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A Systematic Approach to Pricing and Hedging of International Derivatives with Interest Rate Risk

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Abstract

The paper deals with the valuation and the hedging of non path-dependent European options on one or several underlying assets in a model of an international economy allowing for both, interest rate risk and exchange rate risk. Using martingale theory and in particular the change of numeraire technique we provide a unified and easily applicable approach to pricing and hedging exchange options on stocks, bonds, futures, interest rates and exchange rates. We also cover the pricing and hedging of compound exchange options.

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1 Introduction

The present paper deals with the valuation and hedging of non- path-dependent European options on one or several underlyings in a model of an international economy which allows for both, *interest rate risk* and *exchange rate risk*. We study options on stocks, bonds, future contracts, interest rates and exchange rates; their payoff may be in any currency and a relatively complex function of one or several underlyings.

There exists meanwhile a huge number of different contracts on various underlying assets, and it is easy to construct new payoffs by simply combining in a different manner the elements that make up a certain contract. Hence a case by case analysis as has been carried out previously is no longer appropriate. This lead Kat and Roozen (1994) to develop a unified method for the pricing and hedging of non-path-dependent European stock options. They restrict themselves to a model with deterministic interest rates, which allows them to use a partial differential equation (PDE) as main tool of their analysis.

We believe an extension of their study to a framework with stochastic interest rates to be important for a number of reasons. To begin with, introducing interest rate risk opens the possibility to treat a huge number of payoffs not amenable to the analysis of Kat and Roozen. For instance we deal with guaranteed-exchange-rate options on bonds, options on the difference of two LIBOR rates in different countries or with options on the spread between the rate of return on a stock and a LIBOR rate, possibly in different countries. In the last years these and similar payoffs have witnessed a strongly increasing importance in practice. Moreover, is well known that there is a close interaction between currency markets and fixed income markets. Hence a good model for the pricing of derivatives in an international economy setting should be rich enough to incorporate a wide range of different correlations between these assets. Necessarily such a model must allow for stochastic interest rates.

We have therefore chosen an international economy model similar to the one introduced by Amin and Jarrow (1991) as the framework of our analysis. Their model combines a fully developed stochastic theory of the term structure of interest rates in the sense of Heath, Jarrow, and Morton (1992) with models for the valuation of exchange rate and stock options. However, following the approach of El Karoui and Rochet (1989) and El Karoui, Myneni, and Viswanathan (1992a), we use bond prices instead of forward rates as primitives for the modelling of interest rate risk. The main tools of our analysis are stochastic methods and in particular the change of numeraire technique as introduced among others in (El Karoui and Rochet 1989) and (Jamshidian 1989). This enables us to give a unified treatment of all international economy models in the sense of Amin and Jarrow (1991), the only restriction being that the volatility of the underlying assets is deterministic. The PDE-approach of Kat and Roozen works less well in our framework, because under stochastic interest rates the precise form of the PDE for derivative prices depends on the factor structure of the term structure model used in the analysis which in turn depends on the precise form of the bond price volatilities. Moreover, including the possibility of stochastic interest rates means enlarging the state space of the pricing PDE, which renders difficult a numerical treatment of the equation in cases where the boundary conditions are such that an explicit solution cannot be found.

Starting from the concept of a lognormal claim — which includes among others stocks, bonds, future contracts and exchange rates — we derive a generic option pricing formula for options to exchange two lognormal claims. To make this result operational also for options on relatively complex lognormal claims we provide a systematic procedure for calculating the input parameters needed in the generic valuation formula. Since pricing formulas are of little practical use without knowledge of the corresponding hedge portfolio we present a systematic approach to computing hedging strategies. In order to illustrate the flexibility of our method we derive explicit formulas for prices and hedge portfolios for a wide range of examples containing among others currency options, guaranteed-exchange-rate options, options on futures or options on the spread of two LIBOR rates.

We then go on to study the pricing of what we term compound exchange options. This latter class of derivatives contains for instance spread options. As it is well known from the case of deterministic interest rates there are in general no explicit valuation formulas for these contracts. However, by using the change of numeraire technique we are able to reduce the problem of pricing such contracts to the computation of the probability of a well-specified domain in $\mathbb{R}^d$ under certain multivariate normal distributions. Moreover, we demonstrate that the hedge portfolio can be expressed in terms of these probabilities. This is remarkable, since it shows that even in the absence of explicit pricing formulas we are able to compute hedge portfolios without resorting to numerical differentiation. In order to illustrate how these results can be applied we study some concrete examples and sketch along the way some numerical techniques for the evaluation of the probabilities that enter our formulas. This discussion will also show that our approach is far more efficient than direct Monte Carlo simulation. This stems from the fact that with stochastic interest rates the latter technique requires the simulation of whole trajectories even for the pricing of path-independent payoffs.

The pricing of certain types of options belonging to the class of derivatives considered here, and the development of arbitrage free models of international economies have previously been adressed in the literature. Among the early work on correlation dependent options are the papers by Margrabe (1978), Stulz (1982) and Johnson (1987), who all work in the classical Black-Scholes model. A collection of their results can be found in the work of Rubinstein (1990) on exotic options. Papers on currency options are due to Garman and Kohlhagen (1983) and (Grabbe 1983). More recently Amin and Jarrow (1991) have developed the above mentioned arbitrage free model of an international economy. Our treatment of interest rate risk follows (El Karoui, Myneni, and Viswanathan 1992a) and (El Karoui, Myneni, and Viswanathan 1992b).

Besides the already mentioned work of Kat and Roozen the papers (Jamshidian 1993) and (Jamshidian 1994) are most closely related to our analysis. In (Jamshidian 1993) valuation formulas for certain options which fall within our class of options to exchange two lognormal claims are derived. The paper lacks however a general procedure for applying the main valuation result to derivatives which are in principle within the scope of the analysis. The related work (Jamshidian 1994) gives hedge portfolios for certain correlation-dependent securities including Quanto futures, but it does not provide a systematic approach to the computation of such portfolios, which would allow one to deal also with contracts which are not explicitely considered. Moreover, none of the two papers treats the pricing and hedging of options on several underlyings such as our compound exchange options.

The paper is organized as follows:
In Section 2 we present the general N-country model of financial markets and introduce the concept of a lognormal claim. Section 3 contains the results on pricing and hedging exchange options on lognormal claims. In Section 4 we discuss compound exchange options. Section 5 finally concludes.

Notation:

Throughout the paper we denote the inner product of two vectors X, $Y \in \mathbb{R}^d$ by $X \cdot Y := \sum_{i=1}^{d} x_i y_i$; the norm of a vector will be denoted by $|X| := (X \cdot X)^{\frac{1}{2}}$.

2 The Model

In this section we introduce an arbitrage-free model of an international economy that incorporates stochastic interest rates and exchange rates. This model will serve as our framework for the valuation of derivatives. We consider N countries indexed by $n \in \{0, \ldots, N\}$. Country 0 will be the *domestic* country. The exchange rate between country 0 and country $n \in \{1, \ldots, N\}$ will be denoted by e^n, that is e^n units of the domestic currency can be exchanged for one unit of the foreign currency. When working with only two countries we simply talk about the domestic and the foreign country and index them with d and f. The choice of the domestic country is arbitrary and depends on the particular pricing and hedging problem under consideration. We call an asset a *domestic* asset if its payoffs are denominated in the domestic currency. Notice that every asset whose payoffs are not originally denominated in this currency can be transformed into a domestic asset by translating its payoffs into the domestic currency using the corresponding exchange rate.

We assume that in all countries zero coupon bonds of all maturities $T \in [0, T_F]$ are traded. The zero coupon bond in country n with maturity date T shall be denoted by $B^n(t, T)$ for $t \in [0, T]$. By assumption $B^n(T, T) \equiv 1 \;\forall\, T, n$. The short rate in country n, r^n, is given by

$$r_t^n = -\frac{\partial}{\partial T}\Big|_{T=t} \ln B^n(t, T)\,. \tag{1}$$

For an explicit formula deduced from (1) see for instance (El Karoui, Myneni, and Viswanathan 1992a). By $\beta^n_{t,T} := \exp(\int_t^T r_s^n \, ds)$ we denote the savings-account of country n. Apart from zero coupon bonds we consider other primitive assets such as dividend free stocks. They are denoted by $S^{n,j},\ 0 \le n \le N,\ 0 \le j \le i_n$ where $S^{n,j}$ is the price of asset j in country n.

We now introduce our model of asset price dynamics. When modelling asset price processes one usually starts from assumptions on their dynamics under the so-called historical probabilities which govern the actual evolution of asset prices. Since we are only interested in the pricing of derivatives by no-arbitrage arguments it is legitimate to model the asset price dynamics directly under a domestic risk-neutral measure P. Under such a measure all non-dividend paying domestic assets are martingales after discounting with the domestic savings account. This implies that their mean instantaneous growth rate — in the sequel simply referred to as their drift — is equal to r^0.

Assumption 2.1 *Let there be given a filtered probability space* $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t\in[0,T_F]}$ *supporting a d-dimensional Brownian Motion* $W = (W_t)_{0\leq t\leq T_f}$. *We work with the following assumptions on asset price dynamics: We put for the domestic assets*

$$\begin{aligned} dB^0(t,T) &= r_t^0 B^0(t,T)dt + \eta^0(t,T)B^0(t,T)dW_t \\ dS_t^{0,j} &= r_t^0 S_t^{0,j} dt + \eta^{0,j}(t) S_t^{0,j} dW_t\,, \end{aligned} \tag{2}$$

and for the foreign assets

$$\begin{aligned} dB^n(t,T) &= (r_t^n - \eta^n(t,T)\cdot\eta^{e^n}(t))B^n(t,T)dt + \eta^n(t,T)B^n(t,T)dW_t \\ dS_t^{n,j} &= (r_t^n - \eta^{n,j}(t)\cdot\eta^{e^n}(t))S_t^{n,j}dt + \eta^{n,j}(t)S_t^{n,j}dW_t\,. \end{aligned} \tag{3}$$

Finally the dynamics of the exchange rates are given by

$$de^n(t) = (r_t^0 - r_t^n)e^n(t)dt + \eta^{e^n}(t)e^n(t)dW_t\,. \tag{4}$$

Here $\eta^n(t,T), \eta^{n,j}(t), \eta^{e^n}(t) : [0,T_F] \to I\!R^d$ *are* deterministic *square integrable functions of time. For the bonds we require moreover that* $\eta^n(t,T) = 0\,\forall t \geq T$ *and that* $\eta^n(t,T)$ *is smooth in the second argument.*

Remarks: The assumption of deterministic dispersion coefficients is essential as we want to obtain explicit pricing formulas.

As shown by Amin and Jarrow (1991) the dynamics of asset prices and exchange rates given in Assumption 2.1 actually specify an arbitrage-free model of an international economy with P representing a domestic risk-neutral measure. The drift terms of the exchange rate and the foreign assets are determined by absence of arbitrage considerations. As an example we derive the drift of e^n. Consider the domestic asset $Y := e^n\beta^n_{(0,\cdot)}$. By absence of arbitrage its drift must equal r^0. Using Itô's Lemma to compute the dynamics of Y it is immediate that the drift of Y equals r^0 if and only if the drift of the exchange rate equals the interest rate differential.

The volatility of asset $S^{n,j}$ is given by $\sigma^{n,j}(t) := |\eta^{n,j}(t)|$. The instantaneous correlations between the assets in our economy are given by

$$\rho(S^{n_1,j_1}, S^{n_2,j_2}) := \frac{\eta^{n_1,j_1}\cdot\eta^{n_2,j_2}}{\sigma^{n_1,j_1}\sigma^{n_2,j_2}}\,.$$ [1]

Only volatilities and instantaneous correlations matter for the pricing of derivatives, since they determine the law of the asset prices under the domestic risk neutral measure.

[1] Of course similar formulas hold for bonds and exchange rates.

In our analysis this is reflected by the fact that only inner prod-ucts of the dispersion coefficients η and hence instantaneous covariances enter the pricing formulas. Nonetheless we decided to start with independent Brownian motions and to model correlations by means of the dispersion coefficients η because this faciliates the use of stochastic calculus. To compute these coefficients from the estimated instantaneous covariance matrix of the processes one may use the Cholesky decomposition of this matrix as explained for instance in (Hamilton 1994). The calibration of Gaussian term structure models to market data is for instance discussed in (Brace and Musiela 1994); the methods developed in this article are applicable to our model, too.

Finally we note that the price process of a discounted foreign asset is not a martingale under the domestic risk-neutral measure as can be seen from (3); hence this measure must not be used for the valuation of derivatives paying off in foreign currencies.

For our pricing theory we need to assume that the markets in our economy are complete.

Assumption 2.2 *There are d traded domestic assets such that for all $t \in [0, T_F]$ the instantaneous covariance matrix of these assets is strictly positive definite.*

This assumption guarantees that every contingent claim adapted to the filtration generated by the asset prices can be replicated by a dynamic trading strategy in the d assets and the domestic savings-account, see for instance (Duffie 1992, section 6.I). Hence the domestic risk neutral measure is unique and the price at time t of every domestic contingent claim H with single $\mathcal{F}_T$ measurable and integrable payoff H_T at time T is given by

$$H_t := E^P\left[\left(\beta^0_{t,T}\right)^{-1} \cdot H_T \;\middle|\; \mathcal{F}_t\right] ; \tag{5}$$

see e.g. (Harrison and Pliska 1981). This equation is the exact probabilistic analogue of equation (6) of (Kat and Roozen 1994); it will be the starting point for our valuation results in the next sections.

We now introduce the class of admissible underlyings for the derivative contracts considered in the paper. A typical example of the kind of options we want to analyze is the guaranteed-exchange-rate call. This contract is defined by its terminal payoff $[\bar{e}S^f_T - K]^+$, where S^f_T is some primitive foreign asset and $\bar{e}$ is a guaranteed exchange rate which will be applied at time T to convert the price of the foreign asset into domestic currency. Now, $\bar{e}S^f_T$ is not the time T value of a traded domestic asset. However, it defines a domestic contingent claim X whose price $X_t = E^P[(\beta_{t,T})^{-1}\bar{e}S^f_T|\mathcal{F}_t]$ is given by

$$X_t = X_0 \exp\left(\int_0^t \eta^X_s dW_s - \frac{1}{2}\int_0^t |\eta^X_s|^2 ds + \int_0^t r^d_s ds\right) \text{ with } X_0 = E^P\left[(\beta_{0,T})^{-1} X_T\right]$$

and η^X_s a deterministic $\mathbb{R}^d$-valued function of time. We will see in section 3.2 below that this structure is found in many ostensibly complex option contracts. This motivates the following definition.

Definition 2.3 *A domestic contingent-claim X with a single payoff X_T at a certain date T is called a* lognormal claim[2] *if its price process $(X_t)_{0\le t\le T}$ given by*

$$X_t := E^P\left[\left(\beta^d_{t,T}\right)^{-1}\cdot X_T \mid \mathcal{F}_t\right]$$

admits a representation of the form

$$X_t = X_0\cdot\exp\left(\int_0^t \eta^X_s dW_s - \frac{1}{2}\int_0^t |\eta^X_s|^2 ds + \int_0^t r^d_s ds\right) \tag{6}$$

with some constant X_0 and with deterministic *dispersion coefficients $\eta^X : [0,T]\to \mathbb{R}^d$.*

Remarks: The main restriction made in the definition of a lognormal claim is the assumption of η^X being deterministic. In fact, whenever X_T is strictly positive,

$$X_t := E^P\left[\left(\beta^d_{t,T}\right)^{-1}\cdot X_T \mid \mathcal{F}_t\right]$$

is always of the form (6) with possibly stochastic "volatility" η^X, as can easily be shown by means of the martingale representation theorem. Note that the solution of the SDE $dX_t = r^0_t X_t dt + \eta^X_t X_t dW_t$ is given by

$$X_t := X_0\cdot\ \exp\left(\int_0^t \eta^X_s dW_s - 1/2\int_0^t |\eta^X_s|^2 ds + \int_0^t r^0_s ds\right)\ .$$

Hence under our assumption on asset price dynamics every primitive domestic asset, interpreted as contingent claim with payoff equal to the asset's price at time T, is a lognormal claim. However, the class of contingent claims that satisfy Definition 2.3 is much larger. For instance products and quotients of lognormal claims remain lognormal claims.

The next proposition gives a method for computing the initial value and the volatility coefficients of a lognormal claim also in certain cases where these parameters cannot be read off directly from the asset price dynamics. For an application of this proposition we refer the reader to the examples considered in section 3.2.

Proposition 2.4 *Assume that the payoff X_T of a domestic contingent claim X is given by*

$$X_T = \gamma^X\cdot\exp\left(\int_0^T \sum_{n=0}^N \alpha^{n,X} r^n_s\, ds + \int_0^T \mu^X_s ds + \int_0^T v^X_s\, dW_s\right)$$

where $\gamma^X\in\mathbb{R}_+$, $\alpha^{n,X}\in\mathbb{Z}\ \forall n$, and where $\mu^X : [0,T]\to\mathbb{R}$, $v^x : [0,T]\to\mathbb{R}^d$ are deterministic *functions. Then there is a $X_0\in\mathbb{R}_+$ and a* deterministic *function $\eta^X : [0,T]\to\mathbb{R}^d$ defined in the proof below such that X_T admits a representation of the form*

$$X_T = X_0\cdot\exp\left(\int_0^T \eta^X_s dW_s - \frac{1}{2}\int_0^T |\eta^X_s|^2 ds + \int_0^T r^0_s ds\right)\ .$$

In particular X is a lognormal claim.

[2]This name is motivated by the fact that X_T is lognormally distributed. This is immediate if one writes $X_T = X_T/B^0(T,T)$ and then expresses the right hand side using (6) and the corresponding expression for $B^0(\cdot,T)$.

Proof: Since $B^n(T,T) = 1$ we get

$$
\begin{aligned}
X_T &= X_T \cdot B^0(T,T) \cdot \prod_{n=0}^{N} B^n(T,T)^{-\alpha^{n,X}} \\
&= \gamma^X \cdot B^0(T,T) \cdot \prod_{n=0}^{N} B^n(0,T)^{-\alpha^{n,X}} \\
&\cdot \exp\left[\int_0^T r_s^0\, ds + \int_0^T \left(\mu_s^X + \frac{1}{2}\sum_{n=0}^{N} \alpha^{n,X}\left(|\eta^n(s,T)|^2 + \eta^n(s,T)\cdot \eta^{e^n}(s)\right)\right.\right. \\
&- \left.\left.\frac{1}{2}|\eta^0(s,T)|^2\right) ds + \int_0^T \left(v_s^X - \sum_{n=0}^{N} \alpha^{n,X}\eta^n(s,T) + \eta^0(s,T)\right) dW_s\right]
\end{aligned}
$$

If we now define

$$
\eta_s^X \;:=\; v_s^X - \sum_{n=0}^{N} \alpha^{n,X}\eta^n(s,T) + \eta^0(s,T) \tag{7}
$$

$$
X_0 \;:=\; \gamma^X \prod_{n=0}^{N} B^n(0,T)^{-\alpha^{n,X}} B^0(0,T) \cdot \exp\left(\int_0^T \mu_s^X + \right. \tag{8}
$$

$$
+ \left. \frac{1}{2}\sum_{n=0}^{N} \alpha^{n,X}\left(|\eta^n(s,T)|^2 + \eta^n(s,T)\cdot\eta^{e^n}(s)\right) - \frac{1}{2}|\eta^0(s,T)|^2 + \frac{1}{2}|\eta_s^X|^2\, ds\right) \tag{9}
$$

we get $X_T = X_0 \cdot \exp\left(\int_0^T \left(r_s^0 - \frac{1}{2}|\eta_s^X|^2\right) ds + \int_0^T \eta_s^X\, dW_s\right)$. From this representation it is immediate that X is a lognormal claim whose value at time t is given by (8) but with $B^n(t,T)$ replacing $B^n(0,T)$ and with t as lower bound of the integral in the argument of the exponential term. □.

3 Exchange Options on Lognormal Claims

3.1 The Theoretical Result

In this section we give a rather general theorem which leads to a unified treatment of the pricing of European options on various underlyings such as foreign and domestic zero coupon bonds, foreign or domestic stocks or forward and future contracts on foreign and domestic assets. It is similar in spirit to a result by Jamshidian (1993); however, we feel that our theorem is more easily applicable.

Theorem 3.1 *Let X, Y be lognormal claims. Consider an option to exchange X for Y at the maturity date T , i.e. a European option with payoff $[X_T - Y_T]^+$.*

1. The price process $C = (C_t)_{0\le t\le T}$ of this option is given by

$$
C_t = C(t, X_t, Y_t) \;:=\; X_t\mathcal{N}(d_t^1) - Y_t\mathcal{N}(d_t^2)
$$

where $\mathcal{N}$ denotes the one-dimensional standard normal distribution function, and where d_t^1 and d_t^2 are given by

$$d_t^1 = \frac{\ln(X_t/Y_t) + \frac{1}{2}\int_t^T |\eta_s^X - \eta_s^Y|^2 ds}{\sqrt{\int_t^T |\eta_s^X - \eta_s^Y|^2 ds}}, \quad d_t^2 = d_t^1 - \sqrt{\int_t^T |\eta_s^X - \eta_s^Y|^2 ds}\,.$$

2. The hedge portfolio $P^C = (P_t^C)_{0\le t\le T}$ for this option in terms of the lognormal claims X and Y is given by

$$\delta_X^C(t) := \mathcal{N}(d_t^1) \textit{ units of } X \quad \textit{and } \delta_Y^C(t) := -\mathcal{N}(d_t^2) \textit{ units of } Y\,.$$

Proof: The main tool in the proof is the change of numeraire technique developed among others in (El Karoui, Geman, and Rochet 1995). We now recall a few facts from this theory. Define for a lognormal claim X a new equivalent probability measure Q^X on $\mathcal{F}_T$ by

$$\frac{dQ^X}{dP} = \frac{X_T \cdot \left(\beta_{0,T}^d\right)^{-1}}{X_0}.$$

Then for every domestic asset Z whose discounted price process is a martingale under P — that is for every asset that pays no dividends in $[0,T)$ — the process Z/X is a martingale under Q^X, i.e. Q^X is the martingale measure corresponding to the numeraire X. Moreover we have the transition formula

$$E^P\left[\left(\beta_{t,T}^d\right)^{-1} \cdot X_T \cdot Z_T \,|\mathcal{F}_t\right] = X_t \cdot E^{Q^X}[Z_T|\mathcal{F}_t] \tag{10}$$

Remark: If $X = B^d(\cdot,T)$ the measure Q^X is just the *forward risk adjusted measure* associated with $B^d(\cdot,T)$. This measure is well known in the interest rate literature.

In our setup it is easy to determine the law of the asset price processes by means of the Girsanov theorem. Applying this theorem to dQ^X/dP immediately yields that $W_t^X := W_t - \int_0^t \eta_s^X ds$ is a new Brownian Motion under Q^X.

Now it is easy to proof the first part of the theorem. According to (5) the price of the option is given by

$$\begin{aligned}
C_t &= E^P\left[\left(\beta_{t,T}^d\right)^{-1}[X_T - Y_T]^+ \,\Big|\, \mathcal{F}_t\right] \\
&= E^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T \cdot 1_{\{Y_T/X_T<1\}} \,\Big|\, \mathcal{F}_t\right] - E^P\left[\left(\beta_{t,T}^d\right)^{-1} Y_T \cdot 1_{\{X_T/Y_T>1\}} \,\Big|\, \mathcal{F}_t\right] \\
&= X_t \cdot E^{Q^X}\left[1_{\{Y_T/X_T<1\}} \,\Big|\, \mathcal{F}_t\right] - Y_t \cdot E^{Q^Y}\left[1_{\{X_T/Y_T>1\}} \,\Big|\, \mathcal{F}_t\right]
\end{aligned}$$

The last line follows from (10) if we take once X and once Y as numeraire. Now we get under Q^X for Y_T/X_T

$$\frac{Y_T}{X_T} = \frac{Y_t}{X_t} \cdot \exp\left(\int_t^T (\eta_s^Y - \eta_s^X)dW_s^X - \frac{1}{2}\int_t^T |\eta_s^Y - \eta_s^X|^2 ds\right)$$

Hence

$$
\begin{aligned}
Q^X\left[\frac{Y_T}{X_T}<1 \,\middle|\, \mathcal{F}_t\right] &= Q^X\left[\ln Y_T - \ln X_T < 0 \mid \mathcal{F}_t\right] \\
&= Q^X\left[\frac{\int_t^T (\eta_s^Y-\eta_s^X)dW_s^X}{\sqrt{\int_t^T |\eta_s^Y-\eta_s^X|^2 ds}} < \frac{\ln X_T - \ln Y_T + \frac{1}{2}\int_t^T |\eta_s^Y-\eta_s^X|^2 ds}{\sqrt{\int_t^T |\eta_s^Y-\eta_s^X|^2 ds}}\right]
\end{aligned}
$$

Since η^X and η^Y are deterministic, $\int_t^T (\eta_s^Y-\eta_s^X)dW_s^X/\sqrt{\int_t^T |\eta_s^Y-\eta_s^X|^2 ds}$ is a standard normally distributed random variable so that

$$
Q^X\left[\frac{Y_T}{X_T}<1 \,\middle|\, \mathcal{F}_t\right] = \mathcal{N}(d_t^1)\,.
$$

Analogously we get $Q^Y\left[X_T/Y_T > 1 \mid \mathcal{F}_t\right] = \mathcal{N}(d_t^2)$, and the first part of the theorem follows.

To prove the second claim we note that the proposed hedge portfolio duplicates the option if the martingale part of the portfolio's value process is the same as that of the option and if the value of the portfolio equals the option's price for all $0 \le t \le T$. We now check these two conditions.

(i) Let $(Z)^M$ denote the (uniquely determined) martingale part of a continuous semi-martingale Z. As C_t is a function only of X_t and Y_t we get from Itô's Lemma

$$
d\,(C)_t^M = \frac{\partial C}{\partial x}(t,X_t,Y_t)d\,(X)_t^M + \frac{\partial C}{\partial y}(t,X_t,Y_t)d\,(Y)_t^M\,.
$$

Now following El Karoui, Myneni, and Viswanathan (1992b) we may compute the derivatives of the option:

$$
\begin{aligned}
\frac{\partial C}{\partial x}(t,X_t,Y_t) &= E_t^P\left[\frac{\partial}{\partial X_t}\left(\left(\beta_{t,T}^d\right)^{-1}\left[X_T-Y_T\right]^+\right)\right] \\
&= E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} 1_{\{X_T\ge Y_T\}}\frac{\partial X_T}{\partial X_t}\right] \\
&= \frac{1}{X_t}E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} 1_{\{X_T\ge Y_T\}}X_T\right]
\end{aligned}
$$

As shown in the first part of the proof this expression equals $\mathcal{N}(d_t^1)$. Similarly we get $\partial C/\partial y\,(t,X_t,Y_t) = -\mathcal{N}(d_t^2)$.

Remark: While unnecessary in the present proof where explicit pricing formulas are available this technique of exchanging differentiation and expectation will prove very helpful in the absence of explicit pricing formulas in Section 4.

On the other hand we get from the selffinancing condition and from $\left(\beta_{0,\cdot}^d\right)_t^M = 0$ for the value process V of the hedge portfolio

$$
d\,(V)_t^M = \mathcal{N}(d_t^1)d(X)_t^M - \mathcal{N}(d_t^2)d(Y)_t^M \text{ and hence } d\,(V)_t^M = d\,(C)_t^M\,.
$$

(ii) By Euler's Theorem we get from the linear homogeneity of C in X_t and Y_t

$$C_t = \frac{\partial C}{\partial x}(t, X_t, Y_t) \cdot X_t + \frac{\partial C}{\partial y}(t, X_t, Y_t) \cdot Y_t = \mathcal{N}(d_t^1) X_t - \mathcal{N}(d_t^2) Y_t \,,$$

which shows that also the second condition is satisfied. □

Whenever the lognormal claims X and Y are assets for which liquid markets exist, Theorem 3.1 is sufficient for the construction of a hedge portfolio. Otherwise we must go on and duplicate X and Y by a dynamic trading strategy. The existence of such a strategy is guaranteed by Assumption 2.2; it can be computed as in the proof of Theorem 3.1. The following observation then shows how to construct hedging strategies for C from the hedge portfolios for X and Y. Suppose that the hedge portfolios for X and Y in terms of domestic assets H_i^X and H_i^Y for which we assume the existence of liquid markets are given by

$$P_t^X = \sum_{i=1}^{L^X} \delta_i^X(t) H_i^X \quad \text{and} \quad P_t^Y = \sum_{i=1}^{L^Y} \delta_i^Y(t) H_i^Y \,.$$

Then the hedge portfolio for the exchange option on X and Y in terms of H_i^X and H_i^Y is given by

$$P_t = \sum_{i=1}^{L^X} \mathcal{N}(d_t^1) \cdot \delta_i^X(t) H_i^X - \sum_{i=1}^{L^Y} \mathcal{N}(d_t^2) \cdot \delta_i^Y(t) H_i^Y \,.$$

The application of this principle is illustrated in certain examples presented below.

3.2 Examples

Now we want to consider a number of examples which illustrate flexibility and generality of Theorem 3.1.

Currency Options: The payoff of a plain vanilla currency option equals $[e_T - K]^+$. Define the domestic assets $X := e \cdot B^f(\cdot, T)$ and $Y := K B^d(\cdot, T)$; the parameters of their price processes can be read off from the asset price dynamics and are given by $X_0 = e_0 B^f(0, T)$, $\eta^X(t) = \eta^e(t) + \eta^f(t, T)$ and $Y_0 = K B^d(0, T)$, $\eta^Y(t) = \eta^d(t, T)$, respectively. Since $B^d(T, T) = B^f(T, T) = 1$ the option's payoff equals $[X_T - Y_T]^+$, and its price can be computed by means of Theorem 3.1. Since we assume B^f and B^d to be traded assets we can use directly Theorem 3.1 to compute a feasible hedge portfolio.[3]

Currency Converted Options: There are two types of currency converted options. The payoff of a *Foreign Asset/ Domestic Strike Option* equals $[e_T S_T^f - K]^+$. To deal with this claim we set $X := e S^f$ and notice that this is a lognormal claim with $X_t = e_t S_t^f$ and $\eta^X = \eta^e + \eta^{S^f}$. Next set $Y := K \cdot B^d(\cdot, T)$. Theorem 3.1 can now be directly applied to give the price and the hedging strategy of this contract. Similarly for a *Domestic*

[3] In practice liquid markets for zero coupon bonds of arbitrary maturity usually do not exist. However, at least in a one-factor term structure model it is possible to duplicate a zero coupon bond by a dynamic trading strategy in a futures contract on some coupon bond and cash. Details are for instance given in (Frey and Sommer 1995).

Asset/ Foreign Strike Option with payoff $[S_T^d - e_T K]^+$, where K is in foreign currency we use the lognormal claims $X := S^d$ and $Y = KeB^f(\cdot, T)$.

Guaranteed-Exchange-Rate Options: The payoff of this derivative equals $[\bar{e}S_T^f - \bar{e}K]^+$, where S^f is a foreign asset and $\bar{e}$ some predetermined exchange rate. This contract can be interpreted as an option to exchange the lognormal claims X and Y with payoff $X_T = \bar{e}S_T^f$ and $Y_T := \bar{e}K$. Whereas Y_T equals the time T value of $K \cdot \bar{e}$ units of $B^f(\cdot, T)$, there is no traded asset whose value at T is equal to X_T. To price and hedge the option we therefore have to compute the parameters of X using Proposition 2.4. We have

$$\sum_i \alpha^{i,X} r_s^i = r_s^f, \quad \mu_s^X = -\frac{1}{2}|\eta_s^{S^f}|^2, \quad v_s^X = \eta_s^{S^f}.$$

Applying Proposition 2.4 yields

$$\begin{aligned} X_0 &= \bar{e}S_0^f \frac{B^d(0,T)}{B^f(0,T)} \exp\left\{ \int_0^T |\eta^f(s,T)|^2 + \eta^{S^f}(s) \cdot \eta^d(s,T) + \eta^e(s) \cdot \eta^f(s,T) - \right. \qquad (11) \\ &\qquad \left. \eta^{S^f}(s) \cdot \eta^f(s,T) - \eta^f(s,T) \cdot \eta^d(s,T) - \eta^{S^f}(s) \cdot \eta^e(s) ds \right\} \\ \eta_s^X &= \eta^{S^f}(s) + \eta^d(s,T) - \eta^f(s,T) \end{aligned}$$

The price of the option can now be computed by plugging these parameters into the pricing formula of Theorem 3.1. Next we want to determine the hedge portfolio for the option. As X_T is not the terminal value of a traded asset we have to go through the procedure outlined after the proof of Theorem 3.1. To replicate X_T by a dynamic trading strategy we first note that by (11) X_t is given by a function $\tilde{X}$ of the domestic assets $e \cdot S^f$, $B^d(t,T)$ and $e \cdot B^f(t,T)$ with derivatives $\partial\tilde{X}/\partial eS^f = \tilde{X}/(e_t S_t^f)$, $\partial\tilde{X}/\partial eB^f = -\tilde{X}/(e_t B^f(t,T))$ and $\partial\tilde{X}/\partial B^d = \tilde{X}/B^d(t,T)$. As $\tilde{X}$ is linear homogenous in the prices of these assets, an argument similar to the proof of the second part of Theorem 3.1 shows that the hedge portfolio for X equals

$$\delta^X_{e\cdot S^f}(t) = \frac{X_t}{e_t \cdot S_t^f}, \qquad \delta^X_{e\cdot B^f}(t) = -\frac{X_t}{e_t \cdot B^f(t,T)}, \qquad \delta^X_{B^d}(t) = \frac{X_t}{B^d(t,T)}.$$

Remark: Our formula contains the pricing formula of (Kat and Roozen 1994) as a special case. To derive their formula one simply has to set all the bond volatilities to zero. It is of interest to analyze the effect of the additional correlations that enter the pricing formula if interest-rate risk is taken into account. We see that allowing for interest-rate risk does not necessarily raise the price of a GER Option over that of a GER Option in a model with deterministic interest rates, since e.g. $\eta^e \cdot \eta^f$, the covariance between exchange rate and foreign bonds will typically be negative, while $\eta^{S^f} \cdot \eta^f$, that is the covariance between foreign bonds and stocks, will typically be positive. Both effects lead to a reduction in the option price, while the direct effect of stochastic interest rates, namely $|\eta^f|^2$ unambiguously raises the option price.

Options on Futures: Let $(\tilde{X}_t)_{0\le t\le \bar{T}}$ be the price process of a lognormal claim. It is well known that the futures price at time t of a futures contract on $\tilde{X}$ with maturity date T_2 equals $\tilde{X}_t^f := E^P[\tilde{X}_{T_2} \mid \mathcal{F}_t]$, see e.g. (Duffie 1992). A European option on this futures contract with maturity date $T_1 < T_2$ and strike K has payoff $[\tilde{X}^f_{T_1} - K]^+$. We now show that the contingent claim X with payoff $X_{T_1} = \tilde{X}^f_{T_1}$ at date T_1 is a lognormal claim. We get

$$
\begin{aligned}
\tilde{X}^f_{T_1} &= E^P\left[\tilde{X}_{T_2} \,\Big|\, \mathcal{F}_{T_1}\right] = E^P\left[\tilde{X}_{T_2} \cdot \left(B^d(T_2,T_2)\right)^{-1} \,\Big|\, \mathcal{F}_{T_1}\right] \\
&= E^P\left[\frac{\tilde{X}_0}{B^d(0,T_2)} \cdot \exp\left(-\frac{1}{2}\int_0^{T_2} |\eta^{\tilde{X}}|^2 - |\eta^d(t,T_2)|^2 dt + \int_0^{T_2} \eta^{\tilde{X}} - \eta^d(t,T_2) dW_t\right) \Big| \mathcal{F}_{T_1}\right] \\
&= \underbrace{\frac{\tilde{X}_0}{B^d(0,T_2)} \exp\left(\int_0^{T_2} |\eta^d(t,T_2)|^2 - \eta^{\tilde{X}} \cdot \eta^d(t,T_2) dt\right)}_{=:\,\gamma^X} \\
&\quad \cdot \exp\left(\int_0^{T_1} \eta^{\tilde{X}} - \eta^d(t,T_2) dW_t - \frac{1}{2}\int_0^{T_1} |\eta^{\tilde{X}} - \eta^d(t,T_2)|^2 dt\right)
\end{aligned}
$$

To compute the parameters of X we may now apply Proposition 2.4 with $\mu^X_t := -\frac{1}{2}|\eta^{\tilde{X}} - \eta^d(t,T_2)|^2$, $v^X := \eta^{\tilde{X}} - \eta^d(t,T_2)$ and γ^X as above. We leave the computations to the reader. To duplicate X by dynamic trading in the futures contract and in the domestic zero coupon bond with maturity T_1 one has to hold the following hedge portfolio.

$$
\delta^X_{\text{futures}}(t) = \frac{X_t}{\tilde{X}^f_t}, \qquad \delta^X_{B^d(\cdot,T_1)}(t) = \frac{X_t}{B^d(t,T_1)}\,.
$$

Options on Interest Rates: We are mainly interested in contracts where one of the underlying assets is a foreign or domestic LIBOR rate. For a fixed $\alpha > 0$ (in practice usually $\alpha = 0.25$ or $\alpha = 0.5$) the LIBOR rate $L^n(t,\alpha)$ prevailing in country n over the period $[t, t+\alpha]$ is defined by the equation

$$
(1 + \alpha \cdot L^n(t,\alpha))B^n(t,t+\alpha) = 1\,,
$$

that is $L^n(t,\alpha) = \alpha^{-1}(1/B^n(t,t+\alpha) - 1)$.

Caps: Perhaps the most important LIBOR derivatives are caps and floors. A cap is a portfolio of caplets. The payoff of a caplet with face value V, underlying interest rate process $L^d(t,\alpha)$, level K and maturity date $T+\alpha$ equals

$$
V \cdot \alpha \cdot [L^d(T,\alpha) - K]^+ = V \cdot \left[\frac{1}{B^d(T,T+\alpha)} - (\alpha K + 1)\right]^+
$$

As the payoff of this caplet is known already at T we may compute its present value at T which equals $V[1 - (\alpha K + 1) \cdot B^d(T,T+\alpha)]^+$. From this we see that the price and the hedge portfolio for caplets can be inferred directly from Theorem 3.1 if we use the lognormal claims $X = B^d(\cdot,T)$ and $Y = (\alpha K + 1) \cdot B^d(\cdot,T+\alpha)$. Of course this choice of X and Y reflects the well-known fact that caplets can be considered as options on zero coupon bonds.

LIBOR spreads: Next we want to consider an option on the spread between a domestic and a foreign LIBOR rate. The payoff (in domestic currency) in $T+\alpha$ of this option is given by $V\alpha[L^d(T,\alpha) - L^f(T,\alpha)]^+$, i.e. αV units of the positive difference between the domestic and the foreign LIBOR rate. Using the definition of the LIBOR rate we see that the present value at T of this payoff equals $V[1 - B^d(T,T+\alpha)/B^f(T,T+\alpha)]^+$. To value this contract we have to compute the parameters of the lognormal claim Y with payoff $Y_T = B^d(T,T+\alpha)/B^f(T,T+\alpha)$. Applying Proposition 2.4 we get

$$
\begin{aligned}
Y_0 &= \frac{B^d(0,T+\alpha)B^f(0,T)}{B^f(0,T+\alpha)} \\
&\cdot \quad \exp\left[\int_0^T |\eta^f(s,T+\alpha)|^2 + \eta^d(s,T+\alpha)\cdot\eta^f(s,T) - \eta^f(s,T+\alpha)\cdot\eta^f(s,T)\right. \\
&- \quad \left.\eta^f(s,T+\alpha)\cdot\eta^d(s,T+\alpha) - \eta^e(s)\cdot(\eta^f(s,T) - \eta^f(s,T+\alpha))\,ds\right]
\end{aligned}
$$

The hedge portfolio for Y can be computed as in the case of the guaranteed exchange rate option.

Remark: The valuation of interest rate derivatives in a Gaussian framework is somewhat problematic because of the occurrence of negative interest rates. However for reasonable parameter values and not too long times to maturity these problems are rather minor; see e.g. (Rogers 1996).

4 Compound Exchange Options

We define *compound exchange options* as European options with payoff given by

$$
\left[[X_T^1 - Y_T^1]^+ - [X_T^2 - Y_T^2]^+\right]^+ , \tag{12}
$$

where X^i and Y^i, $i = 1,2$ are lognormal claims. While the payoff is interesting in its own right it can, if combined with ordinary exchange options and lognormal claims, also serve as a building block to construct a great number of other payoffs. These include among others options on the maximum or the minimum of two lognormal claims, spread options and dual strike options. In Appendix A we explain how the payoffs of these options are related to equation (12).

We shall first state a general theorem on the pricing and hedging of the payoff in equation (12) and explain then how this theorem can be applied. We have

Theorem 4.1 *Let $(X_t^i)_{0\le t\le T}$, $(Y_t^i)_{0\le t\le T}$, $i = 1,2$ be lognormal claims. Let C_t be the value at time t of a compound exchange option with payoff as in (12).*

1. We have the following near explicit pricing formula

$$
\begin{aligned}
C_t &= C(t, X_t^1, Y_t^1, X_t^2, Y_t^2) := X_t^1\mathcal{N}_2\left(d_t^1, d_t^2, \rho_t\right) - Y_t^1\mathcal{N}_2\left(d_t^3, d_t^4, \rho_t\right) \\
&\quad + X_t^1 Q_t^{X^1}[A] + Y_t^2 Q_t^{Y^2}[A] - \left(X_t^2 Q_t^{X^2}[A] + Y_t^1 Q_t^{Y^1}[A]\right)
\end{aligned}
$$

where the arguments of the normal distributions[4] *are as defined in equations (14), (15), (16) below and where the set* A *is given by*

$$A = \left\{ \omega \in \Omega \mid X_T^2 \geq Y_T^2 \wedge X_T^1 - Y_T^1 \geq X_T^2 - Y_T^2 \right\} . \tag{13}$$

The probability measures $Q_t^{X^i}$, $i = 1, 2$ *and* $Q_t^{Y^i}$, $i = 1, 2$ *are as defined in the proof of Theorem 3.1.*

2. The exercise probabilities $Q_t^{X^i}[A]$ *and* $Q_t^{Y^i}[A]$ *are given by the measure of the domain* $\tilde{A} \subset I\!R^4$ *defined by*

$$\tilde{A} = \left\{ (x_1, \ldots, x_4) \in I\!R^4 \mid \exp(x_3) \geq \exp(x_4) \;\wedge\; \exp(x_1) - \exp(x_2) \geq \exp(x_3) - \exp(x_4) \right\}$$

under a — possibly degenerate — four-dimensional normal distribution whose mean and covariance matrix depend smoothly on t, X_t^i, Y_t^i *and the volatility coefficients* η^{X^i} *and* η^{Y^i} *of the lognormal claims.*[5]

3. The hedge portfolio (P_t) *in terms of the lognormal derivatives* X^i *and* Y^i *is given by*

$$\begin{aligned} \delta^C_{X^1}(t) &= \mathcal{N}_2\left(d_t^1, d_t^2, \rho_t\right) + Q_t^{X^1}[A] , & \delta^C_{X^2}(t) = -Q_t^{X^2}[A] \\ \delta^C_{Y^1}(t) &= -\left(\mathcal{N}_2\left(d_t^3, d_t^4, \rho_t\right) + Q_t^{Y^1}[A]\right) , & \delta^C_{Y^2}(t) = Q_t^{Y^2}[A] . \end{aligned}$$

Remark: This theorem shows that even under stochastic interest rates the valuation of a compound exchange option can be reduced to an integration with respect to the joint conditional terminal distribution of the underlying claims and hence to a problem of numerical integration in $\mathbb{R}^4$. We will see below that there are numerical techniques for the evaluation of the exercise probabilities that are far more efficient than a direct evaluation of the option's price by Monte Carlo simulation. Another advantage of our approach is that we are able to obtain the price *and* the hedge portfolio in one single step and with equal precision. No numerical differentiation is necessary.

PROOF: We consider first the pricing problem. A suitable decomposition of the payoff in (12) is given by:

$$\begin{aligned} &\left[\left[X_T^1 - Y_T^1\right]^+ - \left[X_T^2 - Y_T^2\right]^+\right]^+ \\ &= \left(X_T^1 - Y_T^1\right) 1_{\{X_T^1 \geq Y_T^1\}} 1_{\{X_T^2 \leq Y_T^2\}} + \left(X_T^1 - Y_T^1 - X_T^2 + Y_T^2\right) 1_{\{X_T^2 \geq Y_T^2\}} 1_{\{X_T^1 - Y_T^1 \geq X_T^2 - Y_T^2\}} \end{aligned}$$

Proceeding as in the proof of Theorem 3.1 we get for the first expression

$$E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^1 1_{\{X_T^1 \geq Y_T^1\}} 1_{\{X_T^2 \leq Y_T^2\}}\right] = X_t^1 \cdot Q_t^{X^1}\left[\left\{\ln \frac{Y_T^1}{X_T^1} \leq 0\right\} \cap \left\{\ln \frac{X_T^2}{Y_T^2} \leq 0\right\}\right] .$$

[4]By $\mathcal{N}_2\left(d^1, d^2, \rho\right)$ we denote the probability of the rectangle $(-\infty, d^1] \times -(\infty, d^2] \subset \mathbb{R}^2$ under the centered bivariate normal distribution with covariance matrix $(\sigma_{i,j})_{1 \leq i,j \leq 2}$ given by $\sigma_{1,1} = \sigma_{2,2} = 1$, $\sigma_{1,2} = \sigma_{2,1} = \rho$.

[5]In concrete examples these parameters can easily be computed from the parameters of the lognormal claims involved, see Example 1 below.

Now under $Q_t^{X^1}$ the random variables $\ln \frac{Y_T^1}{X_T^1}$ and $\ln \frac{X_T^2}{Y_T^2}$ are jointly normally distributed with correlation ρ_t given by

$$\rho_t = \frac{\int_t^T \left(\eta_s^{Y^1} - \eta_s^{X^1}\right) \cdot \left(\eta_s^{X^2} - \eta_s^{Y^2}\right)}{\sqrt{\left(\int_t^T |\eta_s^{Y^1} - \eta_s^{X^1}|^2 ds\right) \cdot \left(\int_t^T |\eta_s^{X^2} - \eta_s^{Y^2}|^2 ds\right)}} \,. \tag{14}$$

Hence this expectation equals $X_t^1 \mathcal{N}_2\left(d_t^1, d_t^2, \rho_t\right)$, where d_t^1 and d_t^2 are given by

$$\begin{aligned} d_t^1 &= \frac{\ln\left(\frac{X_t^1}{Y_t^1}\right) + \frac{1}{2}\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}} \\ d_t^2 &= \frac{\ln\left(\frac{Y_t^2}{X_t^2}\right) + \frac{1}{2}\int_t^T |\eta_s^{X^2}|^2 - 2\left(\eta_s^{X^2} - \eta_s^{Y^2}\right)\eta_s^{X^1} - |\eta_s^{Y^2}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^2} - \eta_s^{Y^2}|^2 ds}} \,. \end{aligned} \tag{15}$$

Similarly we get

$$E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} Y_T^1 \cdot 1_{\left\{X_T^1 \geq Y_T^1\right\}} 1_{\left\{X_T^2 \leq Y_T^2\right\}}\right] \;=\; Y_t^1 \mathcal{N}_2\left(d_t^3, d_t^4, \rho_t\right)$$

with ρ_t as before and with

$$\begin{aligned} d_t^3 &= \frac{\ln\left(\frac{X_t^1}{Y_t^1}\right) - \frac{1}{2}\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^1} - \eta_s^{Y^1}|^2 ds}} \\ d_t^4 &= \frac{\ln\left(\frac{Y_t^2}{X_t^2}\right) + \frac{1}{2}\int_t^T |\eta_s^{X^2}|^2 - 2\left(\eta_s^{X^2} - \eta_s^{Y^2}\right)\eta_s^{Y^1} - |\eta_s^{Y^2}|^2 ds}{\sqrt{\int_t^T |\eta_s^{X^2} - \eta_s^{Y^2}|^2 ds}} \,. \end{aligned} \tag{16}$$

For the remaining four expectations for $i = 1, 2$ we immediately get using the appropriate probability measures

$$E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} X_T^i 1_{\{A\}}\right] = X_t^i Q_t^{X^i}[A] \quad \text{and} \quad E_t^P\left[\left(\beta_{t,T}^d\right)^{-1} Y_T^i 1_{\{A\}}\right] = Y_t^i Q_t^{Y^i}[A] \,.$$

Now observe that under each of the four probability measures the four random variables $(\ln X_T^1,\ \ln Y_T^1,\ \ln X_T^2,\ \ln Y_T^2)$, obey a four-dimensional normal distribution. The mean $\mu(Q^{X^i})$ or $\mu(Q^{Y^i})$ of this distribution — which depends on the particular probability measure — is a smooth function of time and the initial values $X_t^i,\ i = 1, 2$ and $Y_t^i,\ i = 1, 2$, whereas Σ, the covariance matrix[6] of this normal distribution, depends only on time and the instantaneous volatilities and correlations of the four assets under consideration. Hence $Q_t^{X^1}[A]$ is given by the measure of the set $\tilde{A}$ defined in the theorem under the four dimensional normal distribution with mean $\mu(Q^{X^1}) \in \mathbb{R}^4$ and covariance matrix Σ.

[6] Σ is the same for all probability measures. We remark that the distribution might be degenerate, i.e. Σ might not have full rank. A concrete example for the computation of Σ is given in Example 1 below.

A similar result holds for the other exercise probabilities which proves the second part of the theorem.

Finally we deal with the hedging problem. By Itô's Lemma we have for the martingale part of C_t

$$d\,(C)_t^M = \sum_{i=1}^{2} \frac{\partial C}{\partial x^i}(t, X_t^1, Y_t^1, X_t^2, Y_t^2) d\left(X^i\right)_t^M + \sum_{i=1}^{2} \frac{\partial C}{\partial y^i}(t, X_t^1, Y_t^1, X_t^2, Y_t^2) d\left(Y^i\right)_t^M$$

Now we get for the partial derivatives

$$\begin{aligned}\frac{\partial C}{\partial x^1}(t, X_t^1, Y_t^1, X_t^2, Y_t^2) &= \frac{\partial}{\partial X_t^1} E_t^P \left[\left(\beta_{t,T}^d\right)^{-1} \left[X_T^1 - Y_T^1\right]^+ 1_{\{X_T^2 \le Y_T^2\}}\right] \\ &\quad + \frac{\partial}{\partial X_t^1} E_t^P \left[\left(\beta_{t,T}^d\right)^{-1} \left[X_T^1 - Y_T^1 - \left(X_T^2 - Y_T^2\right)\right]^+ 1_{\{X_T^2 \ge Y_T^2\}}\right]\end{aligned}$$

Now by arguments similar to those in the proof of part 2 of Theorem 3.1 this equals

$$E_t^P \left[\left(\beta_{t,T}^d\right)^{-1} \frac{X_T^1}{X_t^1} \cdot 1_{\{X_T^1 \ge Y_T^1\}} 1_{\{X_T^2 \le Y_T^2\}}\right] + E_t^P \left[\left(\beta_{t,T}^d\right)^{-1} \frac{X_T^1}{X_t^1} \cdot 1_{\{X_T^1 - Y_T^1 > X_T^2 - Y_T^2\}} 1_{\{X_T^2 > Y_T^2\}}\right].$$

By the same argument as in the the first part of the proof this equals $\mathcal{N}_2\left(d_t^1, d_t^2, \rho_t\right) + Q_t^{X^1}[A]$. Similarly it is easy to show that

$$\frac{\partial C}{\partial y^1} = -\left(\mathcal{N}_2\left(d_t^3, d_t^4, \rho_t\right) + Q_t^{Y^1}[A]\right), \quad \frac{\partial C}{\partial x^2} = -Q_t^{X^2}[A] \text{ and } \frac{\partial C}{\partial y^2} = Q_t^{Y^2}[A]\,.$$

Notice that $C(t, x^1, y^1, x^2, y^2)$ is linear homogeneous in its last four arguments, since the payoff of our option is linear homogeneous in the terminal values X_T^i and Y_T^i, and since X_T^i and Y_T^i are linear functions of the initial values X_t^i and Y_t^i. The remainder of the argument is, therefore, as in the proof of Theorem 3.1 □

Inspection of the set A in equation (13) shows that in general one will not be able to price a compound exchange option explicitely in the framework of our model. This is simply due to the fact that the distribution of a linear combination of lognormal distributed random variables is not known analytically.

There are, however, a number of numerical procedures that can be used to evaluate the respective integrals. In some special cases there are even closed form solutions. In the sequel we shall discuss an example and sketch different numerical procedures for the evaluation of the integrals. Finally, we shall also consider another example where the set A is such that a closed form solution for the option price and the hedge portfolio can be obtained.

Example 1: Consider pricing and hedging an option on the spread between the rate of return over a certain period in the stock market and the fixed income market in a foreign country. Assume that the payoff is received in the currency of another country, the domestic country. As an example think of the following construction: The maturity date of the option is nine month from now. The option is written on the difference between the realization of three months LIBOR on French Francs six months from the present and the annualized rate of return on the CAC40 index over the last three months of the lifetime of the option. The payoff of the entire derivative is received in Deutschmarks.

To be precise for $\alpha > 0$ let $\frac{1}{\alpha}\left(S^f_{T+\alpha}/S^f_T - 1\right)$ be the annualized rate of return on the stock index in the foreign country over the period from T to $T+\alpha$. As in Section 3.2 the LIBOR rate in the foreign country for the period from T to $T+\alpha$ is given by $\frac{1}{\alpha}\left(1/B^f(T, T+\alpha) - 1\right)$. Hence the payoff of our option equals

$$V\left[\frac{S^f_{T+\alpha}}{S^f_T} - \frac{1}{B^f(T,T+\alpha)} - K\right]^+ \;=\; V\left[\left[\frac{S^f_{T+\alpha}}{S^f_T} - \frac{1}{B^f(T,T+\alpha)}\right]^+ - [K-0]^+\right]^+$$

for some $K > 0$. To apply Theorem 4.1 we define the lognormal claims $X^1_{T+\alpha} := S^f_{T+\alpha}/S^f_T$, $Y^1_{T+\alpha} := 1/B^f(T, T+\alpha)$ and $X^2 = K \cdot B^d(\cdot, T+\alpha)$. As a first step we need to determine the price process $(X^1_t)_{0\le t\le T+\alpha}$ and $(Y^1_t)_{0\le t\le T+\alpha}$ under P. Applying Proposition 2.4 we have

$$X^1_t = X^1_0 \exp\left\{\int_0^t r^d_s ds - \frac{1}{2}\int_0^t |\eta^{X^1}_s|^2 ds + \int_0^t \eta^{X^1}_s dW_s\right\}, \tag{17}$$

where

$$\begin{aligned}
X^1_0 &= \frac{B^f(0,T)}{B^f(0,T+\alpha)} B^d(0,T+\alpha)\exp\left\{-\frac{1}{2}\int_0^{T+\alpha}\left(|\eta^f(s,T)1_{\{s\le T\}}|^2\right.\right.\\
&+ \left.|\eta^{Sf}_s 1_{\{s>T\}}|^2 - |\eta^f(s,T+\alpha)|^2|\eta^d(s,T+\alpha)|^2\right) ds\\
&- \left.\int_0^{T+\alpha}\left(\eta^f(s,T)1_{\{s\le T\}} + \eta^{Sf}_s 1_{\{s\ge T\}} - \eta^f(s,T+\alpha)\right)\eta^e_s + \frac{1}{2}|\eta^{X^1}_s|^2 ds\right\} \qquad (18)\\
\eta^{X^1}_s &= \eta^f(s,T)1_{\{s\le T\}} + \eta^{Sf}_s 1_{\{s>T\}} - \eta^f(s,T+\alpha) + \eta^d(s,T+\alpha)\,.
\end{aligned}$$

The corresponding formulas for Y^1 are easily obtained from those for X^1 by simply replacing η^{Sf}_s by $\eta^f(s, T+\alpha)$. The next step is to determine the joint distribution of $(\ln X^1_{T+\alpha}, \ln Y^1_{T+\alpha}, \ln X^2_{T+\alpha})$. Since $X^2_{T+\alpha}$ is constant and equal to K this distribution is a bivariate normal distribution. Again the mean of this distribution depends on the numeraire, while the covariance matrix Σ does not. The parameters $\mu(Q^{X^1})$, $\mu(Q^{Y^1})$, $\mu(Q^{X^2})$ and Σ are given in Appendix B. With this information at hand there are now several possibilities for evaluating the option pricing formula.

First one might evaluate the integral by a *simple Monte-Carlo simulation*. To obtain one simulated value we first make a draw from two independent standard normal variates, denoted by (z_1, z_2). We then transform the result as follows

$$\begin{pmatrix}\tilde{x}^1(z_1,z_2)\\ \tilde{x}^2(z_1,z_2)\end{pmatrix} = \mu(Q^{X^1}) + \tilde{\Sigma}\begin{pmatrix}z_1\\ z_2\end{pmatrix} \tag{19}$$

where $\tilde{\Sigma}\tilde{\Sigma}^T = \Sigma$. We then check, if the vector $(\tilde{x}^1(z_1,z_2), \tilde{y}^1(z_1,z_2))$ belongs to $\tilde{A}$ which in our case reduces to the set $\{(\tilde{x}_1,\tilde{x}_2) \in \mathbb{R}^2 \mid \exp(\tilde{x}_1) - \exp(\tilde{x}_2) \ge K\}$. If so we denote a 1, otherwise we denote a 0. Summing the ones over all simulations and deviding by the total number of simulations gives an estimate of $Q^{X^1}[A]$. Analogously we can use the simulated values (z_1, z_2) to obtain estimates for $Q^{Y^1}[A]$ and $Q^{X^2}[A]$ and hence obtain a simulation of all four exercise probabilities from one draw of (z_1, z_2). Of course variance reduction methods like the antithetic variable technique and modern Monte Carlo methods using quasi random numbers may be used to improve the accuracy of the method. For an account of these techniques see for instance (Boyle, Broadie, and Glasserman 1995).

Alternatively we could conduct a *regular grid search*. Starting from the observation that the density of the normal distribution is approximately zero outside the intervall $I := [-3,\ 3]$ one partitions I into n sub-intervalls. One may consider different ways of partitioning, e.g. such that the n sub-intervals have equal length or such that the n sub-intervals have equal mass under the standard normal distribution. Let $M(n) = \{m(1), \ldots, m(n)\}$ denote the set of the midpoints of the n sub–intervals of I under one of the two partitioning schemes mentioned. Then a two–dimensional grid is defined by $(M(n))^2 := M(n) \times M(n)$. Further denote by $\Xi(n) = \{\xi(1), \ldots, \xi(n)\}$ the set of increments of the standard normal distribution function over the n sub–intervals generated by one of the partitions. Then we associate with every element $m = (m(j_1), m(j_2))$ belonging to $(M(n))^2$ the weight $\pi(m) := \xi(j_1) \cdot \xi(j_2)$. We now transform the elements $m \in (M(n))^2$ as in (19) and define

$$s^1 : (M(n))^2 \to \{0, 1\}\,, \quad m \mapsto s^1(m)$$

by $s^1(m) := 1$ if the vector $(\tilde{x}^1(m(j_1), m(j_2)), \tilde{x}^2(m(j_1), m(j_2)))$ belongs to $\tilde{A}$, $s^1(m) := 0$ otherweise. The approximation for $Q^1[A]$ is then given by

$$Q^{X^1}[A] = \sum_{m \in (M(n))^2} s^1(m)\pi(m)\,.$$

Of course to compute the exercise probabilities under the measures corresponding to the numeraires Y^1 and X^2 one can proceed analogously.

As to a comparison of the two methods we found that for problems where the rank of the covariance matrix Σ exceeds 3 the Monte Carlo approach seems more favourable whereas for rank $(\Sigma) \leq 3$ a grid search approach seems to be advantageous. This is due to the fact that the number of grid points and hence the number of computations increases exponentially with the rank of Σ whereas using a Monte Carlo approach the number of computations grows only linearly with the rank of Σ.

Example 2: As an example of an option whose price and hedge portfolio can be calculated explicitely consider a put option on the maximum of the rates of the fixed income market and the stock market, i.e.

$$V\alpha\left[K - \max\left\{\frac{1}{\alpha}\left(\frac{1}{B^f(T, T+\alpha)} - 1\right); \frac{1}{\alpha}\left(\frac{S^f_{T+\alpha}}{S_T} - 1\right)\right\}\right]^+ =$$

$$V\alpha\left[\left[K - \frac{1}{\alpha}\left(\frac{1}{B^f(T, T+\alpha)} - 1\right)\right]^+ - \frac{1}{\alpha}\left[\frac{S^f_{T+\alpha}}{S_T} - \frac{1}{B^f(T, T+\alpha)}\right]\right]^+ =$$

$$V\left[\left[\tilde{K}-\frac{1}{B^f(T,T+\alpha)}\right]^+ - \left[\frac{S^f_{T+\alpha}}{S_T}-\frac{1}{B^f(T,T+\alpha)}\right]\right]^+ ,$$

where $\tilde{K} = \alpha K - 1$. Again the payoff is assumed to be in domestic currency. The interpretation of this payoff is simple. If both, the fixed income and the stock market in the foreign country underperform over the period from T to $T+\alpha$ with respect to a certain predetermined minimum rate of return this option bails out the portfolio manager.

To develop this example further we introduce the following notation:
$X^1 := \tilde{K}B^d(\cdot, T+\alpha)$, X^2 is the same as the process X^1 in Example 1, and Y is the same process as Y^1 in Example 1. Using this notation the payoff of the above option can be rewritten as follows

$$V\left[\left[X^1_{T+\alpha}-Y_{T+\alpha}\right]^+ - \left[X^2_{T+\alpha}-Y_{T+\alpha}\right]^+\right]^+ .$$

From this formula we immediately get that the set A defined in (13) is given by

$$A = \{\omega \in \Omega \mid (X^1_{T+\alpha} > Y_{T+\alpha}) \cap (X^1_{T+\alpha} > (X^2_{T+\alpha})\}$$

Since this expression does not contain any differences of lognormal claims, the exercise probabilities can be computed explicitely. We remark, however, that the formula obtained in this way contains unnecessarily many bivariate normal distributions. A more efficient decomposition of the payoff is given in (Frey and Sommer 1995).

5 Conclusion

The paper treats the valuation and hedging of non-pathdependent European options on several underlyings with interest rate risk. Using martingale techniques in many cases we are able to provide general closed form solutions together with a procedure for applying these general solutions to some specific payoff at hand. In cases where explicit solutions do not exist we give near explicit solutions for both pricing and hedging.

The main restriction of our approach is the assumption of deterministic volatilities which is particularly bothersome in the case of bonds. However, it is known from the interest rate literature that it is difficult to relax this hypothesis if one is interested in explicit solutions.

6 Appendix

A Payoffprofiles

Options on the maximum of two lognormal claims: We have

$$\left[\max\{X^1_T, X^2_T\} - K\right]^+ = \left[\left[X^1_T - K\right]^+ - \left[X^2_T - K\right]^+\right]^+ + \left[X^2_T - K\right]^+$$

Options on the minimum of two lognormal claims:

$$\left[\min\{X_T^1, X_T^2\} - K\right]^+ = \left[X_T^1 - K\right]^+ - \left[\left[X_T^1 - K\right]^+ - \left[X_T^2 - K\right]^+\right]^+$$

Spread options: We distinguish two cases. First consider a positive strike price K:

$$\left[X_T^1 - X_T^2 - K\right]^+ = \left[\left[X_T^1 - K_T\right]^+ - \left[X_T^2 - 0\right]^+\right]^+$$

Next consider a negative strike price K: Define $\tilde{K} := -K$. The payoff of the option is now

$$\left[X_T^1 - X_T^2 + \tilde{K}\right]^+ = \left[\left[X_T^1 - 0\right]^+ - \left[X_T^2 - \tilde{K}\right]^+\right]^+ + \left[\tilde{K} - X_T^2\right]^+$$

Dual strike options: Again we have two cases.

$$\max\left\{\left[X_T^1 - K^1\right]^+ ; \left[X_T^2 - K^2\right]^+\right\} = \left[\left[X_T^1 - K^1\right]^+ - \left[X_T^2 - K^2\right]^+\right]^+ + \left[X_T^2 - K_T^2\right]^+$$

$$\min\left\{\left[X_T^1 - K^1\right]^+ ; \left[X_T^2 - K^2\right]^+\right\} = \left[X_T^1 - K^1\right]^+ - \left[\left[X_T^1 - K^1\right]^+ - \left[X_T^2 - K^2\right]^+\right]^+ .$$

B Example 1: Distributions

From equation (17) we have for $X_{T+\alpha}^1$

$$\begin{aligned} X_{T+\alpha}^1 &= \frac{X_0^i}{B^d(0, T+\alpha)} \exp\left\{ -\frac{1}{2}\int_0^{T+\alpha} \left(|\eta_s^{X^1}|^2 - |\eta^d(s,T)|^2 \right) ds \right. \\ & \left. + \int_0^{T+\alpha} \left(\eta_s^{X^1} - \eta^d(s, T+\alpha) \right) dW_s \right\} . \end{aligned}$$

A similar equation holds for $Y_{T+\alpha}^1$ if we replace η^{X^1} by η^{Y^1}. Hence we see that

$$\begin{pmatrix} \ln X_{T+\alpha}^1 \\ \ln Y_{T+\alpha}^1 \end{pmatrix} \sim \mathcal{N}_2\left(\mu(\cdot); \Sigma\right) ,$$

where $\mathcal{N}_2\left(\mu(\cdot), \Sigma\right)$ is now a bivariate normal distibution. For Σ we obtain

$$\begin{aligned} \sigma_{1,1} &= \int_0^{T+\alpha} \left(|\eta_s^{X^1} - \eta^0(s, T+\alpha)|^2 \right) ds \\ \sigma_{1,2} &= \sigma_{2,1} = \int_0^{T+\alpha} \left(\eta_s^{X^1} - \eta^0(s, T+\alpha) \right) \left(\eta_s^{Y^1} - \eta^0(s, T+\alpha) \right) ds \\ \sigma_{2,2} &:= \int_0^{T+\alpha} |\eta_s^{Y^1} - \eta^0(s, T+\alpha)|^2 ds \, . \end{aligned}$$

For $\mu(\cdot)$ we obtain

$$\mu(Q^{X^i}) = \bar{\mu} + \begin{pmatrix} \int_0^{T+\alpha} \left(\eta_s^{X^1} - \eta^d(s, T+\alpha) \right) \eta_s^{X^i} ds \\ \int_0^{T+\alpha} \left(\eta_s^{Y^1} - \eta^d(s, T+\alpha) \right) \eta_s^{X^i} ds \end{pmatrix}$$

$$\mu(Q^{Y^1}) = \bar{\mu} + \begin{pmatrix} \int_0^{T+\alpha} \left(\eta_s^{X^1} - \eta^d(s, T+\alpha)\right) \eta_s^{Y^1} ds \\ \int_0^{T+\alpha} \left(\eta_s^{Y^1} - \eta^d(s, T+\alpha)\right) \eta_s^{Y^1} ds \end{pmatrix}$$

where

$$\bar{\mu} = \begin{pmatrix} \ln \dfrac{X_0^1}{B^d(0, T+\alpha)} - \frac{1}{2} \int_0^{T+\alpha} \left(|\eta_s^{X^1}|^2 - |\eta^d(s, T)|^2\right) ds \\ \ln \dfrac{Y_0^1}{B^d(0, T+\alpha)} - \frac{1}{2} \int_0^{T+\alpha} \left(|\eta_s^{Y^1}|^2 - |\eta^d(s, T)|^2\right) ds \end{pmatrix}.$$

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