# Correspondence Analysis and Related Methods


Michael Greenacre
Universitat Pompeu Fabra
Barcelona


www.econ.upf.edu/~michael

www.globalsong.net

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WIRTSCHAFTS UNIVERSITÄT WIEN VIENNA UNIVERSITY OF ECONOMICS AND BUSINESS

## PROGRAM

| | |
|---|---|
| Monday | May 10, 12:00 pm – 4:00 pm – SR Statistik |
| Tuesday | May 11, 2:00 pm – 6:00 pm – 2H564 (PC Labor Statistik) |
| Wednesday | May 12, 12:00 pm – 2:00 pm – SR Statistik |
| Wednesday | May 12, 2:00 pm – 4:00 pm – 2H564 (PC Labor Statistik) |
| Monday | May 17, 12:00 pm – 2:00 pm – SR Statistik |
| Tuesday | May 18, 2:00 pm – 4:00 pm – 2H564 (PC Labor Statistik) |
| Wednesday | May 19, 2:00 pm – 4:00 pm – SR Statistik |
| Tuesday | May 25, 10:00 am – 12:00 – 2H564 (PC Labor Statistik) |
| Wednesday | May 26, 10:00 am – 12:00 – SR Statistik |

## COURSE CONTENTS: main themes

Theme 1: Introduction to multivariate data and multivariate analysis

Theme 2: Geometric concepts of correspondence analysis and related methods

Theme 3: Theory of correspondence analysis and related methods: the SVD

Theme 4: Biplots

Theme 5: Diagnostics for interpretation

Theme 5: Multiple & joint correspondence analysis

Theme 6: Extension to other types of data: ratings, rankings, square matrices

Theme 7: Investigating stability using bootstrap; testing hypotheses using permutation test

## BIBLIOGRAPHY and SUPPORTING MATERIAL


Greenacre, M. and Blasius, J. (2006). Multiple Correspondence Analysis and Related Methods. Chapman & Hall /CRC Press.

Greenacre, M. (2007). Correspondence Analysis in Practice, 2nd edition. Chapman & Hall/ CRC Press.


Some PDFs of selected articles…

Web page of course material and R scripts:

**www.econ.upf.edu/~michael/CARME**

# Introduction to multivariate data and multivariate analysis

# Introduction to multivariate data

- Let's start with some simple trivariate data...

Continuous variables

*X1* – Purchasing power/capita (euros)
*X2* – GDP/capita (index)
*X3* – inflation rate (%)

| | Country | X1 | X2 | X3 |
|---|---|---|---|---|
| **Be** | Belgium | 19200 | 115.2 | 4.5 |
| **De** | Denmark | 20400 | 120.1 | 3.6 |
| **Ge** | Germany | 19500 | 115.6 | 2.8 |
| **Gr** | Greece | 18800 | 94.3 | 4.2 |
| **Sp** | Spain | 17600 | 102.6 | 4.1 |
| **Fr** | France | 19600 | 108.0 | 3.2 |
| **Ir** | Ireland | 20800 | 135.4 | 3.1 |
| **It** | Italy | 18200 | 101.8 | 3.5 |
| **Lu** | Luxembourg | 28800 | 276.4 | 4.1 |
| **Ne** | Netherlands | 20400 | 134.0 | 2.2 |
| **Po** | Portugal | 15000 | 76.0 | 2.7 |
| **UK** | United Kingdom | 22600 | 116.2 | 3.6 |

Count variables

*C1* – Glance reader
*C2* – Fairly thorough reader
*C3* – Very thorough reader

| **Education** | | *C1* | *C2* | *C3* |
|---|---|---|---|---|
| **Some primary** | **E1** | 5 | 7 | 2 |
| **Primary completed** | **E2** | 18 | 46 | 20 |
| **Some secondary** | **E3** | 19 | 29 | 39 |
| **Secondary completed** | **E4** | 12 | 40 | 49 |
| **Some tertiary** | **E5** | 3 | 7 | 16 |

# Visualizing trivariate continuous data

Continuous variables

*X1* – Purchasing power/capita (euros)
*X2* – GDP/capita (index)
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| | Country | X1 | X2 | X3 |
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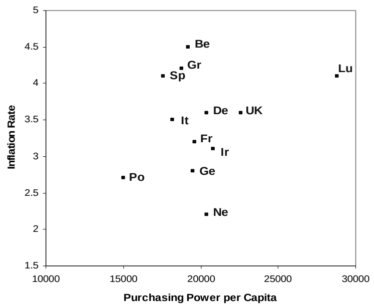


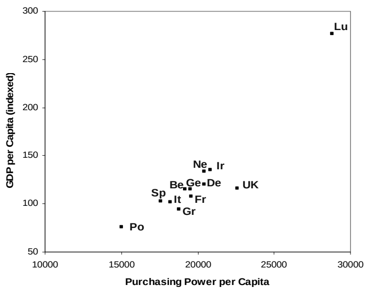


# Visualizing trivariate continuous data

Continuous variables

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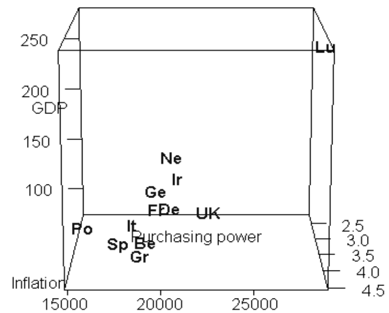

# Visualizing trivariate continuous data

Continuous variables

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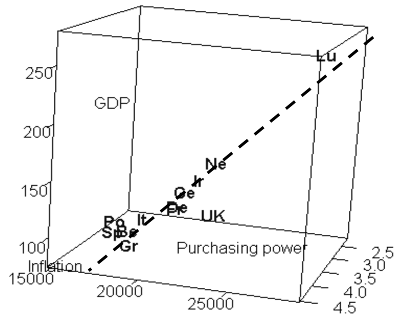


# Visualizing trivariate continuous data

| cor | *X1* | *X2* | *X3* |
|---|---|---|---|
| *X1* | 1.000 | 0.929 | 0.243 |
| *X2* | 0.929 | 1.000 | 0.207 |
| *X3* | 0.243 | 0.207 | 1.000 |

Continuous variables

*X1* – Purchasing power/capita (euro
*X2* – GDP/capita (index)
*X3* – inflation rate (%)

| | Country | *X1* | *X2* | *X3* |
|---|---|---|---|---|
| **Be** | Belgium | 19200 | 115.2 | 4.5 |
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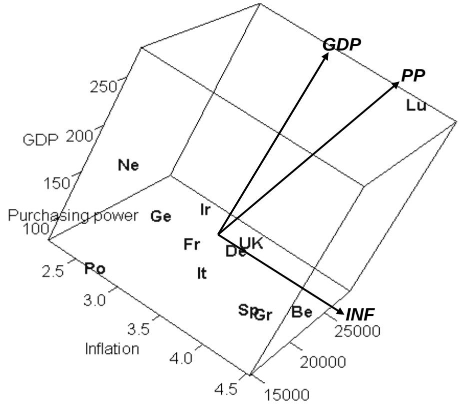



# Visualizing trivariate count data

Count variables

*C1* – Glance reader
*C2* – Fairly thorough reader
*C3* – Very thorough reader

**Education**

| | | *C1* | *C2* | *C3* | |
|---|---|---|---|---|---|
| **Primary incomplete** | E1 | 5 | 7 | 2 | *14* |
| **Primary completed** | E2 | 18 | 46 | 20 | *84* |
| **Secondary incomplete** | E3 | 19 | 29 | 39 | *87* |
| **Secondary completed** | E4 | 12 | 40 | 49 | *101* |
| **Some tertiary** | E5 | 3 | 7 | 16 | *263* |

→ row profiles

| | *C1* | *C2* | *C3* | |
|---|---|---|---|---|
| E1 | .36 | .50 | .14 | *1* |
| E2 | .21 | .55 | .24 | *1* |
| E3 | .22 | .33 | .45 | *1* |
| E4 | .12 | .40 | .49 | *1* |
| E5 | .12 | .27 | .62 | *1* |

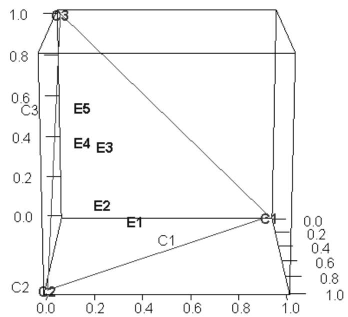
# Visualizing trivariate count data

Count variables

*C1* – Glance reader
*C2* – Fairly thorough reader
*C3* – Very thorough reader

row profiles

| **Education** | | *C1* | *C2* | *C3* | |
|---|---|---|---|---|---|
| **Some primary** | E1 | .36 | .50 | .14 | *1* |
| **Primary completed** | E2 | .21 | .55 | .24 | *1* |
| **Some secondary** | E3 | .22 | .33 | .45 | *1* |
| **Secondary completed** | E4 | .12 | .40 | .49 | *1* |
| **Some tertiary** | E5 | .12 | .27 | .62 | *1* |
| | C1 | 1 | 0 | 0 | *1* |
| | C2 | 1 | 0 | 0 | *1* |
| | C3 | 1 | 0 | 0 | *1* |



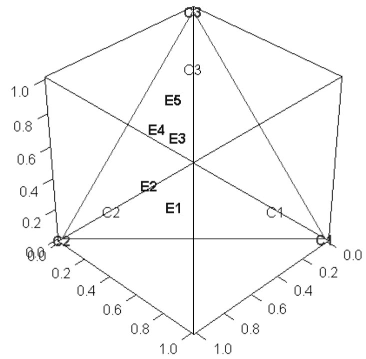
# Visualizing trivariate count data

Count variables

*C1* – Glance reader
*C2* – Fairly thorough reader
*C3* – Very thorough reader

row profiles

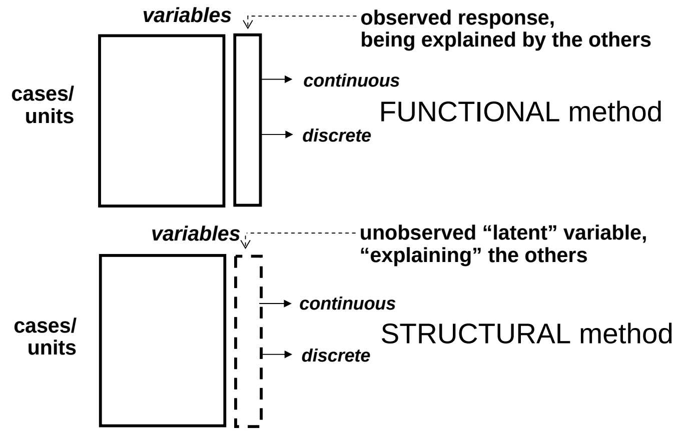
| Education | | *C1* | *C2* | *C3* | |
|---|---|---|---|---|---|
| Some primary | E1 | .36 | .50 | .14 | *1* |
| Primary completed | E2 | .21 | .55 | .24 | *1* |
| Some secondary | E3 | .22 | .33 | .45 | *1* |
| Secondary completed | E4 | .12 | .40 | .49 | *1* |
| Some tertiary | E5 | .12 | .27 | .62 | *1* |
| | C1 | 1 | 0 | 0 | *1* |
| | C2 | 1 | 0 | 0 | *1* |
| | C3 | 1 | 0 | 0 | *1* |




This is almost a correspondence analysis!

# A basic scheme of multivariate analysis

All multivariate methods fall basically into two types, depending on the data structure and the question being asked:




# Four corners of multivariate analysis

**FUNCTIONAL method**

Generalized linear & additive models (GLMs & GAMs)
Classification and regression trees

- **regression**
  - ***regression***
  - ***analysis of (co)variance***
  - ***general linear model***
- **classification**
  - ***discriminant analysis***
  - ***logistic regression***
  - ***pattern recognition***

**CONTINUOUS variable of interest** — **DISCRETE variable of interest**

Structural equation models
Partial least squares
Canonical correspondence analysis

- **scaling**
  - ***principal component analysis***
  - ***factor analysis***
  - ***correspondence analysis***
  - ***multidimensional scaling***
- **cluster analysis**
  - ***hierarchical clustering***
  - ***nonhierarchical clustering***
  - ***latent class analysis***

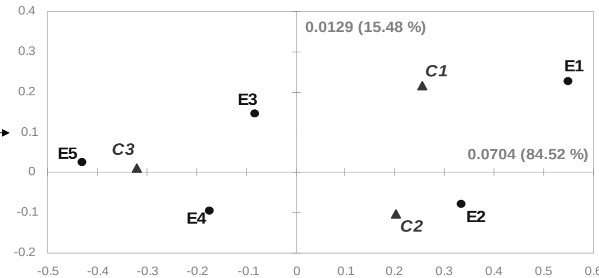
**STRUCTURAL method**

## Basic geometric concepts of correspondence analysis and related methods
(principal component analysis, log-ratio analysis, discriminant analysis, multidimensional scaling…)

# Basic geometric concepts

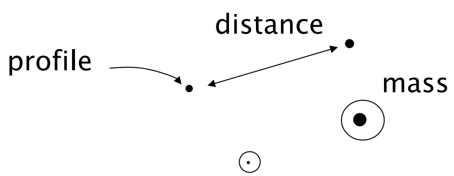
- 312 respondents, all readers of a certain newspaper, cross-tabulated according to their education group and level of reading of the newspaper

| | *C1* | *C2* | *C3* |
|---|---|---|---|
| **E1** | 5 | 7 | 2 |
| **E2** | 18 | 46 | 20 |
| **E3** | 19 | 29 | 39 |
| **E4** | 12 | 40 | 49 |
| **E5** | 3 | 7 | 16 |




- **E1**: some primary **E2**: primary completed **E3**: some secondary **E4**: secondary completed **E5**: some tertiary
- ***C1***: glance ***C2***: fairly thorough ***C3***: very thorough
- We use this simple example to explain the three basic concepts of CA: **profile**, **mass** and (chi-square) **distance**

# Three basic geometric concepts




profile – *the coordinates (position) of the point*

mass – *the weight given to the point*

(chi-square) distance – *the measure of proximity between points*

# Profile

- A profile is a set of relative frequencies, that is a set of frequencies expressed relative to their total (often in percentage form).
- Each row or each column of a table of frequencies defines a different profile.
- It is these profiles which CA visualises as points in a map.

*original data*

| | *C1* | *C2* | *C3* | |
|---|---|---|---|---|
| **E1** | 5 | 7 | 2 | *14* |
| **E2** | 18 | 46 | 20 | *84* |
| **E3** | 19 | 29 | 39 | *87* |
| **E4** | 12 | 40 | 49 | *101* |
| **E5** | 3 | 7 | 16 | *26* |
| | *57* | *129* | *126* | *312* |

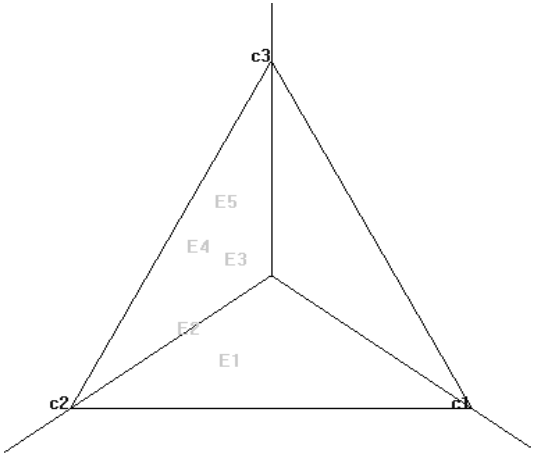
*row profiles*

| | *C1* | *C2* | *C3* | |
|---|---|---|---|---|
| **E1** | .36 | .50 | .14 | *1* |
| **E2** | .21 | .55 | .24 | *1* |
| **E3** | .22 | .33 | .45 | *1* |
| **E4** | .12 | .40 | .49 | *1* |
| **E5** | .12 | .27 | .62 | *1* |
| | *.18* | *.41* | *.40* | *1* |

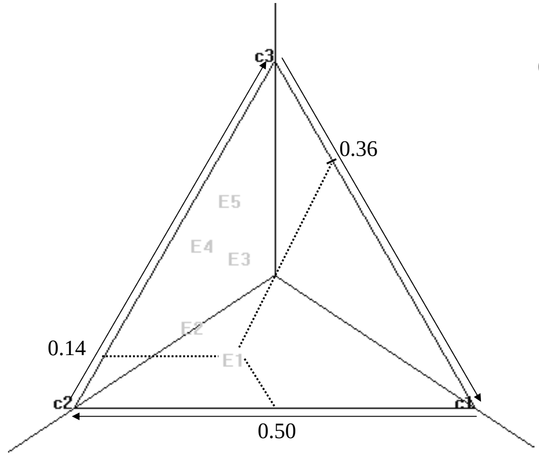
*column profiles*

| | *C1* | *C2* | *C3* | |
|---|---|---|---|---|
| **E1** | .09 | .05 | .02 | *.05* |
| **E2** | .32 | .37 | .16 | *.27* |
| **E3** | .33 | .22 | .31 | *.28* |
| **E4** | .21 | .31 | .39 | *.32* |
| **E5** | .05 | .05 | .13 | *.08* |
| | *1* | *1* | *1* | *1* |

# Row profiles viewed in 3-d



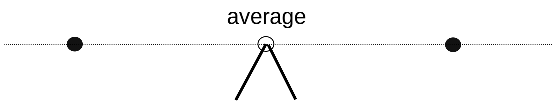

# Plotting profiles in profile space (triangular coordinates)



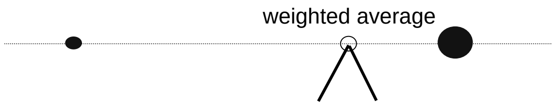

E1 :

0.36 0.50 0.14

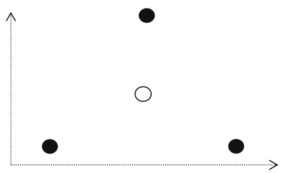
# Weighted average (centroid)

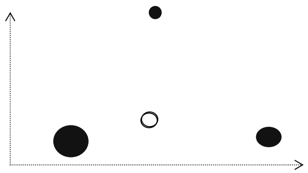



The average is the point at which the two points are balanced.

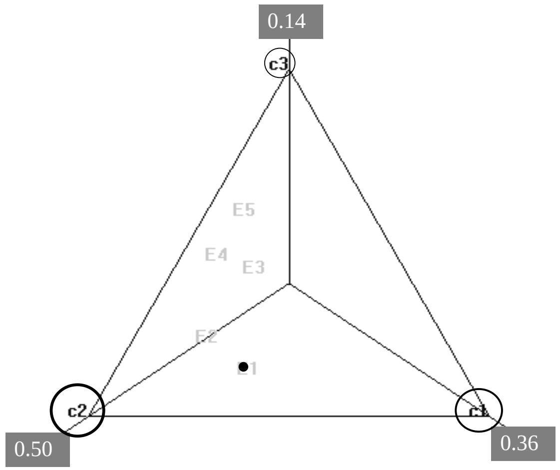



The situation is identical for multidimensional points...





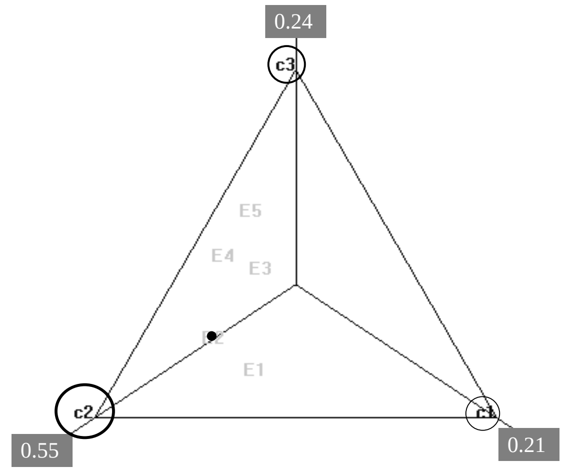
# Plotting profiles in profile space (barycentric – or weighted average – principle)




E1:

0.36 0.50 0.14

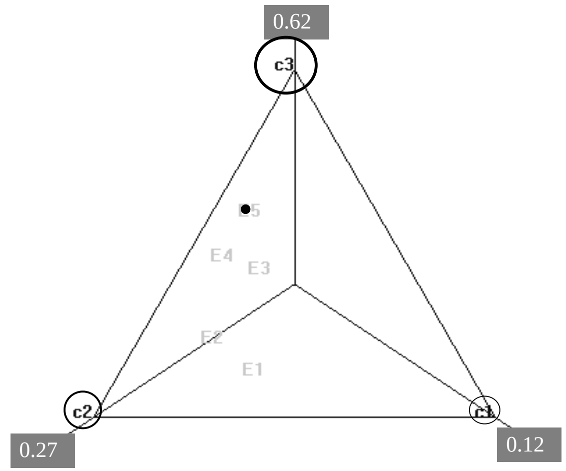
# Plotting profiles in profile space (barycentric – or weighted average – principle)




E2:

0.21 0.55 0.24

# Plotting profiles in profile space (barycentric – or weighted average – principle)



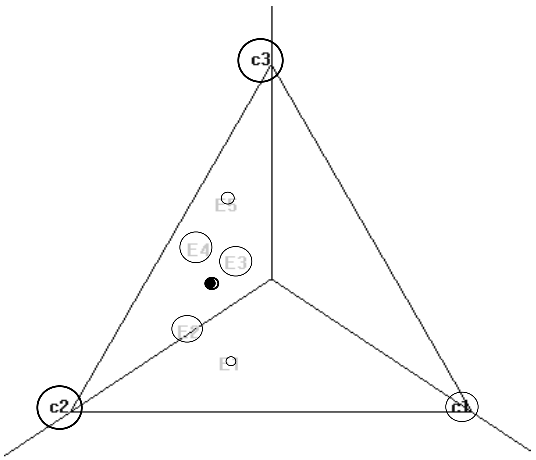

E5:

0.12 0.27 0.62

# Masses of the profiles

*original data*

| | *C1* | *C2* | *C3* | | *masses* |
|---|---|---|---|---|---|
| **E1** | **5** | **7** | **2** | *14* | *.045* |
| **E2** | **18** | **46** | **20** | *84* | *.269* |
| **E3** | **19** | **29** | **39** | *87* | *.279* |
| **E4** | **12** | **40** | **49** | *101* | *.324* |
| **E5** | **3** | **7** | **16** | *26* | *.083* |
| | *57* | *129* | *126* | *312* | *1* |
| ***average row profile*** | *.183* | *.413* | *.404* | | *1* |




# Readership data

| | *Education Group* | *C1* | *C2* | *C3* | *Total* | *Mass* |
|---|---|---|---|---|---|---|
| E1 | Some primary | 5 *(0.357)* | 7 *(0.500)* | 2 *(0.143)* | 14 | ***0.045*** |
| E2 | Primary completed | 18 *(0.214)* | 46 *(0.548)* | 20 *(0.238)* | 84 | ***0.269*** |
| E3 | Some secondary | 19 *(0.218)* | 29 *(0.333)* | 39 *(0.448)* | 87 | ***0.279*** |
| E4 | Secondary completed | 12 *(0.119)* | 40 *(0.396)* | 49 *(0.485)* | 101 | ***0.324*** |
| E5 | Some tertiary | 3 *(0.115)* | 7 *(0.269)* | 16 *(0.615)* | 26 | ***0.083*** |
| | *Total* | 57 ***(0.183)*** | 129 ***(0.413)*** | 126 ***(0.404)*** | 312 | |

***C1**: glance* ***C2**: fairly thorough* ***C3**: very thorough*

# Calculating chi-square

$$\chi^2 = \textit{12 similar terms ....} + \frac{(3 - 4.76)^2}{4.76} + \frac{(7 - 10.74)^2}{10.74} + \frac{(16 - 10.50)^2}{10.50}$$

$$= 26.0$$

| | *Education Group* | **C1** | **C2** | **C3** | *Total* | *Mass* |
|---|---|---|---|---|---|---|
| .... | ………….. | .... | .... | .... | **14** | **....** |
| .... | ………….. | .... | .... | .... | **84** | **....** |
| .... | ………….. | .... | .... | .... | **87** | **....** |
| .... | ………….. | .... | .... | .... | **101** | **....** |
| **E5** | *Observed Frequency* Some tertiary *Expected Frequency* | **3** *(0.115)* ***4.76*** | **7** *(0.269)* ***10.74*** | **16** *(0.615)* ***10.50*** | **26** | ***0.083*** |
| | *Total* | 57 ***(0.183)*** | 129 ***(0.413)*** | 126 ***(0.404)*** | **312** | |

*For example, expected frequency of (***E5,C1***):*

***0.183*** x ***26 = 4.76***

# Calculating chi-square

$$\chi^2 = \textit{12 similar terms ....} + 26\left[\frac{(3/26 - 4.76/26)^2}{4.76/26} + \frac{(7/26 - 10.74/26)^2}{10.74/26} + \frac{(16/26 - 10.50/26)^2}{10.50/26}\right]$$

$$\chi^2 / \mathbf{312} = \textit{12 similar terms ....} + 0.083\left[\frac{(0.115 - 0.183)^2}{0.183} + \frac{(0.269 - 0.413)^2}{0.413} + \frac{(0.615 - 0.404)^2}{0.404}\right]$$

| | *Education Group* | **C1** | **C2** | **C3** | *Total* | *Mass* |
|---|---|---|---|---|---|---|
| .... | ………….. | .... | .... | .... | **14** | **....** |
| .... | ………….. | .... | .... | .... | **84** | **....** |
| .... | ………….. | .... | .... | .... | **87** | **....** |
| .... | ………….. | .... | .... | .... | **101** | **....** |
| **E5** | *Observed Frequency* Some tertiary *Expected Frequency* | **3** *(0.115)* ***4.76*** | **7** *(0.269)* ***10.74*** | **16** *(0.615)* ***10.50*** | **26** | ***0.083*** |
| | *Total* | 57 ***(0.183)*** | 129 ***(0.413)*** | 126 ***(0.404)*** | **312** | |

# Calculating inertia

$$\text{Inertia} = \chi^2 / 312 = \textit{similar terms for first four rows ...}$$

$$+ \underbrace{0.083}_{\text{mass}}\underbrace{\left[\frac{(0.115 - 0.183)^2}{0.183} + \frac{(0.269 - 0.413)^2}{0.413} + \frac{(0.615 - 0.404)^2}{0.404}\right]}_{\text{squared chi-square distance}}$$

***mass*** (of row **E5**)

***squared chi-square distance*** (between the profile of **E5** and the average profile)

$$\text{Inertia} = \sum \textit{mass} \times (\textit{chi-square distance})^2$$

$$\frac{(0.115 - 0.183)^2}{0.183} + \frac{(0.269 - 0.413)^2}{0.413} + \frac{(0.615 - 0.404)^2}{0.404}$$

EUCLIDEAN WEIGHTED

# How can we see chi-square distances?

$$\text{Inertia} = \chi^2 / 312 = \textit{similar terms for first four rows ...}$$

$$+ \underbrace{0.083}_{\text{mass}}\underbrace{\left[\frac{(0.115 - 0.183)^2}{0.183} + \frac{(0.269 - 0.413)^2}{0.413} + \frac{(0.615 - 0.404)^2}{0.404}\right]}_{\text{squared chi-square distance}}$$

***mass*** (of row **E5**)

***squared chi-square distance*** (between the profile of **E5** and the average profile)

$$\frac{(0.115 - 0.183)^2}{0.183} + \frac{(0.269 - 0.413)^2}{0.413} + \frac{(0.615 - 0.404)^2}{0.404}$$

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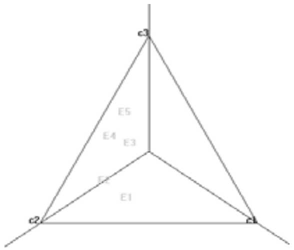
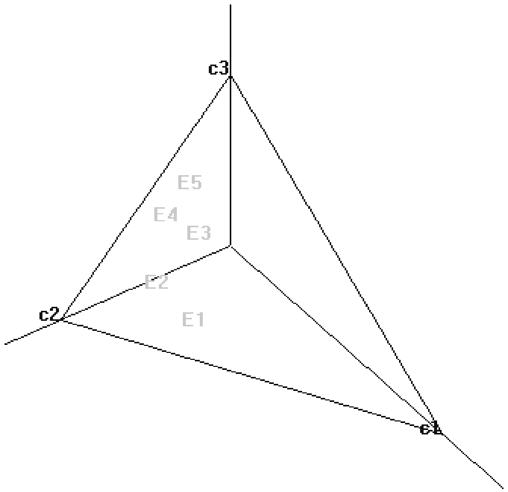
$$\left(\frac{0.115}{\sqrt{0.183}} - \frac{0.183}{\sqrt{0.183}}\right)^2 + \left(\frac{0.269}{\sqrt{0.413}} - \frac{0.413}{\sqrt{0.413}}\right)^2 + \left(\frac{0.615}{\sqrt{0.404}} - \frac{0.404}{\sqrt{0.404}}\right)^2$$

So the answer is to divide all profile elements by the √ of their averages

# “Stretched” row profiles viewed in 3-d chi-squared space

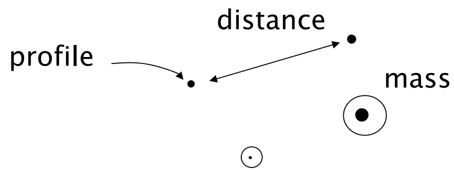


“Pythagorian” – ordinary Euclidean distances




Chi-square distances

# Three basic geometric concepts

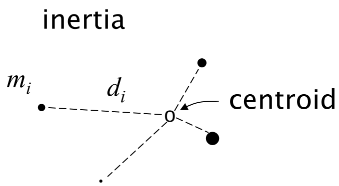


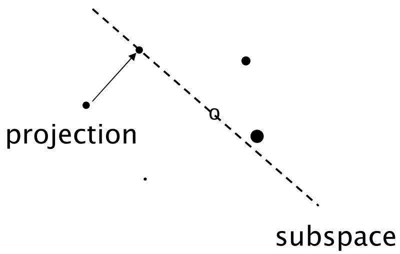

profile – *the coordinates (position) of the point*

mass – *the weight given to the point*

(chi-square) distance – *the measure of proximity between points*

# Four derived geometric concepts




inertia = $\sum_i m_i d_i^2$

centroid – *the weighted average position*

inertia – *the weighted sum-of-squared distances to centroid*

subspace – *space of reduced dimensionality within the space (it will go through the centroid)*

projection – *the closest point in the subspace*

# Summary: Basic geometric concepts

- **Profiles** are rows or columns of relative frequencies, that is the rows or columns expressed relative to their respective marginals, or bases.
- Each profile has a weight assigned to it, called the **mass**, which is proportional to the original marginal frequency used as a base .
- The **average profile** is the the centroid (weighted average) of the profiles.
- **Vertex profiles** are the extreme profiles in the profile space (“simplex”).
- Profiles are weighted averages of the vertices, using the profile elements as weights.
- The **dimensionality** of an $I \times J$ matrix = min{$I - 1$, $J - 1$}
- The **chi-square distance** measures the difference between profiles, using an Euclidean-type function which standardizes each profile element by dividing by the square root of its expected value.
- The **(total) inertia** can be expressed as the weighted average of the squared chi-square distances between the profiles and their average.

# The one-minute CA course

- The 'famous' smoking data.

| staff group | | *smoking class* *none* | *light* | *medium* | *heavy* | sum |
|---|---|---|---|---|---|---|
| Senior managers | **SM** | 4 | 2 | 3 | 2 | *11* |
| Junior managers | **JM** | 4 | 3 | 7 | 4 | *18* |
| Senior employees | **SE** | 25 | 10 | 12 | 4 | *51* |
| Junior employees | **JE** | 18 | 24 | 33 | 13 | *88* |
| Secretaries | **SC** | 10 | 6 | 7 | 2 | *25* |
| | sum | *61* | *45* | *62* | *25* | *193* |

- Now for the one-minute course in correspondence analysis, possible thanks to dynamic graphics!

# One minute CA course: slide 1

3 columns

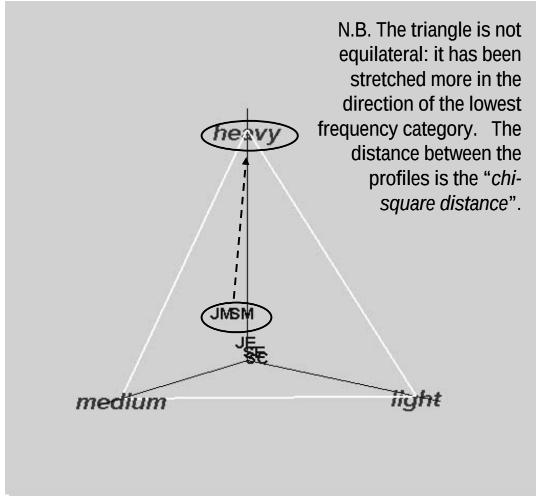
| | *light* | *medium* | *heavy* | sum |
|---|---|---|---|---|
| **SM** | 2 | 3 | 2 | 7 |
| **JM** | 3 | 7 | 4 | 14 |
| **SE** | 10 | 12 | 4 | 26 |
| **JE** | 24 | 33 | 13 | 70 |
| **SC** | 6 | 7 | 2 | 15 |

express relative to row sums

These are called "row profiles"

| | *light* | *medium* | *heavy* | sum |
|---|---|---|---|---|
| **SM** | 0.29 | 0.43 | 0.29 | 1 |
| **JM** | 0.21 | 0.50 | 0.29 | 1 |
| **SE** | 0.38 | 0.46 | 0.15 | 1 |
| **JE** | 0.34 | 0.47 | 0.19 | 1 |
| **SC** | 0.40 | 0.47 | 0.13 | 1 |

plot




# One minute CA course: slide 2

Relative values of row sums are used to weight the row profiles

4 columns

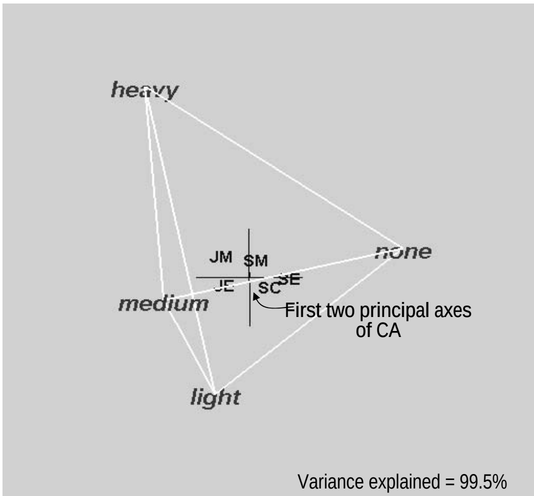
| | *none* | *light* | *medium* | *heavy* | sum |
|---|---|---|---|---|---|
| **SM** | 4 | 2 | 3 | 2 | *11* |
| **JM** | 4 | 3 | 7 | 4 | *18* |
| **SE** | 25 | 10 | 12 | 4 | *51* |
| **JE** | 18 | 24 | 33 | 13 | *88* |
| **SC** | 10 | 6 | 7 | 2 | *25* |
| sum | *61* | *45* | *62* | *25* | *193* |

express relative to row sums

| | *none* | *light* | *medium* | *heavy* | sum |
|---|---|---|---|---|---|
| **SM** | 0.36 | 0.18 | 0.27 | 0.18 | *1* |
| **JM** | 0.22 | 0.17 | 0.39 | 0.22 | *1* |
| **SE** | 0.49 | 0.20 | 0.24 | 0.08 | *1* |
| **JE** | 0.20 | 0.27 | 0.38 | 0.15 | *1* |
| **SC** | 0.40 | 0.24 | 0.28 | 0.08 | *1* |
| sum | *0.32* | *0.23* | *0.32* | *0.13* | *1* |

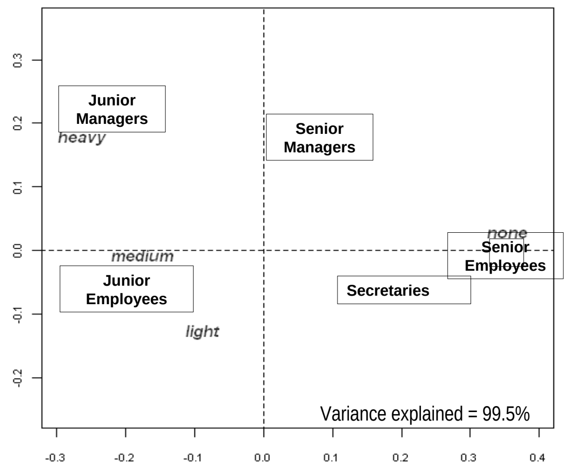
Average profile is the centre of the CA map

plot



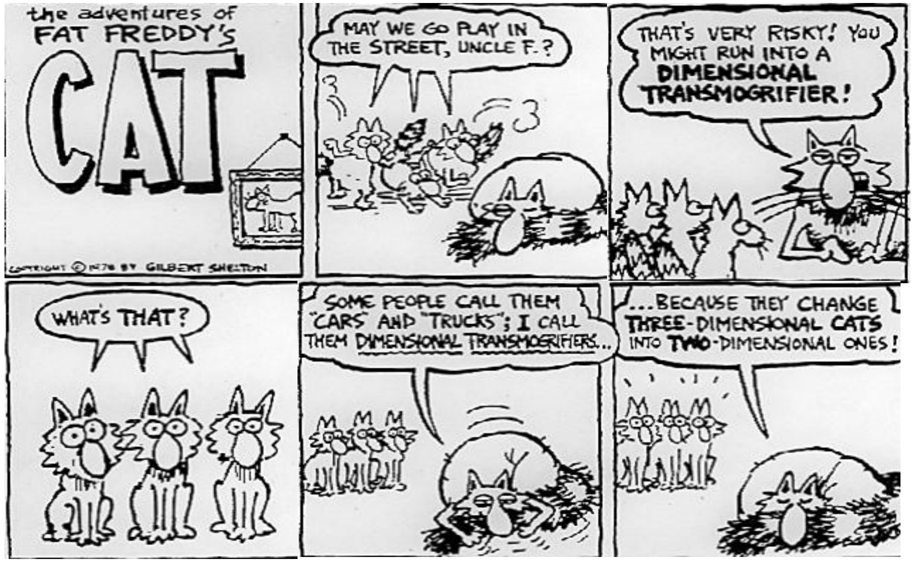

# On minute CA course: slide 3

often rescale result so that rows and columns have same dispersions along the axes



# Dimension reduction
# Joint display of rows and columns

# Dimensional Transmogrifier




*with thanks to Jörg Blasius*

# The "famous" smoking data: row problem

- Artificial example designed to illustrate two-dimensional maps

| | no | li | me | hv |
|---|---|---|---|---|
| Senior managers **SM** | 4 | 2 | 3 | 2 |
| Junior managers **JM** | 4 | 3 | 7 | 4 |
| Senior employees **SE** | 25 | 10 | 12 | 4 |
| Junior employees **JE** | 18 | 24 | 33 | 13 |
| Secretaries **SC** | 10 | 6 | 7 | 2 |

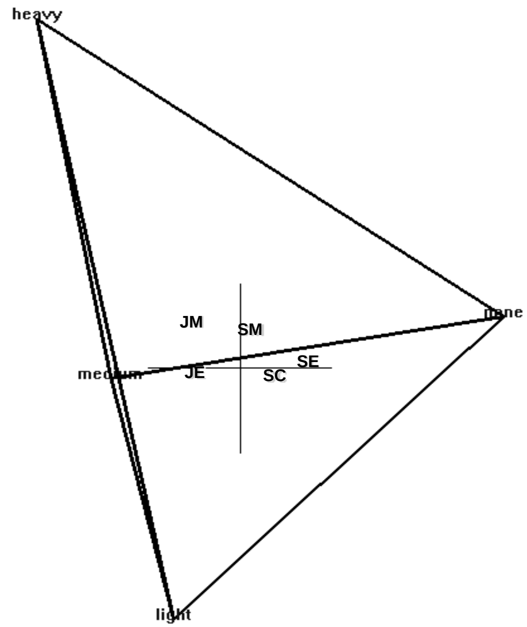
→

*row profiles*

| | | | | |
|---|---|---|---|---|
| **SM** | .36 | .18 | .27 | .18 |
| **JM** | .22 | .17 | .39 | .22 |
| **SE** | .49 | .20 | .24 | .08 |
| **JE** | .20 | .27 | .38 | .15 |
| **SC** | .40 | .24 | .28 | .08 |
| **ave** | .32 | .23 | .32 | .13 |
| **none** | 1 | 0 | 0 | 0 |
| **light** | 0 | 1 | 0 | 0 |
| **medium** | 0 | 0 | 1 | 0 |
| **heavy** | 0 | 0 | 0 | 1 |

- 193 employees of a firm
- 5 categories of staff group
- 4 categories of smoking (none/light/medium/heavy)

# View of row profiles in 3-d



# The “famous” smoking data: column problem

| | no | li | me | hv |
|---|---|---|---|---|
| Senior managers **SM** | 4 | 2 | 3 | 2 |
| Junior managers **JM** | 4 | 3 | 7 | 4 |
| Senior employees **SE** | 25 | 10 | 12 | 4 |
| Junior employees **JE** | 18 | 24 | 33 | 13 |
| Secretaries **SC** | 10 | 6 | 7 | 2 |

↓

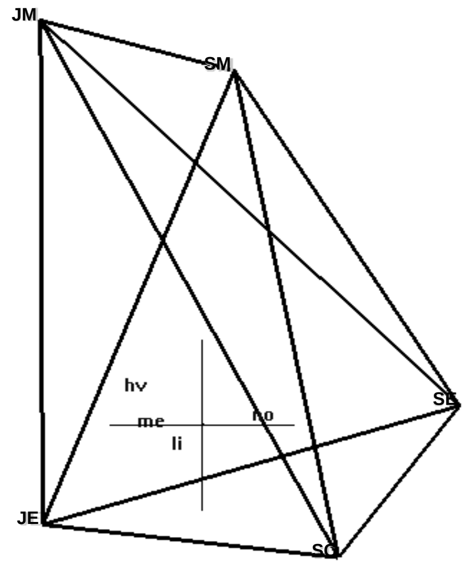
*column profiles*

| no | li | me | hv | ave | SM | JM | SE | JE | SC |
|---|---|---|---|---|---|---|---|---|---|
| .07 | .04 | .05 | .08 | .06 | 1 | 0 | 0 | 0 | 0 |
| .07 | .07 | .11 | .16 | .09 | 0 | 1 | 0 | 0 | 0 |
| .41 | .22 | .19 | .16 | .26 | 0 | 0 | 1 | 0 | 0 |
| .30 | .53 | .53 | .52 | .46 | 0 | 0 | 0 | 1 | 0 |
| .16 | .13 | .11 | .08 | .13 | 0 | 0 | 0 | 0 | 1 |

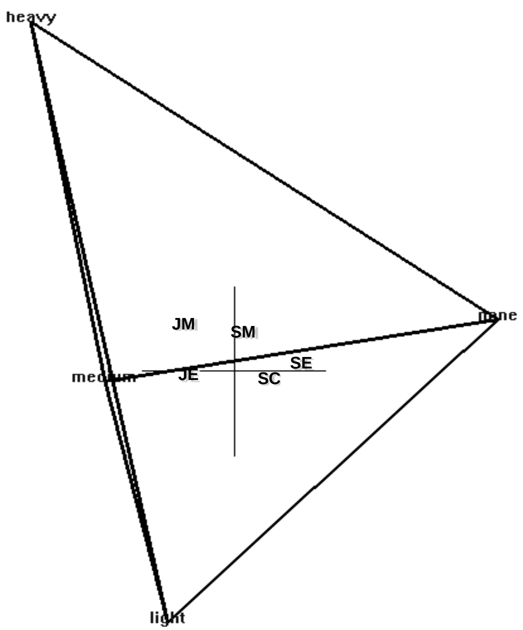
It seems like the column profiles, with 5 elements, are 4-dimensional, BUT there are only 4 points and 4 points lie exactly in 3 dimensions.

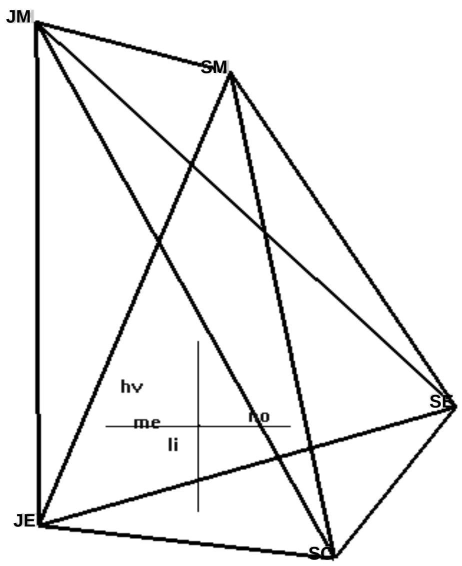
So the dimensionality of the columns is the same as the rows.

# View of column profiles in 3-d




# View of both profiles and vertices in 3-d




# What CA does…

- … centres the row and column profiles with respect to their average profiles, so that the origin represents the average.
- … re-defines the dimensions of the space in an ordered way: first dimension “explains” the maximum amount of inertia possible in one dimension; second adds the maximum amount to first (hence first two explain the maximum amount in two dimensions), and so on… until all dimensions are “explained”.
- … decomposes the total inertia along the principal axes into principal inertias, usually expressed as % of the total.
- … so if we want a low-dimensional version, we just take the first (principal) dimensions

The row and column problem solutions are closely related, one can be obtained from the other; there are simple scaling factors along each dimension relating the two problems.

## Singular value decomposition

## Generalized principal component analysis

## Generalized SVD

We often want to associate weights on the rows and columns, so that the fit is by weighted least-squares, not ordinary least squares, that is we want to minimize

$$\text{RSS} = \sum_{i=1}^{n}\sum_{j=1}^{p} r_i c_j (x_{ij} - x_{ij}^*)^2$$

$$\mathbf{D}_r^{1/2}\mathbf{X}\mathbf{D}_c^{1/2} = \mathbf{U}\mathbf{D}_\alpha\mathbf{V}^\mathsf{T} \quad \text{where} \quad \mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{V}^\mathsf{T}\mathbf{V} = \mathbf{I}\,,\ \alpha_1 \geq \alpha_2 \geq \cdots \geq 0$$

$$\mathbf{X} = \mathbf{D}_r^{-1/2}\mathbf{U}\mathbf{D}_\alpha(\mathbf{D}_c^{-1/2}\mathbf{V})^\mathsf{T}$$

$$\mathbf{X}^* = etc\ldots$$

## Generalized principal component analysis

- Suppose we want to represent the (centred) rows of a matrix $\mathbf{Y}$, weighted by (positive) elements down diagonal of matrix $\mathbf{D}_r$, where distance between rows is in the (weighted) metric defined by matrix $\mathbf{D}_m^{-1}$.
- Total inertia = $\sum_i\sum_j q_i(1/m_j)y_{ij}^2$
- $\mathbf{S} = \mathbf{D}_q^{½}\,\mathbf{Y}\,\mathbf{D}_m^{-½} = \mathbf{U}\,\mathbf{D}_\alpha\mathbf{V}^\mathsf{T}$ where $\mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{V}^\mathsf{T}\mathbf{V} = \mathbf{I}$
- Principal coordinates of rows: $\mathbf{F} = \mathbf{D}_q^{-½}\,\mathbf{U}\,\mathbf{D}_\alpha$
- Principal axes of the rows: $\mathbf{D}_m^{½}\,\mathbf{V}$
- Standard coordinates of columns: $\mathbf{G} = \mathbf{D}_m^{-½}\,\mathbf{V}$
- Variances (inertias) explained: $\lambda_1 = \alpha_1^2,\ \lambda_2 = \alpha_2^2, \ldots$

## Correspondence analysis

Of the rows:

- $\mathbf{Y}$ is the centred matrix of row profiles
- row masses in $\mathbf{D}_q$ are the relative frequencies of the rows
- column weights in $\mathbf{D}_w$ are the inverses of the relative frequencies of the columns
- Total inertia = $\chi^2/n$

Of the columns:

- $\mathbf{Y}$ is the centred matrix of column profiles
- column masses in $\mathbf{D}_q$ are the relative frequencies of the columns
- row weights in $\mathbf{D}_w$ are the inverses of the relative frequencies of the rows
- Total inertia = $\chi^2/n$

Both problems lead to the SVD of the same matrix

# Correspondence analysis

- Table of nonnegative data **N**
- Divide **N** by its grand total $n$ to obtain the so-called *correspondence matrix* $\mathbf{P} = (1/n)\,\mathbf{N}$
- Let the row and column marginal totals of **P** be the vectors **r** and **c** respectively, that is the vectors of row and column *masses*, and $\mathbf{D}_r$ and $\mathbf{D}_c$ be the diagonal matrices of these masses

⋮ *(to be derived algebraically in class)*

$$\mathbf{S} = \mathbf{D}_r^{-1/2}(\mathbf{P} - \mathbf{r}\mathbf{c}^{\mathsf{T}})\mathbf{D}_c^{-1/2} \qquad \boxed{\frac{p_{ij} - r_i c_j}{\sqrt{r_i c_j}}}$$

or equivalently

$$\mathbf{S} = \mathbf{D}_r^{1/2}(\mathbf{D}_r^{-1}\mathbf{P}\mathbf{D}_c^{-1} - \mathbf{1}\mathbf{1}^{\mathsf{T}})\mathbf{D}_c^{1/2} \qquad \boxed{\sqrt{r_i}\left(\frac{p_{ij}}{r_i c_j} - 1\right)\sqrt{c_j}}$$

**Principal coordinates** $\mathbf{F} = \mathbf{D}_r^{-1/2}\mathbf{U}\mathbf{D}_\alpha$, $\mathbf{G} = \mathbf{D}_c^{-1/2}\mathbf{V}\mathbf{D}_\alpha$

**Standard coordinates** $\mathbf{\Phi} = \mathbf{D}_r^{-1/2}\mathbf{U}$, $\mathbf{\Gamma} = \mathbf{D}_c^{-1/2}\mathbf{V}$

# Decomposition of total inertia along principal axes

| | *I* rows | *(smoking I=5)* | | *J* columns | *(smoking J=4)* |
|---|---|---|---|---|---|
| Total inertia | in(*I*) | 0.08519 | | in(*J*) | 0.08519 |
| Inertia axis 1 | $\lambda_1$ | 0.07476 | (87.8%) | $\lambda_1$ | 0.07476 |
| Inertia axis 2 | $\lambda_2$ | 0.01002 | (11.8%) | $\lambda_2$ | 0.01002 |
| Inertia axis 3 | $\lambda_3$ | 0.00041 | ( 0.5%) | $\lambda_3$ | 0.00041 |

# Duality (symmetry) of the rows and columns

| | no | li | me | hv | *sum* |
|---|---|---|---|---|---|
| Senior managers **SM** | 4 | 2 | 3 | 2 | 11 |
| Junior managers **JM** | 4 | 3 | 7 | 4 | 18 |
| Senior employees **SE** | 25 | 10 | 12 | 4 | 51 |
| Junior employees **JE** | 18 | 24 | 33 | 13 | 88 |
| Secretaries **SC** | 10 | 6 | 7 | 2 | 25 |
| *sum* | 61 | 45 | 62 | 25 | |

→

| | *row profiles* | | | | *masses* |
|---|---|---|---|---|---|
| **SM** | .36 | .18 | .27 | .18 | .06 |
| **JM** | .22 | .17 | .39 | .22 | .09 |
| **SE** | .49 | .20 | .24 | .08 | .26 |
| **JE** | .20 | .27 | .38 | .15 | .46 |
| **SC** | .40 | .24 | .28 | .08 | .13 |
| **ave** | .32 | .23 | .32 | .13 | |

| | | | | |
|---|---|---|---|---|
| **no** | 1 | 0 | 0 | 0 |
| **li** | 0 | 1 | 0 | 0 |
| **me** | 0 | 0 | 1 | 0 |
| **hv** | 0 | 0 | 0 | 1 |

↓

*column profiles*

| no | li | me | hv | ave | SM | JM | SE | JE | SC |
|---|---|---|---|---|---|---|---|---|---|
| .07 | .04 | .05 | .08 | .06 | 1 | 0 | 0 | 0 | 0 |
| .07 | .07 | .11 | .16 | .09 | 0 | 1 | 0 | 0 | 0 |
| .41 | .22 | .19 | .16 | .26 | 0 | 0 | 1 | 0 | 0 |
| .30 | .53 | .53 | .52 | .46 | 0 | 0 | 0 | 1 | 0 |
| .16 | .13 | .11 | .08 | .13 | 0 | 0 | 0 | 0 | 1 |

| *masses* | .32 | .23 | .32 | .13 |
|---|---|---|---|---|

# Relationship between row and column solutions

| | rows | columns |
|---|---|---|
| standard coordinates | $\mathbf{\Phi} = [\,\phi_{ik}\,]$ | $\mathbf{\Gamma} = [\,\gamma_{jk}\,]$ |
| principal coordinates | $\mathbf{F} = [\,f_{ik}\,]$ | $\mathbf{G} = [\,g_{jk}\,]$ |
| relationships between | $\mathbf{F} = \mathbf{\Phi}\mathbf{D}_\alpha$ | $\mathbf{G} = \mathbf{\Gamma}\mathbf{D}_\alpha$ |
| coordinates | $f_{ik} = \alpha_k\, x_{ik}$ | $g_{jk} = \alpha_k\, y_{jk}$ |

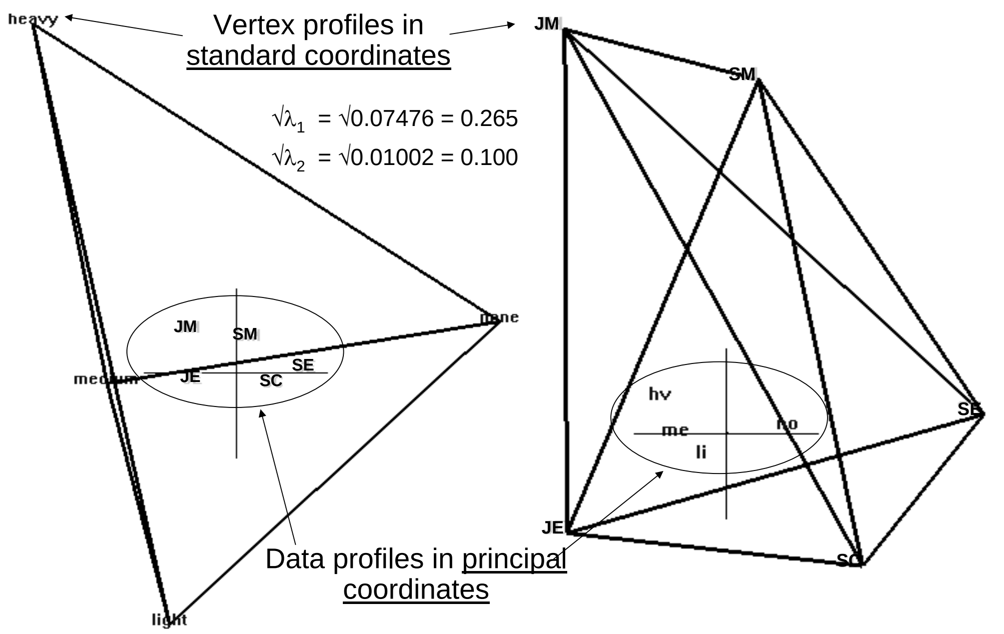
where $\alpha_k = \sqrt{\lambda_k}$ is the square root of the principal inertia on axis $k$

> principal = standard × $\alpha_k$
>
> standard = principal / $\alpha_k$

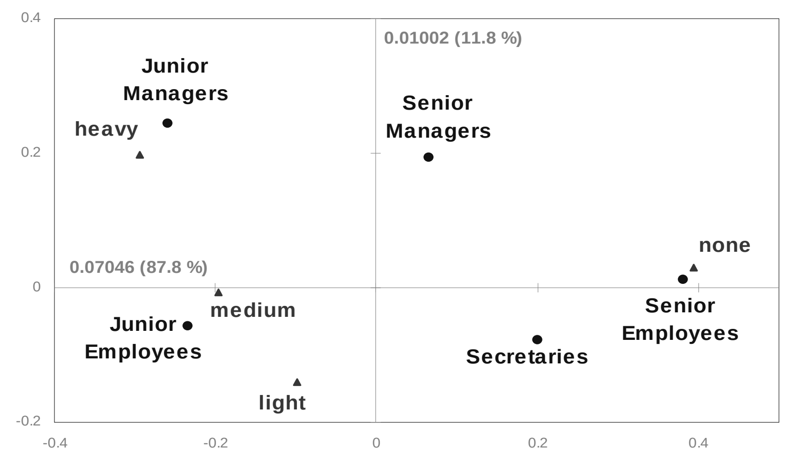
Data profiles in <u>principal coordinates</u>

Vertiex profiles in <u>standard coordinates</u>

## Relationship between row and column solutions




## Symmetric map using XLSTAT




## Summary:
## Relationship between row and column solutions

1. Same dimensionality (*rank*) = min{$I$ –1, $J$ – 1}
2. Same total inertia and same principal inertias $\lambda_1$, $\lambda_2$, …, on each dimension (i.e., same decomposition of inertia along principal axes), hence same percentages of inertia on each dimension
3. "Same" coordinate solutions, up to a scalar constant along each principal axis, which depends on the square root $\sqrt{\lambda_k} = \alpha_k$ of the principal inertia on each axis:

   principal = standard × $\sqrt{\lambda_k}$

   standard = principal / $\sqrt{\lambda_k}$
4. Asymmetric map: one set principal, other standard
5. Symmetric map: both sets principal