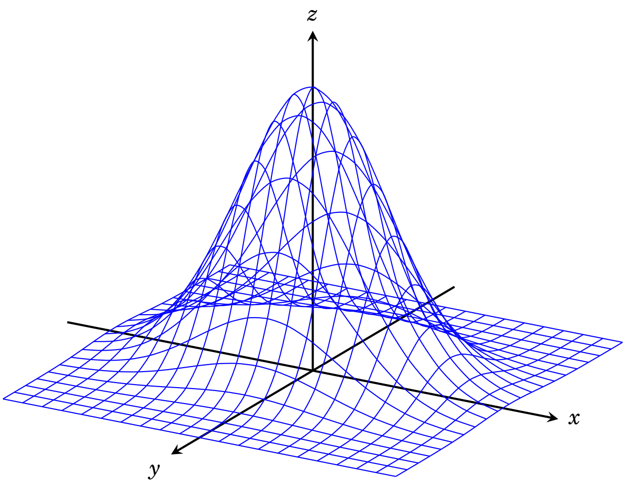
Foundations of Mathematics

Exercises and Problems
Winter Semester 2023/24

Josef Leydold



August 4, 2023

Institute for Statistics and Mathematics · WU Wien

Contents

Logic, Sets and Maps

1. We have the three statements

A = "Vienna is located at the Danube."

B = "Water freezes at 100° Celsius.'"

C = "16 is the square of 4.'"

Which of the following compound statements are true of false?

(a) $A \vee B$, (b) $A \wedge B$ (c) $(A \wedge B) \vee C$

(d) $\neg B \wedge C$ (e) $\neg(A \wedge C)$ (f) $(\neg A \wedge C) \vee (A \wedge \neg C)$

2. Transfer the following sentences (from everyday speech) into symbolic form. Use P for *"It is raining"* and Q for *"We have westerly winds"*, and the connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$.

(a) It is raining and we have westerly winds.

(b) If it is raining, then we have westerly winds.

(c) We have westerly winds if and only if it is raining.

(d) We do not have westerly winds or it is raining.

(e) It is wrong that we have westerly winds or that it is not raining.

3. Statement P reads *"x is a prime number"*, statement Q reads *"$x+1$ is a prime number'"*. Variable x is an arbitrary positive integer greater than 2. Are the statements

(a) $P \Rightarrow \neg Q$

(b) $P \Leftrightarrow \neg Q$

true of false for every value of x?

Remark:
Recall that a positive integer is a *prime number* if it is only divisible by 1 and itself.

4. Assume that statement $P \Rightarrow Q$ is true. What do you know about the truth values of the following statements? Are these always either true or false indpendently of the truth values of P and Q? Give examples.

(a) $Q \Rightarrow P$ (b) $\neg Q \Rightarrow P$

(c) $\neg Q \Rightarrow \neg P$ (d) $\neg P \Rightarrow \neg Q$

5. Compute the truth table for statement $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ and verify that it is an tautology (i.e., always true).

6. Verify that statement $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ (*contraposition*) is always true.

Remark:
This logical equivalence if often used in mathematical proofs. Instead of showing that "every number which is divisible by 6 is also divisible by 3" one can show that "every number which is not divisible by 3 is not divisible by 6, either".

7. The set $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$ has subsets $A = \{1,3,6,9\}$, $B = \{2,4,6,10\}$ and $C = \{3,6,7,9,10\}$. Draw the Venn diagram and give the following sets:

(a) $A \cup C$ (b) $A \cap B$ (c) $A \setminus C$

(d) $\overline{A}$ (e) $(A \cup C) \cap B$ (f) $(\overline{A} \cup B) \setminus C$

(g) $\overline{(A \cup C)} \cap B$ (h) $(\overline{A} \setminus B) \cap (\overline{A} \setminus C)$ (i) $(A \cap B) \cup (A \cap C)$

8. Mark the following set in the corresponding Venn diagram and simplify this set-theoretic expression:

$$(A \cap \overline{B}) \cup (A \cap B).$$

9. Simplify the following set-theoretic expressions:

(a) $\overline{(A \cup B)} \cap \overline{B}$

(b) $(A \cup \overline{B}) \cap (A \cup B)$

(c) $((\overline{A} \cup \overline{B}) \cap \overline{(A \cap \overline{B})}) \cap A$

(d) $(C \cup B) \cap \overline{(\overline{C} \cap \overline{B})} \cap (C \cup \overline{B})$

10. Which of the the following sets is a subset of $A = \{x \mid x \in \mathbb{R} \wedge 10 < x < 200\}$:

(a) $\{x \mid x \in \mathbb{R}$ and $10 < x \leq 200\}$

(b) $\{x \mid x \in \mathbb{R}$ and $x^2 = 121\}$

(c) $\{x \mid x \in \mathbb{R}$ and $4\pi < x < \sqrt{181}\}$

(d) $\{x \mid x \in \mathbb{R}$ and $20 < |x| < 100\}$

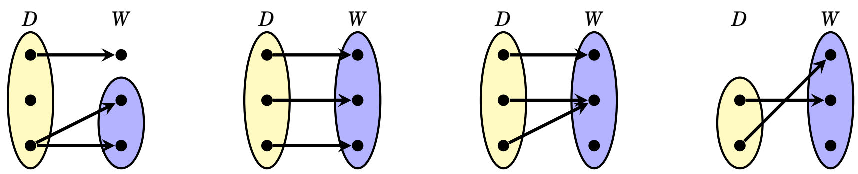
11. We are given map

$$\varphi\colon [0,\infty) \to \mathbb{R},\ x \mapsto y = x^\alpha \qquad \text{for some } \alpha > 0$$

What are

- function name,
- domain and codomain,
- image,
- range,
- function term,
- argument,
- independent and dependent variable?

12. Which of these diagrams represent maps? Which of these maps are one-to-one, onto, both or neither?



(a) (b) (c) (d)

13. Which of the following are proper definitions of mappings?

Which of the maps are one-to-one, onto, both or neither?

(a) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^2$

(b) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^{-2}$

(c) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto x^2$

(d) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto \sqrt{x}$

(e) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \sqrt{x}$

(f) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \{y \in [0,\infty)\colon x = y^2\}$

14. Let $\mathscr{P}_n = \{\sum_{i=0}^n a_i x^i : a_i \in \mathbb{R}\}$ be the set of all polynomials in x of degree less than or equal to n.

Which of the following are proper definitions of mappings?

Which of the maps are one-to-one, onto, both or neither?

(a) $D : \mathscr{P}_n \to \mathscr{P}_n,\ p(x) \mapsto \frac{dp(x)}{dx}$ (derivative of p)

(b) $D : \mathscr{P}_n \to \mathscr{P}_{n-1},\ p(x) \mapsto \frac{dp(x)}{dx}$

(c) $D : \mathscr{P}_n \to \mathscr{P}_{n-2},\ p(x) \mapsto \frac{dp(x)}{dx}$

Matrix Algebra

15. Let

$$\mathbf{A} = \begin{pmatrix} 1 & -6 & 5 \\ 2 & 1 & -3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 4 & 3 \\ 8 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$

Compute

(a) $\mathbf{A}+\mathbf{B}$ (b) $\mathbf{A}\cdot\mathbf{B}$ (c) $3\mathbf{A}^\top$ (d) $\mathbf{A}\cdot\mathbf{B}^\top$

(e) $\mathbf{B}^\top\cdot\mathbf{A}$ (f) $\mathbf{C}+\mathbf{A}$ (g) $\mathbf{C}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{B}$ (h) $\mathbf{C}^2$

16. Demonstrate by means of matrices $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ that matrix multiplication is not commutative in general, i.e., we may find $\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$.

17. Let $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$.

Compute $\mathbf{x}^\top\mathbf{x}$, $\mathbf{x}\mathbf{x}^\top$, $\mathbf{x}^\top\mathbf{y}$, $\mathbf{y}^\top\mathbf{x}$, $\mathbf{x}\mathbf{y}^\top$ and $\mathbf{y}\mathbf{x}^\top$.

18. Let $\mathbf{A}$ be a 3×2 matrix, $\mathbf{C}$ be a 4×3 matrix, and $\mathbf{B}$ a matrix, such that the multiplication $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$ is possible. How many rows and columns must $\mathbf{B}$ have? How many rows and columns does the product $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$ have?

19. Consider the products of

(a) two diagonal matrices,

(b) two upper triangle matrices,

(c) a diagonal matrix and a lower triangular matrix,

(d) an upper and a lower triangular matrix.

What do you know about the kind of matrix (diagonal matrix, triangular matrix, neither) of these products?

Illustrate your observations by examples.

Remark:
Finding good examples can be quite challenging. For example, the zero matrix $\mathbf{O}$ could be used for all these products. However, it always results in $\mathbf{O}$ which is not informative at all.

20. *(optional)***.** Show that the product of two diagonal matrices is again a diagonal matrix.

21. The product of two upper triangular matrices is again an upper triangular matrix.

How can this proposition be used to show that the product of a diagonal matrix and an upper triangular matrices is an upper triangular matrix?

22. Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ be a 2×3 matrix and $\mathbf{e}_i$ the i-th unit vector in $\mathbb{R}^3$. What is the result of $\mathbf{A}\cdot\mathbf{e}_i$? Illustrate your conjecture by means of an example.

23. *(optional)***.** Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be a $m\times n$ matrix and $\mathbf{e}_i$ the i-th unit vector in $\mathbb{R}^n$. Verify your conjecture from Problem 22.

24. Let $\mathbf{A}$ be a 3×2 matrix and $\mathbf{e}_i$ the i-th unit vector in $\mathbb{R}^3$. What is the result of $\mathbf{e}_i^\top\mathbf{A}$? Illustrate your conjecture by means of an example.

25. *(optional)*. Let $\mathbf{A} = (\mathbf{a}_1^\top, \ldots, \mathbf{a}_n^\top)^\top$ be an $m \times n$ matrix (observe that $\mathbf{a}_i^\top$ is the i-th row vector of $\mathbf{A}$) and $\mathbf{e}_i$ the i-th unit vector in $\mathbb{R}^m$.

Verify your conjecture from Problem 24. That is, show that $\mathbf{e}_i^\top \mathbf{A} = \mathbf{a}_i^\top$.

Hint: Observe that $\mathbf{e}_i^\top \mathbf{A} = ((\mathbf{e}_i^\top \mathbf{A})^\top)^\top = (\mathbf{A}^\top \mathbf{e}_i)^\top$.

26. Let $\mathbf{A}$ by an $n \times n$ square matrix and $\mathbf{D}$ an $n \times n$ diagonal matrix. What are the results of $\mathbf{D} \cdot \mathbf{A}$ and $\mathbf{A} \cdot \mathbf{D}$. That is, describe how you obtain the rows or columns in the product from the rows and columns in $\mathbf{A}$ and the diagonal entries in $\mathbf{D}$.

27. Compute $\mathbf{X}$. Assume that all matrices are square matrices and all required inverse matrices exist.

(a) $\mathbf{AX} + \mathbf{BX} = \mathbf{CX} + \mathbf{I}$

(b) $(\mathbf{A} - \mathbf{B})\mathbf{X} = -\mathbf{BX} + \mathbf{C}$

(c) $\mathbf{AXA}^{-1} = \mathbf{B}$

(d) $\mathbf{XAX}^{-1} = \mathbf{C}(\mathbf{XB})^{-1}$

28. Show that the product of two invertible matrices $\mathbf{A}$ and $\mathbf{B}$ is invertible and that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Hint: Recall that $(\mathbf{AB})^{-1}$ is defined as that matrix $\mathbf{C}$ with the property $\mathbf{C} \cdot (\mathbf{AB}) = \mathbf{I}$.

29. *(optional)*. Show that the inverse of an invertible matrix $\mathbf{A}$ is uniquely determined.

Remark:
The following trick can be used to verify uniqueness:
Assume that there are two inverse matrices $\mathbf{B}$ and $\mathbf{C}$ and show that $\mathbf{B} = \mathbf{C}$.

30. Compute norm and inner product of the following vectors:

(a) $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

(b) $\mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ -3 \\ 1 \\ 2 \end{pmatrix}$

31. *(optional)*. Show that $\mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$. Use the calculation rules for matrices: $(\mathbf{A} \cdot \mathbf{B})^\top = \mathbf{B}^\top \cdot \mathbf{A}^\top$ and $(\mathbf{A}^\top)^\top$.

Linear Equations

32. Solve the following system of equations by means of Gaussian elimination:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 2 \\ 4x_1 + 3x_2 + x_3 &= 10 \\ x_1 + 2x_2 + 4x_3 &= 5 \end{aligned}$$

Write this equation in matrix notation.

33. Solve the following system of equations by means of Gaussian elimination:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 + 3x_4 &= 10 \\ 3x_1 + 5x_2 + 2x_3 - x_4 &= 30 \\ x_1 + 2x_2 + x_3 - x_4 &= 12 \end{aligned}$$

Write this equation in matrix notation.

34. Solve the following system of equations by means of Gaussian elimination:

$$\begin{aligned} 2x_1 + 10x_2 + 4x_3 + 9x_4 &= 1 \\ x_1 + 6x_2 + 5x_3 + 3x_4 &= 1 \\ 3x_1 + 16x_2 + 9x_3 + 11x_4 &= -1 \\ x_1 + 5x_2 + 2x_3 + 5x_4 &= 2 \\ x_2 + 3x_3 &= 4 \end{aligned}$$

Write this equation in matrix notation.

35. Solve the following system of equations by means of Gaussian elimination:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 1 \\ x_1 + 4x_2 + 2x_3 + 8x_4 - 3x_5 &= 3 \\ -x_1 - 4x_3 + 3x_4 - 5x_5 &= -2 \end{aligned}$$

Write this equation in matrix notation.

36. Which of the following matrices are invertible?
Compute the inverse matrices.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & -5 & 2 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & -2 \end{pmatrix}$

37. Solve the linear equation $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$ with $\mathbf{A}$ from Problem 36(a) and $\mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$ by means of the inverse of $\mathbf{A}$.

38. We are given the linear equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}$ is a constant vector of length m and $\mathbf{x}$ the vector of unknowns of length n. What can you say about the solution set of this equation if the following information is known:

(a) $m < n$, $\mathbf{b} \neq 0$ (b) $m = n$, $\mathbf{b} \neq 0$ (c) $m > n$, $\mathbf{b} \neq 0$

(d) $m < n$, $\mathbf{b} = 0$ (e) $m = n$, $\mathbf{b} = 0$ (f) $m > n$, $\mathbf{b} = 0$

That is, can you conclude that the equation is inconsistent or has a unique or infinitely many solutions?

Vector Space

39. Give linear combinations of the two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

(a) $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (b) $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

40. Check whether the given vectors are linearly independent.

(a) $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$

(b) $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$

(c) $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{x}_4 = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

41. Compute the ranks of the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 3 & 5 & 2 & -1 \\ 1 & 2 & 1 & -1 \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} 2 & 3 \\ 4 & 3 \\ 1 & 2 \end{pmatrix}$

(e) $\mathbf{E} = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 3 & 2 \end{pmatrix}$

42. Illustrate inequality

$$\operatorname{rank}(\mathbf{A}\cdot\mathbf{B}) \le \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}$$

by means of matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & -5 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & -2 \end{pmatrix}.$$

43. Show that for matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$\operatorname{rank}(\mathbf{A}\cdot\mathbf{B}) < \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}\ .$$

What is the geometrical interpretation of these two matrices and their product?

44. (a) How many solutions does a *homogeneous* equation $\mathbf{A}\cdot\mathbf{x} = 0$ may have?

(b) Is it possible that $\operatorname{rank}(\mathbf{A}) > \operatorname{rank}(\mathbf{A}, \mathbf{b})$?

45. Are the following vectors linearly independent?
Which dimension does the linear span of these vectors have?
In which cases for these vectors a basis of $\mathbb{R}^3$?

(a) $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

(b) $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix}$

(c) $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from (b) and $\mathbf{v}_4 = \begin{pmatrix} -1 \\ 3 \\ 14 \end{pmatrix}$.

46. Compute the respective coordinate vectors of $\mathbf{x} = (2,0,1)^\top$, $\mathbf{y} = (1,1,4)^\top$, and zero vector 0 w.r.t. to the following bases:

(a) canonical basis

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) basis

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

47. Let $B_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$ and $B_2 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ be two bases of $\mathbb{R}^3$. (Verify this property!)

(a) Compute the transformation matrix for the change from basis B_1 to basis B_2.

(b) Let $\mathbf{c}_1(\mathbf{x}) = (1,2,3)^\top$ be the coordinate vector of $\mathbf{x}$ w.r.t. basis B_1. Compute coordinate vector $\mathbf{c}_2(\mathbf{x})$ w.r.t. basis B_2.

(c) Compute coordinate vector $\mathbf{c}_0(\mathbf{x})$ w.r.t. the canonical basis.

(d) Let $\mathbf{c}_2(\mathbf{y}) = (2,3,4)^\top$ be the coordinate vector of $\mathbf{y}$ w.r.t. basis B_2. Compute coordinate vector $\mathbf{c}_1(\mathbf{y})$ w.r.t. basis B_1.

48. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $\varphi_{\mathbf{A}}\colon \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$.
Compute dimensions of image $\operatorname{Im}(\varphi_{\mathbf{A}})$ and kernel $\operatorname{Ker}(\varphi_{\mathbf{A}})$ and give a basis for these subspaces.

49. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3, \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ be a linear map with $\mathbf{A}\mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{A}\mathbf{e}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

Determine $\mathbf{A}$?

50. Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n, \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ be a linear map. How may rows and columns must $\mathbf{A}$ have?

51. *(optional)*. Let $\mathbf{Ax} = \mathbf{b}$ be a (non-homogeneous) linear system of m equations in n unknowns with non-empty solution set L. Assume that $\mathbf{x}_0 \in L$ is a particular solution. Let $\mathscr{L}_0$ denote the solution set of the corresponding homogeneous linear equation $\mathbf{Ax} = 0$.

(a) Show that for every $\mathbf{x}, \mathbf{y} \in L$, the difference $\mathbf{z} = \mathbf{x} - \mathbf{y} \in \mathscr{L}_0$ (i.e., it is a solution of $\mathbf{Ax} = 0$).

(b) Show that $\mathscr{L}_0 = \{\mathbf{x} - \mathbf{x}_0 : \mathbf{x} \in L\}$. That is, show that every solution $\mathbf{y} \in \mathscr{L}_0$ of the homogeneous equation can be written as the difference of some solution $\mathbf{x} \in L$ of the inhomogeneous equation and $\mathbf{x}_0$.

(c) Show that $L = \mathbf{x}_0 + \mathscr{L}_0 = \{\mathbf{x}_0 + \mathbf{y} : \mathbf{y} \in \mathscr{L}_0\}$. This implies that the solution set can be represented by one particular solution and the basis of the nullspace (kernel) of $\mathbf{A}$.

52. Which of the following sets form vector spaces?

Give bases and dimensions for vector spaces.

(a) Set of all vectors with 3 real components.

(b) Set of all vectors with real components.

(c) Set of polynomials in x of degree less than or equal to 3.

(d) Set of polynomials in x of degree 3.

(e) Set of polynomials in x.

(f) Solution set of a homogeneous linear equation $\mathbf{Ax} = 0$.

(g) Solution set of an inhomogeneous linear equation $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{b} \neq 0$.

(h) Set of all vectors $\mathbf{y}$ such that $\mathbf{y} = \mathbf{Ax}$ for some $\mathbf{x}$.

(i) Set of all points $\mathbf{x} = (x_1, x_2, x_3)$ with $\|\mathbf{x}\| \leq 1$.

Determinant

53. Compute the following determinants by means of Sarrus' rule or by transforming into an upper triangular matrix:

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -3 \\ 0 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 1 & -4 \\ 2 & 1 & 4 \\ 3 & 4 & -4 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 1 \\ 4 & -3 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & 3 & -2 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

54. Compute the determinants from Exercise 53 by means of Laplace expansion.

55. (a) Estimate the ranks of the matrices from Exercise 53.

(b) Which of these matrices are regular?

(c) Which of these matrices are invertible?

(d) Are the column vectors of these matrices linear independent?

56. Let

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 2\times 1 & 0 \\ 0 & 2\times 1 & 0 \\ 1 & 2\times 0 & 1 \end{pmatrix} \quad \text{und} \quad \mathbf{C} = \begin{pmatrix} 3 & 5\times 3+1 & 0 \\ 0 & 5\times 0+1 & 0 \\ 1 & 5\times 1+0 & 1 \end{pmatrix}$$

Compute by means of the properties of determinants:

(a) $\det(\mathbf{A})$ (b) $\det(5\mathbf{A})$ (c) $\det(\mathbf{B})$ (d) $\det(\mathbf{A}^\mathsf{T})$

(e) $\det(\mathbf{C})$ (f) $\det(\mathbf{A}^{-1})$ (g) $\det(\mathbf{A}\cdot\mathbf{C})$ (h) $\det(\mathbf{I})$

57. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 & 1 & 1 \\ 1 & 7 & 8 & 5 & 8 \\ 1 & 1 & 4 & 3 & 1 \\ 1 & 2 & 2 & 6 & 1 \\ 1 & 3 & 9 & 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 3 & 4 & 1 & 1 \\ 4 & 7 & 8 & 5 & 8 \\ 4 & 1 & 4 & 3 & 1 \\ 4 & 2 & 2 & 6 & 1 \\ 4 & 3 & 9 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 4 & 4 & 1 & 1 \\ 1 & 8 & 8 & 5 & 8 \\ 1 & 2 & 4 & 3 & 1 \\ 1 & 3 & 2 & 6 & 1 \\ 1 & 4 & 9 & 1 & 2 \end{pmatrix}.$$

The determinant of $\mathbf{A}$ is given by $|\mathbf{A}| = -216$.

Compute by means of the properties of determinants:

(a) $|\mathbf{B}|$ (b) $|\mathbf{C}|$ (c) $|\mathbf{A}^\mathsf{T}|$ (d) $|\mathbf{C}^{-1}|$

(e) $|\mathbf{C}\cdot\mathbf{B}|$ (f) $|\mathbf{A}^{-1}\cdot\mathbf{C}|$ (g) $|(\mathbf{A}^{-1}\cdot\mathbf{C})^\mathsf{T}|$ (h) $|((\mathbf{A}\cdot\mathbf{C})^\mathsf{T})^{-1}|$

58. Let $\mathbf{A}$ and $\mathbf{B}$ be two 8×8 matrices with $\det(\mathbf{A}) = -5$ and $\det(\mathbf{B}) = 34$.

Compute:

(a) $\det(2\mathbf{A})$ (b) $\det(\mathbf{A}^2)$ (c) $\det(\mathbf{A}^\mathsf{T})$

(d) $\det(\mathbf{A}^{-1})$ (e) $\det(\mathbf{A}\cdot\mathbf{B})$

(f) What are the respective ranks of $\mathbf{A}$ and $\mathbf{B}$?

(g) Are these matrices invertible?

(h) Are these matrices singular?

(i) Are these matrices regular?

(j) Which respective dimensions have image and kernel of $\mathbf{A}$?

59. Let $\mathbf{A}$ be a 3×4 matrix. Estimate $\left|\mathbf{A}^\top \cdot \mathbf{A}\right|$ and $\left|\mathbf{A} \cdot \mathbf{A}^\top\right|$.

60. Compute area of the parallelogram and volume of the parallelepiped, respectively, which are created by the following vectors:

(a) $\begin{pmatrix}-2\\3\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$ (b) $\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}$

(c) $\begin{pmatrix}2\\1\\-4\end{pmatrix}, \begin{pmatrix}2\\1\\4\end{pmatrix}, \begin{pmatrix}3\\4\\-4\end{pmatrix}$ (d) $\begin{pmatrix}2\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\4\end{pmatrix}, \begin{pmatrix}-4\\4\\-4\end{pmatrix}$

61. Compute the matrix of cofactors, the adjugate matrix and the inverse of the following matrices:

(a) $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$ (b) $\begin{pmatrix}-2 & 3\\1 & 3\end{pmatrix}$ (c) $\begin{pmatrix}4 & -3\\0 & 2\end{pmatrix}$

(d) $\begin{pmatrix}3 & 1 & 1\\0 & 1 & 0\\0 & 2 & 1\end{pmatrix}$ (e) $\begin{pmatrix}2 & 1 & -4\\2 & 1 & 4\\3 & 4 & -4\end{pmatrix}$ (f) $\begin{pmatrix}0 & -2 & 1\\2 & 2 & 1\\4 & -3 & 3\end{pmatrix}$

62. Compute the inverse of the following matrices:

(a) $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ (b) $\begin{pmatrix}\alpha & \beta\\\alpha^2 & \beta^2\end{pmatrix}$

63. Solve linear equation

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

by means of Cramer's rule for $\mathbf{b} = (1,2)^\top$ and $\mathbf{b} = (1,2,3)^\top$, respectively, and the following matrices:

(a) $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$ (b) $\begin{pmatrix}-2 & 3\\1 & 3\end{pmatrix}$ (c) $\begin{pmatrix}4 & -3\\0 & 2\end{pmatrix}$

(d) $\begin{pmatrix}3 & 1 & 1\\0 & 1 & 0\\0 & 2 & 1\end{pmatrix}$ (e) $\begin{pmatrix}2 & 1 & -4\\2 & 1 & 4\\3 & 4 & -4\end{pmatrix}$ (f) $\begin{pmatrix}0 & -2 & 1\\2 & 2 & 1\\4 & -3 & 3\end{pmatrix}$

64. Solve the following linear equation by means of Cramer's rule:

(a) $\begin{pmatrix}a & b\\c & d\end{pmatrix} \cdot \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ (b) $\begin{pmatrix}\alpha & \beta\\\alpha^2 & \beta^2\end{pmatrix} \cdot \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

65. Show that the determinants of similar square matrices are equal.
Recall that two matrices $\mathbf{A}$ and $\mathbf{B}$ are similar if there exists an invertible matrix $\mathbf{U}$ such that $\mathbf{B} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$.

Eigenvalues

66. Compute eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & 13 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix}$

67. *(optional)***.** Compute eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -9 \end{pmatrix}$

(e) $\mathbf{E} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ (f) $\mathbf{F} = \begin{pmatrix} 11 & 4 & 14 \\ 4 & -1 & 10 \\ 14 & 10 & 8 \end{pmatrix}$

68. Compute eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

69. Estimate the definiteness of the matrices from Exercises 66a, 66c, 67a, 67d, 67f and 68a.

What can you say about the definiteness of the other matrices from Exercises 66, 67 and 68?

70. Show: Matrix $\mathbf{A}$ is negative definite if and only if $-\mathbf{A}$ is positive definite.

71. Let $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Give the quadratic form that is generated by $\mathbf{A}$.

72. Let $q(\mathbf{x}) = 5x_1^2 + 6x_1x_2 - 2x_1x_3 + x_2^2 - 4x_2x_3 + x_3^2$ be a quadratic form. Give its corresponding symmetric matrix $\mathbf{A}$.

73. Compute the eigenspace of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

74. Illustrate the following properties of eigenvalues (by means of examples, e.g. from Exercises 66–68):

(1) Quadratic matrices $\mathbf{A}$ and $\mathbf{A}^\top$ have the same spectrum.
(Do they have the same eigenvectors as well?)

(2) Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. Then $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ have the same eigenvalues.
(Do they have the same eigenvectors as well?)

(3) If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $\mathbf{A}^k$ corresponding to eigenvalue λ^k.

(4) If $\mathbf{x}$ is an eigenvector of regular matrix $\mathbf{A}$ corresponding to eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $\mathbf{A}^{-1}$ corresponding to eigenvalue λ^{-1}.

(5) The determinant of an $n \times n$ matrix $\mathbf{A}$ is equal to the product of all its eigenvalues: $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$.

(6) The trace of an $n \times n$ matrix $\mathbf{A}$ (i.e., the sum of its diagonal entries) is equal to the sum of all its eigenvalues: $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$.

75. *(optional)*. Show:

(1) Matrices $\mathbf{A}$ and $\mathbf{A}^{\mathsf{T}}$ have the same eigenvalues.
(Hint: use the characteristic polynomial.)

(4) If $\mathbf{x}$ is an eigenvector of regular matrix $\mathbf{A}$ corresponding to eigenvalue λ, then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ corresponding to eigenvalue λ^{-1}.
(Hint: use the eigenvalue equation, i.e., $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.)

76. A symmetric 2×2 matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 4$. Compute determinant and trace of $\mathbf{A}$. Which of the following statements are true, false or depend on the particular entries of $\mathbf{A}$.

(a) All entries of $\mathbf{A}$ are positive.

(b) At least one entry of $\mathbf{A}$ is positive.

(c) All diagonal entries of $\mathbf{A}$ are positive.

(d) All off-diagonal entries of $\mathbf{A}$ are positive.

(e) At least one off-diagonal entry of $\mathbf{A}$ is positive.

77. A 2×2 matrix $\mathbf{A}$ has $\det(\mathbf{A}) = -6$ and $\mathrm{tr}(\mathbf{A}) = -1$. Compute the eigenvalues of $\mathbf{A}$.

78. Compute all leading principle minors of the symmetric matrices from Exercises 66, 67 and 68 and determine their definiteness.

79. Compute all principle minors of the symmetric matrices from Exercises 66, 67 and 68 and determine their definiteness.

80. *(optional)*. Derive the conditions for negative definite matrices from the corresponding conditions for positive definite matrices.

81. *(optional)*. Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

with corresponding linear map $\varphi_{\mathbf{A}}$.

(a) Give a basis such that the corresponding map to $\varphi_{\mathbf{A}}$ becomes a diagonal matrix. Give this diagonal matrix $\mathbf{D}$.

(b) Give the transformation matrix that transforms a coordinate vector w.r.t. the basis in (a) into that w.r.t. the canonical basis.

(c) Let $\mathbf{x}$ be the coordinate basis w.r.t. the canoncial basis. What is the coordinate vector $\mathbf{c}$ w.r.t. basis in (a)?

Real Functions

82. Draw the graph of the following functions in interval $[-2,2]$.

(a) $f(x) = -x^4 + 2x^2$

(b) $f(x) = e^{-x^4+2x^2}$

(c) $f(x) = \dfrac{x-1}{|x-1|}$

(d) $f(x) = \sqrt{|1-x^2|}$

83. Draw (sketch) the graph of power function

$$f(x) = x^n$$

in interval $[0,2]$ for

$$n = -4, -2, -1, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4\,.$$

84. Draw the graphs of the following functions and determine whether these functions are one-to-one or onto (or both).

(a) $f\colon [-2,2] \to \mathbb{R},\, x \mapsto 2x+1$

(b) $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R},\, x \mapsto \frac{1}{x}$

(c) $f\colon \mathbb{R} \to \mathbb{R},\, x \mapsto x^3$

(d) $f\colon [2,6] \to \mathbb{R},\, x \mapsto (x-4)^2 - 1$

(e) $f\colon [2,6] \to [-1,3],\, x \mapsto (x-4)^2 - 1$

(f) $f\colon [4,8] \to [-1,15],\, x \mapsto (x-4)^2 - 1$

85. Let $f(x) = x^2 + 2x - 1$ and $g(x) = 1 + |x|^{\frac{3}{2}}$.
Compute

(a) $(f \circ g)(4)$

(b) $(f \circ g)(-9)$

(c) $(g \circ f)(0)$

(d) $(g \circ f)(-1)$

86. Determine $f \circ g$ and $g \circ f$.
What are the domains of f, g, $f \circ g$ and $g \circ f$?

(a) $f(x) = x^2, \quad g(x) = 1 + x$

(b) $f(x) = \sqrt{x} + 1, \quad g(x) = x^2$

(c) $f(x) = \frac{1}{x+1}, \quad g(x) = \sqrt{x} + 1$

(d) $f(x) = 2 + \sqrt{x}, \quad g(x) = (x-2)^2$

87. In a simplistic model we are given utility function U of a household w.r.t. two complementary goods (e.g. left and right shoes):

$$U(x_1, x_2) = \sqrt{\min\{x_1, x_2\}}, \quad x_1, x_2 \geq 0.$$

(a) Sketch the graph of U.

(b) Sketch the contour lines for $U = U_0 = 1$ and $U = U_1 = 2$.

88. Draw the following indifference curves:

(a) $x + y^2 - 1 = 0$

(b) $x^2 + y^2 - 1 = 0$

(c) $x^2 - y^2 - 1 = 0$

89. Sketch the graphs of the following paths:

(a) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

(b) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$

(c) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t\cos(2\pi t) \\ t\sin(2\pi t) \end{pmatrix}$

Limits

90. Compute the following limits:

(a) $\lim_{n\to\infty}\left(7+\left(\frac{1}{2}\right)^n\right)$ (b) $\lim_{n\to\infty}\left(\frac{2n^3-6n^2+3n-1}{7n^3-16}\right)$

(c) $\lim_{n\to\infty}\left(n^2-(-1)^n n^3\right)$ (d) $\lim_{n\to\infty}\left(\frac{n^2+1}{n+1}\right)$

(e) $\lim_{n\to\infty}\left(\frac{n \bmod 10}{(-2)^n}\right)$

91. Compute the following limits:

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$ (b) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ (c) $\lim_{n\to\infty}\left(1+\frac{1}{nx}\right)^n$

92. Draw the graph of function

$$f(x)=\begin{cases} -\frac{x^2}{2}, & \text{for } x\le -2,\\ x+1, & \text{for } -2<x<2,\\ \frac{x^2}{2}, & \text{for } x\ge 2.\end{cases}$$

and determine $\lim_{x\to x_0^+} f(x)$, $\lim_{x\to x_0^-} f(x)$, and $\lim_{x\to x_0} f(x)$ for $x_0=-2$, 0 and 2:

$\lim_{x\to -2^+} f(x)=$ $\lim_{x\to -2^-} f(x)=$ $\lim_{x\to -2} f(x)=$

$\lim_{x\to 0^+} f(x)=$ $\lim_{x\to 0^-} f(x)=$ $\lim_{x\to 0} f(x)=$

$\lim_{x\to 2^+} f(x)=$ $\lim_{x\to 2^-} f(x)=$ $\lim_{x\to 2} f(x)=$

Is function f continuous at these points?

93. Determine the left-handed limit $\lim_{x\to 1^-} f(x)$ and right-handed limits $\lim_{x\to 1^+} f(x)$ for the following functions.

(a) $f(x)=\begin{cases}1, & \text{for } x\ne 1,\\ 0, & \text{for } x=1.\end{cases}$ (b) $f(x)=\frac{x-1}{|x-1|}$

(c) $f(x)=\frac{|x^2-1|}{x-1}$ (d) $f(x)=\frac{x-1}{x^2-1}$

94. Determine

(a) $\lim_{x\to 1^+}\frac{x^{3/2}-1}{x^3-1}$ (b) $\lim_{x\to -2^-}\frac{\sqrt{|x^2-4|^2}}{x+2}$

(c) $\lim_{x\to 0^-}\lfloor x\rfloor$ (d) $\lim_{x\to 1^+}\frac{x-1}{\sqrt{x-1}}$

Remark: $\lfloor x\rfloor$ is the largest integer less than or equal to x.

95. Determine the following limits. Draw the graphs the corresponding functions

(a) $\lim_{x\to\infty}\frac{1}{x+1}$ (b) $\lim_{x\to 0} x^2$ (c) $\lim_{x\to\infty}\ln(x)$ (d) $\lim_{x\to 0}\ln|x|$ (e) $\lim_{x\to\infty}\frac{x+1}{x-1}$

96. Sketch the graph of

$$f(x)=x\sin(1/x)$$

and determine $\lim_{x\to 0} f(x)$.

97. Sketch the graphs of the following functions.
Which of these are continuous (on its domain)?

(a) $D = \mathbb{R}$, $f(x) = x$

(b) $D = \mathbb{R}$, $f(x) = 3x + 1$

(c) $D = \mathbb{R}$, $f(x) = e^{-x} - 1$

(d) $D = \mathbb{R}$, $f(x) = |x|$

(e) $D = \mathbb{R}^+$, $f(x) = \ln(x)$

(f) $D = \mathbb{R}$, $f(x) = \lfloor x \rfloor$

(g) $D = \mathbb{R}$, $f(x) = \begin{cases} 1, & \text{for } x \le 0, \\ x+1, & \text{for } 0 < x \le 2, \\ x^2, & \text{for } x > 2. \end{cases}$

Remark: $\lfloor x \rfloor$ is the largest integer less than or equal to x.

98. Sketch the graph of $f(x) = \frac{1}{x}$. Is it continuous?

99. Determine a value for h, such that function

$$f(x) = \begin{cases} x^2 + 2hx, & \text{for } x \le 2, \\ 3x - h, & \text{for } x > 2, \end{cases}$$

is continuous.

Derivatives

100. Draw (sketch) the graphs of the following functions.
At which points are these function differentiable?

(a) $f(x) = 2x + 2$
(b) $f(x) = 3$
(c) $f(x) = |x|$
(d) $f(x) = \sqrt{|x^2 - 1|}$
(e) $f(x) = \begin{cases} -\frac{1}{2}x^2, & \text{for } x \le -1, \\ x, & \text{for } -1 < x \le 1, \\ \frac{1}{2}x^2, & \text{for } x > 1. \end{cases}$
(f) $f(x) = \begin{cases} 2 + x, & \text{for } x \le -1, \\ x^2, & \text{for } x > -1. \end{cases}$

101. Compute the first and second derivative of the following functions:

(a) $f(x) = 4x^4 + 3x^3 - 2x^2 - 1$
(b) $f(x) = e^{-\frac{x^2}{2}}$
(c) $f(x) = \exp\left(-\frac{x^2}{2}\right)$
(d) $f(x) = \frac{x+1}{x-1}$

102. Compute the first and second derivative of the following functions:

(a) $f(x) = \frac{1}{1+x^2}$
(b) $f(x) = \frac{1}{(1+x)^2}$
(c) $f(x) = x\ln(x) - x + 1$
(d) $f(x) = \ln(|x|)$

103. Compute the first and second derivative of the following functions:

(a) $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$
(b) $f(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$
(c) $f(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$
(d) $f(x) = \cos(1 + x^2)$

104. Derive the quotient rule by means of product rule and chain rule.

105. Let $f(x) = \frac{\ln(x)}{x}$.
Compute $\Delta f = f(3.1) - f(3)$ approximately by means of the differential at point $x_0 = 3$.
Compare your approximation to the exact value.

106. Compute the regions where the following functions are elastic, 1-elastic and inelastic, resp.

(a) $g(x) = x^3 - 2x^2$
(b) $h(x) = \alpha x^\beta, \quad \alpha, \beta \ne 0$

107. Which of the following statements are correct?
Suppose function $y = f(x)$ is elastic in its domain.

(a) If x changes by one unit, then the change of y is greater than one unit.
(b) If x changes by one percent, then the relative change of y is greater than one percent.
(c) The relative rate of change of y is larger than the relative rate of change of x.
(d) The larger x is the larger will be y.

108. Compute the first and second order partial derivatives of

$$f(x,y) = \exp(x^2 + y^2)$$

at point $(0,0)$.

109. Compute the first and second order partial derivatives of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$
(b) $f(x,y) = x\,y$
(c) $f(x,y) = x^2 + y^2$
(d) $f(x,y) = x^2\,y^2$
(e) $f(x,y) = x^\alpha\,y^\beta, \quad \alpha,\beta > 0$

110. Compute the first and second order partial derivatives of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$
(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$
(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

111. Compute the gradients of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$
(b) $f(x,y) = x\,y$
(c) $f(x,y) = x^2 + y^2$
(d) $f(x,y) = x^2\,y^2$
(e) $f(x,y) = x^\alpha\,y^\beta, \quad \alpha,\beta > 0$

112. Compute the gradients of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$
(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$
(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

113. Compute the directional derivatives of the following functions along $\mathbf{h} = (1,2)^\mathsf{T}$ at point $(1,1)$

(a) $f(x,y) = x^2 + y^2$
(b) $f(x,y) = x^2\,y^3$
(c) $f(x,y) = \sqrt{x^2 + y^2}$
(d) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

114. Let $f(\mathbf{x}) = \sum_{i=1}^n x_i^2$. Compute the directional derivative at point $\mathbf{x}$ along $\mathbf{a}$ with $\|\mathbf{a}\| = 1$ by means of

(a) function $g(t) = f(\mathbf{x} + t\mathbf{a})$;
(b) the gradient of f;
(c) the chain rule for the Jacobian matrix.

115. Suppose a differentiable function $f(x,y)$ has in point $(0,0)$ its steepest ascent along direction $(1,3)^\mathsf{T}$ with corresponding directional derivative 4.

Compute the gradient f' in $(0,0)$.

116. A function $f(\mathbf{x})$ is called *homogeneous* of degree k, if $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$ for all $\alpha \in \mathbb{R}$.

(a) Give an example for a homogeneous function of degree 2.
(b) Show that all partial derivative of first order of a differentiable homogeneous function of degree $k \geq 1$ are homogeneous of degree $k - 1$.

117. Compute the Hessian matrix of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$
(b) $f(x,y) = xy$
(c) $f(x,y) = x^2 + y^2$
(d) $f(x,y) = x^2 y^2$
(e) $f(x,y) = x^\alpha y^\beta, \quad \alpha, \beta > 0$

118. Compute the Hessian matrix of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$
(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$
(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

119. Let

$$f(x,y) = x^2 + y^2 \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

Compute the derivative of the composite functions

(a) $h = f \circ \mathbf{g}$, and
(b) $\mathbf{p} = \mathbf{g} \circ f$

by means of the chain rule.

120. Let $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^3 - x_2 \\ x_1 - x_2^3 \end{pmatrix}$ and $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x_2^2 \\ x_1 \end{pmatrix}$.

Compute the derivatives of the composite functions

(a) $\mathbf{g} \circ \mathbf{f}$, and
(b) $\mathbf{f} \circ \mathbf{g}$

by means of the chain rule.

121. Let $Q(K,L,t)$ be a production function, where $L = L(t)$ and $K = K(t)$ are depend on time t. Compute the total derivative $\frac{dQ}{dt}$ by means of the chain rule.

122. Let $\mathbf{A}$ be an $n \times m$ matrix. What is the Jacobian matrix of function $\mathbf{f}\colon \mathbb{R}^m \to \mathbb{R}^n, \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$.

123. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. Compute the gradient of quadratic form $q_{\mathbf{A}}(\mathbf{x})$.

124. Compute the following limits:

(a) $\lim\limits_{x\to 4} \dfrac{x^2 - 2x - 8}{x^3 - 2x^2 - 11x + 12} =$
(b) $\lim\limits_{x\to -1} \dfrac{x^2 - 2x - 8}{x^3 - 2x^2 - 11x + 12} =$
(c) $\lim\limits_{x\to 2} \dfrac{x^3 - 5x^2 + 8x - 4}{x^3 - 3x^2 + 4} =$
(d) $\lim\limits_{x\to 0} \dfrac{1 - \cos(x)}{x^2} =$
(e) $\lim\limits_{x\to 0^+} x \ln(x) =$
(f) $\lim\limits_{x\to \infty} x \ln(x) =$

125. If we apply l'Hôpital's rule on the following limit we obtain

$$\lim_{x\to 1} \frac{x^3 + x^2 - x - 1}{x^2 - 1} = \lim_{x\to 1} \frac{3x^2 + 2x - 1}{2x} = \lim_{x\to 1} \frac{6x + 2}{2} = 4\,.$$

However, the correct value for the limit is 2.
Why does l'Hôpital's rule not work for this problem?
How do you get the correct value?

126. The Box-Cox transformation is a useful data transformation technique and is used to stabilize variance and make the data more normal distribution-like.

It is defined as

$$T^{(\lambda)}(x) = \begin{cases} \frac{x^\lambda - 1}{\lambda}, & \text{if } \lambda \neq 0, \\ \ln(x), & \text{if } \lambda = 0. \end{cases}$$

Show that for fixed x, function $f : \lambda \mapsto f(\lambda) = T^{(\lambda)}(x)$ is continuous in $\lambda = 0$.

Hint: Use l'Hôpital's rule.

Inverse and Implicit Functions

127. Let $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$$\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}) = \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_1 x_2 \\ x_1 x_2 \end{pmatrix}$$

(a) Compute the Jacobian matrix and determinant of $\mathbf{f}$.

(b) Around which points is $\mathbf{f}$ locally invertible?

(c) Compute the Jacobian matrix of the inverse function?

(d) Compute the inverse function (where it does exist).

128. Give a sufficient condition for f und g for which equation

$$u = f(x, y), \quad v = g(x, y)$$

can be solved locally w.r.t. to x and y.
Denote these solutions by $x = F(u, v)$ and $y = G(u, v)$ and compute $\frac{\partial F}{\partial u}$ and $\frac{\partial G}{\partial u}$.

129. Show that for following equations y is implicitly defined as a function of x in some interval around x_0. Compute $y'(x_0)$.

(a) $y^3 + y - x^3 = 0, \quad x_0 = 0$

(b) $x^2 + y + \sin(xy) = 0, \quad x_0 = 0$

130. Compute $\frac{dy}{dx}$ of the implicit function $x^2 + y^3 = 0$.
For which values of x exists the explicit function $y = f(x)$ locally?

131. Compute $\frac{dy}{dx}$ and $\frac{dx}{dy}$ of the indifference curve of utility function

$$u(x, y) = c\,x^\alpha\,y^\beta\,.$$

Assume that $x, y \geq 0$.

132. Compute derivative $\frac{dx_i}{dx_j}$ of the indifference curves of the following utility functions. Assume that $x_1, x_2 > 0$.

(a) $u(x_1, x_2) = \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)^2$

(b) $u(x_1, \ldots, x_n) = \left(\sum_{i=1}^n x_i^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \qquad \theta > 1$

Show by means of the implicit function theorem that x_1 is a strictly monotonically decreasing function of x_2.

133. Verify that the following implicit functions can be locally represented as $z = g(x, y)$ around the given points. Compute $\frac{\partial g}{\partial x}$ und $\frac{\partial g}{\partial y}$.

(a) $x^3 + y^3 + z^3 - xyz - 1 = 0, \quad (x_0, y_0, z_0) = (0, 0, 1)$

(b) $\exp(z) - z^2 - x^2 - y^2 = 0, \quad (x_0, y_0, z_0) = (1, 0, 0)$

134. Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ be locally invertible around $\mathbf{x}_0$. Show by means of the chain rule that

$$(\mathbf{f}^{-1})'(\mathbf{f}(\mathbf{x}_0)) = (\mathbf{f}'(\mathbf{x}_0))^{-1}$$

Hint: Apply the chain rule to the l.h.s. of equation $\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}_0)) = \mathrm{id}(\mathbf{x}_0)$. What is the derivative of $\mathrm{id}(\mathbf{x})$?

Taylor Series

135. Expand $f(x) = \frac{1}{2-x}$ into a Maclaurin polynomial of

(a) first order; (b) second order.

Draw the repective graphs of $f(x)$ and of these two Maclaurin polynomials in interval $[-3,5]$.
Give an estimate for the radius of convergence.

136. Expand $f(x) = (x+1)^{1/2}$ into the 3rd order Taylor polynomial around $x_0 = 0$.
Give an upper bound for the radius of convergence.

137. Expand $f(x) = \sin(x^{10})$ into a Maclaurin polynomial of degree 30.

138. Expand $f(x) = \sin(x^2 - 5)$ into a Maclaurin polynomial of degree 4.

139. Expand $f(x) = 1/(1+x^2)$ into a Maclaurin series.
Give an upper bound for the radius of convergence.

140. The density of the standard normal distribution is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

(a) Expand density $\phi(x)$ into a Maclaurin polynomial of 4-th order.

(b) Expand the distribution function of the standard normal distribution $\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt$ into a Maclaurin polynomial of 6-th order.

141. Consider the following market model:

$$\begin{aligned} q_s(p) &= \sqrt{p+1} && \text{(supply)} \\ q_d(p) &= \tfrac{5}{p} && \text{(demand)} \end{aligned}$$

where q_s, q_d and p with $q_s, q_d, p > 0$ are supply, demand, and price, resp.

Compute market equilibrium ($q_s = q_d$) by means of approximation by Taylor polynomials. First draw the graphs of these two functions and graphically find an integer estimate for the equilibrium point. Compute the Taylor series exansions of first order (i.e., linear approximation) for demand and supply function using your estimate as expansion point. Use these to get a better approximation of the market equilibrium.

142. Use Taylor polynomial of second order in Example 141 for approximating the equilibrium point.

143. Expand function $f(x,y) = e^{x^2+y^2}$ into a Taylor polynomial of order 2 around $\mathbf{x}_0 = (0,0)$.

Integration

144. Determine the antiderivatives of the following functions.

(a) x^3 (b) $\frac{3}{x^2}$ (c) $\sqrt{x^3}$ (d) $\frac{1}{\sqrt{x}}$

(e) e^{2x} (f) 2^{3x} (g) $\frac{1}{2x}$ (h) 5

(i) $\sin(\pi x)$ (j) $\cos(2\pi x)$

145. Determine the antiderivatives of the following functions.

(a) x^4+2x^2-x+3 (b) $x^3+7x+\frac{6}{x+1}$ (c) e^x+x^e+e+x

(d) $\frac{x+1}{\sqrt{x}}$ (e) $4x^3+3x^2+2x+1+\frac{1}{x}+\frac{1}{x^3}$

146. Compute the antiderivatives of the following functions by means of integration by parts.

(a) $f(x)=2x\,e^x$ (b) $f(x)=x^2\,e^{-x}$ (c) $f(x)=x\,\ln(x)$

(d) $f(x)=x^3\ln x$ (e) $f(x)=x(\ln(x))^2$ (f) $f(x)=x^2\sin(x)$

147. Compute the antiderivatives of the following functions by means of integration by substitution.

(a) $\int x\,e^{x^2}\,dx$ (b) $\int 2x\sqrt{x^2+6}\,dx$ (c) $\int \frac{x}{3x^2+4}\,dx$

(d) $\int x\sqrt{x+1}\,dx$ (e) $\int \frac{\ln(x)}{x}\,dx$ (f) $\int \frac{1}{x\ln x}\,dx$

(g) $\int \sqrt{x^3+1}\,x^2\,dx$ (h) $\int \frac{x}{\sqrt{5-x^2}}\,dx$ (i) $\int \frac{x^2-x+1}{x-3}\,dx$

(j) $\int x(x-8)^{\frac{1}{2}}\,dx$ (k) $\int 2^{3x}\,dx$

148. Compute the following definite integrals:

(a) $\int_1^4 2x^2-1\,dx$ (b) $\int_0^2 3e^x\,dx$ (c) $\int_1^4 3x^2+4x\,dx$

(d) $\int_0^{\frac{\pi}{3}} \frac{-\sin(x)}{3}\,dx$ (e) $\int_0^1 \frac{3x+2}{3x^2+4x+1}\,dx$

149. Compute the following definite integrals by means of antiderivatives:

(a) $\int_1^e \frac{\ln x}{x}\,dx$ (b) $\int_0^1 x(x^2+3)^4\,dx$ (c) $\int_0^2 x\sqrt{4-x^2}\,dx$

(d) $\int_1^2 \frac{x}{x^2+1}\,dx$ (e) $\int_0^2 x\exp\left(-\frac{x^2}{2}\right)dx$ (f) $\int_0^3 (x-1)^2\,x\,dx$

(g) $\int_0^1 x\exp(x)\,dx$ (h) $\int_0^2 x^2\exp(x)\,dx$ (i) $\int_1^2 x^2\ln x\,dx$

150. The marginal costs for a cost function $C(x)$ are given by $30-0.05x$. Reconstruct $C(x)$ when the fixed costs are €2000.

151. Compute the following improper integrals.

(a) $\int_0^\infty -e^{-3x}\,dx$ (b) $\int_0^1 \frac{2}{\sqrt[4]{x^3}}\,dx$ (c) $\int_0^\infty \frac{x}{x^2+1}\,dx$

152. Compute the following improper integrals.

(a) $\displaystyle\int_0^\infty \frac{1}{(1+x)^2}\,dx$ (b) $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$

(c) $\displaystyle\int_0^\infty x e^{-\frac{x^2}{2}}\,dx$ (d) $\displaystyle\int_1^\infty \frac{\ln(x)}{(1+x)^2}\,dx$

Hint for (d): Use integration by parts and note that $\frac{1}{x}\cdot\frac{1}{1+x} = \frac{1}{x} - \frac{1}{1+x}$.

153. Do the following improper integrals exist? Compute them.

(a) $\displaystyle\int_2^\infty \frac{1}{x\ln(x)}\,dx$ (b) $\displaystyle\int_1^2 \frac{1}{x\ln(x)}\,dx$ (c) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

(d) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ (e) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (f) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

154. *(optional)*. For which value of $\alpha \in \mathbb{R}$ do the following improper integrals converge? What are their values?

(a) $\displaystyle\int_0^1 x^\alpha\,dx$ (b) $\displaystyle\int_1^\infty x^\alpha\,dx$ (c) $\displaystyle\int_0^\infty x^\alpha\,dx$

155. The expectation of a continuous random variate X with density f is defined as

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx\,.$$

Compute the expectation of a so called *half-normal* distributed random variate which has domain $[0,\infty)$ and probability density function

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\exp\left(-\frac{x^2}{2}\right) & \text{for } x \geq 0\,, \\ 0 & \text{for } x < 0\,. \end{cases}$$

156. *(optional)*. Compute the expectation of a normal distributed random variate with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right).$$

Hint: $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \lim_{t\to-\infty}\int_t^0 x f(x)\,dx + \lim_{s\to\infty}\int_0^s x f(x)\,dx$.

157. *(optional)*. Let X be a so called *Cauchy* distributed random variate with probability density function

$$f(x) = \frac{1}{\pi(1+x^2)}\,.$$

Show that X does not have an expectation.

Why is the following approach incorrect?

$$\mathrm{E}(X) = \lim_{t\to\infty}\int_{-t}^{t} \frac{x}{\pi(1+x^2)}\,dx = \lim_{t\to\infty} 0 = 0\,.$$

158. *(optional)***.** Let f be the probability density function of some absolutely continuous distributed random variate X. The *moment generating function* of f is defined as

$$M(t) = \mathrm{E}\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx\,.$$

Show that $M'(0) = \mathrm{E}(X)$, i.e., the expectation of X.

Hint: Use Leibniz's formula.

159. *(optional)***.** The gamma function $\Gamma(z)$ is an extension of the factorial function. That is, if n is a positive integer, then

$$\Gamma(n) = (n-1)!$$

For positive real numbers z it is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\,dt\,.$$

Use integration by parts and show that

$$\Gamma(z+1) = z\,\Gamma(z)\,.$$

160. Compute the following double integrals.

(a) $\displaystyle\int_0^2 \int_0^1 (2x+3y+4)\,dx\,dy$

(b) $\displaystyle\int_0^a \int_0^b (x-a)(y-b)\,dy\,dx$

(c) $\displaystyle\int_0^{1/2} \int_0^{2\pi} y^3 \sin(xy^2)\,dx\,dy$

161. Compute $\displaystyle\int_{-2}^{2} x^2 f(x)\,dx$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$

162. Compute $F(x) = \displaystyle\int_{-2}^{x} f(t)\,dt$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$

Convex and Concave

163. Determine whether the following functions are concave or convex (or neither).

(a) $\exp(x)$
(b) $\sin(x)$
(c) $\ln(x)$
(d) $\log_{10}(x)$
(e) $x^3 - x$
(f) x^α for $x > 0$ for an $\alpha \in \mathbb{R}$.

164. In which region is function

$$f(x) = x^3 - 3x^2 - 9x + 19$$

monotonically increasing or decreasing?
In which region is it convex or concave?

165. In which region the following functions monotonically increasing or decreasing?
In which region is it convex or concave?

(a) $f(x) = x\,e^{x^2}$
(b) $f(x) = e^{-x^2}$
(c) $f(x) = \dfrac{1}{x^2+1}$

166. Function

$$f(x) = b\,x^{1-a}\,, \qquad 0 < a < 1, b > 0, x \ge 0$$

is an example of a production function.
Production functions usually have the following properties:

(1) $f(0) = 0, \quad \lim_{x\to\infty} f(x) = \infty$
(2) $f'(x) > 0, \quad \lim_{x\to\infty} f'(x) = 0$
(3) $f''(x) < 0$

(a) Verify these properties for the given function.
(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$.
(Use appropriate values for a and b.)
(c) What is the economic interpretation of these properties?

167. Function

$$f(x) = b\,\ln(ax+1)\,, \qquad a, b > 0, x \ge 0$$

is an example of a utility function.
Utility functions have the same properties as production functions.

(a) Verify the properties from Problem 166.
(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$.
(Use appropriate values for a and b.)
(c) What is the economic interpretation of these properties?

168. Show:
If $f(x)$ is a two times differentiable concave function, then $g(x) = -f(x)$ convex.

169. *(optional)*. Show:

If $f(x)$ is a concave function, then $g(x) = -f(x)$ convex.

You may not assume that f is differentiable.

170. *(optional)*. Let $f(x)$ and $g(x)$ be two differentiable concave functions.

Show that

$$h(x) = \alpha f(x) + \beta g(x), \qquad \text{for } \alpha, \beta > 0,$$

is a concave function.

What happens, if $\alpha > 0$ and $\beta < 0$?

171. Which of the following functions are strictly convex, convex, strictly concave, or concave?

(a) $f(x, y) = x^2 - 2xy + 2y^2 + 4x - 8$

(b) $g(x, y) = 2x^2 - 3xy + y^2 + 2x - 4y - 2$

(c) $h(x, y) = -x^2 + 4xy - 4y^2 + 1$

Extrema

172. Compute all local and global extremal points of the functions

(a) $f(x) = (x-3)^6$

(b) $g(x) = \frac{x^2+1}{x}$

173. Compute the local and global extremal points of the functions

(a) $f: (0,\infty) \to \mathbb{R}, x \mapsto \frac{1}{x} + x$

(b) $f: [0,\infty) \to \mathbb{R}, x \mapsto \sqrt{x} - x$

(c) $g: \mathbb{R} \to \mathbb{R}, x \mapsto e^{-2x} + 2x$

174. The profit of a company for given price p and wage w is

$$\pi(x) = p \cdot f(x) - w \cdot x\,.$$

Let $f(x) = 4x^{\frac{1}{2}}$ be the production function from Exercise 166 with $a = \frac{1}{2}$ and $b = 4$.

(a) Draw $\pi(x)$ and $\pi'(x)$ for $p = 1$ und $w = 1$.

(b) Estimate from your graphs the optimal production that maximizes the profit $\pi(x)$.

(c) Solve the optimization problem analytically.

(d) What happens if wages are doubled, $w = 2$? (Drawing, computation of maximum)

175. Compute all local extremal points and saddle points of the following functions. Are the local extremal points also globally extremal.

(a) $f(x,y) = -x^2 + xy + y^2$

(b) $f(x,y) = \frac{1}{x}\ln(x) - y^2 + 1$

(c) $f(x,y) = 100(y-x^2)^2 + (1-x)^2$

(d) $f(x,y) = 3x + 4y - e^x - e^y$

176. Compute all local extremal points and saddle points of the following functions. Are the local extremal points also globally extremal.

$$f(x_1,x_2,x_3) = (x_1^3 - x_1)x_2 + x_3^2\,.$$

Is this function convex or concave?

177. Let

$$f(x_1,x_2) = 3x_1 + 4x_2 - e^{x_1} - e^{x_2}$$

(a) Compute all local extrema of f.

(b) Determine whether f is (strictly) convex or concave.

(c) Compute all global extrema of f.

Lagrange Function

178. We are given the following constraint optimization problem

$$\max(\min) \quad f(x,y) = x^2 y \quad \text{subject to} \quad x + y = 3.$$

(a) Solve the problem graphically.

(b) Compute all stationary points.

(c) Use the bordered Hessian to determine whether these stationary points are (local) maxima or minima.

179. *(optional)*. Compute all stationary points of the constraint optimization problem

$$\max\ (\min) \quad f(x_1,x_2,x_3) = \frac{1}{3}(x_1-3)^3 + x_2 x_3$$
$$\text{subject to} \quad x_1 + x_2 = 4 \text{ and } x_1 + x_3 = 5.$$

180. A household wants to minize its expenditure E for a given utility level $U = U_0$. Let $U = U_0 = 1 = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ and $E = 4x_1 + x_2$. Compute the optimal consumptions x_1 and x_2 of two goods.

181. A household has an income m and can buy two commodities with prices p_1 and p_2. We have

$$p_1 x_1 + p_2 x_2 = m$$

where x_1 and x_2 denote the quantities. Assume that the household has a utility function

$$u(x_1,x_2) = \alpha \ln(x_1) + (1-\alpha)\ln(x_2)$$

where $\alpha \in (0,1)$.

(a) Solve this constraint optimization problem.

(b) Compute the change of the optimal utility function when the price of commodity 1 changes.

(c) Compute the change of the optimal utility function when the income m changes.

Kuhn-Tucker Condition

182. Can the Kuhn-Tucker Theorem be applied ti the following problems?

(a) $\max x_1 \qquad \text{subject to} x_1^2 + x_2^2 \leq 1, \quad x_1, x_2 \geq 0$

(b) $\max -(x_1 - 3)^2 + (x_2 - 4)^2 \qquad \text{subject to} \qquad x_1 + x_2 \geq 4, \quad x_1, x_2 \geq 0$

(c) $\max 2x_1 + x_2 \qquad \text{subject to} \qquad x_1^2 - 4x_1 + x_2^2 \geq 0, \quad x_1, x_2 \geq 0$

183. Solve the following optimization problem graphically:

$$\max f(x, y) = -(x-2)^2 - y \qquad \text{subject to} \qquad x + y \leq 1, \qquad x, y \geq 0$$

184. Solve the optimization problem from Exercise 183 numerically.

Solutions

1. (a) T; (b) F; (c) T; (d) T; (e) F; (f) F.

2. (a) $P \wedge Q$; (b) $P \Rightarrow Q$; (c) $Q \Leftrightarrow P$; (d) $\neg Q \vee P$; (e) $\neg(Q \vee \neg P)$.

3. (a) true; (b) false.

4. (c) is always true; the logical values of (a),(b) and (d) are unknown and depend on the particular truth values of P and Q.

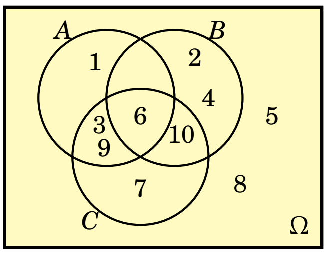
5.

P	Q	$\neg P$	$(\neg P \vee Q)$	$(P \Rightarrow Q)$	$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

6.

P	Q	$\neg P$	$\neg Q$	$(\neg Q \Rightarrow \neg P)$	$(P \Rightarrow Q)$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

7. (a) $\{1,3,6,7,9,10\}$; (b) $\{6\}$; (c) $\{1\}$; (d) $\{2,4,5,7,8,10\}$; (e) $\{6,10\}$; (f) $\{2,4,5,8\}$; (g) $\{2,4\}$; (h) $\{5,8\}$; (i) $\{3,6,9\}$.



8. A.

9. (a) $\overline{A} \cap \overline{B}$; (b) A; (c) $\emptyset$; (d) C.

10. (a) no subset; (b) no subset, set equals $\{-11, 11\}$; (c) subset; (d) no subset.

11. function: φ; domain: $[0,\infty)$; codomain: $\mathbb{R}$; image (range): $[0,\infty)$; function term: $\varphi(x) = x^\alpha$; independent variable (argument): x; dependent variable: y.

12. (a) no map; (b) bijective map; (c) map, neither one-to-one nor onto; (d) one-to-one map, not onto.

13. (a) one-to-one; (b) no map, as 0^{-2} is not defined; (c) bijective; (d) one-to-one, but not onto; (e) bijective; (f) no map, as $\{y \in [0,\infty) : x = y^2\}$ is a set and not an element of $\mathbb{R}$.

14. (a) map, neither one-to-one ($5' = 3' = 0$) nor onto; (b) map, onto but not one-to-one; (c) no map, as $(x^n)' = nx^{n-1} \notin \mathcal{P}_{n-2}$.

15. (a) $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 2 & -2 & 8 \\ 10 & 1 & -1 \end{pmatrix}$; (b) not possible since the number of columns of $\mathbf{A}$ does not coincide with the number of rows of $\mathbf{B}$; (c) $3\mathbf{A}^\top = \begin{pmatrix} 3 & 6 \\ -18 & 3 \\ 15 & -9 \end{pmatrix}$; (d) $\mathbf{A}\cdot\mathbf{B}^\top = \begin{pmatrix} -8 & 18 \\ -3 & 10 \end{pmatrix}$; (e) $\mathbf{B}^\top\cdot\mathbf{A} = \begin{pmatrix} 17 & 2 & -19 \\ 4 & -24 & 20 \\ 7 & -16 & 9 \end{pmatrix}$; (f) not possible; (g) $\mathbf{C}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{B} = \mathbf{C}\cdot(\mathbf{A}+\mathbf{B}) = \begin{pmatrix} -8 & -3 & 9 \\ 22 & 0 & 6 \end{pmatrix}$; (h) $\mathbf{C}^2 = \mathbf{C}\cdot\mathbf{C} = \begin{pmatrix} 0 & -3 \\ 3 & 3 \end{pmatrix}$.

16. $\mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix} \neq \mathbf{B}\cdot\mathbf{A} = \begin{pmatrix} 5 & 1 \\ -1 & 1 \end{pmatrix}$.

17. $\mathbf{x}^\top\mathbf{x} = 21$; $\mathbf{x}\mathbf{x}^\top = \begin{pmatrix} 1 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -8 & 16 \end{pmatrix}$; $\mathbf{x}^\top\mathbf{y} = -1$; $\mathbf{y}^\top\mathbf{x} = -1$;
$\mathbf{x}\mathbf{y}^\top = \begin{pmatrix} -3 & -1 & 0 \\ 6 & 2 & 0 \\ -12 & -4 & 0 \end{pmatrix}$; $\mathbf{y}\mathbf{x}^\top = \begin{pmatrix} -3 & 6 & -12 \\ -1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}$.

18. $\mathbf{B}$ must be a 2×4 matrix. $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$ is then a 3×3 matrix.

19. (a) diagonal matrix; (b) upper triangular matrix; (c) lower triangular matrix; (d) can be any square matrix.

20. Let $\mathbf{A}$ and $\mathbf{B}$ be two diagonal matrices. Thus $[A]_{ij} = a_{ij} = 0$ and $[B]_{ij} = b_{ij} = 0$ if $i \neq j$. By means of the Kronecker delta this also can be written as $[A]_{ij} = a_{ij}\delta_{ij}$ and $[B]_{ij} = b_{ij}\delta_{ij}$. We then find $[\mathbf{AB}]_{ij} = \sum_{s=1}^n a_{is}b_{sj} = \sum_{s=1}^n a_{is}b_{sj}\delta_{is}\delta_{sj}$. Observe that $\delta_{is}\delta_{sj} = 0$ whenever $i \neq s$ or $s \neq j$ which implies that $\delta_{is}\delta_{sj} = 0$ for all s whenever $i \neq j$. Thus $[\mathbf{AB}]_{ij} = 0$ for $i \neq j$, i.e., $\mathbf{AB}$ is a diagonal matrix.

21. Every diagonal matrix is an upper triangular matrix. So this product is an upper triangular matrix, too.

22. $\mathbf{A}\mathbf{e}_i = \mathbf{a}_i$, i.e., we obtain the i-th column $\mathbf{a}_i$ of $\mathbf{A}$.

23. We show that $\mathbf{A}\mathbf{e}_i = \mathbf{a}_i$: $[\mathbf{A}\mathbf{e}_i]_{j1} = \sum_{s=1}^n [\mathbf{A}]_{js}[\mathbf{e}_i]_{s1} = [\mathbf{A}]_{ji}$ as $[\mathbf{e}_i]_{s1} = \delta_{is}$. However, for fixed i, $[\mathbf{A}]_{ji}$ is the j-th component of column vector $\mathbf{a}_i$.

24. We obtain the i-th row vector of $\mathbf{A}$.

25. Solution 1: Analogously to the solution of Problem 23.
Solution 2: Use the proposition from Problem 23. $\mathbf{e}_i^\top\mathbf{A} = ((\mathbf{e}_i^\top\mathbf{A})^\top)^\top = (\mathbf{A}^\top\mathbf{e}_i)^\top = (\mathbf{a}_i)^\top = \mathbf{a}_i^\top$.

26. We obtain the i-th row of $\mathbf{D}\cdot\mathbf{A}$ by multiplying the i-th row of $\mathbf{A}$ by d_{ii}. We obtain the j-th column of $\mathbf{A}\cdot\mathbf{D}$ by multiplying the j-th columns of $\mathbf{A}$ by d_{jj}.

27. (a) $\mathbf{X} = (\mathbf{A}+\mathbf{B}-\mathbf{C})^{-1}$; (b) $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}$; (c) $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}$; (d) $\mathbf{X} = \mathbf{C}\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{C}(\mathbf{AB})^{-1}$.

28. $(\mathbf{B}^{-1}\mathbf{A}^{-1})\cdot(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\cdot\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{I}\mathbf{B} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$. Thus by the definition of the inverse matrix, $(\mathbf{B}^{-1}\mathbf{A}^{-1})$ is the inverse of $(\mathbf{AB})$.

29. Suppose that $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$, i.e., $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$. Thus we find $\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}$.

30. (a) $\|\mathbf{x}\| = \sqrt{21}$, $\|\mathbf{y}\| = \sqrt{10}$, $\mathbf{x}^\top\mathbf{y} = -1$; (b) $\|\mathbf{x}\| = \sqrt{14}$, $\|\mathbf{y}\| = \sqrt{15}$, $\mathbf{x}^\top\mathbf{y} = 10$.

31. Let $a = \mathbf{x}^\top\mathbf{y}$. As $a \in \mathbb{R} = \mathbb{R}^{1\times 1}$ (1×1 matrix) we trivially have $a^\top = a$. Therefore $\mathbf{x}^\top\mathbf{y} = a = a^\top = (\mathbf{x}^\top\mathbf{y})^\top = \mathbf{y}^\top(\mathbf{x}^\top)^\top = \mathbf{y}^\top\mathbf{x}$, as claimed.

32. $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 10 \\ 5 \end{pmatrix}$, solution: $x_1 = 13$, $x_2 = -16$, $x_3 = 6$; or in vector notation $\mathbf{x} = (13, -16, 6)^\top$.

33. $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 3 & 5 & 2 & -1 \\ 1 & 2 & 1 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 10 \\ 30 \\ 12 \end{pmatrix}$,

solution set $L = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ -2 \\ 0 \end{pmatrix} + \alpha \cdot \begin{pmatrix} -4 \\ 3 \\ -1 \\ 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$.

Remark: The representation of L is not unique.

34. $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 2 & 10 & 4 & 9 \\ 1 & 6 & 5 & 3 \\ 3 & 16 & 9 & 11 \\ 1 & 5 & 2 & 5 \\ 0 & 1 & 3 & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \\ 4 \end{pmatrix}$,

The system of equations is inconsistent, $L = \emptyset$.

35. $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 1 & 4 & 2 & 8 & -3 \\ -1 & 0 & -4 & 3 & -5 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$,

solution set: $L = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \alpha_1 \cdot \begin{pmatrix} -3 \\ \frac{1}{6} \\ 0 \\ \frac{2}{3} \\ 1 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} -4 \\ \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} \,\middle|\, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$.

Remark: The representation of L is not unique.

36. (a) invertible, $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -19 & -3 \\ 0 & 1 & 0 \\ -1 & 12 & 2 \end{pmatrix}$; (b) not invertible ($\Leftrightarrow$ singular), the inverse matrix $\mathbf{B}^{-1}$ does not exist.

37. $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \begin{pmatrix} 72 \\ -4 \\ -45 \end{pmatrix}$.

38. If $\mathbf{b} = 0$, then equation $\mathbf{Ax} =$ is called **homogeneous**nd has at least one solution a
$\mathbf{x} = 0$. If $m < n$, then the solution cannot be unique. No further conclusions can be drawn in any of these cases.

39. For example: (a) $2\mathbf{x}_1 + 0\mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$; (b) $3\mathbf{x}_1 - 2\mathbf{x}_2 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$.

40. (a) linearly independent; (b) linearly independent; (c) linearly dependent (At most three vectors with three components can be linearly independent. Further computation is not required.)

41. (a) $\text{rank}(\mathbf{A}) = 3$; (b) $\text{rank}(\mathbf{B}) = 3$; (c) $\text{rank}(\mathbf{C}) = 3$; (d) $\text{rank}(\mathbf{D}) = 2$, (e) $\text{rank}(\mathbf{E}) = \text{rank}(\mathbf{D}^\mathsf{T}) = 2$.

42. $\text{rank}(\mathbf{A}) = 3$, $\text{rank}(\mathbf{B}) = 2$, $\text{rank}(\mathbf{A}\cdot\mathbf{B}) = \text{rank}\begin{pmatrix} 14 & 8 & 2 \\ 3 & 2 & 1 \\ -10 & -8 & -6 \end{pmatrix} = 2$.

43. $\text{rank}(\mathbf{A}) = 2$, $\text{rank}(\mathbf{B}) = 2$, $\text{rank}(\mathbf{A}\cdot\mathbf{B}) = \text{rank}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1$.

44. (a) at least one solution (zero vector $\mathbf{0}$); (b) no (if we augment a matrix by a column vector the rank cannot decrease.

45. (a) $\dim = 2$; (b) $\dim = 3$; (c) $\dim = 3$. Only vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from (b) for a basis.

46. (a) $\mathbf{c}(\mathbf{x}) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{c}(\mathbf{y}) = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{c}(\mathbf{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; (b) $\mathbf{c}(\mathbf{x}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{c}(\mathbf{y}) = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{c}(\mathbf{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

47. Let $\mathbf{V} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ und $\mathbf{W} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.
(a) $\mathbf{U} = \mathbf{W}^{-1}\cdot\mathbf{V} = \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; (b) $\mathbf{c}_2(\mathbf{x}) = \mathbf{U}\mathbf{c}_1(\mathbf{x}) = (-10,5,3)^\mathsf{T}$; (c) $\mathbf{c}_0(\mathbf{x}) = \mathbf{V}\mathbf{c}_1(\mathbf{x}) = (2,3,8)^\mathsf{T}$; (d) $\mathbf{c}_1(\mathbf{y}) = \mathbf{U}^{-1}\mathbf{c}_2(\mathbf{y}) = (13,-1,4)^\mathsf{T}$.

48. $\dim(\text{Im}(\varphi_\mathbf{A})) = \text{rank}(\mathbf{A}) = 2$. $\dim(\text{Ker}(\varphi_\mathbf{A})) = 3 - \text{rank}(\mathbf{A}) = 1$. Example of a basis of $\text{Im}(\varphi_\mathbf{A})$: two linearly independent columns of $\mathbf{A}$; example of a basis of $\text{Ker}(\varphi_\mathbf{A})$: $(1,-2,1)^\mathsf{T}$.

49. $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

50. $\mathbf{A}$ is an $n \times m$ matrix.

51. (a) $\mathbf{A}\mathbf{z} = \mathbf{A}(\mathbf{x}-\mathbf{y}) = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{y} = \mathbf{b} - \mathbf{b} = 0$.
(b) Let $\mathbf{y} \in \mathscr{L}_0$, i.e., $\mathbf{A}\mathbf{y} = 0$. Let $\mathbf{x} = \mathbf{y} + \mathbf{x}_0$. Then $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$ and $\mathbf{A}\mathbf{x} = \mathbf{A}(\mathbf{y}+\mathbf{x}_0) = \mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{x}_0 = 0 + \mathbf{b} = \mathbf{b}$, i.e., $\mathbf{x} \in L$.
(c) Follows from $\mathscr{L}_0 = \{\mathbf{x} - \mathbf{x}_0 : \mathbf{x} \in L\} = L - \mathbf{x}_0$.

52. (a) vector space, $\dim = 3$, canonical basis $\{(1,0,0)^\mathsf{T}, (0,1,0)^\mathsf{T}, (0,0,1)\}$; (b) no vector space, as $(1,2)^\mathsf{T} + (1,2,3)^\mathsf{T}$ is not defined; (c) vector space, $\dim = 4$, basis $\{1, x, x^2, x^3\}$; (d) no vector space, as $(x^3 + x) - (x^3 - 1) = x + 1$ is not a polynomial of degree 3; (e) vector space, $\dim = \infty$ (such vector spaces are out of scope for this course); (f) vector space, $\dim = n - \text{rank}(\mathbf{A})$ for an $m \times n$ matrix $\mathbf{A}$, basis by means of Gaussian elimination; (g) no vector space, as $\mathbf{A}\mathbf{x}_1 = \mathbf{b}$ and $\mathbf{A}\mathbf{x}_2 = \mathbf{b}$ implies $\mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = 2\mathbf{b} \neq \mathbf{b}$; (h) vector space (as this is just $\text{Im}(\mathbf{A})$, $\dim = \text{rank}(\mathbf{A})$, basis is any maximal set of linearly independent columns of $\mathbf{A}$; (i) no vector space, as $\|2\cdot(1,0,0)^\mathsf{T}\| = 2 > 1$ while $\|(1,0,0)^\mathsf{T}\| = 1 \leq 1$.

53. (a) -3; (b) -9; (c) 8; (d) 0; (e) -40; (f) -10; (g) 48; (h) -49; (i) 0.

54. See Exercise 53.

55. All matrices except those in Exercise 53(d) and (i) are regular and thus invertible and have linear independent column vectors.

Ranks of the matrices: (a)–(d) rank 2; (e)–(f) rank 3; (g)–(h) rank 4; (i) rank 1.

56. (a) $\det(\mathbf{A}) = 3$; (b) $\det(5\mathbf{A}) = 5^3\det(\mathbf{A}) = 375$ (D1); (c) $\det(\mathbf{B}) = 2\det(\mathbf{A}) = 6$ (D1); (d) $\det(\mathbf{A}^\mathsf{T}) = \det(\mathbf{A}) = 3$ (D6); (e) $\det(\mathbf{C}) = \det(\mathbf{A}) = 3$ (D5); (f) $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} = \frac{1}{3}$ (D9); (g) $\det(\mathbf{A}\cdot\mathbf{C}) = \det(\mathbf{A})\cdot\det(\mathbf{C}) = 3\cdot 3 = 9$ (D8); (h) $\det(\mathbf{I}) = 1$ (D3).

57. (a) -864; (b) -216; (c) -216; (d) $-1/216$; (e) 186624; (f) 1; (g) 1; (h) $1/46656$.

58. (a) -1280; (b) 25; (c) -5; (d) $-1/5$; (e) -170; (f) 8; (g) yes; (h) no; (i) yes; (j) $\dim(\mathrm{Im}) = 8$, $\dim(\mathrm{Ker}) = 0$.

59. $\left|\mathbf{A}^\mathsf{T}\cdot\mathbf{A}\right| = 0$; $\left|\mathbf{A}\cdot\mathbf{A}^\mathsf{T}\right|$ depends on matrix $\mathbf{A}$.

60. (a) 9; (b) 9; (c) 40; (e) 40.

61. $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^{*\mathsf{T}}$.

(a) $\mathbf{A}^* = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, $\mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, $|\mathbf{A}| = -3$;

(b) $\mathbf{A}^* = \begin{pmatrix} 3 & -1 \\ -3 & -2 \end{pmatrix}$, $\mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} 3 & -3 \\ -1 & -2 \end{pmatrix}$, $|\mathbf{A}| = -9$;

(c) $\mathbf{A}^* = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$, $\mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $|\mathbf{A}| = 8$;

(d) $\mathbf{A}^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -6 \\ -1 & 0 & 3 \end{pmatrix}$, $\mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & -6 & 3 \end{pmatrix}$, $|\mathbf{A}| = 3$;

(e) $\mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} -20 & -12 & 8 \\ 20 & 4 & -16 \\ 5 & -5 & 0 \end{pmatrix}$, $|\mathbf{A}| = -40$;

(f) $\mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} 9 & 3 & -4 \\ -2 & -4 & 2 \\ -14 & -8 & 4 \end{pmatrix}$, $|\mathbf{A}| = -10$.

62. (a) $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; (b) $\mathbf{A}^{-1} = \frac{1}{\alpha\beta^2-\alpha^2\beta}\begin{pmatrix} \beta^2 & -\beta \\ -\alpha^2 & \alpha \end{pmatrix}$.

63. (a) $\mathbf{x} = (1,0)^\mathsf{T}$; (b) $\mathbf{x} = (1/3, 5/9)^\mathsf{T}$; (c) $\mathbf{x} = (1,1)^\mathsf{T}$; (d) $\mathbf{x} = (0,2,-1)^\mathsf{T}$; (e) $\mathbf{x} = (1/2, 1/2, 1/8)^\mathsf{T}$; (f) $\mathbf{x} = (-3/10, 2/5, 9/5)^\mathsf{T}$.

64. (a) $x = (d-2b)/(ad-bc)$, $y = (2a-c)/(ad-bc)$;
(b) $x = (\alpha\beta - b)/(\alpha(\beta-\alpha))$, $y = (b-\alpha\alpha)/(\beta(\beta-\alpha))$.

65. Assume that $\mathbf{A}$ and $\mathbf{B}$ are similar. Then there exists an invertible matrix $\mathbf{U}$ such that $\mathbf{B} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$. Then by the properties of the determinant we find $\det(\mathbf{B}) = \det(\mathbf{U}^{-1}\mathbf{A}\mathbf{U}) = \det(\mathbf{U}^{-1})\det(\mathbf{A})\det(\mathbf{U}) = \frac{1}{\det(\mathbf{U})}\det(\mathbf{A})\det(\mathbf{U}) = \det(\mathbf{A})$.

66. (a) $\lambda_1 = 7$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$; (b) $\lambda_1 = 14$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$; $\lambda_2 = 1$, $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$; (c) $\lambda_1 = -6$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\lambda_2 = 4$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

67. (a) $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\lambda_3 = 2$, $\mathbf{x}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

(b) $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{x}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

(c) $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $\lambda_2 = 3$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(d) $\lambda_1 = -3$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $\lambda_2 = -5$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_3 = -9$, $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(e) $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$; $\lambda_3 = 4$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(f) $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$; $\lambda_2 = 27$, $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$; $\lambda_3 = -9$, $\mathbf{x}_3 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$.

68. (a) $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; (b) $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

69. 66a: positive definite; 66c: indefinite; 67a: positive semidefinite; 67d: negative definite; 67f: indefinite; 68a: positive definite.
The other matrices are not symmetric. So our criteria cannot be applied.

70. For every vector $\mathbf{x} \neq 0$ we have: $-\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = \mathbf{x}^\mathsf{T}(-\mathbf{A})\mathbf{x} > 0$. So $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} < 0$ if and only if $\mathbf{x}^\mathsf{T}(-\mathbf{A})\mathbf{x} > 0$.

71. $q_\mathbf{A}(\mathbf{x}) = 3x_1^2 + 4x_1x_2 + 2x_1x_3 - 2x_2^2 - x_3^2$.

72. $\mathbf{A} = \begin{pmatrix} 5 & 3 & -1 \\ 3 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$.

73. Eigenspace corresponding to eigenvalue $\lambda_1 = 0$: $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$;

Eigenspace corresponding to eigenvalues $\lambda_2 = \lambda_3 = 2$: $\operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

74. Give examples.

75. (1) We find for the characteristic polynomial of $\mathbf{A}^\mathsf{T}$, $|\mathbf{A}^\mathsf{T} - \lambda\mathbf{I}| = |(\mathbf{A}-\lambda\mathbf{I})^\mathsf{T}| = |\mathbf{A}-\lambda\mathbf{I}|$, i.e., λ is an eigenvalue of $\mathbf{A}$ if and only if it is an eigenvalue of $\mathbf{A}^\mathsf{T}$. (4) We can transform $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ into $\mathbf{A}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$.

76. $\det(\mathbf{A}) = 12$; $\operatorname{tr}(\mathbf{A}) = 7$; (a) depends; (b) true; (c) true; (d) depends; (e) depends.

77. Eigenvalues -3 and 2.

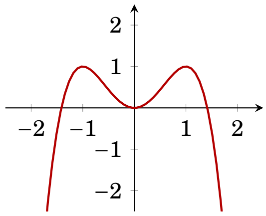
78. 66a: $M_1 = 3$, $M_2 = 14$, positive definite; 66c: $M_1 = -1$, $M_2 = -24$, indefinite; 67a: $M_1 = 1$, $M_2 = 0$, $M_3 = 0$, cannot be applied; 67d: $M_1 = -3$, $M_2 = 15$, $M_3 = -135$, negative definite; 67f: $M_1 = 11$, $M_2 = -27$, $M_3 = 0$, cannot be applied; 68a: $M_1 = 1$, $M_2 = 1$, $M_3 = 1$, positive definite.
All other matrices are not symmetric.

79. 66a: $M_1 = 3$, $M_2 = 6$, $M_{1,2} = 14$, positive definite; 66c: $M_1 = -1$, $M_2 = -1$, $M_{1,2} = -24$, indefinite; 67a: $M_1 = 1$, $M_2 = 1$, $M_3 = 2$, $M_{1,2} = 0$, $M_{1,3} = 2$, $M_{2,3} = 2$, $M_{1,2,3} = 0$, positive semidefinite. 67d: $M_1 = -3$, $M_2 = -5$, $M_3 = -9$, $M_{1,2} = 15$, $M_{1,3} = 27$, $M_{2,3} = 45$, $M_{1,2,3} = -135$, negative definite. 67f: $M_1 = 11$, $M_2 = -1$, $M_3 = 8$, $M_{1,2} = -27$, $M_{1,3} = -108$, $M_{2,3} = -108$, $M_{1,2,3} = 0$, indefinite.

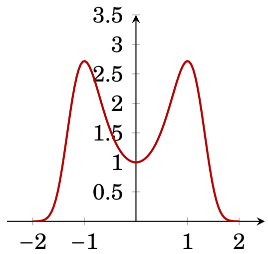
80. Matrix $\mathbf{A}$ is negativ definite if and only if $\mathbf{B} = -\mathbf{A}$ is positive definite. Matrix $\mathbf{B}$ is positive definite if all its eigenvalues are positive. Hence all eigenvalues of $\mathbf{A} = -\mathbf{B}$ are negative. Matrix $\mathbf{B}$ is positive definite if all its leading principle minors are positive. The leading principles minors of $\mathbf{A} = -\mathbf{B}$ are obtained by $A_k = (-1)^k B_k$. Hence the leading principles minors of $\mathbf{A} = -\mathbf{B}$ of even order must be positive as well while that of negative order are negative.

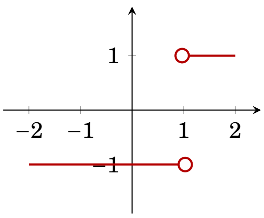
81. (a) We need a basis of orthonormal eigenvectors of $\mathbf{A}$: $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^\mathsf{T}, (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)^\mathsf{T}, (0,0,1)^\mathsf{T}\}$, $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; (b) $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (c) $\mathbf{c} = \mathbf{U}^{-1}\mathbf{x} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$.

82. (a)

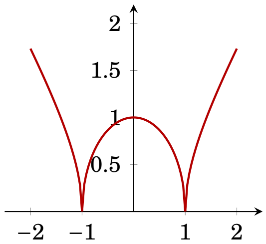


(b)





(c)



(d)

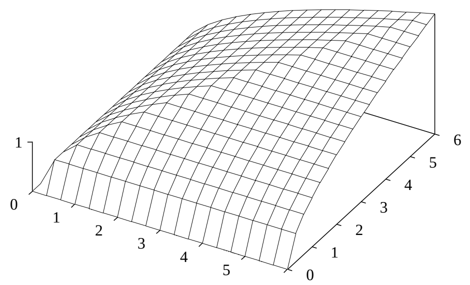
83. —

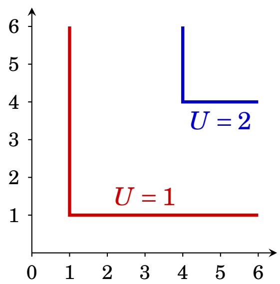
84. (a) one-to-one, not onto; (b) one-to-one, not onto; (c) bijective; (d) not one-to-one, not onto; (e) not one-to-one, onto; (f) bijective. Beware! Domain and codomain are part of the function.

85. (a) $(f \circ g)(4) = f(g(4)) = f(9) = 98$; (b) $(f \circ g)(-9) = f(g(-9)) = f(28) = 839$; (c) $(g \circ f)(0) = g(f(0)) = g(-1) = 2$; (d) $(g \circ f)(-1) = g(f(-1)) = g(-2) \approx 3.828$.

86. (a) $(f \circ g)(x) = (1+x)^2$, $(g \circ f)(x) = 1+x^2$, $D_f = D_g = D_{f\circ g} = D_{g\circ f} = \mathbb{R}$; (b) $(f \circ g)(x) = |x|+1$, $(g \circ f)(x) = (\sqrt{x}+1)^2$, $D_f = D_{g\circ f} = [0,\infty)$, $D_g = D_{f\circ g} = \mathbb{R}$; (c) $(f \circ g)(x) = \frac{1}{\sqrt{x}+2}$, $(g \circ f)(x) = \sqrt{\frac{1}{x+1}+1}$, $D_f = \mathbb{R} \setminus \{-1\}$, $D_g = D_{f\circ g} = [0,\infty)$, $D_{g\circ f} = (-1,\infty)$; (d) $(f \circ g)(x) = 2 + |x-2|$, $(g \circ f)(x) = x$, $D_f = D_{g\circ f} = [0,\infty)$, $D_g = D_{f\circ g} = \mathbb{R}$.

87.





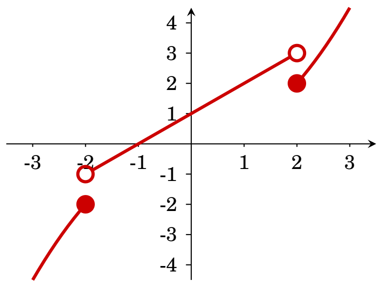
88. —

89. —

90. (a) 7; (b) $\frac{2}{7}$; (c) divergent; (d) divergent but tends to ∞; (e) 0.

91. (a) e^x; (b) e^x; (c) $e^{1/x}$.

92.



$\lim_{x\to-2^+} f(x) = -1$, $\lim_{x\to-2^-} f(x) = -2$, $\lim_{x\to-2} f(x)$ does not exist; $\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) = \lim_{x\to0} f(x) = 1$; $\lim_{x\to2^+} f(x) = 2$, $\lim_{x\to2^-} f(x) = 3$, $\lim_{x\to2} f(x)$ does not exist;f is continuous at 0 but not in -2 and 2.

93. (a) $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 1$; (b) $\lim_{x\to1^-} f(x) = -1$, $\lim_{x\to1^+} f(x) = 1$; (c) $\lim_{x\to1^-} f(x) = -2$, $\lim_{x\to1^+} f(x) = 2$; (d) $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = \frac{1}{2}$.

94. (a) $\frac{1}{2}$; (b) -4; (c) -1; (d) 0.

95. (a) 0; (b) 0; (c) ∞; (d) $-\infty$; (e) 1.

96. $\lim_{x\to0} f(x) = 0$.

97. The respective functions are continuous in (a) D; (b) D; (c) D; (d) D; (e) D;(f) $\mathbb{R}\setminus\mathbb{Z}$; (g) $\mathbb{R}\setminus\{2\}$.

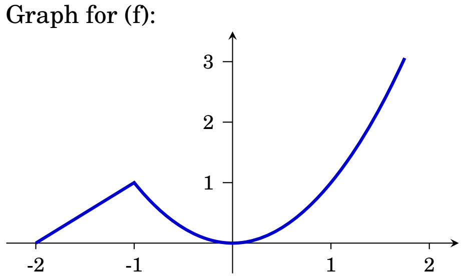
98. Yes.

99. $h = \frac{2}{5}$.

100. Differentiable in

(a) $\mathbb{R}$; (b) $\mathbb{R}$; (c) $\mathbb{R}\setminus\{0\}$; (d) $\mathbb{R}\setminus\{-1,1\}$; (e) $\mathbb{R}\setminus\{-1,1\}$; (f) $\mathbb{R}\setminus\{-1\}$.

Graph for (f):



101. (a) $f'(x) = 16x^3 + 9x^2 - 4x$, $f''(x) = 48x^2 + 19x - 4$; (b) $f'(x) = -x\,e^{-\frac{x^2}{2}}$, $f''(x) = (x^2 - 1)e^{-\frac{x^2}{2}}$; (c) = (b); (d) $f'(x) = \frac{-2}{(x-1)^2}$, $f''(x) = \frac{4}{(x-1)^3}$.

102. (a) $f'(x) = -\frac{2x}{(1+x^2)^2}$, $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$; (b) $f'(x) = -\frac{2}{(1+x)^3}$, $f''(x) = \frac{6}{(1+x)^4}$; (c) $f'(x) = \ln(x)$, $f''(x) = \frac{1}{x}$; (d) $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$.

103. (a) $f'(x) = \frac{1}{cos(x)^2}$, $f''(x) = \frac{2\sin(x)}{\cos(x)^3}$; (b) $f'(x) = \sinh(x)$, $f''(x) = \cosh(x)$; (c) $f'(x) = \cosh(x)$, $f''(x) = \sinh(x)$; (d) $f'(x) = -2x\sin(1+x^2)$, $f''(x) = -2\sin(1+x^2) - 4x^2\cos(1+x^2)$.

104. $\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot(g(x))^{-1}\right)' = \ldots = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}.$

105. By means of the differential: $f(3.1) - f(3) \approx -0.001096$,

exact value: $f(3.1) - f(3) = -0.00124\ldots$

106. (a) $\varepsilon_g(x) = \frac{3x^3-4x^2}{x^3-2x^2}$, 1-elastic for $x = 1$ and $x = \frac{3}{2}$, elastic for $x < 1$ and $x > \frac{3}{2}$, inelastic for $1 < x < \frac{3}{2}$; (b) $\varepsilon_h(x) = \beta$, the elasticity of $h(x)$ only depends on parameter β and is constant on the domain of h.

107. (a) wrong; (b) approximately; (c) correct; (d) wrong.

108. Derivatives: $f_x(x,y) = 2x\exp(x^2+y^2)$; $f_y(x,y) = 2y\exp(x^2+y^2)$; $f_{xx}(x,y) = (2+4x^2)\exp(x^2+y^2)$; $f_{xy}(x,y) = f_{yx}(x,y) = 4xy\exp(x^2+y^2)$; $f_{yy}(x,y) = (2+4y^2)\exp(x^2+y^2)$.
Derivatives at point $(0,0)$: $f_x(0,0) = 0$; $f_y(0,0) = 0$; $f_{xx}(0,0) = 2$; $f_{xy}(0,0) = f_{yx}(0,0) = 0$; $f_{yy}(0,0) = 2$.

109. Derivatives:

	(a)	(b)	(c)	(d)	(e)
f_x	1	y	$2x$	$2xy^2$	$\alpha x^{\alpha-1}y^\beta$
f_y	1	x	$2y$	$2x^2y$	$\beta x^\alpha y^{\beta-1}$
f_{xx}	0	0	2	$2y^2$	$\alpha(\alpha-1)x^{\alpha-2}y^\beta$
$f_{xy} = f_{yx}$	0	1	0	$4xy$	$\alpha\beta x^{\alpha-1}y^{\beta-1}$
f_{yy}	0	0	2	$2x^2$	$\beta(\beta-1)x^\alpha y^{\beta-2}$

Derivatives at point $(1,1)$:

	(a)	(b)	(c)	(d)	(e)
f_x	1	1	2	2	α
f_y	1	1	2	2	β
f_{xx}	0	0	2	2	$\alpha(\alpha-1)$
$f_{xy} = f_{yx}$	0	1	0	4	$\alpha\beta$
f_{yy}	0	0	2	2	$\beta(\beta-1)$

110. Derivatives:

	(a)
f_x	$x(x^2+y^2)^{-1/2}$
f_y	$y(x^2+y^2)^{-1/2}$
f_{xx}	$(x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2}$
$f_{xy} = f_{yx}$	$-xy(x^2+y^2)^{-3/2}$
f_{yy}	$(x^2+y^2)^{-1/2} - y^2(x^2+y^2)^{-3/2}$

	(b)
f_x	$x^2(x^3+y^3)^{-2/3}$
f_y	$y^2(x^3+y^3)^{-2/3}$
f_{xx}	$2x(x^3+y^3)^{-2/3} - 2x^4(x^3+y^3)^{-5/3}$
$f_{xy} = f_{yx}$	$-2x^2y^2(x^3+y^3)^{-5/3}$
f_{yy}	$2y(x^3+y^3)^{-2/3} - 2y^4(x^3+y^3)^{-5/3}$

	(c)
f_x	$x^{p-1}(x^p+y^p)^{(1-p)/p}$
f_y	$y^{p-1}(x^p+y^p)^{(1-p)/p}$
f_{xx}	$(p-1)x^{p-2}(x^p+y^p)^{(1-p)/p} - (p-1)x^{2(p-1)}(x^p+y^p)^{(1-2p)/p}$
$f_{xy} = f_{yx}$	$-(p-1)x^{p-1}y^{p-1}(x^p+y^p)^{(1-2p)/p}$
f_{yy}	$(p-1)y^{p-2}(x^p+y^p)^{(1-p)/p} - (p-1)y^{2(p-1)}(x^p+y^p)^{(1-2p)/p}$

Derivatives at point $(1,1)$:

	(a)	(b)	(c)
f_x	$\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt[3]{4}}$	$2^{(1-p)/p}$
f_y	$\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt[3]{4}}$	$2^{(1-p)/p}$
f_{xx}	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt[3]{4}}$	$(p-1)2^{(1-2p)/p}$
$f_{xy}=f_{yx}$	$-\frac{1}{\sqrt{8}}$	$-\frac{1}{\sqrt[3]{4}}$	$-(p-1)2^{(1-2p)/p}$
f_{yy}	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt[3]{4}}$	$(p-1)2^{(1-2p)/p}$

111. Gradient: (a) $\nabla f(x,y)=(1,1)$; (b) $\nabla f(x,y)=(y,x)$; (c) $\nabla f(x,y)=(2x,2y)$; (d) $\nabla f(x,y)=(2xy^2,2x^2y)$; (e) $\nabla f(x,y)=(\alpha x^{\alpha-1}y^{\beta},\beta x^{\alpha}y^{\beta-1})$.
Gradient at $(1,1)$: (a) $\nabla f(1,1)=(1,1)$; (b) $\nabla f(1,1)=(1,1)$; (c) $\nabla f(1,1)=(2,2)$; (d) $\nabla f(1,1)=(2,2)$; (e) $\nabla f(1,1)=(\alpha,\beta)$.

112. Gradient: (a) $\nabla f(x,y)=(x(x^2+y^2)^{-1/2},\ y(x^2+y^2)^{-1/2})$, (b) $\nabla f(x,y)=(x^2(x^3+y^3)^{-2/3},\ y^2(x^3+y^3)^{-2/3})$, (c) $\nabla f(x,y)=(x^{p-1}(x^p+y^p)^{(1-p)/p},\ y^{p-1}(x^p+y^p)^{(1-p)/p})$,
Gradient at $(1,1)$: (a) $\nabla f(1,1)=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$, (b) $\nabla f(1,1)=(\frac{1}{\sqrt[3]{4}},\frac{1}{\sqrt[3]{4}})$, (c) $\nabla f(1,1)=(2^{(1-p)/p},2^{(1-p)/p})$.

113. (a) $\frac{\partial f}{\partial \mathbf{h}}=\frac{6}{\sqrt{5}}$; (b) $\frac{\partial f}{\partial \mathbf{h}}=\frac{8}{\sqrt{5}}$; (c) $\frac{\partial f}{\partial \mathbf{h}}=\sqrt{\frac{18}{20}}$; (d) $\frac{\partial f}{\partial \mathbf{h}}=\frac{3}{\sqrt{5}\sqrt[3]{4}}$.

114. (a) $g(t)=f(\mathbf{x}+t\mathbf{a})=\sum_{i=1}^n(x_i+ta_i)^2$, $g'(t)=\sum_{i=1}^n 2(x_i+ta_i)\cdot a_i$, $\frac{\partial f}{\partial \mathbf{a}}=g'(0)=\sum_{i=1}^n 2x_ia_i$; (b) $\frac{\partial f}{\partial \mathbf{a}}=f'(\mathbf{x})\mathbf{a}=2\mathbf{x}^{\mathsf{T}}\mathbf{a}=\sum_{i=1}^n 2x_ia_i$; (c) using $\mathbf{s}(t)=\mathbf{x}+t\mathbf{a}$ we find $\frac{\partial f}{\partial \mathbf{a}}=(f\circ\mathbf{s})'(0)=f'(\mathbf{s}(0))\mathbf{s}'(0)=f'(\mathbf{x})\mathbf{a}=$ (b).

115. The directional derivative must be normalized:
$\mathbf{a}=(1,3)^{\mathsf{T}}/\|(1,3)^{\mathsf{T}}\|=(1/\sqrt{10},3/\sqrt{10})^{\mathsf{T}}$. Recall that $\frac{\partial f}{\partial \mathbf{a}}=f'(0)\mathbf{a}\le 4$, where equality holds if $f'(0)$ and $\mathbf{a}$ are linearly dependent. Thus $f'(0)=4\mathbf{a}=(4/\sqrt{10},12/\sqrt{10})^{\mathsf{T}}$.

116. (a) For example function f from Exercise 114. (b) Partial derivative $\frac{\partial}{\partial x_i}$ of both sides of equation $f(\alpha\mathbf{x})=\alpha^k f(\mathbf{x})$ we find $f_{x_i}(\alpha\mathbf{x})\cdot\alpha=\alpha^k f_{x_i}(\mathbf{x})$. That is, $f_{x_i}(\alpha\mathbf{x})=\alpha^{k-1}f_{x_i}(\mathbf{x})$, as claimed.

117. (a) $\mathbf{H}_f(1,1)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$; (b) $\mathbf{H}_f(1,1)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$; (c) $\mathbf{H}_f(1,1)=\begin{pmatrix}2&0\\0&2\end{pmatrix}$;
(d) $\mathbf{H}_f(1,1)=\begin{pmatrix}2&4\\4&2\end{pmatrix}$; (e) $\mathbf{H}_f(1,1)=\begin{pmatrix}\alpha(\alpha-1)&\alpha\beta\\\alpha\beta&\beta(\beta-1)\end{pmatrix}$.

118. (a) $\mathbf{H}_f(1,1)=\begin{pmatrix}\frac{\sqrt{8}}{8}&-\frac{\sqrt{8}}{8}\\-\frac{\sqrt{8}}{8}&\frac{\sqrt{8}}{8}\end{pmatrix}$; (b) $\mathbf{H}_f(1,1)=\begin{pmatrix}\frac{\sqrt[3]{2}}{2}&-\frac{\sqrt[3]{2}}{2}\\-\frac{\sqrt[3]{2}}{2}&\frac{\sqrt[3]{2}}{2}\end{pmatrix}$;
(c) $\mathbf{H}_f(1,1)=\begin{pmatrix}(p-1)2^{(1-2p)/p}&-(p-1)2^{(1-2p)/p}\\-(p-1)2^{(1-2p)/p}&(p-1)2^{(1-2p)/p}\end{pmatrix}$.

119. (a) $h'(t)=f'(\mathbf{g}(t))\cdot\mathbf{g}'(t)=(2g_1(t),2g_2(t))\cdot\begin{pmatrix}1\\2t\end{pmatrix}=(2t,2t^2)\cdot\begin{pmatrix}1\\2t\end{pmatrix}=2t+4t^3$.
(Composite function: $h(t)=f(\mathbf{g}(t))=t^2+t^4$.)
(b) $\mathbf{p}'(x,y)=\mathbf{g}'(f(x,y))\cdot f'(x,y)=\begin{pmatrix}1\\2(x^2+y^2)\end{pmatrix}\cdot(2x,2y)=\begin{pmatrix}2x&2y\\4x(x^2+y^2)&4y(x^2+y^2)\end{pmatrix}$
(Composite function: $\mathbf{p}(x,y)=\mathbf{g}(f(x,y))=\begin{pmatrix}x^2+y^2\\(x^2+y^2)^2\end{pmatrix}$)

120. $\mathbf{f}'(\mathbf{x})=\begin{pmatrix}3x_1^2&-1\\1&-3x_2^2\end{pmatrix}$, $\mathbf{g}'(\mathbf{x})=\begin{pmatrix}0&2x_2\\1&0\end{pmatrix}$,
(a) $(\mathbf{g}\circ\mathbf{f})'(\mathbf{x})=\mathbf{g}'(\mathbf{f}(\mathbf{x}))\mathbf{f}'(\mathbf{x})=\begin{pmatrix}0&2(x_1-x_2^3)\\1&0\end{pmatrix}\cdot\begin{pmatrix}3x_1^2&-1\\1&-3x_2^2\end{pmatrix}=\begin{pmatrix}2(x_1-x_2^3)&6(-x_1x_2^2+x_2^5)\\3x_1^2&-1\end{pmatrix}$

(b) $(\mathbf{f}\circ\mathbf{g})'(\mathbf{x}) = \mathbf{f}'(\mathbf{g}(\mathbf{x}))\,\mathbf{g}'(\mathbf{x}) = \begin{pmatrix} 3x_2^4 & -1 \\ 1 & -3x_1^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2x_2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 6x_2^5 \\ -3x_1^2 & 2x_2 \end{pmatrix}$.

121. Let $\mathbf{s}(t) = \begin{pmatrix} K(t) \\ L(t) \\ t \end{pmatrix}$. Then $\dfrac{dQ}{dt} = Q'(\mathbf{s}(t)) \cdot \mathbf{s}'(t) = \ldots = Q_K K'(t) + Q_L L'(t) + Q_t$.

122. $\mathbf{f}'(\mathbf{x}) = \mathbf{A}$.

123. $\mathbf{f}'(\mathbf{x}) = 2(\mathbf{A}\mathbf{x})^\top = 2\mathbf{x}^\top\mathbf{A}$.

124. (a) $\frac{2}{7}$; (b) $-\frac{1}{4}$; (c) $\frac{1}{3}$; (d) $\frac{1}{2}$; (e) $= \lim\limits_{x\to 0^+} \frac{\ln x}{x^{-1}} = 0$; (f) ∞. In (b) and (f) we cannot apply l'Hôpital's rule.

125. Look at numerator and denominator.

126. Hint: $\lim_{\lambda\to 0} f(\lambda) = \ln(x) = f(0)$.

127. (a) $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 1-x_2 & -x_1 \\ x_2 & x_1 \end{pmatrix}$, $\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} = x_1$; (b) $x_1 \neq 0$;

(c) $D(\mathbf{f}^{-1})(\mathbf{y}) = (D\mathbf{f}(\mathbf{x}))^{-1} = \begin{pmatrix} 1-x_2 & -x_1 \\ x_2 & x_1 \end{pmatrix}^{-1} = \frac{1}{x_1}\begin{pmatrix} x_1 & x_1 \\ -x_2 & 1-x_2 \end{pmatrix}$;

(d) we obtain the inverse transformation by solving the equation w.r.t. variables x_1 and x_2: $x_1 = y_1 + y_2$ and $x_2 = y_2/(y_1+y_2)$ (if $y_1 + y_2 \neq 0$).

128. Let $J = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$ be the Jacobian determinant. The equation can be solved locally if $J \neq 0$. We then have $\frac{\partial F}{\partial u} = \frac{1}{J}\frac{\partial g}{\partial y}$ and $\frac{\partial G}{\partial u} = -\frac{1}{J}\frac{\partial g}{\partial x}$.

129. (a) $F_y = 3y^2+1 \neq 0$, $y' = -F_x/F_y = 3x^2/(3y^2+1) = 0$ for $x = 0$; (b) $F_y = 1 + x\cos(xy) = 1 \neq 0$ for $x = 0$, $y'(0) = 0$.

130. $\frac{dy}{dx} = -\frac{2x}{3y^2}$, $y = f(x)$ exists locally in an interval around $\mathbf{x}_0 = (x_0, y_0)$ if $y_0 \neq 0$.

131. $\frac{dy}{dx} = -\frac{u_x}{u_y} = -\frac{\alpha c x^{\alpha-1} y^\beta}{\beta c x^\alpha y^{\beta-1}} = -\frac{\alpha y}{\beta x}$; $\frac{dx}{dy} = -\frac{\beta x}{\alpha y}$.

132. (a) $\dfrac{dx_i}{dx_j} = -\dfrac{u_{x_j}}{u_{x_i}} = -\dfrac{\left(x_1^{\frac12}+x_2^{\frac12}\right)x_j^{-\frac12}}{\left(x_1^{\frac12}+x_2^{\frac12}\right)x_i^{-\frac12}} = -\dfrac{x_i^{\frac12}}{x_j^{\frac12}} < 0$;

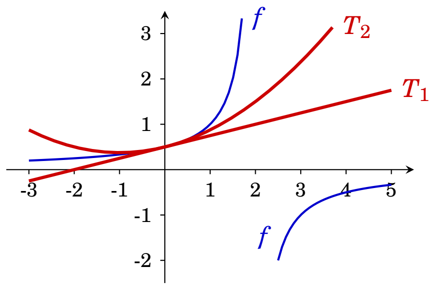
(b) $\dfrac{dx_i}{dx_j} = -\dfrac{u_{x_j}}{u_{x_i}} = -\dfrac{\frac{\theta}{\theta-1}\left(\sum_{i=1}^n x_i^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}} \frac{\theta-1}{\theta} x_j^{-\frac{1}{\theta}}}{\frac{\theta}{\theta-1}\left(\sum_{i=1}^n x_i^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}} \frac{\theta-1}{\theta} x_i^{-\frac{1}{\theta}}} = -\dfrac{x_i^{\frac{1}{\theta}}}{x_j^{\frac{1}{\theta}}} < 0$.

133. (a) $z = g(x,y)$ can be locally represented as $F_z = 3z^2 - xy$ and $F_z(0,0,1) = 3 \neq 0$; $\frac{\partial g}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2-yz}{3z^2-xy} = -\frac{0}{3} = 0$ for $(x_0,y_0,z_0) = (0,0,1)$; $\frac{\partial g}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2-xz}{3z^2-xy} = -\frac{0}{3} = 0$.

(b) $z = g(x,y)$ can be locally represented as $F_z = \exp(z) - 2z$ and $F_z(1,0,0) = 1 \neq 0$; $\frac{\partial g}{\partial x} = -\frac{F_x}{F_z} = -\frac{-2x}{\exp(z)-2z} = 2$ for $(x_0,y_0,z_0) = (1,0,0)$; $\frac{\partial g}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2y}{\exp(z)-2z} = 0$ for $(x_0,y_0,z_0) = (1,0,0)$.

134. $\mathbf{I} = \mathrm{id}'(\mathbf{x}) = (\mathbf{f}^{-1}\circ\mathbf{f})'(\mathbf{x}) = (\mathbf{f}^{-1})'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x})$. Thus the proposition follows.

135.



(a) $f(x) \approx T_1(x) = \frac{1}{2} + \frac{1}{4}x$,

(b) $f(x) \approx T_2(x) = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2$.

radius of convergence $\rho = 2$.

136. $f(x) \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$; radius of convergence $\rho = 1$.

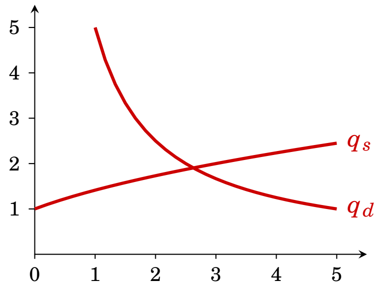
137. Substitute the MacLaurin series for the sine function
$f(x) \approx \sin(x^{10}) \approx (x^{10}) - \frac{1}{6}(x^{10})^3 = x^{10} - \frac{1}{6}x^{30}$.

138. $f(x) \approx 0.959 + 0.284x^2 - 0.479x^4$.

139. By Substitution: $f(x) = 1 - x^2 + x^4 - x^6 + x^8 + O(x^9)$; radius of convergence $\rho = 1$.

140. (a) $\phi(x) = \frac{1}{\sqrt{2\pi}}\left(1 - x^2/2 + x^4/8\right)$; (b) $\Phi(x) = 1/2 + 1/\sqrt{2\pi}\left(x - x^3/6 + x^5/40\right)$; radius of convergence $\rho = \infty$.

141.



equilibrium price
according to drawing $\approx 2{,}6$.

Taylor polynomial of first order with expansion point $p_0 = 3$:
$q_s(p) \approx 2 + \frac{1}{4}(p-3)$, $q_d(p) \approx \frac{5}{3} - \frac{5}{9}(p-3)$.
Equilibrium price by Taylor polynomial approximation: $\bar{p} = \frac{75}{29} \approx 2.586$.

142. $q_s(p) \approx 2 + \frac{1}{4}(p-3) - \frac{1}{64}(p-3)^2$, $q_d(p) \approx \frac{5}{3} - \frac{5}{9}(p-3) + \frac{5}{27}(p-3)^2$. Equilibrium price by Taylor polynomial approximation: $\bar{p} \approx 2.62$ (solution 7.39 is is too far from expansion point $p_0 = 3$.).

143. $f(0,0) = 1$, $\nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{H}_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$,

$$f(x,y) \approx f(0,0) + (0,0)\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\cdot\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = 1 + x^2 + y^2.$$

144. (a) $\frac{1}{4}x^4 + c$; (b) $-\frac{3}{x} + c$; (c) $\frac{2}{5}\sqrt{x^5} + c$; (d) $2\sqrt{x} + c$; (e) $\frac{1}{2}e^{2x} + c$; (f) $\frac{1}{3\ln(2)}2^{3x} + c$; (g) $\frac{1}{2}\ln|x| + c$; (h) $5x + c$; (i) $-\frac{1}{\pi}\cos(\pi x) + c$; (j) $\frac{1}{2\pi}\sin(2\pi x) + c$.

145. (a) $\frac{1}{5}x^5 + \frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x + c$; (b) $\frac{7}{2}x^2 + \frac{1}{4}x^4 + 6\ln|x+1| + c$; (c) $e^x + \frac{1}{e+1}x^{e+1} + ex + \frac{1}{2}x^2 + c$; (d) $\frac{2}{3}x^{3/2} + 2\sqrt{x} + c$; (e) $x^4 + x^3 + x^2 + x + \ln|x| - \frac{1}{2x^2} + c$.

146. (a) $2(x-1)e^x + c$; (b) $-(x^2+2x+2)e^{-x} + c$; (c) $\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + c$; (d) $\frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4 + c$; (e) $\frac{1}{2}x^2(\ln(x))^2 - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + c$; (f) $2x\sin(x) + (2-x^2)\cos(x)$.

147. (a) $z = x^2$: $\frac{1}{2}e^{x^2} + c$; (b) $z = x^2 + 6$:$\frac{2}{3}\left(x^2+6\right)^{\frac{3}{2}} + c$; (c) $z = 3x^2 + 4$: $\frac{1}{6}\ln|3x^2+4| + c$; (d) $z = x + 1$: $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$; (e) $z = \ln(x)$: $\frac{1}{2}(\ln(x))^2 + c$; (f) $z = \ln(x)$: $\ln|\ln(x)| + c$; (g) $z =$

x^3+1: $\frac{2}{9}(x^3+1)^{3/2}+c$; (h) $z=5-x^2$: $-\sqrt{5-x^2}+c$; (i) $z=x-3$: $7\ln|x-3|+\frac{1}{2}x^2+2x+c$; (j) $z=x-8$: $\frac{2}{5}(x-8)^{5/2}+\frac{16}{3}(x-8)^{3/2}+c$; (k) $\frac{1}{3\ln(2)}2^{3x}+c$.

148. (a) 39; (b) $3e^2-3\approx 19.16717$; (c) 93; (d) $-\frac{1}{6}$ (use radian!); (e) $\frac{1}{2}\ln(8)\approx 1.0397$.

149. (a) $\frac{1}{2}$; (b) $\frac{781}{10}$; (c) $\frac{8}{3}$; (d) $\frac{1}{2}\ln(5)-\frac{1}{2}\ln(2)\approx 0.4581$; (e) $1-e^{-2}\approx 0.8647$; (f) $\frac{27}{4}$; (g) 1; (h) $2e^2-2\approx 12.778$; (i) $\frac{8}{3}\ln(2)-\frac{7}{9}\approx 1.07061$.

150. We need the antiderivative $C(x)$ of $C'(x)=30-0.05x$ with $C(0)=2000$: $C(x)=2000+30x-0{,}025x^2$.

151. (a) $\int_0^\infty -e^{-3x}\,dx=\lim\limits_{t\to\infty}\int_0^t -e^{-3x}\,dx=\lim\limits_{t\to\infty}\frac{1}{3}e^{-3t}-\frac{1}{3}=-\frac{1}{3}$;
(b) $\int_0^1 \frac{2}{\sqrt[4]{x^3}}\,dx=\lim\limits_{t\to 0}\int_t^1 \frac{2}{\sqrt[4]{x^3}}\,dx=\lim\limits_{t\to 0}8-8t^{\frac{1}{4}}=8$;
(c) $=\lim\limits_{t\to\infty}\int_0^t \frac{x}{x^2+1}\,dx=\lim\limits_{t\to\infty}\frac{1}{2}\int_1^{t^2+1}\frac{1}{z}\,dz=\lim\limits_{t\to\infty}\frac{1}{2}(\ln(t^2+1)-\ln(1))=\infty$,
the improper integral does not exist.

152. (a) $\lim_{t\to\infty}-\frac{1}{1+t}+1=1$;
(b) $\lim_{t\to\infty}-(2+2t+t^2)e^{-t}+2=2$;
(c) $\lim_{t\to\infty}-e^{-t^2/2}+1=1$;
(d) $\lim_{t\to\infty}\ln(t)-\ln(1+t)-\frac{\ln(t)}{1+t}+\ln(2)=\lim_{t\to\infty}\ln\left(\frac{t}{1+t}\right)-\frac{\ln(t)}{1+t}+\ln(2)=\ln(2)$.

153. (a) $\lim_{t\to\infty}\ln(\ln(t))-\ln(\ln(2))=\infty$, the improper integral does not exist;
(b) $\lim_{t\to 1}\ln(\ln(2))-\ln(\ln(t))=-\infty$, the improper integral does not exist;
(c) $\lim_{t\to 0}-1+\frac{1}{t}=\infty$, the improper integral does not exist;
(d) $\lim_{t\to\infty}-\frac{1}{t}+1=1$; (e) $\lim_{t\to 0}2-2\sqrt{t}=2$;
(f) $\lim_{t\to\infty}2\sqrt{t}-2=\infty$, the improper integral does not exist.

154. (a) We have to distinguish between three cases:

$\alpha<-1$: $\displaystyle\int_0^1 x^\alpha\,dx=\lim_{t\to 0}\int_t^1 x^\alpha\,dx=\lim_{t\to 0}\left(\frac{1}{\alpha+1}-\frac{1}{\alpha+1}t^{\alpha+1}\right)=\infty$, as $\alpha+1<0$;

$\alpha=-1$: $\displaystyle\int_0^1 x^{-1}\,dx=\lim_{t\to 0}(\ln(1)-\ln(t))=\infty$;

$\alpha>-1$: $\displaystyle\int_0^1 x^\alpha\,dx=\lim_{t\to 0}\left(\frac{1}{\alpha+1}-\frac{1}{\alpha+1}t^{\alpha+1}\right)=\frac{1}{\alpha+1}$;

the improper integral exists if and only if $\alpha>-1$.
(b) analogously:

$\displaystyle\int_1^\infty x^\alpha\,dx=\lim_{t\to\infty}\int_1^t x^\alpha\,dx=-\frac{1}{1+\alpha}$ if $\alpha<-1$ and ∞ otherwise.

(c) $\displaystyle\int_0^\infty x^\alpha\,dx=\lim_{t\to 0}\int_t^1 x^\alpha\,dx+\lim_{s\to\infty}\int_1^s x^\alpha\,dx=\infty$ for all α.

155. $\mathrm{E}(x)=\int_0^\infty x\,f(x)\,dx=\sqrt{\frac{2}{\pi}}\lim\limits_{t\to\infty}\int_0^t \exp\left(-\frac{x^2}{2}\right)x\,dx=\sqrt{\frac{2}{\pi}}$. (Integral by substitution)

156. $\mathrm{E}(X)=-\sqrt{\frac{2}{\pi}}+\sqrt{\frac{2}{\pi}}=0$.

157. Analogously to Exercise 151(c). The two intervals $(-\infty,0)$ and $(0,\infty)$ have to be computed separately.

158. Leibniz's formula: $M'(t)=\int_{-\infty}^\infty x\,e^{tx}f(x)\,dx$.
Hence: $M'(0)=\int_{-\infty}^\infty x\,f(x)\,dx=\mathrm{E}(X)$.

159. $z\,\Gamma(z)=z\int_0^\infty t^{z-1}e^{-t}\,dt=\int_0^\infty t^z\,e^{-t}\,dt=t^z(-e^{-t})\big|_0^\infty-\int_0^\infty t^t(-e^{-t})\,dt=\Gamma(z+1)$.

160. (a) 16; (b) $\frac{a^2b^2}{4}$; (c) $\frac{\pi-2}{8\pi}$.

161. $\displaystyle\int_{-2}^2 x^2 f(x)\,dx=\frac{1}{6}$.

162. $F(x)=\begin{cases}0, & \text{for } x\le -1,\\ \frac{1}{2}+\frac{2x+x^2}{2}, & \text{for } -1<x\le 0,\\ \frac{1}{2}+\frac{2x-x^2}{2}, & \text{for } 0<x\le 1,\\ 1, & \text{for } x>1.\end{cases}$

163. (a) convex; (b) neither; (c) concave; (d) concave; (e) neither; (f) convex if $\alpha\ge 1$ and $\alpha\le 0$, concave if $0\le\alpha\le 1$.

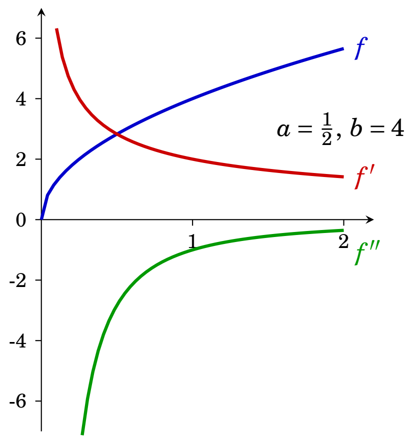
164. Monotonically decreasing in $[-1,3]$, monotonically increasing in $(-\infty,-1]$ and $[3,\infty)$; concave in $(-\infty,1]$, convex in $[1,\infty)$.

165. (a) monotonically increasing (in $\mathbb{R}$), concave in $(-\infty,0]$, convex in $[0,\infty)$;

(b) monotonically increasing $(-\infty,0]$, decreasing in $[0,\infty)$, concave in $[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}]$, and convex in $(-\infty,-\frac{\sqrt{2}}{2}]$ and $[\frac{\sqrt{2}}{2},\infty)$;

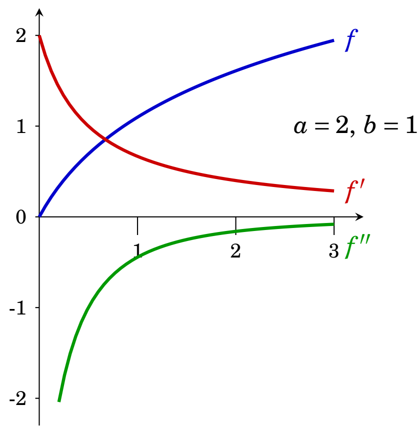
(c) monotonically increasing $(-\infty,0]$, decreasing in $[0,\infty)$, concave in $[-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}]$, and convex in $(-\infty,-\frac{\sqrt{3}}{3}]$ and $[\frac{\sqrt{3}}{3},\infty)$.

166.



Compute all derivatives and verify properties (1)–(3).

167.



Compute all derivatives and verify properties (1)–(3).

168. As f is concave, $f''(x)\le 0$ for all x.
Hence $g''(x)=(-f(x))''=-f''(x)\ge 0$ for all x, i.e., g is convex.

169. —

170. $h''(x) = \alpha f''(x) + \beta g''(x) \leq 0$, i.e., h is concave. If $\beta < 0$ then $\beta g''(x)$ is positive and the sign of $\alpha f''(x) + \beta g''(x)$ cannot be estimated any more.

171. (a) $\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$, $H_1 = 2 > 0$, $H_2 = 4 > 0$, f is strictly convex;

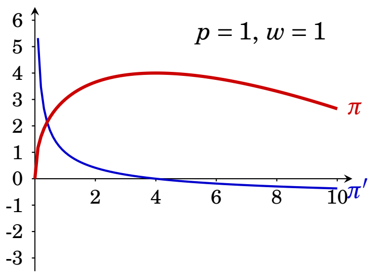
(b) $\mathbf{H}_g(\mathbf{x}) = \begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix}$, $H_1 = 4 > 0$, $H_2 = -1 < 0$, f is neither convex nor concave;

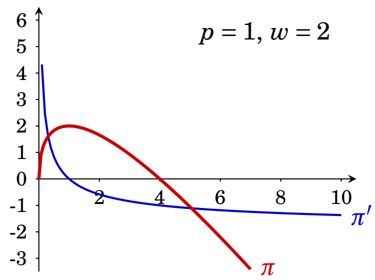
(c) $\mathbf{H}_h(\mathbf{x}) = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}$, $\tilde{H}_1 = -2 < 0$, $\tilde{H}_2 = -8 < 0$, $\tilde{H}_{12} = 0 \geq 0$, f is concave but not stricIty concave.

172. (a) global minimum at $x = 3$ ($f''(x) \geq 0$ for all $x \in \mathbb{R}$), no local maximum; (b) local minimum at $x = 1$, local maximum at $x = -1$, no global extrema.

173. (a) global minimum in $x = 1$, no local maximum; (b) global maximum in $x = \frac{1}{4}$, no local minimum; (c) global minimum in $x = 0$, no local maximum.

174.





(c) $\pi(x) = 4p\sqrt{x} - wx$. critical points: $x_0 = 4\frac{p^2}{w^2}$, $\pi''(x) < 0$, $\Rightarrow x_0$ is maximum, for $p = 1$ and $w = 1$: $x_0 = 4$. (d) analogous, the maximum is at $x = 1$.

175. (a) stationary point: $\mathbf{p}_0 = (0,0)$, $\mathbf{H}_f = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$, $H_2 = -5 < 0$, $\Rightarrow \mathbf{p}_0$ is a saddle point;

(b) stationary point: $\mathbf{p}_0 = (e,0)$, $\mathbf{H}_f(\mathbf{p}_0) = \begin{pmatrix} -e^{-3} & 0 \\ 0 & -2 \end{pmatrix}$, $H_1 = -e^{-3} < 0$, $H_2 = 2e^{-3} > 0$, $\Rightarrow \mathbf{p}_0$ is local maximum; (c) stationary point: $\mathbf{p}_0 = (1,1)$, $\mathbf{H}_f(\mathbf{p}_0) = \begin{pmatrix} 802 & -400 \\ -400 & 200 \end{pmatrix}$, $H_1 = 802 > 0$, $H_2 = 400 > 0$, $\Rightarrow \mathbf{p}_0$ is local minimum; (d) stationary point: $\mathbf{p}_0 = (\ln(3), \ln(4))$, $\mathbf{H}_f = \begin{pmatrix} -e^x & 0 \\ 0 & -e^y \end{pmatrix}$, $H_1 = -e^x < 0$, $H_2 = e^x \cdot e^y > 0$, $\Rightarrow$ local maximum in $\mathbf{p}_0 = (\ln(3), \ln(4))$.

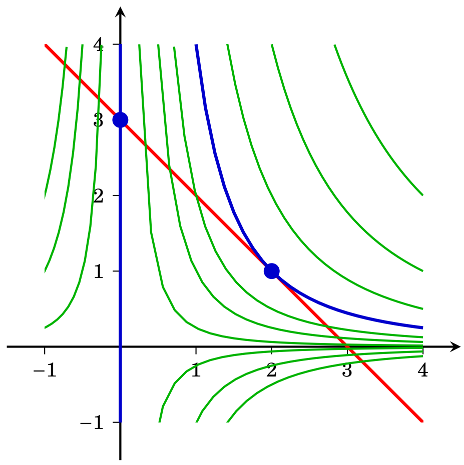
176. stationary points: $\mathbf{p}_1 = (0,0,0)$, $\mathbf{p}_2 = (1,0,0)$, $\mathbf{p}_3 = (-1,0,0)$,

$$\mathbf{H}_f = \begin{pmatrix} 6x_1x_2 & 3x_1^2 - 1 & 0 \\ 3x_1^2 - 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

leading principle minors: $H_1 = 6x_1x_2 = 0$, $H_2 = -(3x_1^2 - 1)^2 < 0$ (da $x_1 \in \{0, -1, 1\}$), $H_3 = -2(3x_1^2 - 1)^2 < 0$,
$\Rightarrow$ all three stationary points are saddle points. The function is neither convex nor concave.

177. critical Punkt: $\mathbf{p}_0 = (\ln(3), \ln(4))$, $\mathbf{H}_f = \begin{pmatrix} -e^{x_1} & 0 \\ 0 & -e^{x_2} \end{pmatrix}$, leading principle minors: $H_1 = -e^{x_1} < 0$, $H_2 = e^{x_1} \cdot e^{x_2} > 0$,
$\Rightarrow$ (a) local maximum at $\mathbf{p}_0 = (\ln(3), \ln(4))$, (b) f is strictly concave, (c) $\mathbf{p}_0$ is global maximum, no global minimum.

178.



(b) Lagrange function: $\mathscr{L}(x,y;\lambda) = x^2 y + \lambda(3-x-y)$,
stationary points $\mathbf{x}_1 = (2,1;4)$ and $\mathbf{x}_2 = (0,3;0)$,

(c) bordered Hessian: $\bar{\mathbf{H}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2y & 2x \\ 1 & 2x & 0 \end{pmatrix}$,

$\bar{\mathbf{H}}(\mathbf{x}_1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 0 \end{pmatrix}$, $\det(\bar{\mathbf{H}}(\mathbf{x}_1)) = 6 > 0$, $\Rightarrow$ $\mathbf{x}_1$ is a local maximum,

$\bar{\mathbf{H}}(\mathbf{x}_2) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 6 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\det(\bar{\mathbf{H}}(\mathbf{x}_2)) = -6$ $\Rightarrow$ $\mathbf{x}_2$ is a local minimum.

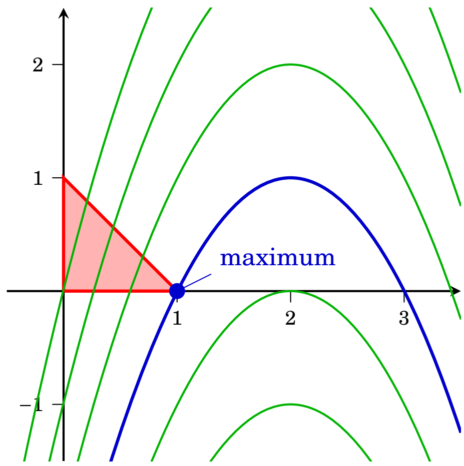
179. Lagrange function: $\mathscr{L}(x_1,x_2,x_3;\lambda_1,\lambda_2) = f(x_1,x_2,x_3) = \frac{1}{3}(x_1-3)^3 + x_2 x_3 + \lambda_1(4-x_1-x_2) + \lambda_2(5-x_1-x_3)$,
stationary points: $\mathbf{x}_1 = (0,4,5;5,4)$ and $\mathbf{x}_2 = (4,0,1;1,0)$.

180. $x_1 = \frac{1}{2}$, $x_2 = 2$; $\lambda = 4$

181. (a) $x_1 = \alpha\frac{m}{p_1}$, $x_2 = (1-\alpha)\frac{m}{p_2}$ and $\lambda = \frac{1}{m}$, (c) marginal change for optimum: $\frac{1}{m}$.

182. (a) yes; (b) no; (c) no.

183.



184. Solution by means of the Kuhn-Tucker theorem: $L(x,y;\lambda) = -(x-2)^2 - y + \lambda(1-x-y)$,
$x = 1$, $y = 0$, $\lambda = 2$.