Chapter 18

Difference Equation

Rules for Differences
For differences similar rules can be applied as for derivatives:

- $\Delta (c \cdot y_t) = c \Delta y_t$
- $\Delta (y_t + z_t) = \Delta y_t + \Delta z_t$ \hspace{1cm} \textbf{Summation rule}
- $\Delta (y_t \cdot z_t) = y_{t+1} \Delta z_t + z_t \Delta y_t$ \hspace{1cm} \textbf{Product rule}
- $\Delta \left( \frac{y_t}{z_t} \right) = \frac{z_t \Delta y_t - y_t \Delta z_t}{z_t z_{t+1}}$ \hspace{1cm} \textbf{Quotient rule}

Differences of Higher Order
The $k$-th derivative $\frac{d^k y}{dt^k}$ has to be replaced by the \textbf{difference of order} $k$:

$$\Delta^k y_t = \Delta (\Delta^{k-1} y_t) = \Delta^{k-1} y_{t+1} - \Delta^{k-1} y_t$$

For example the \textbf{second difference} is then

$$\Delta^2 y_t = \Delta (\Delta y_t) = \Delta y_{t+1} - \Delta y_t = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) = y_{t+2} - 2y_{t+1} + y_t$$

Equivalent Representation

Difference equations can equivalently written without $\Delta$-notation.

- $\Delta y_t = 3 \Leftrightarrow y_{t+1} - y_t = 3 \Leftrightarrow y_{t+1} = y_t + 3$
- $\Delta^2 y_t = \frac{1}{2} y_t \Leftrightarrow y_{t+1} - y_t = \frac{1}{2} y_t \Leftrightarrow y_{t+1} = \frac{3}{2} y_t$
- $\Delta^3 y_t = 2 \Delta y_t = -3 \Leftrightarrow (y_{t+2} - 2y_{t+1} + y_t) + 2 (y_{t+1} - y_t) = -3 \Leftrightarrow y_{t+2} = y_{t+1} - 3$

These can be seen as \textbf{recursion formula} for sequences.

Problem:
Find a sequence $y_t$ which satisfies the given recursion formula for all $t \in \mathbb{N}$.

Example – Iterations

Compute the solution of $y_{t+1} = \frac{2}{3} y_t$ with initial value $y_0$.

$$y_1 = \frac{2}{3} y_0$$
$$y_2 = \frac{2}{3} y_1 = \frac{2}{3} \left( \frac{2}{3} y_0 \right) = \frac{4}{9} y_0$$
$$y_3 = \frac{2}{3} y_2 = \frac{2}{3} \left( \frac{4}{9} y_0 \right) = \frac{8}{27} y_0$$

...  

$$y_t = \left( \frac{2}{3} \right)^{t-1} y_0$$

For initial value $y_0 = 5$ we obtain $y_t = 5 \cdot \left( \frac{2}{3} \right)^t$.

Initial Value Problem and Iterations

Difference equations of first order can be solved by iteratively computing the elements of the sequence if the initial value $y_0$ is given.

Compute the solution of $y_{t+1} = y_t + 3$ with initial value $y_0$.

$$y_1 = y_0 + 3$$
$$y_2 = y_1 + 3 = (y_0 + 3) + 3 = y_0 + 2 \cdot 3$$
$$y_3 = y_2 + 3 = (y_0 + 2 \cdot 3) + 3 = y_0 + 3 \cdot 3$$

...  

$$y_t = y_0 + 3 \cdot t$$

For initial value $y_0 = 5$ we obtain $y_t = 5 + 3 \cdot t$.

First Difference

Suppose a state variable $y$ can only be estimated at \textbf{discrete} time points $t_1, t_2, t_3, \ldots$ In particular we assume that $t_j \in \mathbb{N}$. Thus we can describe the behavior of such a variable by means of a map

$$\mathbb{N} \rightarrow \mathbb{R}, \ t \mapsto y(t)$$

i.e., a sequence. We write $y_i$ instead of $y(t_i)$.

For the marginal changes of $y$ we have to replace the differential quotient $\frac{dy}{dt}$ by the \textbf{difference quotient} $\frac{\Delta y}{\Delta t}$.

So if $\Delta t = 1$ this reduces to the \textbf{first difference}

$$\Delta y_t = y_{t+1} - y_t$$

Difference Equation
A \textbf{difference equation} is an equation that contains the differences of a sequence. It is of order $n$ if it contains a difference of order $n$ (but not higher).

- $\Delta y_t = 3$ \hspace{1cm} \text{difference equation of first order}
- $\Delta^2 y_t = \frac{1}{2} y_t$ \hspace{1cm} \text{difference equation of first order}
- $\Delta^3 y_t + 2 \Delta y_t = -3$ \hspace{1cm} \text{difference equation of second order}

If in addition an initial value $y_0$ is given we have a so called \textbf{initial value problem}.
Homogeneous Linear Difference Equation of First Order

A homogeneous linear difference equation of first order is of form

\[ y_{t+1} + a y_t = 0 \]

Ansatz for general solution:

\[ y_t = C \beta^t, \quad C, \beta \neq 0, \quad \text{for some fixed } C \in \mathbb{R}. \]

It has to satisfy the difference equation for all \( t \):

\[ y_{t+1} + a y_t = C \beta^{t+1} + a C \beta^t = 0. \]

Division by \( C \beta^t \) yields \( \beta + a = 0 \) and thus \( \beta = -a \) and

\[ y_t = C (-a)^t. \]

Properties of Solutions

The behavior of solution

\[ y_t = C \beta^t = C (-a)^t \]

obviously depends on parameter \( \beta = -a \) which can be summarized as following:

- Oscillating \( \iff \beta < 0 \)
- Convergent \( \iff |\beta| < 1 \)

We want to note that \( \beta \) is the root of the characteristic equation \( \beta + a = 0 \).

Inhomogeneous Linear Difference Equation

The general solution of inhomogeneous linear difference equation

\[ y_{t+1} + a y_t = s \]

can be written as

\[ y_t = y_h + y_p, \]

where

- \( y_h \) is the general solution of the corresponding homogeneous equation \( y_{t+1} + a y_t = 0 \), and
- \( y_p \) is some particular solution of the inhomogeneous equation.

How can we find \( y_p \)?

Example – Homogeneous Equation

Homogeneous linear difference equation

\[ y_{t+1} - \frac{3}{2} y_t = 0 \]

has general solution

\[ y_t = C \left( \frac{3}{2} \right)^t. \]

Properties of Solutions

\[ \beta > 1 \]

\[ \beta = 1 \]

\[ 0 < \beta < 1 \]

\[ -1 < \beta < 0 \]

\[ \beta = -1 \]

\[ \beta < -1 \]

Inhomogeneous Linear Difference Equation

As parameters \( a \) and \( s \) are constant we may set \( y_{p,t} = c = \text{const} \)

Then

\[ y_{p,t+1} + a y_{p,t} = c + a c = s \]

which implies

\[ y_{p,t} = c = \frac{s}{1 + a} \quad \text{if } a \neq -1. \]

If \( a = -1 \) we set \( y_{p,t} = c t \). Then

\[ c (t+1) + (-1) c t = s \]

which implies \( c = s \) and

\[ y_{p,t} = s t. \]

Inhomogeneous Linear Difference Equation

An inhomogeneous linear difference equation of first order with constant coefficients is of form

\[ y_{t+1} + a y_t = s \]

The general solution is given by

\[ y_t = \begin{cases} 
C (-a)^t + \frac{s}{1 + a} & \text{if } a \neq -1, \\
C + s t & \text{if } a = -1.
\end{cases} \]

Observe that \( C (-a)^t \) is just the solution of the corresponding homogeneous difference equation \( y_{t+1} + a y_t = 0 \).

Asymptotically Stable

Observe that \( y_{p,t} = \bar{y} = \frac{-a}{1+\bar{a}} \) is a fixed point (or equilibrium point) of the inhomogeneous equation \( y_{t+1} + a y_t = s \).

Obviously solution

\[ y_t = C (-a)^t + \bar{y} \quad (C \neq 0) \]

converges to \( \bar{y} \) if and only if \( |a| < 1 \).

In this case \( \bar{y} \) is (locally) asymptotically stable.

Otherwise if \( |a| > 1 \), \( y_t \) diverges and \( \bar{y} \) is called unstable.
We start in period 0 with price $p_0$ and get supply $q_1 = S(p_0)$ in period 1.

Market equilibrium implies new price $p_1$, given implicitly by $D(p_1) = q_1$.

In period 2 price $p_1$ yields supply $q_2 = S(p_1)$.

Market equilibrium implies new price $p_2$, given implicitly by $D(p_2) = q_2$.

Iterating this procedure spins a cobweb around equilibrium point $(\bar{p}, \bar{q})$ with $\bar{q} = S(\bar{p}) = D(\bar{p})$. 

Cobweb Model
Cobweb Model – Nonlinear Functions

Cobweb models also work when functions $D(p)$ and $S(p)$ are nonlinear.

Then there may not exist a solution in closed form.

However, we still have an equilibrium point $\bar{p}$ with $D(\bar{p}) = S(\bar{p})$,

Linearized versions of $D$ and $S$:

$$D'(\bar{p}) + S'(\bar{p})$$

Equilibrium point $\bar{p}$ is locally asymptotically stable if:

1. $D'(\bar{p}) < 0 < S'(\bar{p})$, and
2. $|S'(\bar{p})| < |D'(\bar{p})|$. 

Homogeneous Linear Difference Equation

We obtain the general solution of the homogeneous linear ODE

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

by means of the ansatz

$$y_t = C \beta^t,$$ 

which has to satisfy the difference equation:

$$C \beta^{t+2} + a_1 C \beta^{t+1} + a_2 C \beta^t = 0.$$ 

Hence $\beta$ has to satisfy the characteristic equation

$$\beta^2 + a_1 \beta + a_2 = 0.$$ 

Case: $a_1^2 - 4 a_2 > 0$

The general solution of the homogeneous difference equation

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

is given by

$$y(t) = C_1 \beta_1^t + C_2 \beta_2^t,$$ 

with $\beta_{1,2} = \frac{-a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}$

where $C_1$ and $C_2$ are arbitrary real numbers.

Case: $a_1^2 - 4 a_2 = 0$

The general solution of the homogeneous difference equation

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

is given by

$$y_t = C_1 \beta^t + C_2 t \beta^t,$$ 

with $\beta = \frac{-a_1}{2}$

We can verify the validity of solution $t \beta^t$ by a simple (but tedious) straight-forward computation.

Linear Difference Equation of Second Order

A difference equation is an equation that contains the differences of second order of a sequence.

We restrict our interest to linear difference equations of second order with constant coefficients:

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = \delta$$

Example: $a_1^2 - 4 a_2 > 0$

Compute the general solution of difference equation

$$y_{t+2} - 3 y_{t+1} + 2 y_t = 0.$$ 

Characteristic equation

$$\beta^2 - 3 \beta + 2 = 0$$

has distinct real solutions

$$\beta_1 = 1 \quad \text{and} \quad \beta_2 = 2.$$ 

Thus the general solution of the homogeneous equation is given by

$$y_t = C_1 1^t + C_2 2^t.$$ 

Example: $a_1^2 - 4 a_2 = 0$

Compute the general solution of difference equation

$$y_{t+2} - 4 y_{t+1} + 4 y_t = 0.$$ 

Characteristic equation

$$\beta^2 - 4 \beta + 4 = 0$$

has the unique solution

$$\beta = 2.$$ 

Thus the general solution of the homogeneous equation is given by

$$y_t = C_1 2^t + C_2 t 2^t.$$
Case: $\frac{a_1^2}{4} - a_2 < 0$

In this case root $\sqrt{\frac{a_1^2}{4} - a_2}$ is a non-real (imaginary) number:

$$\beta_{1,2} = a \pm b i$$

where

- $a = -\frac{a_2}{a_1}$ is called the real part, and
- $b = \sqrt{a_1^2 - \frac{a_2^2}{4}}$ the imaginary part of root $\beta$.

Alternatively $\beta$ can be represented by so-called polar coordinates

$$\beta_{1,2} = r \cos(\theta) \pm i \sin(\theta)$$

where

- $r = |\beta| = \sqrt{a_1^2 - \frac{a_2^2}{4}}$ is called the modulus (or absolute value) of $\beta$, and
- $\theta = \arg(\beta)$ the argument of $\beta$.

Case: $\frac{a_1^2}{4} - a_2 < 0$

From the rules for complex numbers, one can derive purely real solutions of the homogeneous difference equation

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

given by

$$y_t = r^t \left[ C_1 \cos(\theta t) + C_2 \sin(\theta t) \right]$$

with $r = |\beta| = \sqrt{a_1^2 - \frac{a_2^2}{4}}$ and $\theta = \arg(\beta)$.

Argument $\arg(\beta)$ is given by

$$\cos \theta = \frac{a}{r} = \frac{-a_2}{\sqrt{a_1^2 - a_2^2}}$$
$$\sin \theta = \frac{b}{r} = \frac{\sqrt{a_1^2 - a_2^2}}{\sqrt{a_1^2 - a_2^2}}$$

Example: $\frac{a_1^2}{4} - a_2 < 0$

Complex root $\beta = a + b i$ with $a = -1$ and $b = \sqrt{3}$ has polar coordinates:

- $r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (\sqrt{3})^2} = 2$, and
- $\theta = \frac{\pi}{3}$, as $\sin \theta = \frac{b}{r} = \frac{\sqrt{3}}{2}$ and $\cos \theta = \frac{a}{r} = \frac{-1}{2}$.

Thus the general solution of the homogeneous equation is given by

$$y_t = 2^t \left[ C_1 \cos \left( \frac{2\pi}{3} t \right) + C_2 \sin \left( \frac{2\pi}{3} t \right) \right].$$

Argument $\theta$ can be computed by means of the arcus tangens function $\arctan(b/a)$.

A more convenient way is to use function atan2 which is available in programs like R.

Inhomogeneous Linear Difference Equation

By assumption all coefficients $a_1$, $a_2$, and $s$. So we may assume that

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = s$$

which implies

$$y_{t+1} = c = \frac{s}{1 + a_1 + a_2}$$

If $a_1 + a_2 \neq -1$ we may use $y_{t+1} = ct$ and get

$$y_{t+1} = \frac{s}{a_1 + a_2} t $$

if $a_1 + a_2 = -1$ and $a_1 \neq -2$.

Modulus and Argument

A complex number $z = a + b i$ can be interpreted as point $(a, b)$ in the (real) plane. This point can also be given by polar coordinates with radius $r = |z|$ (absolute value or modulus), and angle $\theta$ (called the argument of $z$).

We then have

$$r = |z| = \sqrt{a^2 + b^2}$$
$$\tan \theta = \frac{b}{a} \quad \text{because} \quad \cos \theta = \frac{a}{r} \quad \text{and} \quad \sin \theta = \frac{b}{r}.$$

Example: $\frac{a_1^2}{4} - a_2 < 0$

Compute the general solution of difference equation

$$y_{t+2} + 2 y_{t+1} + 4 y_t = s$$

Characteristic equation

$$\beta^2 + 2 \beta + 4 = 0$$

has the complex solutions

$$\beta_{1,2} = -1 \pm \sqrt{3} i$$

i.e., $a = -1$ and $b = \sqrt{3}$.

Inhomogeneous Linear Difference Equation

The general solution of inhomogeneous linear difference equation

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = s$$

can be written as

$$y_t = y_{t,1} + y_{t,2}$$

where

- $y_{t,1}$ is the general solution of the corresponding homogeneous equation $y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$, and
- $y_{t,2}$ is some particular solution of the inhomogeneous equation.

How can we find $y_{t,2}$?

Example – Inhomogeneous Equation

Compute the general solution of difference equation

$$y_{t+2} + 2 y_{t+1} + 4 y_t = 14 .$$

General solution of homogeneous equation $y_{t+2} + 2 y_{t+1} + 4 y_t = 0$:

$$y_{t,1} = 2^t \left[ C_1 \cos \left( \frac{2\pi}{3} t \right) + C_2 \sin \left( \frac{2\pi}{3} t \right) \right].$$

As $a_1 + a_2 = 2 + 4 \neq -1$ we use $y_{t,1} = \frac{14}{8} 2^t = 2$ and obtain the general solution of the inhomogeneous equation as

$$y_t = y_{t,1} + y_{t,2} = 2^t \left[ C_1 \cos \left( \frac{2\pi}{3} t \right) + C_2 \sin \left( \frac{2\pi}{3} t \right) \right] + 2.$$
Example – Inhomogeneous Equation

Compute the general solution of difference equation

\[ y_{t+2} - 3y_{t+1} + 2y_t = 2. \]

General solution of homogeneous equation \( y_{t+2} - 3y_{t+1} + 2y_t = 0: \)

\[ y_{h,t} = C_1 + C_22^t. \]

As \( a_1 + a_2 = -3 + 2 = -1 \) and \( a_1 \neq -2 \) we use \( y_{p,t} = \frac{2}{a_1 + a_2}t = -2t \)

and obtain the general solution of the inhomogeneous equation as

\[ y_t = y_{h,t} + y_{p,t} = C_1 + C_22^t - 2t. \]

Fixed Point of a Difference Equation

The inhomogeneous linear difference equation

\[ y_{t+2} + a_1 y_{t+1} + a_2 y_t = s \]

has the special constant solution (for \( a_1 + a_2 \neq -1 \))

\[ y_{p,t} = g = \frac{s}{1 + a_1 + a_2} \quad (\text{a constant}) \]

Point \( g \) is called fixed point, or equilibrium point of the difference equation.

Stable and Unstable Fixed Points

When we review general solutions of linear difference equations (with constant coefficients) we observe that these solutions converge to a fixed point \( \bar{y} \) for all choices of constants \( C \) if the absolute values of the roots \( \beta \) of the characteristic equation are less than one:

\[ y_t \rightarrow \bar{y} \quad \text{for} \quad t \rightarrow \infty \quad \text{if} \quad |\beta| < 1. \]

In this case \( \bar{y} \) is called an asymptotically stable fixed point.

Summary

▶ differences of sequences
▶ difference equation
▶ homogeneous and inhomogeneous linear difference equation of first order with constant coefficients
▶ cobweb model
▶ homogeneous and inhomogeneous linear difference equation of second order with constant coefficients
▶ stable and unstable fixed points