

# Testing, Monitoring, and Dating Structural Changes in FX Regimes

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## Overview

- Motivation
  - What is the new Chinese exchange rate regime?
  - Exchange rate regimes
  - Exchange rate regression
- Structural change tools
  - Model frame
  - Testing
  - Monitoring
  - Dating
- Software: strucchange, zoo
- Application: Indian exchange rate regimes
- Summary

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**Initial impulse:** Ajay Shah, long time *R-help* and *R-SIG-Finance* contributor, contacts Achim Zeileis, *strucchange* package maintainer.

Date: Thu, 28 Jul 2005 21:57:10 +0530
From: Ajay Narottam Shah <ajayshah@mayin.org>
To: Achim Zeileis <Achim.Zeileis@wu-wien.ac.at>
Subject: Wonder if this fits (structural breaks work in
a currency regime context)

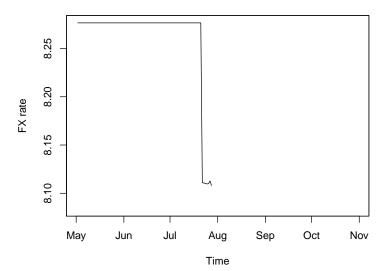
The issues are like this. Many central banks SAY that a currency regime is X. But they routinely lie. Economists would like to know the true currency regime. And, we would like to know the date when something changed.

**Of particular interest:** China gave up on a fixed exchange rate to the US dollar (USD) on 2005-07-21. The People's Bank of China announced that the Chinese yuan (CNY) would no longer be pegged to the USD but to a basket of currencies with greater flexibility.

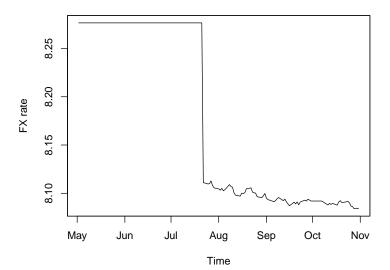
**Collaboration:** Ajay Shah, Ila Patnaik, and Achim Zeileis start to investigate the question *What is the new Chinese exchange rate regime?* 

**First step:** Collect foreign exchange (FX) rates for various currencies for three months up to 2005-10-31.









#### **Exchange rate regimes**

The FX regime of a country determines how it manages its currency wrt foreign currencies. Broadly, it can be

- floating: currency is allowed to fluctuate based on market forces,
- *pegged:* currency has limited flexibility when compared with a basket of currencies or a single currency,
- *fixed:* direct convertibility to another currency.

**Problem:** The *de facto* and *de jure* FX regime in operation in a country often differ. ( $\approx$  *politically correct version of Ajay's original e-mail*)

 $\Rightarrow$  Data-driven classification of FX regimes

The workhorse for de facto FX regime classification is a linear regression model based on log-returns of cross-currency exchange rates (with respect to some floating reference currency). In the literature, this is also known as *Frankel-Wei regression*.

For modeling the log-returns of CNY a basket of regressors USD, JPY, EUR, and GBP (all log-returns wrt CHF) is employed.

Fitting the model for the first three months (up to 2005-10-31, n = 68) shows that a plain USD peg is still in operation.

Ordinary least squares (OLS) estimation gives:

$$CNY_{i} = \underbrace{0.005}_{(0.004)} + \underbrace{0.9997}_{(0.009)} USD_{i} + \underbrace{0.005}_{(0.011)} JPY_{i}$$
$$- \underbrace{0.014}_{(0.027)} EUR_{i} - \underbrace{0.008}_{(0.015)} GBP_{i} + \widehat{\varepsilon}_{i}$$

Only the USD coefficient is significantly different from 0 (but not from 1). The error standard deviation is tiny with  $\hat{\sigma} = 0.028$  leading to  $R^2 = 0.998$ .

#### Questions:

- Is this model for the period 2005-07-26 to 2005-10-31 stable or is there evidence that China kept changing its FX regime after 2005-07-26? (*testing*)
- Oppending on the answer to the first question:
  - Does the CNY stay pegged to the USD in the future (starting from November 2005? (*monitoring*)
  - When and how did the Chinese FX regime change? (*dating*)

**In practice:** Rolling regressions are often used to answer these questions by tracking the evolution of the FX regime in operation.

**More formally:** Structural change techniques can be adapted to the FX regression to estimate and test the stability of FX regimes.

**Problem:** Unlike many other linear regression models, the stability of the error variance (fluctuation band) is of interest as well.

**Solution:** Employ an (approximately) normal regression estimated by ML where the variance is a full model parameter.

**Generic idea:** Consider a regression model for *n* ordered observations  $y_i | x_i$  with *k*-dimensional parameter  $\theta$ . Ordering is typically with respect to time in time-series regressions, but could also be with respect to income, age, etc. in cross-section regressions.

To fit the model to observations i = 1, ..., n an objective function  $\Psi(y, x, \theta)$  is used such that

$$\widehat{\theta} = \operatorname{argmin}_{\theta} \sum_{i=1}^{n} \Psi(y_i, x_i, \theta).$$

This can also be defined implicitly based on the corresponding score function (or estimating function)  $\psi(y, x, \theta) = \partial \Psi(y, x, \theta) / \partial \theta$ :

$$\sum_{i=1}^{n} \psi(\mathbf{y}_i, \mathbf{x}_i, \widehat{\theta}) = \mathbf{0}.$$

This class of M-estimators includes OLS and maximum likelihood (ML) estimation as well as IV, Quasi-ML, robust M-estimation etc.

Under parameter stability and some mild regularity conditions, a central limit theorem holds

$$\sqrt{n}(\widehat{\theta} - \theta_0) \stackrel{d}{\longrightarrow} \mathcal{N}(0, V(\theta_0)),$$

where the covariance matrix is

$$V(\theta_0) = \{A(\theta_0)\}^{-1}B(\theta_0)\{A(\theta_0)\}^{-1}$$

and A and B are the expectation of the derivative of  $\psi$  and its variance respectively.

For the standard linear regression model

$$\mathbf{y}_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \boldsymbol{\varepsilon}_i$$

with coefficients  $\beta$  and error variance  $\sigma^2$  one can either treat  $\sigma^2$  as a nuisance parameter  $\theta = \beta$  or include it as  $\theta = (\beta, \sigma^2)$ .

In the former case, the estimating functions are  $\psi=\psi_{eta}$ 

$$\psi_{\beta}(\boldsymbol{y},\boldsymbol{x},\beta) = (\boldsymbol{y} - \boldsymbol{x}^{\top}\beta) \boldsymbol{x}$$

and in the latter case, they have an additional component

$$\psi_{\sigma^2}(\mathbf{y}, \mathbf{x}, \beta, \sigma^2) = (\mathbf{y} - \mathbf{x}^\top \beta)^2 - \sigma^2.$$

and  $\psi = (\psi_{\beta}, \psi_{\sigma^2})$ . This is used for FX regressions.

**Testing:** Given that a model with parameter  $\hat{\theta}$  has been estimated for these *n* observations, the question is whether this is appropriate or: Are the parameters stable or did they change through the sample period i = 1, ..., n?

**Monitoring:** Given that a stable model could be established for these *n* observations, the question is whether it remains stable in the future or: Are incoming observations for i > n still consistent with the established model or do the parameters change?

**Dating:** Given that there is evidence for a structural change in i = 1, ..., n, it might be possible that stable regression relationships can be found on subsets of the data. *How many segments are in the data? Where are the breakpoints?* 

To assess the stability of the fitted model with  $\widehat{\theta},$  we want to test the null hypothesis

$$H_0: \ \theta_i = \theta_0 \qquad (i = 1, \ldots, n)$$

against the alternative that  $\theta_i$  varies over "time" *i*.

Various patterns of deviation from  $H_0$  are conceivable: single/multiple break(s), random walks, etc.

To test this null hypothesis, the basic idea is to assess wether the empirical estimating functions  $\hat{\psi}_i = \psi(y_i, x_i, \hat{\theta})$  deviate systematically from their theoretical zero mean.

To capture systematic deviations the empirical fluctuation process of scaled cumulative sums of empirical estimating functions is computed:

$$efp(t) = \widehat{B}^{-1/2} n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} \widehat{\psi}_i \qquad (0 \le t \le 1).$$

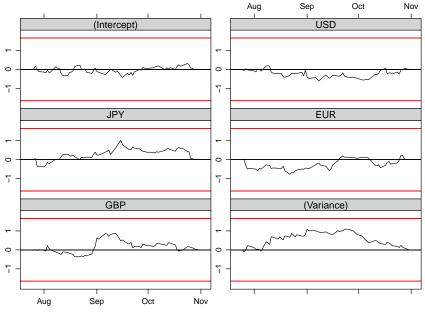
Under H<sub>0</sub> the following functional central limit theorem (FCLT) holds:

$$efp(\cdot) \stackrel{d}{\longrightarrow} W^0(\cdot),$$

where  $W^0$  denotes a standard *k*-dimensional Brownian bridge.

#### **Testing procedure:**

- empirical fluctuation processes captures fluctuation in estimating functions
- theoretical limiting process is known
- choose boundaries which are crossed by the limiting process (or some functional of it) only with a known probability  $\alpha$ .
- if the empirical fluctuation process crosses the theoretical boundaries the fluctuation is improbably large ⇒ reject the null hypothesis.



**More formally:** These boundaries correspond to critical values for a double maximum test statistic

```
\max_{j=1,\ldots,k} \max_{i=1,\ldots,n} |efp_j(i/n)|
```

which is 1.097 for the Chinese FX regression (p = 0.697).

Alternatively: Employ other test statistics for aggregation.

**Special cases:** This class contains various well-known tests from the statistics and econometrics literature, e.g., Andrews' sup*LM* test, Nyblom-Hansen test, OLS-based CUSUM/MOSUM tests.

In empirical samples,  $efp(\cdot)$  is a  $k \times n$  array. For significance testing, aggregate it to a scalar test statistic by a functional  $\lambda(\cdot)$ 

$$\lambda\left(efp_{j}\left(\frac{i}{n}\right)\right),$$

where j = 1, ..., k and i = 1, ..., n.

Typically,  $\lambda(\cdot)$  can be split up into

- λ<sub>comp</sub>(·) aggregating over components *j* (e.g., absolute maximum, Euclidian norm),
- $\lambda_{\text{time}}(\cdot)$  aggregating over time *i* (e.g., max, mean, range).

The limiting distribution is given by  $\lambda(W^0)$  and can easily be simulated (or some closed form results are also available).

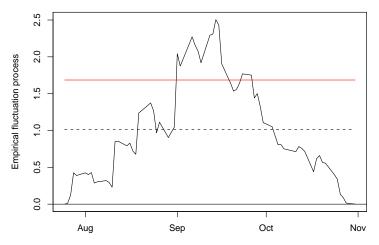
**Nyblom-Hansen test:** The test was designed for a random-walk alternative and employs a Cramér-von Mises functional.

$$\frac{1}{n}\sum_{i=1}^{n}\left\|\left|efp\left(\frac{i}{n}\right)\right\|\right\|_{2}^{2}.$$

It aggregates  $efp(\cdot)$  over the components first, using the squared Euclidian norm, and then over time, using the mean.

For the Chinese FX regression this is 1.012 (p = 0.364).

Nyblom-Hansen test



**Andrews' sup***LM* **test:** This test is designed for a single shift alternative (with unknown timing) and employs the supremum of *LM* statistics for this alternative.

$$\sup_{t\in\Pi} LM(t) = \sup_{t\in\Pi} \frac{||efp(t)||_2^2}{t(1-t)}.$$

It aggregates  $efp(\cdot)$  over the components first, using a weighted squared Euclidian norm, and then over time, using the maximum (over a compact interval  $\Pi \subset [0, 1]$ ).

For the Chinese FX regression this is 10.055 (p = 0.766), using  $\Pi = [0.1, 0.9]$ .

20 Empirical fluctuation process 15 10 S Aug Sep Oct Nov

supLM test

**Idea:** Fluctuation tests can be applied sequentially to monitor regression models.

More formally: Sequentially test the null hypothesis

 $H_0: \ \theta_i = \theta_0 \qquad (i > n)$ 

against the alternative that  $\theta_i$  changes at some time in the future i > n (corresponding to t > 1).

**Basic assumption:** The model parameters are stable  $\theta_i = \theta_0$  in the history period i = 1, ..., n ( $0 \le t \le 1$ ).

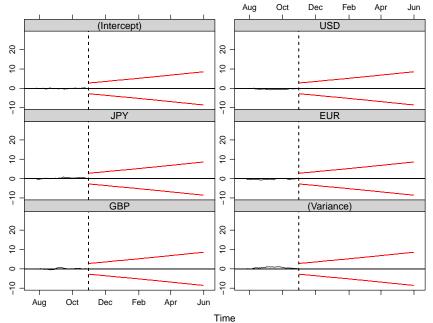
**Test statistics:** Update *efp*(*t*), and re-compute  $\lambda(efp(t))$  in the monitoring period  $1 \le t \le T$ .

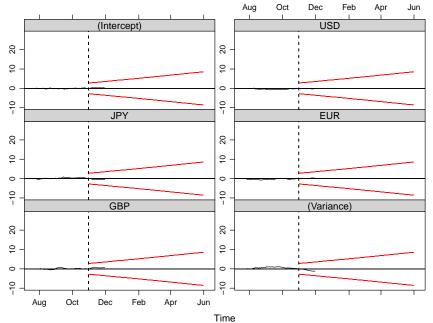
**Critical values:** For sequential testing not only a single critical value is needed, but a full boundary function b(t) that satisfies

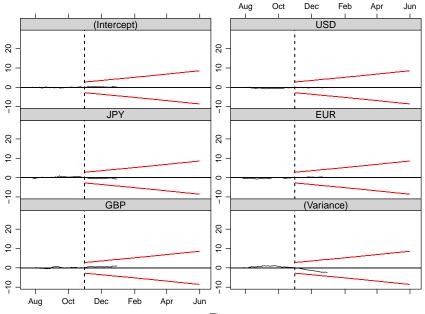
$$1 - \alpha = \mathsf{P}(\lambda(\mathsf{W}^{\mathsf{0}}(t)) \leq b(t) \mid t \in [1, T])$$

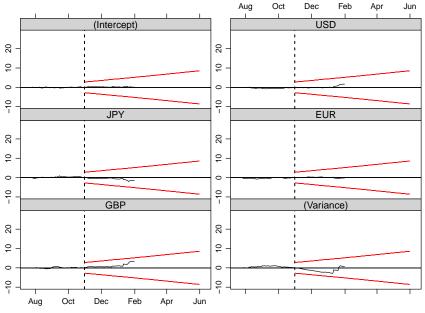
Various boundary (or weighting) functions are conceivable that can direct power to early or late changes or try to spread the power evenly.

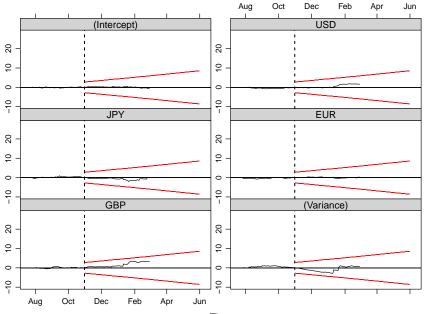
**In 2005:** Ajay Shah, Ila Patnaik, and Achim Zeileis establish a webpage and start monitoring the CNY regime. A double maximum functional with boundary  $b(t) = c \cdot t$  is employed (where *c* controls the significance level, using T = 4 and  $\alpha = 0.05$ ).

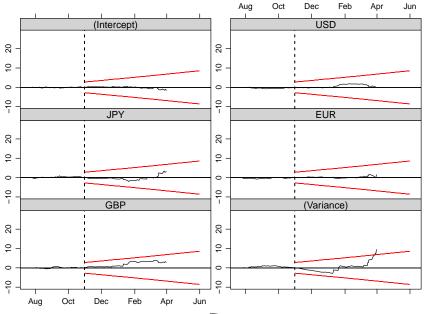


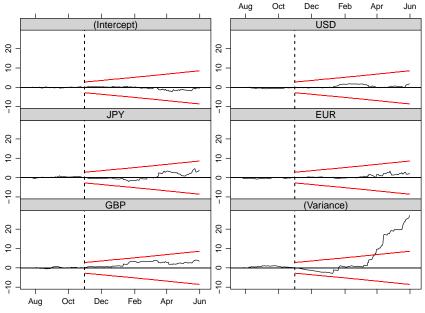












This signals a clear increase in the error variance.

The change is picked up by the monitoring procedure on 2006-03-27.

The other regression coefficients did not change significantly, signalling that they are not part of the basket peg.

Using data from an extended period up to 2009-07-31, we fit a segmented model to determine where and how the model parameters changed.

#### Dating

**Segmented regression model:** A stable model with parameter vector  $\theta^{(j)}$  holds for the observations in  $i = i_{j-1} + 1, \dots, i_j$ . The segment index is  $j = 1, \dots, m + 1$ .

The set of *m* breakpoints  $\mathcal{I}_{m,n} = \{i_1, \ldots, i_m\}$  is called *m*-partition. Convention:  $i_0 = 0$  and  $i_{m+1} = n$ .

The value of the segmented objective function  $\Psi$  is

$$PSI(i_1,...,i_m) = \sum_{j=1}^{m+1} psi(i_{j-1}+1,i_j),$$
  
$$psi(i_{j-1}+1,i_j) = \sum_{i=i_{j-1}+1}^{i_j} \Psi(y_i,x_i,\hat{\theta}^{(j)}).$$

Thus,  $psi(i_{j-1} + 1, i_j)$  is the minimal value of the objective function for the model fitted on the *j*th segment.

Dating tries to find

$$(\hat{\imath}_1,\ldots,\hat{\imath}_m) = \underset{(i_1,\ldots,i_m)}{\operatorname{argmin}} PSI(i_1,\ldots,i_m)$$

over all partitions  $(i_1, \ldots, i_m)$  with  $i_j - i_{j-1} + 1 \ge \lfloor nh \rfloor \ge k$ .

Bellman principle of optimality:

$$PSI(\mathcal{I}_{m,n}) = \min_{mn_h \le i \le n-n_h} [PSI(\mathcal{I}_{m-1,i}) + psi(i+1,n)]$$

It is well-known that this problem can be solved by a dynamic programming algorithm of order  $O(n^2)$  that essentially relies on a triangular matrix of psi(i, j) for all  $1 \le i < j \le n$ .

In linear regressions this approach has been popularized by Bai & Perron and it is common practice to use the residual sum of squares as objective function:

$$\Psi_{\mathsf{RSS}}(y_i, x_i, \beta) = (y_i - x_i^\top \beta)^2.$$

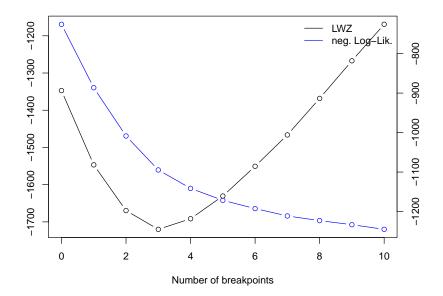
To capture changes in the variances as well the (negative) log-likelihood from a normal model can be employed:

$$\Psi_{\text{NLL}}(y_i, x_i, \beta, \sigma) = -\log\left(\sigma^{-1}\phi\left(\frac{y_i - x_i^\top\beta}{\sigma}\right)\right).$$

Thus, for a given number of breaks *m*, the optimal breaks  $\hat{\imath}_1, \ldots, \hat{\imath}_m$  be found.

To determine the number of breaks, some model selection has to be done, e.g., via information criteria or sequential tests. Here, we use the LWZ criterion (modified BIC):

$$\begin{split} IC(m) &= 2 \cdot NLL(\mathcal{I}_{m,n}) + \text{pen} \cdot ((m+1)k + m), \\ \text{pen}_{\text{BIC}} &= \log(n), \\ \text{pen}_{\text{LWZ}} &= 0.299 \cdot \log(n)^{2.1}. \end{split}$$



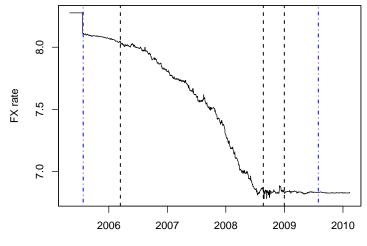
#### The estimated breakpoints and parameters are:

start/end	$\beta_0$	$\beta_{\rm USD}$	$\beta_{JPY}$	$\beta_{\rm EUR}$	$\beta_{\rm GBP}$	σ	$R^2$
2005-07-26	-0.005	0.999	0.005	-0.015	0.007	0.028	0.998
2006-03-14	(0.002)	(0.005)	(0.005)	(0.017)	(0.008)		
2006-03-15	-0.025	0.969	-0.009	0.026	-0.013	0.106	0.965
2008-08-22	(0.004)	(0.012)	(0.010)	(0.023)	(0.012)		
2008-08-25	-0.015	1.031	-0.026	0.049	0.007	0.263	0.956
2008-12-31	(0.030)	(0.044)	(0.030)	(0.059)	(0.035)		
2009-01-02	0.001	0.981	0.008	-0.008	0.009	0.044	0.998
2009-07-31	(0.004)	(0.005)	(0.004)	(0.009)	(0.004)		

#### corresponding to

- tight USD peg with slight appreciation,
- Isightly relaxed USD peg with some more appreciation,
- Slightly relaxed USD peg without appreciation,
- tight USD peg without appreciation.

CNY/USD



Time

**Epilogue:** What happened since summer 2009?

Estimation based on 2009-08-04 through 2010-01-29 (n = 122) gives:

$$CNY_{i} = 0.001 + 0.9953 USD_{i} + 0.002 JPY_{i}$$
  
+ 0.007 EUR<sub>i</sub> + 0.004 GBP<sub>i</sub> +  $\hat{\varepsilon}_{i}$ 

Only the USD coefficient is significantly different from 0 (but not from 1).

The error standard deviation became even smaller with  $\hat{\sigma} = 0.018$  leading to  $R^2 = 0.999$ .

#### Software

All methods are implemented in the R system for statistical computing and graphics and are freely available in the contributed packages *strucchange* and *fxregime* from the Comprehensive R Archive Network:

> http://www.R-project.org/ http://CRAN.R-project.org/

### Software: strucchange

Classical structural change tools for OLS regression:

- Testing: efp(), Fstats(), sctest().
- Monitoring: mefp(), monitor().
- Dating: breakpoints().
- Vignette: "strucchange-intro".

Object-oriented structural change tools:

- Testing: gefp(), efpFunctional() (including special cases: maxBB, meanL2BB, supLM, ...).
- Monitoring: Object-oriented implementation still to do.
- Dating: Some currently unexported support in gbreakpoints() in *fxregime*.
- Vignette: None, but CSDA paper.

# Software: *fxregime*

Structural change tools for exchange rate regression based on normal (quasi-)ML:

- Data: FXRatesCHF ("zoo" series with US Federal Reserve exchange rates in CHF for various currencies).
- Preprocessing: fxreturns().
- Model fitting: fxlm().
- Testing: gefp() from *strucchange*.
- Monitoring: fxmonitor().
- Dating: fxregimes() based on currently unexported gbreakpoints(); refit() method for fitting segmented regression.
- Vignettes: "CNY", "INR".

India also has an expanding economy with a currency receiving increased interest over the last years. We track the evolution of the INR FX regime since trading in the INR began.

```
R> head(FXRatesCHF[, c(1:6, 13)], 3)
```

USD JPY DUR EUR DEM GBP INR 1971-01-04 0.232 82.8 0.429 NA 0.844 0.0967 NA 1971-01-05 0.232 83.0 0.429 NA 0.845 0.0968 NA 1971-01-06 0.232 83.0 0.429 NA 0.845 0.0968 NA

```
R> inr <- fxreturns("INR", data = FXRatesCHF,
+ other = c("USD", "JPY", "DUR", "GBP"), frequency = "weekly",
+ start = as.Date("1993-04-01"), end = as.Date("2008-01-04"))
R> head(inr, 3)
```

INR USD JPY DUR GBP 1993-04-09 0.9773 0.9773 0.0977 0.567 -0.02236 1993-04-16 -0.0339 -0.0339 -0.5387 0.625 0.14295 1993-04-23 3.2339 3.2339 1.4331 1.264 0.00876

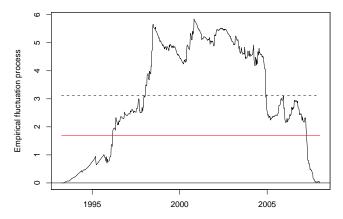
Using weekly returns from 1993-04-09 through to 2008-01-04 (yielding n = 770 observations), we fit a single FX regression using the same basket as above.

As we would expect multiple changes, we assess its stability with the Nyblom-Hansen test. Alternatively, a MOSUM test could be used. The double maximum test has less power.

```
R> inr_efp <- gefp(inr_lm, fit = NULL)
R> sctest(inr_efp, functional = meanL2BB)
M-fluctuation test
data: inr_efp
f(efp) = 3.11, p-value = 0.005
R> sctest(inr_efp, functional = maxBB)
M-fluctuation test
data: inr_efp
f(efp) = 1.72, p-value = 0.03099
```

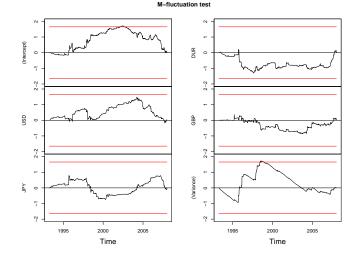
R> plot(inr\_efp, functional = meanL2BB)





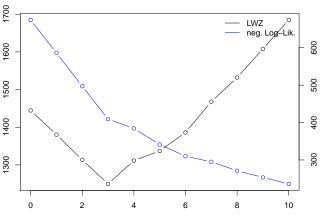
Time

R> plot(inr\_efp, functional = maxBB, aggregate = FALSE, + ylim = c(-2, 2))



Dating is computationally more demanding. The dynamic programming algorithm can be parallelized, though. This is easily available (thanks to Anmol Sethy) by means of optional *foreach* support in *fxregime*.

R> plot(inr\_reg)



LWZ and Negative Log-Likelihood

Number of breakpoints

Various methods for extracting information can be applied directly. Otherwise, refitting of FX regressions gives access to all quantities that might be of interest.

```
R> coef(inr_reg)[, 1:5]
                       (Intercept) USD
                                            JPY
                                                     DUR.
                                                              GBP
1993-04-09--1995-03-03
                          -0.00574 0.972 0.02347
                                                 0.0113
                                                          0.02037
1995-03-10--1998-08-21
                          0.16113 \ 0.943 \ 0.06692 \ -0.0261
                                                         0.04236
1998-08-28--2004-03-19 0.01861 0.993 0.00976 0.0983 -0.00322
2004-03-26--2008-01-04 -0.05761 0.746 0.12561
                                                 0.4354 0.12137
R> inr_rf <- refit(inr_reg)</pre>
R> sapply(inr_rf, function(x) summary(x)$r.squared)
1993-04-09--1995-03-03 1995-03-10--1998-08-21 1998-08-28--2004-03-19
                                        0.729
                0.989
                                                              0.969
2004-03-26--2008-01-04
                 0.800
```

start/end	β <sub>0</sub>	$\beta_{\rm USD}$	$\beta_{JPY}$	$\beta_{\sf DUR}$	$\beta_{GBP}$	σ	$R^2$
1993-04-09	-0.006	0.972	0.023	0.011	0.020	0.157	0.989
1995-03-03	(0.017)	(0.018)	(0.014)	(0.032)	(0.024)		
1995-03-10	0.161	0.943	0.067	-0.026	0.042	0.924	0.729
1998-08-21	(0.071)	(0.074)	(0.048)	(0.155)	(0.080)		
1998-08-28	0.019	0.993	0.010	0.098	-0.003	0.275	0.969
2004-03-19	(0.016)	(0.016)	(0.010)	(0.034)	(0.021)		
2004-03-26	-0.058	0.746	0.126	0.435	0.121	0.579	0.800
2008-01-04	(0.042)	(0.045)	(0.042)	(0.116)	(0.056)		

Somewhat more compactly:

#### corresponding to



- tight USD peg,
- flexible USD peg,
- tight USD peg,
- flexible basket peg.

#### Next steps

Current activities: Application to wider range of currencies.

**Of particular interest:** Classification of exchange rate regimes and monitoring.

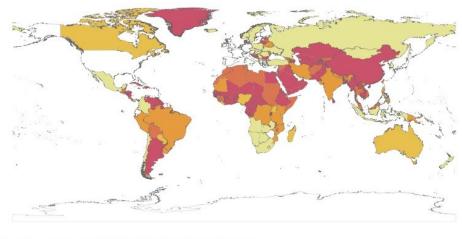
#### **Open problems:**

- Fully automatic selection of breakpoints.
- Sequential usage of BIC/LWZ, i.e., with growing sample size n.
- Differences between subsequent regimes that are statistically significant but not practically relevant (or vice versa).

**First steps:** Anmol Sethy started to build infrastructure for larger FX rates database from mixed sources.

**First results:** World map of  $R^2$  from FX regressions (basket: USD, EUR, JPY, GBP), November 2009, based on segmented weekly data.

# Next steps



[0,0.45] (0.45,0.55] (0.55,0.75] (0.75,0.9] (0.9,1]

### Summary

- Exchange rate regime analysis can be complemented by structural change tools.
- Both coefficients (currency weights) and error variance (fluctuation band) can be assessed using an (approximately) normal regression model.
- Estimation, testing, monitoring, and dating are all based on the same model, i.e., the same objective function.
- Traditional significance tests can be complemented by graphical methods conveying timing and component affected by a structural change.
- Software is freely available, both for the general method and the application to FX regimes.

#### References

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