (row) vector which is called the **gradient**[3,4,5] of f at point $\mathbf{x}$.

gradient

$$\nabla f(\mathbf{x}) = \left(f_{x_1}(\mathbf{x}), \dots, f_{x_n}(\mathbf{x})\right)$$

The gradient is also denoted by $f'(\mathbf{x})$ as it is the analog of the first derivative of univariate functions.

The gradient has the following important properties:

- The gradient of f always points in the direction of *steepest ascent*.
- Its length is equal to the slope at this point.
- The gradient is *normal* (i.e. in right angle) to the corresponding *contour line* (level set).

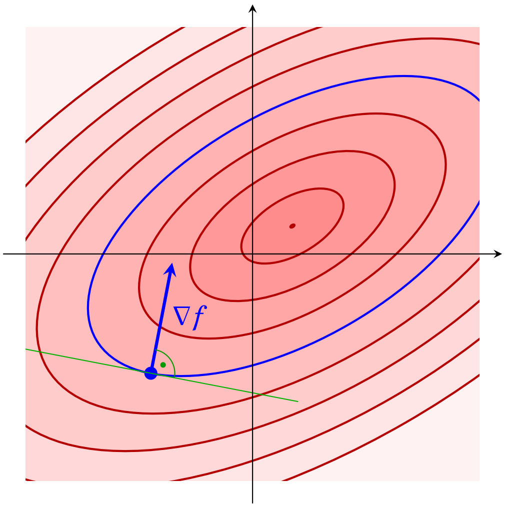


Figure 7.18

Gradient

Example 7.19

Gradient of $f(x,y) = x^2 + 3xy$ at point $\mathbf{x} = (3,2)$:

The first partial derivatives are

$$f_x = 2x + 3y \qquad \text{and} \qquad f_y = 0 + 3x\,.$$

Hence the gradient is $f'(x,y) = (2x+3y,\, 3x)$ and consequently

$$f'(3,2) = (12,9)\,.$$

---

[3] Read: “gradient of f” or “nabla f”.

[4] Symbol “∇” is not a Greek letter. Its name comes from the Hellenistic Greek word νάβλα for a Phoenician harp with a similar shape.

[5] In literature the gradient may also be defined as a column vector.

7.7 Hessian Matrix

Let $f(\mathbf{x}) = f(x_1,\dots,x_n)$ be two a times differentiable function, i.e., we can compute all partial derivatives of second order. Then matrix

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \dots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \dots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \dots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}$$

is called the **Hessian matrix** of f at $\mathbf{x}$.

Hessian matrix

The gradient is also denoted by $f''(\mathbf{x})$ as it is the analog of the second derivative of univariate functions.

Notice that the Hessian matrix is symmetric[6], i.e., $f_{x_ix_k}(\mathbf{x}) = f_{x_kx_i}(\mathbf{x})$.

Example 7.20

Hessian matrix of $f(x,y) = x^2 + 3xy$ at point $\mathbf{x} = (1,2)$:

First partial derivatives:

$$f_x = 2x + 3y \qquad \text{and} \qquad f_y = 0 + 3x\,.$$

Second order partial derivatives:

$$\begin{aligned} f_{xx} &= 2 & f_{xy} &= 3 \\ f_{yx} &= 3 & f_{yy} &= 0 \end{aligned}$$

Hence the Hessian matrix (at $\mathbf{x} = (1,2)$) is given by

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}$$

7.8 Jacobian Matrix

Consider a vector-valued function

$$\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}$$

The $m \times n$ matrix

$$D\mathbf{f}(\mathbf{x}_0) = \mathbf{f}'(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

is called the **Jacobian matrix** of $\mathbf{f}$ at point $\mathbf{x}_0$.

Jacobian matrix

It is the generalization of *derivatives* (and gradients) for vector-valued functions. Observe that the k-th row of this matrix is just the gradient ∇f_k of the k-th component f_k of vector-valued function $\mathbf{f}$.

Example 7.21

Let $f(\mathbf{x}) = f(x_1,x_2) = \exp(-x_1^2 - x_2^2)$. Then

$$\begin{aligned} f'(\mathbf{x}) &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = \nabla f(\mathbf{x}) \\ &= \left(-2x_1 \exp(-x_1^2 - x_2^2), -2x_2 \exp(-x_1^2 - x_2^2)\right) \end{aligned}$$

Let $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1,x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \end{pmatrix}$. Then

$$\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{pmatrix}$$

Let $\mathbf{s}(t) = \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$. Then

$$\mathbf{s}'(t) = \begin{pmatrix} \frac{ds_1}{dt} \\ \frac{ds_2}{dt} \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

Chain Rule

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g}\colon \mathbb{R}^m \to \mathbb{R}^k$. Then

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x})$$

Beware: The r.h.s. of this equation is a multiplication of matrices. ⚠

Example 7.22

Let $\mathbf{f}(x,y) = \begin{pmatrix} e^x \\ e^y \end{pmatrix}$ and $\mathbf{g}(x,y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$. Then

$$\mathbf{f}'(x,y) = \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} \qquad \text{and} \qquad \mathbf{g}'(x,y) = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

and we find by means of the chain rule:

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 2e^x & 2e^y \\ 2e^x & -2e^y \end{pmatrix} \cdot \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} = \begin{pmatrix} 2e^{2x} & 2e^{2y} \\ 2e^{2x} & -2e^{2y} \end{pmatrix}$$

and

$$(\mathbf{f} \circ \mathbf{g})'(\mathbf{x}) = \mathbf{f}'(\mathbf{g}(\mathbf{x})) \cdot \mathbf{g}'(\mathbf{x}) = \begin{pmatrix} e^{x^2+y^2} & 0 \\ 0 & e^{x^2-y^2} \end{pmatrix} \cdot \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix} = \begin{pmatrix} 2x\, e^{x^2+y^2} & 2y\, e^{x^2+y^2} \\ 2x\, e^{x^2-y^2} & 2y\, e^{x^2-y^2} \end{pmatrix}$$

Indirect Dependency. Let $f(x_1,x_2,t)$ where $x_1(t)$ and $x_2(t)$ also depend on t. What is the total derivative of f w.r.t. t?

Example 7.23

We introduce auxiliary function

$$\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} x_1(t) \\ x_2(t) \\ t \end{pmatrix}$$

and apply the chain rule:

$$\begin{aligned}
\frac{df}{dt} &= (f \circ \mathbf{x})'(t) = f'(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \\
&= \nabla f(\mathbf{x}(t)) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} = (f_{x_1}(\mathbf{x}(t)), f_{x_2}(\mathbf{x}(t)), f_t(\mathbf{x}(t)) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} \\
&= f_{x_1}(\mathbf{x}(t)) \cdot x_1'(t) + f_{x_2}(\mathbf{x}(t)) \cdot x_2'(t) + f_t(\mathbf{x}(t)) \\
&= f_{x_1}(x_1,x_2,t) \cdot x_1'(t) + f_{x_2}(x_1,x_2,t) \cdot x_2'(t) + f_t(x_1,x_2,t)
\end{aligned}$$

— Summary

- difference quotient and differential quotient
- differential quotient and derivative
- derivatives of elementary functions
- differentiation rules
- higher order derivatives
- total differential
- elasticity
- partial derivatives
- gradient and Hessian matrix
- Jacobian matrix and chain rule

---

[6] Provided that all second order derivatives are continuous, see page 106.

— Exercises

7.1 Draw (sketch) the graphs of the following functions.

At which points are these function differentiable?

(a) $f(x) = 2x + 2$

(b) $f(x) = 3$

(c) $f(x) = |x|$

(d) $f(x) = \sqrt{|x^2 - 1|}$

(e) $f(x) = \begin{cases} -\frac{1}{2}x^2, & \text{for } x \le -1, \\ x, & \text{for } -1 < x \le 1, \\ \frac{1}{2}x^2, & \text{for } x > 1. \end{cases}$

(f) $f(x) = \begin{cases} 2 + x, & \text{for } x \le -1, \\ x^2, & \text{for } x > -1. \end{cases}$

7.2 Compute the first and second derivative of the following functions:

(a) $f(x) = 4x^4 + 3x^3 - 2x^2 - 1$

(b) $f(x) = e^{-\frac{x^2}{2}}$

(c) $f(x) = \exp\left(-\dfrac{x^2}{2}\right)$

(d) $f(x) = \dfrac{x+1}{x-1}$

7.3 Compute the first and second derivative of the following functions:

(a) $f(x) = \dfrac{1}{1+x^2}$

(b) $f(x) = \dfrac{1}{(1+x)^2}$

(c) $f(x) = x\ln(x) - x + 1$

(d) $f(x) = \ln(|x|)$

7.4 Compute the first and second derivative of the following functions:

(a) $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

(b) $f(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

(c) $f(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$

(d) $f(x) = \cos(1 + x^2)$

7.5 Derive the quotient rule by means of product rule and chain rule.

7.6 Let $f(x) = \dfrac{\ln(x)}{x}$.

Compute $\Delta f = f(3.1) - f(3)$ approximately by means of the differential at point $x_0 = 3$.

Compare your approximation to the exact value.

7.7 Compute the regions where the following functions are elastic, 1-elastic and inelastic, resp.

(a) $g(x) = x^3 - 2x^2$

(b) $h(x) = \alpha x^\beta, \quad \alpha, \beta \neq 0$

7.8 Which of the following statements are correct?

Suppose function $y = f(x)$ is elastic in its domain.

(a) If x changes by one unit, then the change of y is greater than one unit.
(b) If x changes by one percent, then the relative change of y is greater than one percent.
(c) The relative rate of change of y is larger than the relative rate of change of x.
(d) The larger x is the larger will be y.

7.9 Compute the first and second order partial derivatives of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$

(b) $f(x,y) = xy$

(c) $f(x,y) = x^2 + y^2$

(d) $f(x,y) = x^2 y^2$

(e) $f(x,y) = x^\alpha y^\beta, \quad \alpha, \beta > 0$

7.10 Compute the first and second order partial derivatives of

$$f(x,y) = \exp(x^2 + y^2)$$

at point $(0,0)$.

7.11 Compute the first and second order partial derivatives of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

7.12 Compute the gradients of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$ (b) $f(x,y) = xy$

(c) $f(x,y) = x^2 + y^2$ (d) $f(x,y) = x^2 y^2$

(e) $f(x,y) = x^\alpha y^\beta, \quad \alpha, \beta > 0$

7.13 Compute the gradients of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

7.14 Compute the Hessian matrix of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$ (b) $f(x,y) = xy$

(c) $f(x,y) = x^2 + y^2$ (d) $f(x,y) = x^2 y^2$

(e) $f(x,y) = x^\alpha y^\beta, \quad \alpha, \beta > 0$

7.15 Compute the Hessian matrix of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

7.16 Let

$$f(x,y) = x^2 + y^2 \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

Compute the derivative of the composite functions

(a) $h = f \circ \mathbf{g}$, and (b) $\mathbf{p} = \mathbf{g} \circ f$

by means of the chain rule.

7.17 Let $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^3 - x_2 \\ x_1 - x_2^3 \end{pmatrix}$ and $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x_2^2 \\ x_1 \end{pmatrix}$.

Compute the derivatives of the composite functions

(a) $\mathbf{g} \circ \mathbf{f}$, and (b) $\mathbf{f} \circ \mathbf{g}$

by means of the chain rule.

7.18 Let $Q(K,L,t)$ be a production function, where $L = L(t)$ and $K = K(t)$ also depend on time t. Compute the total derivative $\frac{dQ}{dt}$ by means of the chain rule.

8

Extrema

In this chapter we apply derivatives for analyzing univariate functions. In particular determine whether functions are monotone, convex and where they have extremal points.

8.1 Monotone Functions

A function f is called **monotonically increasing**, if

monotonically increasing

$$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$

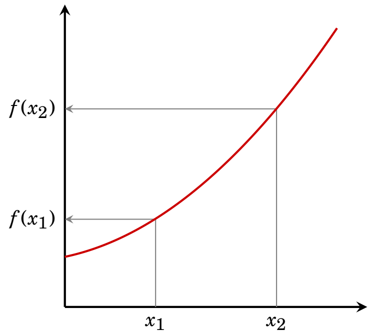


Figure 8.1

Monotonically increasing function

Function f is called **monotonically decreasing**, if

monotonically decreasing

$$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

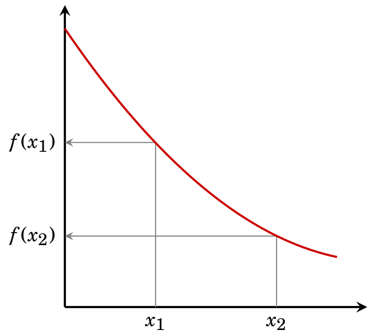


Figure 8.2

Monotonically decreasing function

When we want to determine whether a function is monotonically increasing we have to check that $f(x_1) \leq f(x_2)$ holds for $x_1, x_2 \in D_f$ with $x_1 < x_2$. For differentiable functions we have a much simpler tool based on their first derivatives:

$$\begin{array}{lcll} f \text{ monotonically increasing} & \iff & f'(x) \geq 0 & \text{for all } x \in D_f \\ f \text{ monotonically decreasing} & \iff & f'(x) \leq 0 & \text{for all } x \in D_f \end{array}$$

Example 8.3

Function $f(x) = x^3$ is monotonically increasing as $f'(x) = 3x^2 \geq 0$ for all $x \in D_f = \mathbb{R}$.

Function $g : (0, \infty) \to \mathbb{R}$, $x \mapsto g(x) = \frac{1}{x^2}$ is monotonically decreasing as $g'(x) = -\frac{2}{x^3} < 0$ for all $x \in D_g = (0, \infty)$.

Example 8.4

Function $k(x) = 5$ is both monotonically increasing *and* monotonically decreasing as $k'(x) = 0$ for which both inequalities $k'(x) \geq 0$ and $k'(x) \leq 0$ hold for all $x \in D_k = \mathbb{R}$.

The last example shows that by our definition every *constant* function is both monotonically increasing and monotonically decreasing at the same time. So this condition might not be useful in some situations.

The notion of *strictly* monotone functions is stronger and avoids this issue (but is a bit harder to handle) as the weak inequalities are replaced by strict ones.

strictly monotonically increasing

Function f called **strictly monotonically increasing**, if

$$x_1 < x_2 \iff f(x_1) < f(x_2)$$

strictly monotonically decreasing

It is called **strictly monotonically decreasing**, if

$$x_1 < x_2 \iff f(x_1) > f(x_2)$$

The characterization of strict monotonicity by means of first derivatives now provides a criterion that is sufficient but not necessary[1]:

$$\begin{array}{lcll} f \text{ strictly monotonically increasing} & \Longleftarrow & f'(x) > 0 & \text{for all } x \in D_f \\ f \text{ strictly monotonically decreasing} & \Longleftarrow & f'(x) < 0 & \text{for all } x \in D_f \end{array}$$

Example 8.5

Function $f : (0, \infty)$, $x \mapsto \ln(x)$ is strictly monotonically increasing, as

$$f'(x) = (\ln(x))' = \frac{1}{x} > 0 \quad \text{for all } x > 0.$$

---

[1] That is, there exist strictly monotonically increasing functions where the condition $f'(x) > 0$ is violated for at least one point. For example $f(x) = x^3$ is strictly monotonically increasing but $f'(0) = 0 \not> 0$.

Observe that every *strictly* monotonically increasing function is *monotonically increasing* but not vice versa.

Remark: If f is *strictly monotonically increasing*, then

$$x_1 < x_2 \iff f(x_1) < f(x_2)$$

immediately implies

$$x_1 \neq x_2 \iff f(x_1) \neq f(x_2)$$

That is, f is one-to-one. So if f is onto and strictly monotonically increasing (or decreasing), then f is invertible.

Locally Monotone Functions

A function f can be monotonically increasing in some interval and decreasing in some other interval. Then we have to solve the inequalities $f'(x) \geq 0$ and $f'(x) \leq 0$ using the methods described in Section 3.2 on page 39.

For *continuously* differentiable functions (i.e., when $f'(x)$ is continuous) we can use the following recipe:

1. Compute first derivative $f'(x)$.
2. Determine all roots of $f'(x)$.
3. We thus obtain intervals where $f'(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f'(x_i)$.

Example 8.6

In which region is function $f(x) = 2x^3 - 12x^2 + 18x - 1$ monotonically increasing?

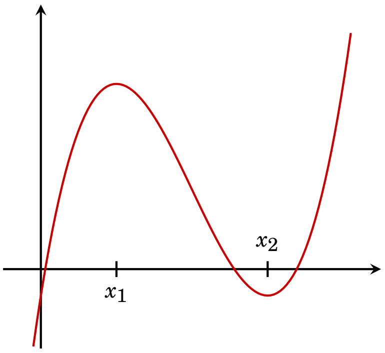


Figure 8.7

We have to solve inequality $f'(x) \geq 0$:

1. First derivative: $f'(x) = 6x^2 - 24x + 18$

2. Roots of $x^2-4x+3=0$: $x_1=1$ and $x_2=3$
3. Obtain 3 intervals: $(-\infty,1]$, $[1,3]$, and $[3,\infty)$
4. Sign of $f'(x)$ at appropriate points in each interval:
   $f'(0)=3>0$, $f'(2)=-1<0$, and $f'(4)=3>0$.
5. $f'(x)$ cannot change sign in each interval:
   $f'(x)\geq 0$ in $(-\infty,1]$ and $[3,\infty)$.

Function $f(x)$ is monotonically increasing in $(-\infty,1]$ and in $[3,\infty)$.

8.2 Convex and Concave

convex

A function f is called **convex**, if its domain D_f is an interval and

$$f((1-h)x_1+hx_2)\leq(1-h)f(x_1)+hf(x_2)$$

for all $x_1,x_2\in D_f$ and all $h\in[0,1]$.

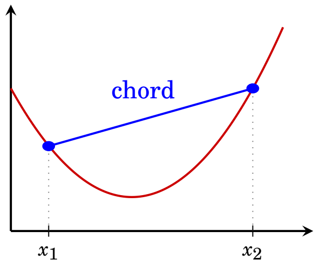


Figure 8.8

Convex function

concave

It is called **concave**, if

$$f((1-h)x_1+hx_2)\geq(1-h)f(x_1)+hf(x_2)$$

for all $x_1,x_2\in D_f$ and all $h\in[0,1]$.

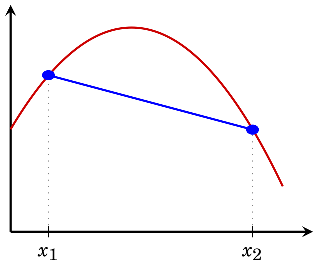


Figure 8.9

Concave function

The above inequality is a formal description of the following – verbal – definition of "concave": Function f is *concave* if all chords are always below the graph of the function, see Figure 8.10. Term "$(1-h)x_1+hx_2$" is a point p between x_1 and x_2 where the distances between p and x_1 and x_2 are $h|x_2-x_1|$ and $(1-h)|x_2-x_1|$, resp. Consequently, the l.h.s.

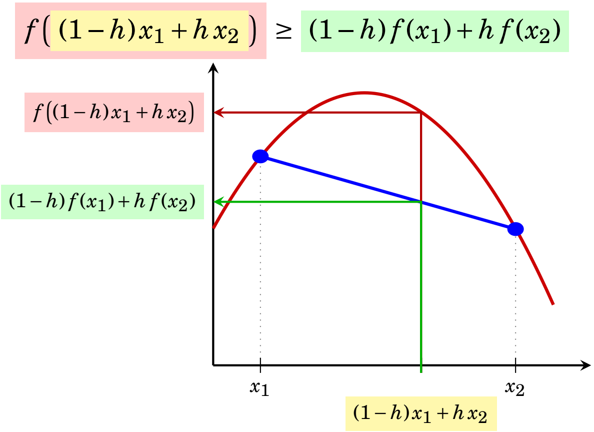


Figure 8.10

Inequality for concave functions

of see above inequality is function value (the point on the graph) at p. Analogously the r.h.s. is then the corresponding point on the chord.

When we want to determine whether a function is concave we have to check that $f((1-h)x_1+hx_2) \geq (1-h)f(x_1)+hf(x_2)$ holds for $x_1, x_2 \in D_f$ and all $h \in (0,1)$. Fortunately, for two times differentiable functions we have a much simpler tool based on their *second* derivatives. As illustrated in Figure 8.11 below the slope of the tangent (i.e., its first derivative) of a concave function is an increasing function of its point of contact. Thus the derivative of the first derivative of f, i.e., its second derivative $f''(x)$, must be non-positive.

$$\begin{array}{llll} f \text{ convex} & \iff & f''(x) \geq 0 & \text{for all } x \in D_f \\ f \text{ concave} & \iff & f''(x) \leq 0 & \text{for all } x \in D_f \end{array}$$

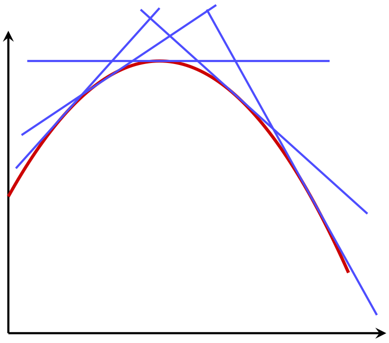


Figure 8.11

Tangents of concave functions

Example 8.12

Function $f(x) = 1 - x^4$ is concave as $f''(x) = -12x^2 \leq 0$ for all $x \in D_f = \mathbb{R}$.

Function $g : (0,\infty) \to \mathbb{R}$, $x \mapsto g(x) = \frac{1}{x^2}$ is convex as $g''(x) = \frac{6}{x^4} > 0$ for all $x \in D_g = (0,\infty)$.

Example 8.13

Function $l(x) = 5x - 3$ is both convex *and* concave $l''(x) = 0$ for which both inequalities $l''(x) \geq 0$ and $l''(x) \leq 0$ hold for all $x \in D_l = \mathbb{R}$.

The last example shows that by our definition every *linear* function is both convex and concave at the same time. So this condition might not be useful in some situations.

The notion of *strictly* convex functions is stronger and avoids this issue (but is a bit harder to handle) as the weak inequalities are replaced by strict ones.

Function f is called **strictly convex**, if its domain D_f is an interval and

strictly convex

$$f((1-h)x_1 + h x_2) < (1-h) f(x_1) + h f(x_2)$$

for all $x_1, x_2 \in D_f$, $x_1 \neq x_2$, and all $h \in (0,1)$.

It is called **strictly concave**, if its domain D_f is an interval and

strictly concave

$$f((1-h)x_1 + h x_2) > (1-h) f(x_1) + h f(x_2)$$

for all $x_1, x_2 \in D_f$, $x_1 \neq x_2$, and all $h \in (0,1)$.

The characterization of strict convexity by means of second derivatives now provides a criterion that is sufficient but not necessary[2]:

$$\begin{array}{lcll} f \text{ strictly convex} & \Longleftarrow & f''(x) > 0 & \text{for all } x \in D_f \\ f \text{ strictly concave} & \Longleftarrow & f''(x) < 0 & \text{for all } x \in D_f \end{array}$$

Example 8.14

The exponential function $f(x) = e^x$ is strictly convex as $f''(x) = e^x > 0$ for all $x \in \mathbb{R}$.

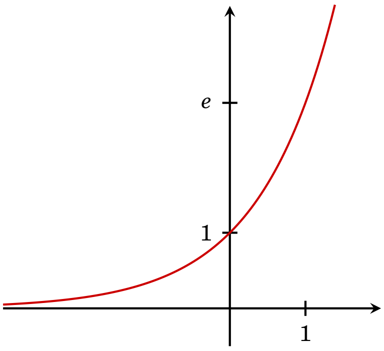


Figure 8.15

Exponential functions are convex

Example 8.16

The logarithm $f(x) = \ln(x)$ is strictly concave as $f''(x) = -\frac{1}{x^2} < 0$ for all $x \in D_f = (0, \infty)$.

---

[2] That is, there exist strictly convex functions where the condition $f''(x) > 0$ is violated for at least one point. For example $f(x) = x^4$ is strictly convex but $f''(0) = 0 \not> 0$.

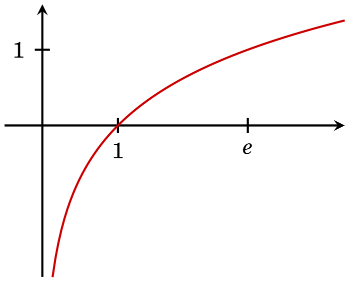


Figure 8.17

Logarithms are concave

Observe that every *strictly* concave function is *concave* but not vice versa.

Locally Convex Functions

A function f can be convex in some interval and concave in some other interval. Then we have to solve the inequalities $f''(x) \geq 0$ and $f''(x) \leq 0$ using the methods described in Section 3.2 on page 39 analogously to that locally monotone functions.

For two times *continuously* differentiable functions (i.e., when $f''(x)$ is continuous) we can use the following recipe:

1. Compute second derivative $f''(x)$.
2. Determine all roots of $f''(x)$.
3. We thus obtain intervals where $f''(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f''(x_i)$.

Example 8.18

In which region is function $f(x) = 2x^3 - 12x^2 + 18x - 1$ concave?

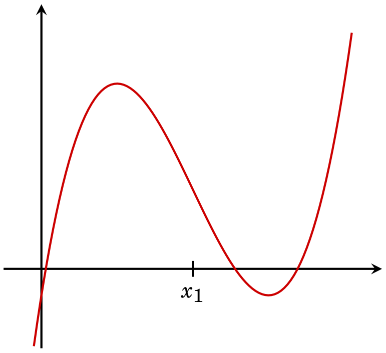


Figure 8.19

We have to solve inequality $f''(x) \leq 0$:

1. Second derivative: $f''(x) = 12x - 24$
2. Roots of $12x - 24 = 0$: $\quad x_1 = 2$

3. Obtain 2 intervals: $(-\infty,2]$ and $[2,\infty)$
4. Sign of $f''(x)$ at appropriate points in each interval:
   $f''(0) = -24 < 0$ and $f''(4) = 24 > 0$.
5. $f''(x)$ cannot change sign in each interval: $f''(x) \leq 0$ in $(-\infty,2]$.

Function $f(x)$ is concave in $(-\infty,2]$.

8.3 Extrema

global maximum

A point x^* is called **global maximum** (or *absolute maximum*) of f, if for all $x \in D_f$,

$$f(x^*) \geq f(x).$$

global minimum

Point x^* is called **global minimum** (or *absolute minimum*) of f, if for all $x \in D_f$,

$$f(x^*) \leq f(x).$$

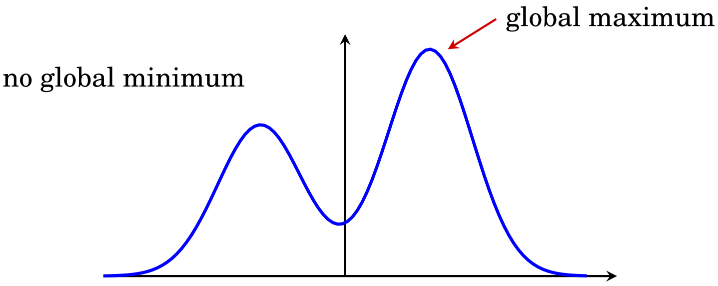


Figure 8.20

Global extrema

local maximum

A point x_0 is called **local maximum** (or *relative maximum*) of f, if for all x in some *neighborhood*[3] of x_0,

$$f(x_0) \geq f(x).$$

local minimum

Point x_0 is called **local minimum** (or *relative minimum*) of f, if for all x in some *neighborhood* of x_0,

$$f(x_0) \leq f(x).$$

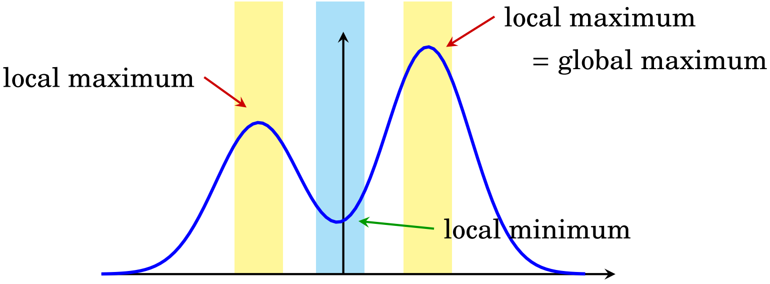


Figure 8.21

Local extrema

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[3] That is, there exists a small open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ around x_0 where x_0 has maximal value. Notice, however, we have no idea how small such an interval has to be.

Every minimization problem can be transformed into a maximization problem and vice versa. ⚠

Point x_0 is a minimum of f
if and only if
x_0 is a maximum of $-f$.

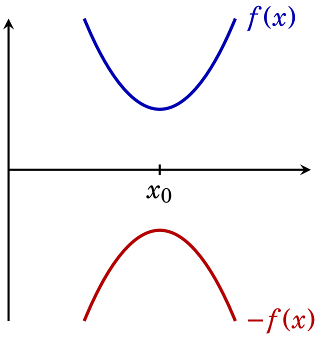


Figure 8.22

Minima and maxima

Critical Points

At a (local) maximum or minimum the first derivative of the function must vanish (i.e., must be equal to 0).

Definition 8.23

critical point

stationary point

Stationary Point. A point x_0 is called a **critical point** (or **stationary point**) of function f, if

$$f'(x_0) = 0$$

Thus we get a *necessary condition* in the case of differentiable functions:

Each extremum of f is a critical point of f.

Global Extremum

Obviously a stationary point can be a maximum or a minimum or neither. So we need further criteria for detecting extremal points.

Here is a *sufficient condition* for (global) extrema: Let x_0 be a critical point of f.

- If f is *concave* then x_0 is a *global maximum* of f.
- If f is *convex* then x_0 is a *global minimum* of f.
- If f is *strictly* concave (or convex), then the extremum is *unique*.

So find get the following recipe for finding (at least some) extremal points:

1. Compute $f'(x)$ and find all stationary points x_i of f.
2. Compute $f''(x)$ and check its sign for **all** $x \in D_f$.
    - If $f''(x) \leq 0$ for all $x \in D_f$, then x_0 is a *global maximum*.
    - If $f''(x) \geq 0$ for all $x \in D_f$, then x_0 is a *global minimum*.

Example 8.24

Find all global extrema of $f(x) = e^x - 2x$.

1. Critical points: $f'(x) = e^x - 2 = 0$ implies $x_0 = \ln 2$.
2. Convexity: $f''(x) = e^x > 0$ **for all** $x \in \mathbb{R}$

   Thus f is strictly convex and consequently $x_0 = \ln 2$ is the (unique) global minimum of f.

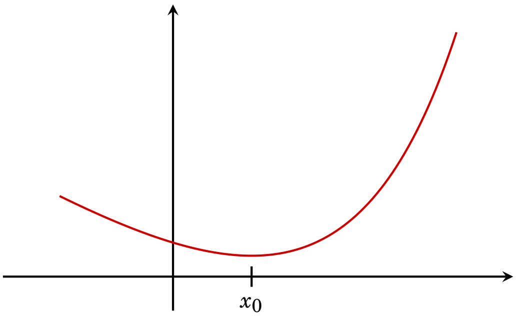


Figure 8.25

⚠

Beware: As we want to find *global* extrema, we have to look at the entire domain D_f. That is, we have to verify that f is convex in D_f. It is not sufficient to look only at $f''(\ln 2)$ at the above example.

Local Extremum

A point x_0 is a *local maximum* (or *local minimum*) of f, if

- x_0 is a *critical point* of f,
- f is *locally concave* (and *locally convex*, resp.) around x_0.

So we get the following *sufficient condition* for two times differentiable functions: Let x_0 be a critical point of f.

- If $f''(x_0) < 0$, then x_0 is local maximum.
- If $f''(x_0) > 0$, then x_0 is local minimum.

It is sufficient to evaluate $f''(x)$ at the critical point x_0 (in opposition to the condition for global extrema).

The following recipe allows to find (at least some) local extrema:

1. Compute $f'(x)$ and find all stationary points x_i of f.
2. Compute $f''(x)$ and check its sign for *stationary* points x_i.
   - If $f''(x_i) < 0$, then x_i is a *local maximum*.
   - If $f''(x_i) > 0$, then x_i is a *local minimum*.
   - If $f''(x_i) = 0$, then no conclusion is possible[4].

---

[4] That is, stationary point x_i can be a local maximum, a local minimum, or neither.

Example 8.26

Find all local extrema of $f(x) = \frac{1}{12}x^3 - x^2 + 3x + 1$.

1. Critical points: $\frac{1}{4}x^2 - 2x + 3 = 0$ has roots $x_1 = 2$ and $x_2 = 6$.
2. Second derivative: $f''(x) = \frac{1}{2}x - 2$. We find

$$f''(2) = -1 < 0 \text{ and hence } x_1 \text{ is a local maximum.}$$
$$f''(6) = 1 > 0 \text{ and hence } x_2 \text{ is a local minimum.}$$

Function f has a local maximum in $x_1 = 2$ and a local minimum in $x_2 = 6$.

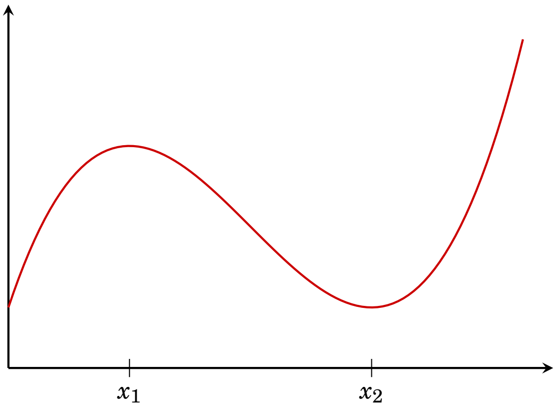


Figure 8.27

Necessary and Sufficient

We want to explain two important concepts using the example of local minima.

necessary

- Condition "$f'(x_0) = 0$" is **necessary** for a local minimum:

  Every local minimum must have this properties.

  However, not every point with such a property is a local minimum (e.g. $x_0 = 0$ in $f(x) = x^3$).

  Stationary points are thus *candidates* for local extrema.

sufficient

- Condition "$f'(x_0) = 0$ and $f''(x_0) > 0$" is **sufficient** for a local minimum.

  If it is satisfied, then x_0 is a local minimum.

  However, there are local minima where this condition is not satisfied (e.g. $x_0 = 0$ in $f(x) = x^4$).

  If it is *not* satisfied, we cannot draw *any conclusion*.

Sources of Errors

Example 8.28

Consider the following computations.

Find all global minima of $f(x) = \dfrac{x^3 + 2}{3x}$.

1. Critical points: $f'(x) = \frac{2(x^3-1)}{3x^2} = 0$ has single root $x_0 = 1$.

2. Second derivative: $f''(x) = \frac{2x^3+4}{3x^3}$. We find $f''(1) = 2 > 0$. Hence $x_0 = 1$ is a *global* minimum.

However, as can be easily be seen from its graph f cannot have a global minimum. We just have verified that $x_0 = 1$ is a *local* minimum. Looking merely at $f''(1)$ is not sufficient as we are looking for the largest value among all points in domain D_f.

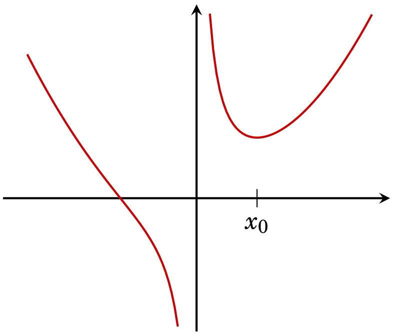


Figure 8.29

Beware!

We have to look at $f''(x)$ for **all** $x \in D_f$. Indeed we find $f''(-1) = -\frac{2}{3} < 0$ and thus f cannot be convex. Moreover, domain $D_f = \mathbb{R} \setminus \{0\}$ is not an interval.

So f is not convex and we cannot apply our theorem.

Consider another computations.

Example 8.30

Find all global maxima of $f(x) = \exp(-x^2/2)$.

1. Critical points: $f'(x) = x\exp(-x^2) = 0$ has single root $x_0 = 0$.

2. Second derivative: $f''(x) = (x^2 - 1)\exp(-x^2)$.

   We find $f''(0) = -1 < 0$ and $f''(2) = 2e^{-2} > 0$.

   Consequently, f is neither convex nor concave.

   However, we cannot conclude that $x_0 = 0$ is *not* a global maximum.

   Indeed in this example $x_0 = 0$ is a global maximum[5].

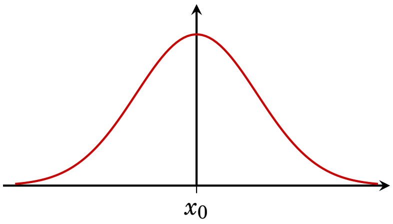


Figure 8.31

Beware! ⚠

We are checking a *sufficient* condition. Since an assumption does not hold (f is not concave), we simply *cannot apply* the theorem. We *cannot* conclude that f does not have a global maximum.

Global Extrema in $[a,b]$

We again want to find global extrema of $f(x)$. However, now the domain of f is a *closed* interval $D_f = [a,b]$. It may happen, that f attains is extrema in a boundary point. Then it need not be a stationary point.

The following procedure works for differentiable functions:

1. Compute $f'(x)$.
2. Find all stationary points x_i (i.e., $f'(x_i) = 0$).
3. Evaluate $f(x)$ for all *candidates*:
   - all stationary points x_i,
   - boundary points a and b.
4. The largest of these values is a *global maximum*,
   the smallest of these values is a *global minimum*.

It is *not* necessary to compute $f''(x_i)$.

Example 8.32

Find all *global* extrema of function

$$f\colon [0{,}5;8{,}5] \to \mathbb{R}, x \mapsto \frac{1}{12}x^3 - x^2 + 3x + 1$$

1. $f'(x) = \frac{1}{4}x^2 - 2x + 3$.
2. Stationary points: $\frac{1}{4}x^2 - 2x + 3 = 0$ has roots $x_1 = 2$ and $x_2 = 6$.
3. Evaluate function for all candidates:
   $$\begin{aligned} f(0.5) &= 2.260 & & \\ f(2) &= 3.667 & & \\ f(6) &= 1.000 & \Rightarrow \quad & \text{global minimum} \\ f(8.5) &= 5.427 & \Rightarrow \quad & \text{global maximum} \end{aligned}$$
4. $x_2 = 6$ is the global minimum and
   $b = 8.5$ is the global maximum of f.

---

[5] In other examples it may not be global maximum.

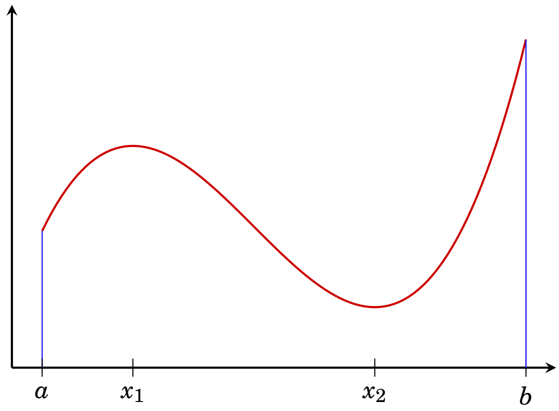


Figure 8.33

Global Extrema in (a,b)

Finally we look at the case when the domain of f is an *open* interval $D_f = (a,b)$ or $D_f = (-\infty,\infty)$.

The following procedure works for differentiable functions:

1. Compute $f'(x)$.
2. Find all stationary points x_i (i.e., $f'(x_i) = 0$).
3. Evaluate $f(x)$ for all *stationary* points x_i.
4. Determine $\lim_{x \to a} f(x)$ and $\lim_{x \to b} f(x)$.
5. The largest of these values is a *global maximum*,
   the smallest of these values is a *global minimum*.
6. A global extremum exists *only if* the largest (smallest) value is obtained in a *stationary point* (and not as a limit)!

Example 8.34

Compute all *global* extrema of

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto e^{-x^2}$$

1. $f'(x) = -2x\,e^{-x^2}$.
2. Stationary points: $f'(x) = -2x\,e^{-x^2} = 0$ has unique root $x_1 = 0$.
3. Evaluate function for all stationary points and determine limits:
$$\begin{array}{rcl} f(0) = 1 & \Rightarrow & \text{global maximum} \\ \lim_{x \to -\infty} f(x) = 0 & \Rightarrow & \text{no global minimum} \\ \lim_{x \to \infty} f(x) = 0 & & \end{array}$$
4. The function has a global maximum in $x_1 = 0$,
   but no global minimum.

Existence and Uniqueness

- A function need not have maxima or minima:

  $$f\colon (0,1)\to\mathbb{R},\ x\mapsto x$$

  (Points 1 and -1 are not in domain $(0,1)$.)

- (Global) maxima need not be unique:

  $$f\colon \mathbb{R}\to\mathbb{R},\ x\mapsto x^4-2x^2$$

  has two global minima at -1 and 1.

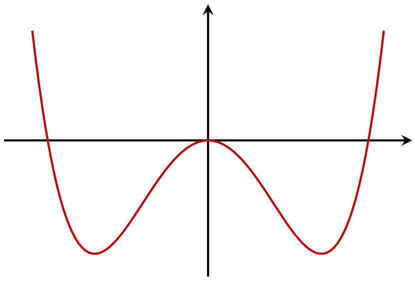


Figure 8.35

— Summary

- monotonically increasing and decreasing
- convex and concave
- global and local extrema

— Exercises

8.1 Determine whether the following functions are concave or convex (or neither).

(a) $\exp(x)$

(b) $\ln(x)$

(c) $\log_{10}(x)$

(d) x^{α} for $x > 0$ for an $\alpha \in \mathbb{R}$.

8.2 In which region is function

$$f(x) = x^3 - 3x^2 - 9x + 19$$

monotonically increasing or decreasing?

In which region is it convex or concave?

8.3 In which region the following functions monotonically increasing or decreasing?

In which region is it convex or concave?

(a) $f(x) = x\,e^{x^2}$

(b) $f(x) = e^{-x^2}$

(c) $f(x) = \dfrac{1}{x^2+1}$

8.4 Function

$$f(x) = b\,x^{1-\alpha}\,, \qquad 0 < \alpha < 1, b > 0, x \geq 0$$

is an example of a *production function*.

Production functions usually have the following properties:

(1) $f(0) = 0, \quad \lim_{x\to\infty} f(x) = \infty$

(2) $f'(x) > 0, \quad \lim_{x\to\infty} f'(x) = 0$

(3) $f''(x) < 0$

(a) Verify these properties for the given function.

(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$. (Use appropriate values for a and b.)

(c) What is the economic interpretation of these properties?

8.5 Function

$$f(x) = b\,\ln(ax+1)\,, \qquad a, b > 0, x \geq 0$$

is an example of a utility function.

Utility functions have the same properties as production functions.

(a) Verify the properties from Problem 8.4.

(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$.
(Use appropriate values for a and b.)

(c) What is the economic interpretation of these properties?

8.6 Use the definition of convexity and show that $f(x) = x^2$ is strictly convex.

Hint: Show that inequality $\left(\frac{1}{2}x + \frac{1}{2}y\right)^2 - \left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) < 0$ holds for all $x \neq y$.

8.7 Show:

If $f(x)$ is a two times differentiable concave function, then $g(x) = -f(x)$ convex.

8.8 Show:

If $f(x)$ is a concave function, then $g(x) = -f(x)$ convex.

You may not assume that f is differentiable.

8.9 Let $f(x)$ and $g(x)$ be two differentiable concave functions.

Show that

$$h(x) = \alpha f(x) + \beta g(x)\,, \qquad \text{for } \alpha, \beta > 0,$$

is a concave function.

What happens, if $\alpha > 0$ and $\beta < 0$?

8.10

Sketch the graph of a function $f\colon [0,2] \to \mathbb{R}$ with the properties:

- continuous,
- monotonically decreasing,
- strictly concave,
- $f(0) = 1$ and $f(1) = 0$.

In addition find a particular term for such a function.

8.11 Suppose we relax the condition *strict concave* into *concave* in Problem 8.10.

Can you find a much simpler example?

8.12 Find all local extrema of the following functions.

(a) $f(x) = e^{-x^2}$ (b) $g(x) = \frac{x^2+1}{x}$ (c) $h(x) = (x-3)^6$

8.13 Find all global extrema of the following functions.

(a) $f:(0,\infty)\to\mathbb{R}, x\mapsto \frac{1}{x}+x$

(b) $f:[0,\infty)\to\mathbb{R}, x\mapsto \sqrt{x}-x$

(c) $f:\mathbb{R}\to\mathbb{R}, x\mapsto e^{-2x}+2x$

(d) $f:(0,\infty)\to\mathbb{R}, x\mapsto x-\ln(x)$

(e) $f:\mathbb{R}\to\mathbb{R}, x\mapsto e^{-x^2}$

8.14 Compute all global maxima and minima of the following functions.

(a) $f(x)=\dfrac{x^3}{12}-\dfrac{5}{4}x^2+4x-\dfrac{1}{2}$ in interval $[1,12]$

(b) $f(x)=\dfrac{2}{3}x^3-\dfrac{5}{2}x^2-3x+2$ in interval $[-2,6]$

(c) $f(x)=x^4-2x^2$ in interval $[-2,2]$

9

Integration

In this chapter we discuss two concepts that are seemingly quite different but which are closely connected by the miraculous Fundamental Theorem of Calculus: the antiderivatives and areas of the region below the graph of a function.

9.1 Antiderivative

Definition 9.1

antiderivative

A function $F(x)$ is called an **antiderivative** (or *primitive* function) of function $f(x)$, if

$$F'(x) = f(x)$$

Thus computing antiderivatives can be seen as the inverse operation to computing the derivative of a function. The most general method for finding antiderivatives is

Guess and verify

Example 9.2

We want the antiderivative of $f(x) = \ln(x)$.

$$\begin{array}{lll} \text{Guess:} & F(x) & = x(\ln(x)-1) \\ \text{Verify:} & F'(x) & = (x(\ln(x)-1)' = \\ & & = 1\cdot(\ln(x)-1) + x\cdot\frac{1}{x} = \ln(x) \\ \text{But also:} & F(x) & = x(\ln(x)-1)+5 \end{array}$$

The antiderivative is denoted by symbol

$$\int f(x)\,dx + c$$

indefinite integral

integration constant

and is also called the **indefinite integral** of function f. Number $c \in \mathbb{R}$ is called the **integration constant**.

Unfortunately, there are no *"recipes"* for computing antiderivatives. Even worse, there are functions where antiderivatives cannot be expressed by means of elementary functions.

Table 9.4

Integrals of elementary functions

$f(x)$	$\int f(x)\,dx$
0	c
x^a	$\frac{1}{a+1}\cdot x^{a+1}+c$
e^x	e^x+c
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
$\cos(x)$	$\sin(x)+c$
$\sin(x)$	$-\cos(x)+c$

Example 9.3

The antiderivative of $\exp(-\frac{1}{2}x^2)$ cannot be written in closed form. Nevertheless, it is an extremely important in probability theory and statistics as it is strongly correlated to the normal distribution.

Fortunately there exist quite a couple of tools that help us to find antiverivatives. First of all there exist tables of primitive functions. Table 9.4 lists the most important ones[1].

Indeed in the last century when one needs a (non-elementary) integral a common approach was to walk to the library and looked it up in a thick book called "Gradshteyn and Ryzhik" (GR)[2]. Today one simply starts an appropriate computer algebra system[3] which does this look up in an digital copy of this list.

integration rules

In addition to these lists of integral so called **integration rules** allow (sometimes) to reduce an unknown integral into one with known solution, see Table 9.5 for the three most important ones.

However, finding antiderivatives is a combination of art and skill and requires some experiences.

Example 9.6

Summation rule. Antiderivative of $f(x)=4x^3-x^2+3x-5$.

$$\begin{aligned}\int f(x)\,dx &= \int 4x^3-x^2+3x-5\,dx\\ &= 4\int x^3\,dx-\int x^2\,dx+3\int x\,dx-5\int dx\\ &= 4\frac{1}{4}x^4-\frac{1}{3}x^3+3\frac{1}{2}x^2-5x+c\\ &= x^4-\frac{1}{3}x^3+\frac{3}{2}x^2-5x+c\end{aligned}$$

---

[1] This table has been created by exchanging the columns in our list of derivatives in Table 7.8 on page 100.

[2] This is the informal name of a comprehensive table of integrals originally compiled by the Russian mathematicians I. S. Gradshteyn and I. M. Ryzhik with full title *Table of Integrals, Series, and Products.*

[3] For example: Mathematica – `https://en.wikipedia.org/wiki/Wolfram_Mathematica`, Maple – `https://en.wikipedia.org/wiki/Maple_(software)`, Maxima – `https://en.wikipedia.org/wiki/Maxima_(software)`.

- *Summation rule*

$$\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

- *Integration by parts*

$$\int f \cdot g'\,dx = f \cdot g - \int f' \cdot g\,dx$$

- *Integration by substitution*

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(z)\,dz$$

$$\text{with } z = g(x) \text{ and } dz = g'(x)\,dx$$

Table 9.5

Integration rules

Example 9.7

Integration by parts. Antiderivative of $f(x) = x \cdot e^x$.

$$\int \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'}\,dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g} - \int \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g}\,dx = x \cdot e^x - e^x + c$$

$$\begin{aligned} f &= x & \Rightarrow\quad & f' = 1 \\ g' &= e^x & \Rightarrow\quad & g = e^x \end{aligned}$$

It may be necessary to apply an integration rule two or more times.

Example 9.8

Antiderivative of $f(x) = x^2 \cos(x)$.

$$\int \underbrace{x^2}_{f} \cdot \underbrace{\cos(x)}_{g'}\,dx = \underbrace{x^2}_{f} \cdot \underbrace{\sin(x)}_{g} - \int \underbrace{2x}_{f'} \cdot \underbrace{\sin(x)}_{g}\,dx$$

Integration by parts of the second terms yields:

$$\begin{aligned} \int \underbrace{2x}_{f} \cdot \underbrace{\sin(x)}_{g'}\,dx &= \underbrace{2x}_{f} \cdot \underbrace{(-\cos(x))}_{g} - \int \underbrace{2}_{f'} \cdot \underbrace{(-\cos(x))}_{g}\,dx \\ &= -2x \cdot \cos(x) - 2 \cdot (-\sin(x)) + c \end{aligned}$$

Thus the antiderivative of f is given by

$$\int x^2 \cos(x)\,dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + c$$

Example 9.9

Integration by substitution. Antiderivative of $f(x) = 2x \cdot e^{x^2}$.

$$\int \exp(\underbrace{x^2}_{g(x)}) \cdot \underbrace{2x}_{g'(x)}\,dx = \int \exp(z)\,dz = e^z + c = e^{x^2} + c$$

$$z = g(x) = x^2 \quad\Rightarrow\quad dz = g'(x)\,dx = 2x\,dx$$

The integration rules from Table 9.5 can be derived from the corresponding rules for derivatives.

The product rule implies integration by parts as

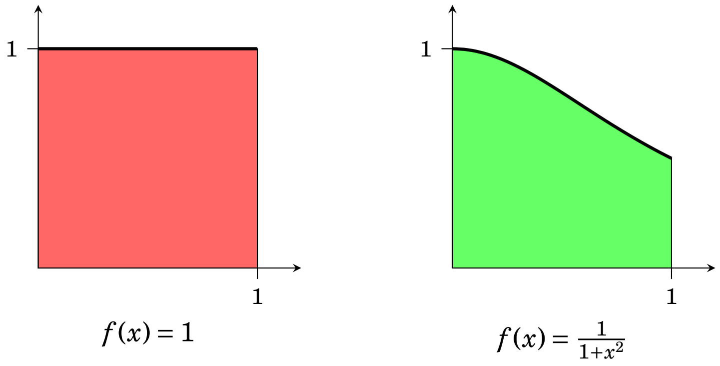
$$\begin{aligned} f(x)\cdot g(x) &= \int (f(x)\cdot g(x))'\,dx = \int \big(f'(x)g(x)+f(x)g'(x)\big)\,dx \\ &= \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \end{aligned}$$

Integration by substitution follows from the chain rule: Let F be an antiderivative of f and let $z = g(x)$. Then

$$\begin{aligned} \int f(z)\,dz = F(z) = F(g(x)) &= \int (F(g(x)))'\,dx \\ &= \int F'(g(x))g'(x)\,dx = \int f(g(x))g'(x)\,dx \end{aligned}$$

9.2 Riemann Integral

We want to compute the area of the region between graph of function f and x-axis. Consider the following two functions:



$f(x) = 1$ $\qquad$ $f(x) = \frac{1}{1+x^2}$

Figure 9.10

Area of region between graph and axis

For the function $f(x) = 1$ (l.h.s.) this task is quite simple. The region is a square and its area is a 1. For function $f(x) = \frac{1}{1+x^2}$ the problem is more challenging as the boundary is not a straight line segment. We can try to approximate f by a piece-wise constant function (step function). It seems that the approximation error becomes smaller when we partition the domain into more subintervals.

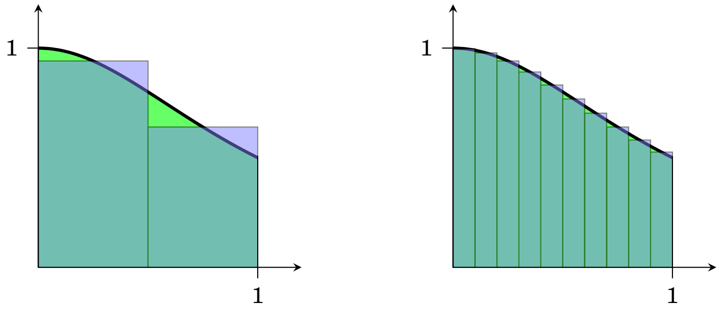


Figure 9.11

Approximation by step function

For such a step function computations of the area is quite straight forward as we can partition the region between graph and x-axis into rectangles. The area then is just the sum of the areas of the rectangles.

We can construct such a piece-wise constant function and approximate the area in the following way: We partition the domain $D = [a,b]$ of f into subintervals using points $a = x_0 < x_1 < x_2 \ldots < x_n = b$. For each subinterval we select a point $\xi_i \in [x_{i-1}, x_i]$, e.g., $\xi_i = \frac{1}{2}(x_{i-1} + x_i)$. Then we use $f(\xi_i)$ as values for the approximating piece-wise constant function on the respective intervals.

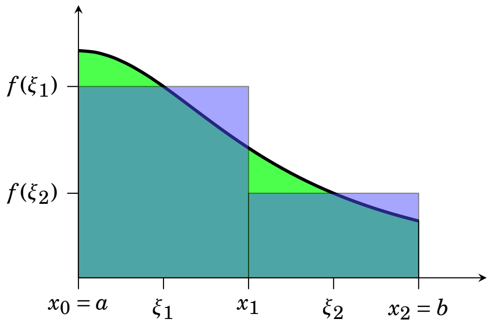


Figure 9.12

Riemann sum

We now obtain an approximation of the integral as

$$A = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

where the sum on the r.h.s. simply give s the area of the approximating step function and is called a **Riemann sum** for f. It can be shown that in many cases these Riemann sums converge when the length of the longest interval tends to 0. This limit then is called the **Riemann integral** of f.

Riemann sum

Riemann integral

Observe that in our construction the Riemann integral may also be negative. So our motivation with areas only works for functions which never have negative values.

Table 9.13 lists computation rules for Riemann integrals.

In Section 9.3 we see that there is no necessity to deal with Riemann sums at all in order to compute Riemann integrals. By the miraculous

Table 9.13

Properties if Riemann integrals

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \qquad \text{if } f(x) \le g(x) \text{ for all } x \in [a,b]$$

Fundamental Theorem of Calculus this can be done by means of antiderivatives.

9.3 Fundamental Theorem of Calculus

Computing limits of Riemann sums is a quite challenging task. However, there is a simpler method for continuous functions.

Let $F(x)$ be an antiderivative of a *continuous* function $f(x)$, then we find

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

By this theorem we can compute Riemann integrals by means of antiderivatives!

For that reason $\int_a^b f(x)\,dx$ is called a **definite integral** of f while $\int f(x)\,dx$ is called the **indefinite integral** of f.

definite integral

indefinite integral

Example 9.14

Compute the integral of $f(x) = x^2$ over interval $[0,1]$.

$$\int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \frac{1}{3}$$

An immediate consequence of the Fundamental Theorem is that the rules for antiderivatives (Table 9.5) hold analogously for definite integrals, see Table 9.15.

Example 9.16

Compute the definite integral $\int_0^2 x \cdot e^x\,dx$.

Table 9.15

Integration rules for definite integrals

- *Summation rule*

$$\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

- *Integration by parts*

$$\int_a^b f \cdot g'\,dx = f \cdot g\Big|_a^b - \int_a^b f' \cdot g\,dx$$

- *Integration by substitution*

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(z)\,dz$$

with $z = g(x)$ and $dz = g'(x)\,dx$

Using integration by parts yields

$$\begin{aligned}\int_0^2 \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'}\,dx &= \underbrace{x}_{f} \cdot \underbrace{e^x}_{g}\Big|_0^2 - \int_0^2 \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g}\,dx \\ &= x \cdot e^x\Big|_0^2 - e^x\Big|_0^2 = (2 \cdot e^2 - 0 \cdot e^0) - (e^2 - e^0) \\ &= e^2 + 1\end{aligned}$$

Note that we also could have used our result from Example 9.7:

$$\int_0^2 x \cdot e^x\,dx = \left(x \cdot e^x - e^x\right)\Big|_0^2 = (2 \cdot e^2 - e^2) - (0 \cdot e^0 - e^0) = e^2 + 1$$

Next there is a more sophisticated example for integration by substition.

Example 9.17

Compute the definite integral $\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx$.

By integration by substitution using $z = \ln(x)$ and thus $dz = \frac{1}{x}\,dx$ we obtain

$$\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx = \int_1^{\ln(10)} \frac{1}{z}\,dz = \ln(z)\Big|_1^{\ln(10)} = \ln(\ln(10)) - \ln(1) \approx 0.834$$

where $z = \ln(x)$ and thus $dz = \frac{1}{x}\,dx$.

Observe that the integration boundaries change when we use integration by substitution. However, this can be avoided by simply replacing z in the indefinite integral and substitute the original integration boundaries.

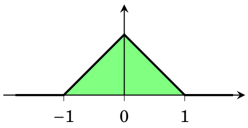
$$\begin{aligned}\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx &= \int_{\dots}^{\dots} \frac{1}{z}\,dz = \ln(z)\Big|_{\dots}^{\dots} \\ &= \ln(\ln(x)\Big|_e^{10} = \ln(\ln(10)) - \ln(\ln(e)) \approx 0.834\end{aligned}$$

Finally there is an example of a definite integral where the domain of integration has to be split into several parts.

Compute $\int_{-2}^{2} f(x)\,dx$ for function

Example 9.18

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$



We have

$$\begin{aligned} \int_{-2}^{2} f(x)\,dx &= \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx \\ &= \int_{-2}^{-1} 0\,dx + \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx + \int_{1}^{2} 0\,dx \\ &= (x + \tfrac{1}{2}x^2)\Big|_{-1}^{0} + (x - \tfrac{1}{2}x^2)\Big|_{0}^{1} \\ &= \tfrac{1}{2} + \tfrac{1}{2} = 1 \end{aligned}$$

— Summary

- antiderivate
- Riemann sum and Riemann integral
- indefinite and definite integral
- Fundamental Theorem of Calculus
- integration rules

— Exercises

9.1 Compute the antiderivatives of the following functions by means of integration by parts.

(a) $f(x) = 2x\,e^{x}$ (b) $f(x) = x^2\,e^{-x}$ (c) $f(x) = x\,\ln(x)$

(d) $f(x) = x^3\ln x$ (e) $f(x) = x(\ln(x))^2$ (f) $f(x) = x^2\,\sin(x)$

9.2 Compute the antiderivatives of the following functions by means of integration by substitution.

(a) $\int x\,e^{x^2}\,dx$ (b) $\int 2x\sqrt{x^2+6}\,dx$ (c) $\int \frac{x}{3x^2+4}\,dx$

(d) $\int x\sqrt{x+1}\,dx$ (e) $\int \frac{\ln(x)}{x}\,dx$

9.3 Compute the antiderivatives of the following functions by means of integration by substitution.

(a) $\int \frac{1}{x\ln x}\,dx$ (b) $\int \sqrt{x^3+1}\,x^2\,dx$ (c) $\int \frac{x}{\sqrt{5-x^2}}\,dx$

(d) $\int \frac{x^2-x+1}{x-3}\,dx$ (e) $\int x(x-8)^{\frac{1}{2}}\,dx$

9.4 Compute the following definite integrals:

(a) $\int_1^4 2x^2-1\,dx$ (b) $\int_0^2 3e^x\,dx$ (c) $\int_1^4 3x^2+4x\,dx$

(d) $\int_0^{\frac{\pi}{3}} \frac{-\sin(x)}{3}\,dx$ (e) $\int_0^1 \frac{3x+2}{3x^2+4x+1}\,dx$

9.5 Compute the following definite integrals by means of antiderivatives:

(a) $\int_1^e \frac{\ln x}{x}\,dx$ (b) $\int_0^1 x(x^2+3)^4\,dx$ (c) $\int_0^2 x\sqrt{4-x^2}\,dx$

(d) $\int_1^2 \frac{x}{x^2+1}\,dx$

9.6 Compute the following definite integrals by means of antiderivatives:

(a) $\int_0^2 x \exp\left(-\frac{x^2}{2}\right) dx$ (b) $\int_0^3 (x-1)^2 x \, dx$ (c) $\int_0^1 x \exp(x) \, dx$

(d) $\int_0^2 x^2 \exp(x) \, dx$ (e) $\int_1^2 x^2 \ln x \, dx$

9.7 Compute $\int_{-2}^{2} x^2 f(x) \, dx$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$

9.8 Compute $F(x) = \int_{-2}^{x} f(t) \, dt$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$

Solutions

1.1 (a) no subset;

(b) no subset, set equals $\{-11, 11\}$;

(c) subset;

(d) no subset.

1.2 (a) $\{1,3,6,7,9,10\}$;

(b) $\{6\}$;

(c) $\{1\}$;

(d) $\{2,4,5,7,8,10\}$;

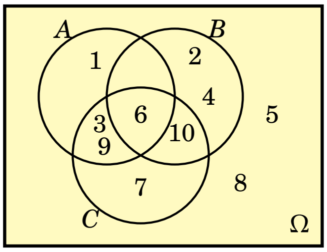
(e) $\{6,10\}$;

(f) $\{2,4,5,8\}$;

(g) $\{2,4\}$;

(h) $\{5,8\}$;

(i) $\{3,6,9\}$.



1.3 A.

1.4 A.

1.5 (a) $\overline{A} \cap \overline{B}$;

(b) A;

(c) $\emptyset$;

(d) C.

1.6 (a) the straight line segment between $(0,2)$ and $(1,2)$ in $\mathbb{R}^2$;

(b) the closed cube $[0,1]^3$ in $\mathbb{R}^3$;

(c) the half-open cube $[0,1] \times (0,1)^2$ in $\mathbb{R}^3$;

(d) cylinder in $\mathbb{R}^3$;

(e) unbounded stripe in $\mathbb{R}^2$;

(f) the closed cube $[0,1]^4$ in $\mathbb{R}^4$.

1.7 function: φ;

domain: $[0,\infty)$;

codomain: $\mathbb{R}$;

image (range): $[0,\infty)$;

function term: $\varphi(x) = x^\alpha$;

independent variable (argument): x;

dependent variable: y.

1.8 (a) no map;

(b) bijective map;

(c) map, neither one-to-one nor onto;

(d) one-to-one map, not onto.

1.9 (a) one-to-one;

(b) no map, as 0^{-2} is not defined;

(c) bijective;

(d) one-to-one, but not onto;

(e) bijective;

(f) no map, as $\{y \in [0,\infty) \colon x = y^2\}$ is a set and not an element of $\mathbb{R}$.

1.10 (a) map, neither one-to-one ($5' = 3' = 0$) nor onto;

(b) map, onto but not one-to-one;

(c) no map, as $(x^n)' = nx^{n-1} \notin \mathscr{P}_{n-2}$.

2.1 (b) and (d).

2.2 (a) $b^5 + ab^4 + a^2b^3 + a^3b^2 + a^4b + a^5$;

(b) $a_1 - a_6$;

(c) $a_1 - a_{n+1}$.

2.3 (a) $2\sum_{i=1}^n a_i b_i$;

(b) 0;

(c) $2\sum_{i=1}^n (x_i^2 + y_i^2)$;

(d) $x_0 - x_n$.

2.4 (a)

$$\begin{aligned}
&\big((x_1-\bar{x})^2 + (x_2-\bar{x})^2\big) - \big((x_1^2+x_2^2) - 2\bar{x}^2\big)\\
&= x_1^2 - 2x_1\bar{x} + \bar{x}^2 + x_2^2 - 2x_2\bar{x} + \bar{x}^2 - x_1^2 - x_2^2 + 2\bar{x}^2\\
&= -2x_1\bar{x} + \bar{x}^2 - 2x_2\bar{x} + \bar{x}^2 + 2\bar{x}^2\\
&= -2(x_1+x_2)\bar{x} + 4\bar{x}^2\\
&= -2(2\bar{x})\bar{x} + 4\bar{x}^2 = 0
\end{aligned}$$

(c)

$$\begin{aligned}
&\sum_{i=1}^n (x_i-\bar{x})^2 - (\sum_{i=1}^n x_i^2 - n\bar{x}^2)\\
&= \sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n x_i\bar{x} + n\bar{x}^2 - \sum_{i=1}^n x_i^2 + n\bar{x}^2\\
&= -2(n\bar{x})\bar{x} + 2n\bar{x}^2 = 0
\end{aligned}$$

2.5

$$\begin{aligned}
&\tfrac{1}{n}\textstyle\sum_{i=1}^{n}(x_i-\mu)^2-\left(\sigma^2+(\bar{x}-\mu)^2\right)\\
&=\tfrac{1}{n}\textstyle\sum_{i=1}^{n}((x_i-\bar{x})+(\bar{x}-\mu))^2-\sigma^2-(\bar{x}-\mu)^2\\
&=\tfrac{1}{n}\textstyle\sum_{i=1}^{n}(x_i-\bar{x})^2+\tfrac{1}{n}\sum_{i=1}^{n}(\bar{x}-\mu)^2\\
&\quad-\tfrac{2}{n}\textstyle\sum_{i=1}^{n}(x_i-\bar{x})(\bar{x}-\mu)-\sigma^2-(\bar{x}-\mu)^2\\
&=-\tfrac{2}{n}\textstyle\sum_{i=1}^{n}(x_i-\bar{x})(\bar{x}-\mu)\\
&=-\tfrac{2}{n}(\bar{x}-\mu)\textstyle\sum_{i=1}^{n}(x_i-\bar{x})\\
&=-2(\bar{x}-\mu)\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n}x_i-\tfrac{n}{n}\bar{x}\right)\\
&=-2(\bar{x}-\mu)(\bar{x}-\bar{x})\\
&=0
\end{aligned}$$

2.6 (a) x^2+1;

(b) x^4 if $x \geq 0$ and $-x^4$ if $x<0$;

(c) $x\,|x|$;

(d) $x\,|x|^{\alpha-1}$.

2.7 (a) $x^{\frac{1}{6}}y^{-\frac{1}{3}}$;

(b) $x^{\frac{3}{4}}$;

(c) $|x|$.

2.8 (a) monomial (polynomial) of degree 2;

(b) neither monomial nor polynomial;

(c) polynomial of degree 2;

(d) polynomial of degree 4;

(e) polynomial of degree 3;

(f) neither monomial nor polynomial;

(g) monomial (polynomial) of degree 0, i.e., a constant;

(h) polynomial of degree 1.

2.9 (a) $x^4+4x^3y+6x^2y^2+4xy^3+y^4$;

(b) $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

2.10 $2^n=(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}\,1^k=\sum_{k=0}^{n}\binom{n}{k}$

2.11 (a) $4xh$;

(b) $a(c-1)$;

(c) A^3-B^3;

(d) $8xy(x^2+y^2)$.

2.12 All polynomials of degree 4 with these roots have the form $c(x+1)(x-2)(x-3)(x-4)$ with $c \neq 0$.
It cannot have more than 4 roots.

2.13 (a) $\frac{2x}{x^2-1}$;

(b) $\frac{1}{st}$;

(c) $\frac{1}{xyz}$;

(d) $\frac{(x^2+y^2)(x+y)}{xy(x-y)}$.

2.14 (a) $1+x$;

(b) $\frac{1+x^2}{1-x}$ (cannot be simplified);

(c) x^3;

(d) $x^3(1+x) = x^3+x^4$;

(e) $x^2(1+x)(1+x^2) = x^2+x^3+x^4+x^5$;

(f) $1+x+x^2$.

2.15 (a) xy;

(b) $\frac{(x^2+y)(2x+y)}{(2x+1)2xy}$;

(c) $\frac{x(a-b)+a}{x(a+b)+b}$;

(d) x.

2.16 (a) $x^{\frac{1}{8}} - y^{\frac{1}{6}}$;

(b) $x^{\frac{1}{4}} + 2$;

(c) $2x^{\frac{51}{14}}$.

2.17 (a) $(1+\frac{y}{\sqrt{x}})^{\frac{1}{3}}$;

(b) $\frac{1}{3x-\frac{1}{3}}$;

(c) $\frac{1}{(xy)^{\frac{1}{6}}+3}$;

(d) $\sqrt{x}+\sqrt{y}$.

2.18 (a) 1;

(b) 2;

(c) 4;

(d) not defined;

(e) 0;

(f) -2;

(g) $\frac{1}{2}$;

(h) $-\frac{1}{2}$;

(i) not real.

2.19 (a) 2.47712;

(b) 4.7712.

2.20 (a) 10 000;

(b) -4;

(c) 27;

(d) $-\frac{\log_{10}(2)}{4} - \frac{1}{2}$;

(e) -3;

(f) -4.

2.21 (a) $y = \frac{1}{10}\, e^{x \ln 10}$;
(b) $y = 16\, e^{x \ln 4}$;
(c) $y = e^{x(\ln 3 + \ln 25)}$;
(d) $y = e^{x \ln \sqrt{1.08}}$;
(e) $y = 0.9\, e^{x\, \frac{1}{10} \ln 1.1}$;
(f) $y = \sqrt{q}\, e^{x \ln \sqrt{2}}$.

3.1 (a) $\{1-\sqrt{2}, 1, 1+\sqrt{2}\}$;
(b) $\{-\sqrt{2}, 0, \sqrt{2}\}$;
(c) $\{3\}$ (-1 not in domain).

3.2 (a) $x = \frac{\ln 3}{\ln 3 - \ln 2} \approx 2.710$;
(b) $x = \frac{\ln 9}{\ln 6} \approx 1.226$;
(c) $x = \frac{\ln 100}{\ln 5} \approx 2.861$;
(d) $x = \frac{\ln 50}{4 \ln 10} \approx 0.425$;
(e) $x = -\frac{\ln 2 + 4 \ln 10}{\ln 0{,}4} \approx 10.808$;
(f) $x = \frac{\ln 4}{\ln 9 - \ln 125} \approx -0.527$.

3.3 $x = \ln\left(a \pm \sqrt{a^2 - 1}\right)$

3.4 $x_1 = 0$,
$x_2 = \frac{27}{32} + \frac{\sqrt{217}}{32} \approx 1.304$,
$x_3 = \frac{27}{32} - \frac{\sqrt{217}}{32} \approx 0.383$.

3.5 (a) 1, (-2 does not satisfy the equation);
(b) 3.

3.6 (a) $(x+1)(x+3)$;
(b) $3\left(x - \frac{9+\sqrt{57}}{6}\right)\left(x - \frac{9-\sqrt{57}}{6}\right)$;
(c) $x(x+1)(x-1)$;
(d) $x(x-1)^2$;
(e) $(x+1)(x-1)^3$.

3.7 We get most of the solutions by means of the solution formula for quadratic equations or by factorizing the equation (displayed in square brackets [...]).

(a) $x = \frac{y}{y+1}$, $y = \frac{x}{1-x}$,
[$x(y+1) = y$ and $y(x-1) = -x$, resp.];

(b) $x = \frac{4y+1}{3y+2}$, $y = \frac{1-2x}{3x-4}$,
[$x(3y+2) = 1+4y$ and $y(3x-4) = 1-2x$, resp.];

(c) $x = y-1$ and $x = -y$; $y = x+1$ and $y = -x$,
[$(x-y+1)(x+y) = 0$];

(d) $x = -y$ and $x = \frac{1}{y}$; $y = -x$ and $y = \frac{1}{x}$,
[$(x+y)\cdot(xy-1) = 0$];

(e) $x = 2-y$ and $x = -2-y$; $y = 2-x$ and $y = -2-x$,
[$(x+y)^2 = 4$];

(f) $x = -\frac{1}{3}y + \frac{5}{3}$ and $x = -\frac{1}{3}y - \frac{5}{3}$; $y = 5-3x$ and $y = -5-3x$,

$[(3x+y)^2 = 25]$;

(g) $x = \pm\frac{3}{2}\sqrt{4-y^2}$, $y = \pm\frac{2}{3}\sqrt{9-x^2}$;

(h) $x = \pm\frac{3}{2}\sqrt{4+y^2}$, $y = \pm\frac{2}{3}\sqrt{x^2-9}$;

(i) $x = y - 2\sqrt{y} + 1$, $y = x - 2\sqrt{x} + 1$.

3.8 We get most of the solutions by means of the solution formula for quadratic equations or by factorizing the equation (displayed in square brackets [...]).

(a) $x = -\frac{1}{2}y \pm \frac{1}{2y}\sqrt{y^4+24y}$, $y = -\frac{1}{2}x \pm \frac{1}{2x}\sqrt{x^4+24x}$;

(b) $x = -y$ and $x = \frac{1}{y}$, $\quad y = -x$ and $y = \frac{1}{x}$,
$[(x+y)\cdot(xy-1) = 0]$;

(c) $x = (1\pm\sqrt{2})y$, $y = \frac{1}{1\pm\sqrt{2}}x$,
$[x^2 - 2xy - y^2 = 0]$;

(d) $x \in \mathbb{R}\setminus\{-y, \pm\sqrt{-y}\}$ arbitrary if $y = -1$ and $x = 0$ else, $y \in \mathbb{R}\setminus\{-x, -x^2\}$ arbitrary if $x = 0$ and $y = -1$ else,
$[x^2(1+y) = 0]$;

(e) $x = \frac{1+3y-y^2}{y-1}$, $y = \frac{1}{2}(3-x) \pm \frac{1}{2}\sqrt{(x-3)^2+4(x+1)}$,
$[y^2 - 3y + xy - x - 1 = 0]$;

(f) $x = \frac{y}{y^2-1}$, $y = \frac{1\pm\sqrt{1+4x^2}}{2x}$,
$[x(y^2-1) = y$ and $xy^2 - y - x = 0$, resp.];

(g) $x = -\frac{1}{8}(y+4) \pm \sqrt{\frac{1}{64}(y+4)^2 - \frac{1}{4}\left(y+\frac{1}{y}\right)}$, $y = -2x \pm \sqrt{4x^2 + \frac{1}{x+1}}$,
$[4x^2y + 4xy + xy^2 + y^2 - 1 = 0]$;

(h) $x = \frac{1}{2}y + \frac{1}{2}$ and $x = y - 1$, $\quad y = 2x - 1$ and $y = y + 1$,
$[(2x-y-1)\cdot(x-y+1) = 0]$;

(i) $x = \frac{-3+\sqrt{17}}{4}y$ and $x = \frac{-3-\sqrt{17}}{4}y$, $\quad y = \frac{3+\sqrt{17}}{2}x$ and $y = \frac{3-\sqrt{17}}{2}x$,
$[2x^2 + 3xy - y^2 = 0]$.

3.9 (a) $a = 1$, $b = 0$, $c = 1$;

(b) $a = 3$, $b = 1$, $c = 3$.

3.10 (a) $L = [-1,0] \cup [3,\infty)$;

(b) $L = (-1,0) \cup (3,\infty)$;

(c) $L = \{1\}$;

(d) $L = \mathbb{R}$;

(e) $L = \emptyset$.

3.11 (a) $(-\infty, -\frac{2}{3}] \cup [\frac{1}{2}, \infty)$;

(b) $(-\infty, -2) \cup (0, \infty)$;

(c) $(-\infty, \frac{22}{5}] \cup (5, \infty)$;

(d) $\left[\frac{3}{2} - \sqrt{\frac{25}{2}}, -1\right) \cup \left(4, \frac{3}{2} + \sqrt{\frac{25}{2}}\right]$;

(e) $(-\frac{13}{2}, -4] \cup [3, \frac{11}{2})$;

(f) $[-14, -9] \cup [1, 6]$.

4.1 (a) increasing but neither alternating nor bounded;

(b) decreasing and bounded but not alternating;

(c) bounded but neither alternating nor monotone;

(d) increasing but neither alternating nor bounded;

(e) alternating and bounded but not monotone.

4.2 (a) $\big(2,6,12,20,30\big)$;

(b) $\big(0.333,0.583,0.783,0.95,1.093\big)$;

(c) $\big(1.072,2.220,3.452,4.771,6.185\big)$.

4.3 $a_n = 2\cdot 1.1^{n-1}$;

$a_7 = 3.543$.

4.4 (a) $\big(s_n\big) = \left(\frac{n(n-1)}{2}\right) = \big(0,1,3,\ldots,45\big)$;

(b) $\big(s_n\big) = \big(n^2\big) = \big(1,4,9,\ldots,100\big)$.

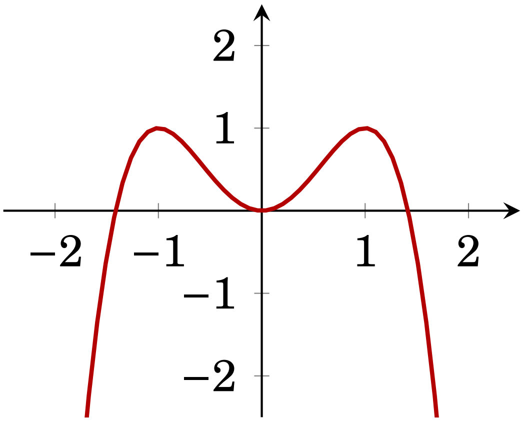
4.5 (a) $s_7 = \frac{1}{3}\cdot\frac{3^7-1}{3-1} = 364.33$;

(b) $s_7 = -\frac{1}{2}\cdot\frac{(-1/4)^7-1}{-1/4-1} = -0.400$.

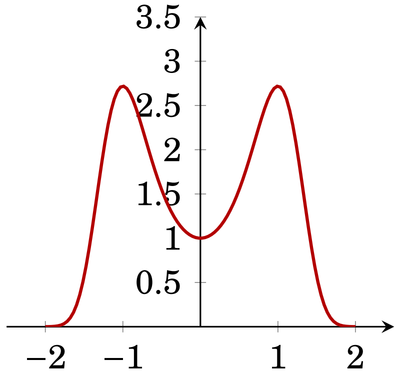
5.1 (a) $D_h = \mathbb{R}\setminus\{2\}$;
(b) $D_D = \mathbb{R}\setminus\{1\}$;
(c) $D_f = [2,\infty)$;
(d) $D_g = (\frac{3}{2},\infty)$;
(e) $D_f = [-3,3]$;
(f) $D_f = \mathbb{R}$;
(g) $D_f = \mathbb{R}$.

5.2 (a) $D_f = \mathbb{R}\setminus\{3\}$;
(b) $D_f = (-1,\infty)$;
(c) $D_f = \mathbb{R}$;
(d) $D_f = (-1,1)$;
(e) $D_f = \mathbb{R}$;
(f) $D_f = \mathbb{R}\setminus\{0\}$.

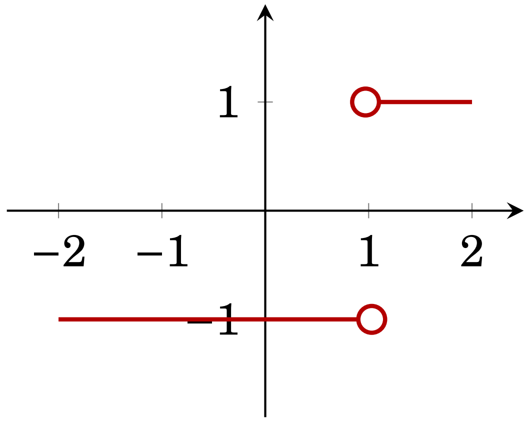
5.3



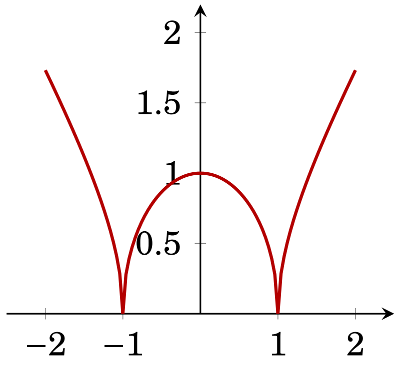
5.4



5.5



5.6



5.7 (a) one-to-one, not onto;

(b) one-to-one, not onto;

(c) bijective;

(d) not one-to-one, not onto;

(e) not one-to-one, onto;

(f) bijective.

Beware! Domain and codomain are part of the function.

5.8 (a) $(f \circ g)(4) = f(g(4)) = f(9) = 98$;

(b) $(f \circ g)(-9) = f(g(-9)) = f(28) = 839$;

(c) $(g \circ f)(0) = g(f(0)) = g(-1) = 2$;

(d) $(g \circ f)(-1) = g(f(-1)) = g(-2) \approx 3.828$.

5.9 (a) $(f\circ g)(x)=(1+x)^2$,
$(g\circ f)(x)=1+x^2$,
$D_f=D_g=D_{f\circ g}=D_{g\circ f}=\mathbb{R}$;

(b) $(f\circ g)(x)=|x|+1$,
$(g\circ f)(x)=(\sqrt{x}+1)^2$,
$D_f=D_{g\circ f}=[0,\infty)$, $D_g=D_{f\circ g}=\mathbb{R}$;

(c) $(f\circ g)(x)=\frac{1}{\sqrt{x}+2}$,
$(g\circ f)(x)=\sqrt{\frac{1}{x+1}}+1$,
$D_f=\mathbb{R}\setminus\{-1\}$, $D_g=D_{f\circ g}=[0,\infty)$, $D_{g\circ f}=(-1,\infty)$;

(d) $(f\circ g)(x)=2+|x-2|$,
$(g\circ f)(x)=|x|$,
$D_f=D_{g\circ f}=[0,\infty)$, $D_g=D_{f\circ g}=\mathbb{R}$;

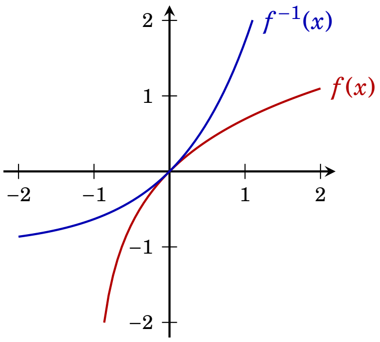
(e) $(f\circ g)(x)=(x-3)^2+2$,
$(g\circ f)(x)=x^2-1$,
$D_f=D_g=D_{f\circ g}=D_{g\circ f}=\mathbb{R}$;

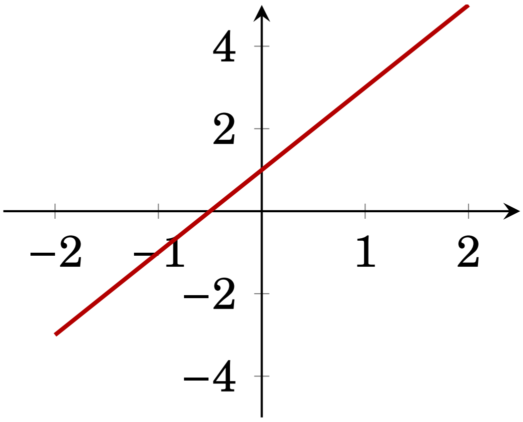
(f) $(f\circ g)(x)=\frac{1}{1+(\frac{1}{x})^2}$,
$(g\circ f)(x)=1+x^2$,
$D_f=D_{g\circ f}=\mathbb{R}$, $D_g=D_{f\circ g}=\mathbb{R}\setminus\{0\}$;

(g) $(f\circ g)(x)=x^2$,
$(g\circ f)(x)=\exp(\ln(x)^2)=x^{\ln x}$,
$D_f=D_{g\circ f}=(0,\infty)$, $D_g=D_{f\circ g}=\mathbb{R}$;

(h) $(f\circ g)(x)=\ln(x^3)$,
$(g\circ f)(x)=(\ln(x-1))^3+1$,
$D_f=D_{g\circ f}=(1,\infty)$, $D_g=\mathbb{R}$, $D_{f\circ g}=(0,\infty)$.

5.10 $f^{-1}(x)=e^x-1$





5.11

5.16 (a) $\boxed{18}$;

(b) $\boxed{11}$;

(c) $\boxed{1}$;

(d) $\boxed{12}$;

(e) $\boxed{9}$;

(f) $\boxed{4}$;

(g) $\boxed{6}$;

(h) $\boxed{16}$;

(i) $\boxed{10}$;

(j) $\boxed{3}$;

(k) $\boxed{8}$;

(l) $\boxed{13}$;

(m) $\boxed{15}$;

(n) $\boxed{5}$;

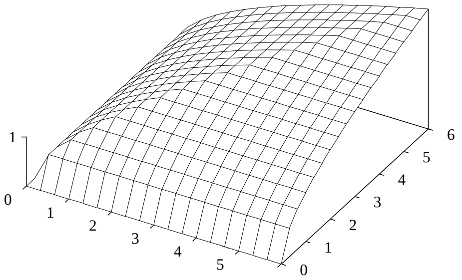
(o) $\boxed{14}$;

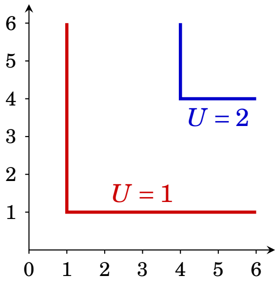
(p) $\boxed{17}$;

(q) $\boxed{2}$;

(r) $\boxed{7}$.

5.17





6.1 (a) 7;

(b) $\frac{2}{7}$;

(c) divergent;

(d) divergent but tends to ∞;

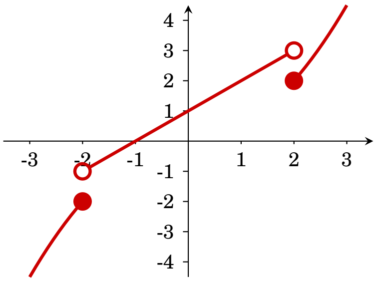
(e) 0.

6.2 (a) e^x;

(b) e^x;

(c) $e^{1/x}$.

6.3



$\lim_{x\to -2^+} f(x) = -1, \quad \lim_{x\to -2^-} f(x) = -2, \quad \lim_{x\to -2} f(x)$ does not exist;

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = 1$;

$\lim_{x\to 2^+} f(x) = 2, \quad \lim_{x\to 2^-} f(x) = 3, \quad \lim_{x\to 2} f(x)$ does not exist.

6.4 (a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$;

(b) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = \infty$;

(c) $\lim_{x\to 1^-} x = \lim_{x\to 1^+} x = 1$.

6.5 (a) 0;

(b) 0;

(c) ∞;

(d) $-\infty$;

(e) 1.

6.6 (a) $\frac{1}{2}$;

(b) -4;

(c) -1;

(d) 0.

6.7 (a) 5;

(b) -5;

(c) 0;

(d) $-\infty$;

(e) $-\frac{1}{5}$.

6.8 (a) 1;

(b) $2x$;

(c) $3x^2$;

(d) nx^{n-1}.

6.9 (a) $\frac{2}{7}$;

(b) $-\frac{1}{4}$;

(c) $\frac{1}{3}$;

(d) $\frac{1}{2}$;

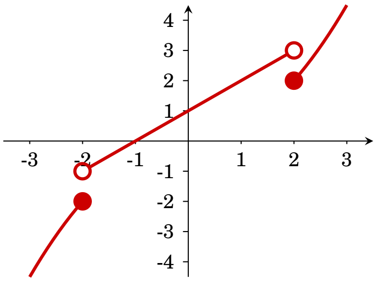
(e) $= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = 0$;

(f) ∞.

In (b) and (f) we cannot apply l'Hôpital's rule.

6.10 Look at numerator and denominator.

6.11



$\lim_{x\to -2^+} f(x) = -1$, $\lim_{x\to -2^-} f(x) = -2$, $\lim_{x\to -2} f(x)$ does not exist;

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = 1$;

$\lim_{x\to 2^+} f(x) = 2$, $\lim_{x\to 2^-} f(x) = 3$, $\lim_{x\to 2} f(x)$ does not exist;

f is continuous at 0 but not in -2 and 2.

6.12 $\lim_{x\to 0^-} f(x) = -1$, $\lim_{x\to 0^+} f(x) = 1$.

Not continuous at 0 and thus not differentiable.

6.13 $\lim_{x\to 1} f(x) = 2$.
Continuous at 1, but not differentiable.

6.14 The respective functions are continuous in
(a) D;
(b) D;
(c) D;
(d) D;
(e) D;
(f) $\mathbb{R}\setminus\mathbb{Z}$;
(g) $\mathbb{R}\setminus\{2\}$.

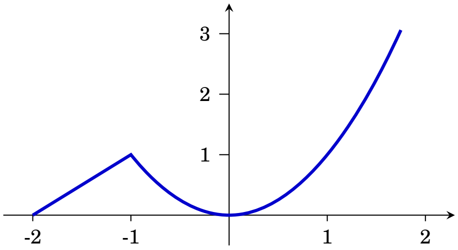
6.15 Yes.

6.16 $h = \frac{2}{5}$.

7.1 differentiable in

(a) $\mathbb{R}$, (b) $\mathbb{R}$, (c) $\mathbb{R}\setminus\{0\}$, (d) $\mathbb{R}\setminus\{-1,1\}$, (e) $\mathbb{R}\setminus\{-1,1\}$, (f) $\mathbb{R}\setminus\{-1\}$.

Graph for (f):



7.2 (a) $f'(x) = 16x^3 + 9x^2 - 4x, \quad f''(x) = 48x^2 + 19x - 4;$

(b) $f'(x) = -x\,e^{-\frac{x^2}{2}}, \quad f''(x) = (x^2 - 1)\,e^{-\frac{x^2}{2}};$

(c) = (b);

(d) $f'(x) = \frac{-2}{(x-1)^2}, \quad f''(x) = \frac{4}{(x-1)^3}.$

7.3 (a) $f'(x) = -\frac{2x}{(1+x^2)^2}, \quad f''(x) = \frac{6x^2-2}{(1+x^2)^3};$

(b) $f'(x) = -\frac{2}{(1+x)^3}, \quad f''(x) = \frac{6}{(1+x)^4};$

(c) $f'(x) = \ln(x), \quad f''(x) = \frac{1}{x};$

(d) $f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}.$

7.4 (a) $f'(x) = \frac{1}{cos(x)^2}, \quad f''(x) = \frac{2\sin(x)}{\cos(x)^3};$

(b) $f'(x) = \sinh(x), \quad f''(x) = \cosh(x);$

(c) $f'(x) = \cosh(x), \quad f''(x) = \sinh(x);$

(d) $f'(x) = -2x\sin(1+x^2), \quad f''(x) = -2\sin(1+x^2) - 4x^2\cos(1+x^2).$

7.5

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\cdot(g(x))^{-1}\right)' \\
&= f'(x)\cdot(g(x))^{-1} + f(x)\cdot\left((g(x))^{-1}\right)' \\
&= f'(x)\cdot(g(x))^{-1} + f(x)(-1)(g(x))^{-2}g'(x) \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\
&= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}
\end{aligned}$$

7.6 By means of the differential: $f(3.1) - f(3) \approx -0.001096$,

exact value: $f(3.1) - f(3) = -0.00124\ldots$

7.7 (a) $\varepsilon_g(x) = \frac{3x^3-4x^2}{x^3-2x^2},$
1-elastic for $x = 1$ and $x = \frac{3}{2}$,
elastic for $x < 1$ and $x > \frac{3}{2}$,
inelastic for $1 < x < \frac{3}{2}$.

(b) $\varepsilon_h(x) = \beta$,
the elasticity of $h(x)$ only depends on parameter β and is constant on the domain of h.

7.8 (a) wrong;

(b) approximately;

(c) is correct;

(d) wrong.

7.9 Derivatives:

	(a)	(b)	(c)	(d)	(e)
f_x	1	y	$2x$	$2xy^2$	$\alpha x^{\alpha-1} y^{\beta}$
f_y	1	x	$2y$	$2x^2 y$	$\beta x^{\alpha} y^{\beta-1}$
f_{xx}	0	0	2	$2y^2$	$\alpha(\alpha-1)x^{\alpha-2} y^{\beta}$
$f_{xy} = f_{yx}$	0	1	0	$4xy$	$\alpha\beta x^{\alpha-1} y^{\beta-1}$
f_{yy}	0	0	2	$2x^2$	$\beta(\beta-1)x^{\alpha} y^{\beta-2}$

Derivatives at point $(1,1)$:

	(a)	(b)	(c)	(d)	(e)
f_x	1	1	2	2	α
f_y	1	1	2	2	β
f_{xx}	0	0	2	2	$\alpha(\alpha-1)$
$f_{xy} = f_{yx}$	0	1	0	4	$\alpha\beta$
f_{yy}	0	0	2	2	$\beta(\beta-1)$

7.10 Derivatives:
$f_x(x,y) = 2x\exp(x^2+y^2)$,
$f_y(x,y) = 2y\exp(x^2+y^2)$,
$f_{xx}(x,y) = (2+4x^2)\exp(x^2+y^2)$,
$f_{xy}(x,y) = f_{yx}(x,y) = 4xy\exp(x^2+y^2)$,
$f_{yy}(x,y) = (2+4y^2)\exp(x^2+y^2)$.

Derivatives at point $(0,0)$:
$f_x(0,0) = 0$,
$f_y(0,0) = 0$,
$f_{xx}(0,0) = 2$,
$f_{xy}(0,0) = f_{yx}(0,0) = 0$,
$f_{yy}0,0) = 2$.

7.11 Derivatives:

	(a)
f_x	$x(x^2+y^2)^{-1/2}$
f_y	$y(x^2+y^2)^{-1/2}$
f_{xx}	$(x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2}$
$f_{xy} = f_{yx}$	$-xy(x^2+y^2)^{-3/2}$
f_{yy}	$(x^2+y^2)^{-1/2} - y^2(x^2+y^2)^{-3/2}$

	(b)
f_x	$x^2(x^3+y^3)^{-2/3}$
f_y	$y^2(x^3+y^3)^{-2/3}$
f_{xx}	$2x(x^3+y^3)^{-2/3} - 2x^4(x^3+y^3)^{-5/3}$
$f_{xy} = f_{yx}$	$-2x^2y^2(x^3+y^3)^{-5/3}$
f_{yy}	$2y(x^3+y^3)^{-2/3} - 2y^4(x^3+y^3)^{-5/3}$

	(c)
f_x	$x^{p-1}(x^p+y^p)^{(1-p)/p}$
f_y	$y^{p-1}(x^p+y^p)^{(1-p)/p}$
f_{xx}	$(p-1)x^{p-2}(x^p+y^p)^{(1-p)/p}$ $-(p-1)x^{2(p-1)}(x^p+y^p)^{(1-2p)/p}$
$f_{xy}=f_{yx}$	$-(p-1)x^{p-1}y^{p-1}(x^p+y^p)^{(1-2p)/p}$
f_{yy}	$(p-1)y^{p-2}(x^p+y^p)^{(1-p)/p}$ $-(p-1)y^{2(p-1)}(x^p+y^p)^{(1-2p)/p}$

Derivatives at point $(1,1)$:

	(a)	(b)	(c)
f_x	$\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt[3]{4}}$	$2^{(1-p)/p}$
f_y	$\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt[3]{4}}$	$2^{(1-p)/p}$
f_{xx}	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt[3]{4}}$	$(p-1)2^{(1-2p)/p}$
$f_{xy}=f_{yx}$	$-\frac{1}{\sqrt{8}}$	$-\frac{1}{\sqrt[3]{4}}$	$-(p-1)2^{(1-2p)/p}$
f_{yy}	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt[3]{4}}$	$(p-1)2^{(1-2p)/p}$

7.12 Gradient:
(a) $\nabla f(x,y)=(1,1)$,
(b) $\nabla f(x,y)=(y,x)$,
(c) $\nabla f(x,y)=(2x,2y)$,
(d) $\nabla f(x,y)=(2xy^2,2x^2y)$,
(e) $\nabla f(x,y)=(\alpha x^{\alpha-1}y^{\beta},\beta x^{\alpha}y^{\beta-1})$,

Gradient at $(1,1)$:
(a) $\nabla f(1,1)=(1,1)$,
(b) $\nabla f(1,1)=(1,1)$,
(c) $\nabla f(1,1)=(2,2)$,
(d) $\nabla f(1,1)=(2,2)$,
(e) $\nabla f(1,1)=(\alpha,\beta)$.

7.13 Gradient:
(a) $\nabla f(x,y)=(x(x^2+y^2)^{-1/2},\ y(x^2+y^2)^{-1/2})$,
(b) $\nabla f(x,y)=(x^2(x^3+y^3)^{-2/3},\ y^2(x^3+y^3)^{-2/3})$,
(c) $\nabla f(x,y)=(x^{p-1}(x^p+y^p)^{(1-p)/p},\ y^{p-1}(x^p+y^p)^{(1-p)/p})$,

Gradient at $(1,1)$:
(a) $\nabla f(1,1)=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$,
(b) $\nabla f(1,1)=(\frac{1}{\sqrt[3]{4}},\frac{1}{\sqrt[3]{4}})$,
(c) $\nabla f(1,1)=(2^{(1-p)/p},2^{(1-p)/p})$.

7.14 (a) $\mathbf{H}_f(1,1)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$;

(b) $\mathbf{H}_f(1,1)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$;

(c) $\mathbf{H}_f(1,1)=\begin{pmatrix}2&0\\0&2\end{pmatrix}$;

(d) $\mathbf{H}_f(1,1)=\begin{pmatrix}2&4\\4&2\end{pmatrix}$;

(e) $\mathbf{H}_f(1,1) = \begin{pmatrix} \alpha(\alpha-1) & \alpha\beta \\ \alpha\beta & \beta(\beta-1) \end{pmatrix}$.

7.15 (a) $\mathbf{H}_f(1,1) = \begin{pmatrix} \frac{\sqrt{8}}{8} & -\frac{\sqrt{8}}{8} \\ -\frac{\sqrt{8}}{8} & \frac{\sqrt{8}}{8} \end{pmatrix}$;

(b) $\mathbf{H}_f(1,1) = \begin{pmatrix} \frac{\sqrt[3]{2}}{2} & -\frac{\sqrt[3]{2}}{2} \\ -\frac{\sqrt[3]{2}}{2} & \frac{\sqrt[3]{2}}{2} \end{pmatrix}$;

(c) $\mathbf{H}_f(1,1) = \begin{pmatrix} (p-1)2^{(1-2p)/p} & -(p-1)2^{(1-2p)/p} \\ -(p-1)2^{(1-2p)/p} & (p-1)2^{(1-2p)/p} \end{pmatrix}$.

7.16 (a) $h'(t) = f'(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = (2g_1(t), 2g_2(t)) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} = (2t, 2t^2) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} = 2t + 4t^3$.

(Composite function: $h(t) = f(\mathbf{g}(t)) = t^2 + t^4$.)

(b) $\mathbf{p}'(x,y) = \mathbf{g}'(f(x,y)) \cdot f'(x,y) = \begin{pmatrix} 1 \\ 2(x^2+y^2) \end{pmatrix} \cdot (2x, 2y) = \begin{pmatrix} 2x & 2y \\ 4x(x^2+y^2) & 4y(x^2+y^2) \end{pmatrix}$

(Composite function: $\mathbf{p}(x,y) = \mathbf{g}(f(x,y)) = \begin{pmatrix} x^2+y^2 \\ (x^2+y^2)^2 \end{pmatrix}$)

7.17 $\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 3x_1^2 & -1 \\ 1 & -3x_2^2 \end{pmatrix}$, $\mathbf{g}'(\mathbf{x}) = \begin{pmatrix} 0 & 2x_2 \\ 1 & 0 \end{pmatrix}$,

(a) $(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x}))\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 0 & 2(x_1 - x_2^3) \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3x_1^2 & -1 \\ 1 & -3x_2^2 \end{pmatrix}$
$= \begin{pmatrix} 2(x_1 - x_2^3) & 6(-x_1 x_2^2 + x_2^5) \\ 3x_1^2 & -1 \end{pmatrix}$

(b) $(\mathbf{f} \circ \mathbf{g})'(\mathbf{x}) = \mathbf{f}'(\mathbf{g}(\mathbf{x}))\mathbf{g}'(\mathbf{x}) = \begin{pmatrix} 3x_2^4 & -1 \\ 1 & -3x_1^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2x_2 \\ 1 & 0 \end{pmatrix}$
$= \begin{pmatrix} -1 & 6x_2^5 \\ -3x_1^2 & 2x_2 \end{pmatrix}$.

7.18 Let $\mathbf{s}(t) = \begin{pmatrix} K(t) \\ L(t) \\ t \end{pmatrix}$. Then

$$\begin{aligned} \frac{dQ}{dt} &= \nabla Q(\mathbf{s}(t)) \cdot \mathbf{s}'(t) \\ &= (Q_K(\mathbf{s}(t)), Q_L(\mathbf{s}(t)), Q_t(\mathbf{s}(t))) \cdot \begin{pmatrix} K'(t) \\ L'(t) \\ 1 \end{pmatrix} \\ &= Q_K K'(t) + Q_L L'(t) + Q_t\,. \end{aligned}$$

8.1 (a) convex;

(b) concave;

(c) concave;

(d) convex if $\alpha \geq 1$ and $\alpha \leq 0$, concave if $0 \leq \alpha \leq 1$.

8.2 Monotonically decreasing in $[-1,3]$,
monotonically increasing in $(-\infty,-1]$ and $[3,\infty)$;

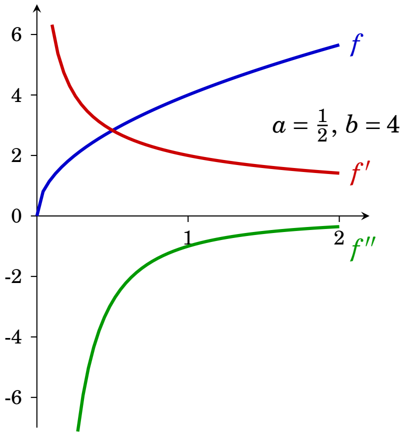
concave in $(-\infty,1]$,
convex in $[1,\infty)$.

8.3 (a) monotonically increasing (in $\mathbb{R}$),
concave in $(-\infty,0]$, convex in $[0,\infty)$;

(b) monotonically increasing $(-\infty,0]$, decreasing in $[0,\infty)$,
concave in $[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}]$, and convex in $(-\infty,-\frac{\sqrt{2}}{2}]$ and $[\frac{\sqrt{2}}{2},\infty)$;

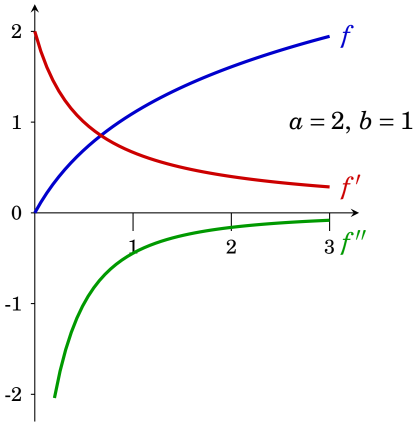
(c) monotonically increasing $(-\infty,0]$, decreasing in $[0,\infty)$,
concave in $[-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}]$, and convex in $(-\infty,-\frac{\sqrt{3}}{3}]$ and $[\frac{\sqrt{3}}{3},\infty)$.

8.4



Compute all derivatives and verify properties (1)–(3).

8.5



Compute all derivatives and verify properties (1)–(3).

8.6 $\left(\frac{1}{2}x+\frac{1}{2}y\right)^2-\left(\frac{1}{2}x^2+\frac{1}{2}y^2\right)=\frac{1}{4}x^2+\frac{1}{2}xy+\frac{1}{4}y^2-\frac{1}{2}x^2-\frac{1}{2}y^2=-\frac{1}{4}x^2+\frac{1}{2}xy-\frac{1}{4}y^2=$ $-\left(\frac{1}{4}x^2-\frac{1}{2}xy+\frac{1}{4}y^2\right)=-\left(\frac{1}{2}x-\frac{1}{2}y\right)^2<0$.

8.7 As f is concave, $f''(x)\le 0$ for all x.
Hence $g''(x)=(-f(x))''=-f''(x)\ge 0$ for all x, i.e., g is convex.

8.8 As f is concave we find for all $x,y\in D_f$ and all $h\in[0,1]$,

$$f((1-h)x+h\,y)\ge(1-h)f(x)+h\,f(y)\,.$$

But then

$$\begin{aligned}g((1-h)x+h\,y)&=-f((1-h)x+h\,y)\\&\le-((1-h)f(x)+h\,f(y))\\&=(1-h)(-f(x))+h(-f(y))\\&=(1-h)g(x)+h\,g(y)\end{aligned}$$

i.e., g is convex by definition.

8.9 $h''(x) = \alpha f''(x) + \beta g''(x) \leq 0$, i.e., h is concave. If $\beta < 0$ then $\beta g''(x)$ is positive and the sign of $\alpha f''(x) + \beta g''(x)$ cannot be estimated any more.

8.10 Please find your own example.

8.11 Please find your own example.

8.12 (a) local maximum in $x = 0$;

(b) local minimum in $x = 1$, local maximum in $x = -1$;

(c) local minimum in $x = 3$.

8.13 (a) global minimum in $x = 1$, no global maximum;

(b) global maximum in $x = \frac{1}{4}$, no global minimum;

(c) global minimum in $x = 0$, no global maximum;

(d) global minimum in $x = 1$, no global maximum;

(e) global maximum in $x = 1$, no global minimum.

8.14 (a) global maximum in $x = 12$, global minimum in $x = 8$;
(b) global maximum in $x = 6$, global minimum in $x = 3$;
(c) global maxima in $x = -2$ and $x = 2$,
global minima in $x = -1$ and $x = 1$.

9.1 (a) $2(x-1)e^x + c$;

(b) $-(x^2 + 2x + 2)e^{-x} + c$;

(c) $\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + c$;

(d) $\frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + c$;

(e) $\frac{1}{2}x^2(\ln(x))^2 - \frac{1}{2}x^2 \ln(x) + \frac{1}{4}x^2 + c$;

(f) $2x\sin(x) + (2 - x^2)\cos(x)$.

9.2 (a) $z = x^2$: $\frac{1}{2}e^{x^2} + c$;

(b) $z = x^2 + 6$: $\frac{2}{3}\left(x^2 + 6\right)^{\frac{3}{2}} + c$;

(c) $z = 3x^2 + 4$: $\frac{1}{6}\ln|3x^2 + 4| + c$;

(d) $z = x + 1$: $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$;

(e) $z = \ln(x)$: $\frac{1}{2}(\ln(x))^2 + c$.

9.3 (a) $z = \ln(x)$: $\ln|\ln(x)| + c$;

(b) $z = x^3 + 1$: $\frac{2}{9}(x^3 + 1)^{3/2} + c$;

(c) $z = 5 - x^2$: $-\sqrt{5 - x^2} + c$;

(d) $z = x - 3$: $7\ln|x - 3| + \frac{1}{2}x^2 + 2x + c$;

(e) $z = x - 8$: $\frac{2}{5}(x-8)^{5/2} + \frac{16}{3}(x-8)^{3/2} + c$.

9.4 (a) 39;

(b) $3e^2 - 3 \approx 19.16717$;

(c) 93;

(d) $-\frac{1}{6}$ (use radian!);

(e) $\frac{1}{2}\ln(8) \approx 1.0397$.

9.5 (a) $\frac{1}{2}$;

(b) $\frac{781}{10}$;

(c) $\frac{8}{3}$;

(d) $\frac{1}{2}\ln(5) - \frac{1}{2}\ln(2) \approx 0.4581$.

9.6 (a) $1 - e^{-2} \approx 0.8647$;

(b) $\frac{27}{4}$;

(c) 1;

(d) $2e^2 - 2 \approx 12.778$;

(e) $\frac{8}{3}\ln(2) - \frac{7}{9} \approx 1.07061$.

9.7 $\displaystyle\int_{-2}^{2} x^2 f(x)\,dx = \frac{1}{6}$.

9.8 $F(x) = \begin{cases} 0, & \text{for } x \le -1, \\ \frac{1}{2} + \frac{2x+x^2}{2}, & \text{for } -1 < x \le 0, \\ \frac{1}{2} + \frac{2x-x^2}{2}, & \text{for } 0 < x \le 1, \\ 1, & \text{for } x > 1. \end{cases}$

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