Corporate Security Prices in Structural Credit Risk Models with Incomplete Information

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Abstract

The paper studies structural credit risk models with incomplete information of the asset value. It is shown that the pricing of typical corporate securities such as equity, corporate bonds or CDSs leads to a nonlinear filtering problem. This problem cannot be tackled with standard techniques as the default time does not have an intensity under full information. We therefore transform the problem to a standard filtering problem for a stopped diffusion process. This problem is analyzed via SPDE results from the filtering literature. In particular we are able to characterize the default intensity under incomplete information in terms of the conditional density of the asset value process. Moreover, we give an explicit description of the dynamics of corporate security prices. Finally, we explain how the model can be applied to the pricing of bond and equity options and we present results from a number of numerical experiments.

Keywords. Structural credit risk models, incomplete information, nonlinear filtering

1 Introduction

Structural credit risk models and in particular the first-passage-time models studied for instance by Black and Cox (1976) or Leland (1994) are widely used in the analysis of defaultable corporate securities. In these models a firm defaults if a random process V representing the firm's asset value hits some barrier K that is often interpreted as value of the firm's liabilities. First-passage-time models offer an intuitive economic interpretation of the default event. However, in the practical application of these models a number of difficulties arise: To begin with, it might be difficult for investors in secondary markets to assess precisely the value of the firm's assets. Moreover, for tractability reasons V is frequently modelled as a diffusion process. In that case the default time τ is a predictable stopping time, leading to unrealistically low values for short-term credit spreads.

For these reasons Duffie and Lando (2001) proposed a model where secondary markets have only incomplete information on the asset value V. More precisely they consider the situation where the market obtains at discrete time points t_n a noisy accounting report of the form $Z_n = \ln V_{t_n} + \varepsilon_n$; moreover, the market knows the default history of the firm. Duffie and Lando show that in this setting the default time τ admits an intensity λ_t that is proportional to the derivative of the conditional density of V_t at the default barrier K. This well-known result

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provides an interesting link between structural and reduced-form models. Moreover, the result shows that by introducing incomplete information it is possible to construct structural models with a fairly realistic behavior of short-term credit spreads. The subsequent work of Frey and Schmidt (2009) discusses the pricing of the firm's equity in the context of the Duffie-Lando model. Moreover, it is shown that the pricing of typical corporate securities (equity and debt) leads to a nonlinear filtering problem: one needs to determine the conditional distribution of the current asset value V_t given the the σ field $\mathcal{F}_t^{\mathbb{M}}$ representing the information available to the market up to time t. This problem is addressed by a Markov-chain approximation for the asset value process. In particular, a recursive updating rule for the conditional distribution of the approximating Markov chain is derived via elementary Bayesian updating arguments. Both papers do not give results on the dynamics of corporate security price information under incomplete information. In fact, the noisy accounting considered by Duffie and Lando implies a very unrealistic dynamics of credit spreads: spreads evolve deterministically between the news-arrival dates t_n and jump only when a new noisy accounting report is received so that credit spread volatility is zero.

These issues are addressed in the present paper. We model noisy asset observation by a continuous time process of the form $Z_t = \int_0^t a(V_s)ds + W_t$ for some Brownian motion W independent of V. This assumption is more in line with the standard literature on stochastic filtering (see for instance Bain and Crisan (2009)) than the discrete noisy accounting information of Duffie and Lando (2001). More importantly, we show that with this type of noisy asset information asset prices follow processes of standard jump-diffusion type

As in Frey and Schmidt (2009), in this setup the pricing of corporate securities leads to the task of determining the conditional distribution of V_t given the market information $\mathcal{F}_t^{\mathbb{M}}$. This is a challenging stochastic filtering problem, since under full observation, that is with observable asset value process, the default time τ is predictable. Hence standard filtering techniques for point process observations cannot be used. We therefore transform the original problem to a new filtering problem where the observations consist only of the noisy asset information via the hazard-rate approach to credit risk modelling (see for instance Blanchet-Scalliet and Jeanblanc (2004)); the signal process in this new problem is on the other hand given by the asset value process stopped at the first exit time of the solvency region (K, ∞). Using results of Pardoux (1978) on the filtering of stopped diffusion processes we derive a stochastic partial differential equation (SPDE) for the conditional density $\pi(t, \cdot)$ of V_t given $\mathcal{F}_t^{\mathbb{M}}$, and we discuss approaches for the numerical solution of this SPDE. Extending the Duffie and Lando (2001)-result, we show that τ admits an intensity λ_t that is proportional to the spatial derivative of $\pi(t, \cdot)$ at $v = K$. These results permit us to derive the nonlinear filtering equations for the market filtration. As a corollary we identify the price dynamics of equity and debt in the market filtration. Understanding the price dynamics of corporate securities is a prerequisite for any kind of derivative asset analysis in structural models with incomplete information so that these are important new results. Finally we turn to more applied issues: we consider the pricing of options on equity and debt, we briefly discuss model calibration and we present several numerical experiments that further illustrate our results.

Incomplete information and stochastic filtering has been used frequently for the analysis of credit risk. Structural models with incomplete information were considered among others by Kusuoka (1999), Duffie and Lando (2001), Nakagawa (2001), Jarrow and Protter (2004), Coculescu, Geman, and Jeanblanc (2008), Frey and Schmidt (2009) and Cetin (2012). The last paper works in a similar setup as the one considered here. Cetin uses purely probabilistic arguments to establish the existence of a default intensity intensity λ_t in the market filtration, and he derives the corresponding filter equations. His results are however less explicit than ours; among others, Cetin (2012) does not give a characterization of λ_t in terms of the derivative of the conditional density at the default barrier, there are no results on asset price dynamics under incomplete information and no numerical experiments.

Reduced-form credit risk models with incomplete information have been considered previously by Duffie, Eckner, Horel and Saita (2009), Frey and Runggaldier (2010) and Frey and Schmidt (2012), among others. The modelling philosophy of the present paper is inspired by the analysis of Frey and Schmidt (2012), but the mathematical arguments used in the two papers differ substantially.

The remainder of the paper is organized as follows: in Section 2 we introduce the model; the pricing of basic corporate securities is discussed in Section 3; Section 4 is concerned with the stochastic filtering of the asset value; in Section 5 we derive the filter equations and discuss the dynamics of corporate securities; Section 6 is concerned with derivative pricing; the results of numerical experiments are given in Section 7.

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2 The model.

We work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}, P)$ and we assume that all processes introduced below are $\mathbb{G}$-adapted. We consider a company with asset value process $V = (V_t)_{t\geq 0}$. The company is subject to default risk and the default time is given by

$$\tau = \inf\{t \geq 0 \colon V_t \leq K\} \ \text{ for some } K > 0\,. \tag{2.1}$$

In practice the default barrier K might represent debt covenants as in Black and Cox (1976) or, in the case of financial institutions, solvency capital requirements imposed by regulators. It is well known that absence of arbitrage implies the existence of a probability measure $Q \sim P$ such that for any traded security the corresponding discounted gains from trade are Q-martingales. Since we are mainly interested in pricing, in the following assumptions it is thus sufficient to specify the Q-dynamics of all economic variables introduced.

Assumption 2.1 (Dividends and asset value process)**.** (i) The risk free rate of interest is constant and equal to $r \geq 0$.

(ii) *Dividends* are paid at discrete time points $T_1, T_2, \ldots$; the expected size of the dividend payment is proportional to the asset value at the dividend date. More precisely, let d_n be the dividend payment at T_n. We assume that

$$d_n = \delta_n V_{T_n}$$

for a iid sequence of noise variables $(\delta_n)_{n=1,2,\ldots}$, independent of V, taking values in $(0, +\infty)$, with density function f_δ and mean $\bar{\delta} = E^Q(\delta_1)$. We denote the cumulative dividend process by $D_t = \sum_{\{n\colon T_n \le t\}} d_n$. The the conditional distribution of d_n given the history of the asset value process is thus of the form

$$\varphi(y, V_{T_n-})dy \quad \text{where } \varphi(y,v) = v^{-1} f_\delta\big(y/v\big)\,. \tag{2.2}$$

We assume that for all $y \in \mathbb{R}^+$ the map $v \mapsto \varphi(y,v)$ is bounded and twice continuously differentiable on $[K,\infty)$. We consider two different models for the timing of the dividends. On the one hand the T_n might be deterministic and spaced equally in time (for instance to semi-annual dividend payments); the number of dividend payments per year is denoted by $\lambda^D > 0$. Alternatively we assume that T_n is the nth jump time of a Poisson process with intensity λ^D. This case is introduced for convenience: with Poissonian dividend dates an easy closed-form solution for the full-information value of the firm's equity can be derived, see (3.8) below.

(iii) The *asset value process* V solves the following SDE

$$dV_t = (r - \lambda^D \bar\delta)V_t dt + \sigma V_t dB_t, \quad V_0 = V \tag{2.3}$$

for a constant $\sigma > 0$ and a standard Q-Brownian motion B. Moreover, V has Lebesgue density $\pi_0(v)$ for a continuously differentiable function $\pi_0\colon [K,\infty) \to \mathbb{R}^+$ with $\pi_0(K) = 0$.

We denote the random measure associated with the marked point process $(T_n, d_n)_{n\in\mathbb{N}}$ by $\mu^D(dy,dt)$. With Poissonian dividend dates the $\mathbb{G}$-compensator of μ^D is given by $\gamma^D(dy,dt) = \varphi(y,V_t)dy\,\lambda dt$. If the dividend dates are deterministic we will typically use lower case letters for dividend dates. In that case the compensator of μ^D is $\gamma^D(dy,dt) = \sum_{n=1}^\infty \varphi(y, V_{t_n})dy\,\delta_{t_n}(dt)$.

Comments. The assumption that the asset value is a geometric Brownian motion is routinely made in the literature on structural credit risk models such as Leland (1994) or Duffie and Lando (2001). For empirical support for the assumption of geometric Brownian motion as a model for the asset price dynamics we refer to Sun, Munves and Hamilton (2012). Note that the assumption that V follows a geometric Brownian motion does not imply that the *equity value* or stock price follows a geometric Brownian motion. In fact, our analysis in Section 5 shows that in our setup the stock price dynamics can be much 'wilder' than geometric Brownian motion.

Assumption 2.2. The following pieces of information are used by the market in the pricing of corporate securities.

(i) *Default information.* The market observes the default state $N_t = 1_{\{\tau \le t\}}$ of the firm. We denote the default history by $\mathbb{F}^N = (\mathcal{F}_t^N)_{t\ge 0}$.

(ii) *Dividend information.* The market observes the dividend payments or equivalently the cumulative dividend process process D; the corresponding filtration is denoted by $\mathbb{F}^D = (\mathcal{F}_t^D)_{t\ge 0}$. Note that dividends carry information on V_t as the distribution of the dividend size depends on the asset value at the dividend date.

(iii) *Noisy asset observation.* The market observes functions of V in additive Gaussian noise. Formally, this bit of information is modelled via some process Z of the form

$$Z_t = \int_0^t a(V_s)ds + W_t, . \tag{2.4}$$

Here W is an l-dimensional standard $\mathbb{G}$-Brownian motion independent of B, and we assume that a is a continuously differentiable function from $\mathbb{R}^+$ to $\mathbb{R}^l$ with $a(K) = 0$.[4] Finally, $\mathbb{F}^Z = (\mathcal{F}_t^Z)_{t \geq 0}$ represents the filtration generated by Z. We view the process Z as an abstract representation of all economic information on V that is used by the market in addition to the publicly observed dividend payments.

Summarizing, the information set of the market at time t is given by the σ-field $\mathcal{F}_t^{\mathbb{M}} = \mathcal{F}_t^N \vee \mathcal{F}_t^Z \vee \mathcal{F}_t^D$; the corresponding filtration is denoted by $\mathbb{F}^{\mathbb{M}}$. Note that $\mathbb{F}^{\mathbb{M}} \subset \mathbb{G}$ and that V is not adapted to $\mathbb{F}^{\mathbb{M}}$.

There are many possibilities for the form of the function a. A natural choice is $a(v) = c(\ln v - \ln K)$; this corresponds to a continuous-time version of the noisy asset information considered in Duffie and Lando (2001). Here the parameter $c \geq 0$ models the information contained in Z; for c large the asset value can be observed with high precision whereas for c close to zero the process Z conveys almost no information. Alternatively we consider a two-dimensional specification of the form $a_1(v) = c_1(\ln v - \ln K)$ and $a_2(v) = c_2\big((\tilde{K} - \ln v)^+ - (\tilde{K} - \ln K)\big)$ for some threshold $\tilde{K}$ that is close to the default barrier. Here the function a_2 models the case where the market receives additional information if a firm is close to default, perhaps due to additional monitoring activity by the stakeholders.

3 Pricing basic corporate securities and nonlinear filtering

In this section we discuss the pricing of basic corporate securities whose associated cash flow stream depends on future dividend payments and on the occurrence of default and is thus $\mathbb{F}^N \vee \mathbb{F}^D$-adapted. Examples include bonds, credit default swaps (CDS) and the equity value of the firm. In particular, we will see that the pricing of these basic securities leads to a nonlinear filtering problem in a straightforward way. The pricing of securities whose payoff depends on the price process of basic corporate securities - such as equity- or bond options - is more involved and will be discussed in Section 6.

Since Q represents the martingale measure used for pricing, the ex-dividend price of a generic security with $\mathbb{F}^{\mathbb{M}}$-adapted cash flow stream $(H_t)_{0 \leq t \leq T}$ and maturity date T is given by

$$\Pi_t^H = E^Q\Big(\int_t^T e^{-r(s-t)} dH_s \mid \mathcal{F}_t^{\mathbb{M}} \Big), \quad t \leq T. \tag{3.1}$$

Note that Π_t^H is defined as conditional expectation with respect to the σ-field $\mathcal{F}_t^{\mathbb{M}}$ that describes the information available to the market at time t.

In the sequel we mostly consider the pre-default value of the security given by $1_{\{\tau > t\}} \Pi_t^H$ (pricing for $\tau \leq t$ is largely related to the modelling of recovery rates which is of no concern to us here). Using iterated conditional expectations we get that

$$1_{\{\tau > t\}} \Pi_t^H = E^Q\Big(E^Q\big(1_{\{\tau > t\}} \int_t^T e^{-r(s-t)} dH_s \mid \mathcal{G}_t\big) \mid \mathcal{F}_t^{\mathbb{M}}\Big).$$

[4] The assumption $a(K) = 0$ is no real restriction as the function a can be replaced with $a - a(K)$ without altering the information content of $\mathbb{F}^{\mathbb{M}}$.

By the Markov property of V, for typical corporate securities the inner conditional expectation can be expressed as a function of time and of the current asset value, that is

$$E^Q\big(1_{\{\tau>t\}}\int_t^T e^{-r(s-t)}dH_s \mid \mathcal{G}_t\big) = 1_{\{\tau>t\}}h(t,V_t). \tag{3.2}$$

The function h will be called *full-information value* of the claim. Below we compute this function for several important examples. We thus get from (3.2) that

$$1_{\{\tau>t\}}\Pi_t^H = 1_{\{\tau>t\}}E^Q\big(h(t,V_t) \mid \mathcal{F}_t^{\mathbb{M}}\big). \tag{3.3}$$

Since V is not $\mathbb{F}^{\mathbb{M}}$ adapted, the evaluation of this conditional expectation is a nonlinear filtering problem that is discussed in Section 4. Note that martingale pricing generally leads to nonlinear filtering problems under the martingale measure Q rather than the physical measure P.

In the remainder of this section we explain how the full information value h can be computed for debt-related securities such as bonds or CDSs and for the equity of the firm.

Full-information value of debt securities. It is well-known that the valuation of debt securities of the firm can be reduced to the pricing of two building blocks, namely a so-called *survival claim* and a so-called *payment-at-default* claim. A survival claim pays one unit of account at the maturity date T provided that $\tau > T$. By standard results the corresponding full-information value h^{surv} solves the boundary value problem $\frac{d}{dt}h^{\text{surv}} + \mathcal{L}_V h^{\text{surv}} = rh^{\text{surv}}$, $t \in [0,t) \times (K,\infty)$, with boundary and terminal conditions $h^{\text{surv}}(t,K) = 0$, $0 \le t \le T$ and $h^{\text{surv}}(T,v) = 1$, $v > K$. Here

$$\mathcal{L}_V f(v) = (r - \lambda^D\bar{\delta})v\frac{df}{dv}(v) + \frac{1}{2}\sigma^2v^2\frac{d^2f}{dv^2}(v) \tag{3.4}$$

is the generator of V. A payment-at-default claim with maturity T pays one unit directly at τ, provided that $\tau \le T$. The corresponding full information value h^{def} solves the boundary value problem $\frac{d}{dt}h^{\text{def}} + \mathcal{L}_V h^{\text{def}} = rh^{\text{def}}$ for $t \in [0,t) \times (K,\infty)$, with boundary value $h^{\text{def}}(t,K) = 1$, $0 \le t \le T$ and terminal value $h^{\text{def}}(T,v) = 0$, $v > K$.

Since V is modeled as a geometric Brownian motion, $\log V_t$ satisfies $\log V_t = \mu t + \sigma B_t$ with $\mu := r - \lambda^D\bar{\delta} - \frac{1}{2}\sigma^2$. Hence h^{surv} and h^{def} can be computed using results for the first passage time of Brownian motion with drift. Denote by

$$f(t;b) := \frac{|b|}{\sqrt{2\pi t^3}\sigma}\exp\Big(-\frac{(b-\mu t)^2}{2\sigma^2 t}\Big) \tag{3.5}$$

the density function of the first passage time of the process $\sigma B_t + \mu t$ to the level b, see for instance Karatzas and Shreve (1988), Section 3.5.C. Then we have for $v > K$

$$h^{\text{surv}}(t,v) = e^{-r(T-t)}\Big(1 - \int_0^{T-t} f\big(s,\log\frac{K}{v}\big)\,ds\Big) \ \text{ and } h^{\text{def}}(t,v) = \int_0^{T-t} e^{-rs}f\big(s,\log\frac{K}{v}\big)\,ds\,.$$

Full information value of equity. In our setup the value of the firm's equity is given by the expected discounted value of future dividend payments up to the default time τ, that is

$$h^{\text{eq}}(v) = E^Q\Big(\int_0^\infty 1_{\{\tau>s\}}e^{-rs}dD_s \mid V_0 = v\Big) \tag{3.6}$$

We now give an explicit formula for h^{eq} for the case of Poissonian dividends. In that case $D_t - \int_0^t \bar\delta V_s \lambda^D\, ds$ is a martingale and it holds that

$$h^{\mathrm{eq}}(v) = E^Q\Big(\int_0^\infty 1_{\{\tau > s\}} e^{-rs} \bar\delta V_s \lambda^D \, ds \mid V_0 = v \Big)$$

The full-information value $h^{\mathrm{eq}}(v)$ is thus time-independent (this is the main advantage of assuming Poissonian dividend dates) and it solves the ordinary differential equation $\mathcal{L}_V h^{\mathrm{eq}}(v) + \bar\delta \lambda^D v = r h^{\mathrm{eq}}(v)$, $v > K$. In the special case where $K = 0$ and hence $\tau = \infty$ we get

$$h^{\mathrm{eq}}(v) = \lambda^D \bar\delta \int_0^\infty e^{-rs} E^Q\big(V_s \mid V_0 = v\big)\, ds = \lambda^D \bar\delta \int_0^\infty e^{-rs} v e^{(r - \bar\delta \lambda^D)s}\, ds = v\,. \tag{3.7}$$

This implies in particular that the asset value can be interpreted as present value of all future dividend payments (up to $t = \infty$). For $K > 0$ it holds that

$$h^{\mathrm{eq}}(v) = v - K(\frac{v}{K})^{\alpha^*}, \tag{3.8}$$

where α^* is the negative root of the equation $(r - \lambda^D \bar\delta)\alpha + \frac{1}{2}\sigma^2 \alpha(\alpha - 1) - r = 0$. This can be shown by standard arguments, see for instance Proposition 2.4 of Frey and Schmidt (2009).

With deterministic dividend dates $t_1, t_2, \ldots$ it holds that

$$\begin{aligned} h^{\mathrm{eq}}(t, v) &= \sum_{n\colon t_n > t} E^Q\left(e^{-r(t_n - t)} d_n 1_{\{\tau > t_n\}} \mid V_t = v \right) \\ &= \sum_{n\colon t_n > t} E^Q\left(e^{-r(t_n - t)} \bar\delta V_{t_n} 1_{\{\min_{s \le t_n} V_s > K\}} \mid V_t = v \right). \end{aligned}$$

It follows that h^{eq} is given by a sum of barrier option prices and is thus computable as well; we omit the details. Note finally that for finely spaced dividend dates the equity value computed with Poissonian dividend dates is a good approximation to the equity value computed for deterministic dividend dates, which is why we work with h^{eq} as given in (3.8) in the sequel.

4 Stochastic Filtering of the Asset Value

The pricing of corporate securities leads to the task of computing for $g \in L^\infty\big([K, \infty)\big)$ the conditional expectation

$$1_{\{\tau > t\}} E^Q\big(g(V_t) \mid \mathcal{F}_t^{\mathbb{M}}\big), \quad t \le T\,. \tag{4.1}$$

This problem is studied in the present section.

4.1 Preliminaries

The inclusion of the default information $\mathbb{F}^N$ in the investor information creates problems for the analysis of (4.1): since the default indicator N does not admit an intensity under full information, standard filtering techniques for point process observations as in Brémaud (1981) do not apply. This difficulty is addressed in the following proposition.

Proposition 4.1. *Denote by* $V^\tau = (V_{t\wedge\tau})_{t\geq 0}$ *the asset value process stopped at the default boundary, by* $\tilde{Z}_t = \int_0^t a(V_s^\tau)ds + W_t$ *the noisy asset information corresponding to the signal process* V^τ *and by* $\tilde{D}_t = \sum_{\{n:\, T_n \leq t\}} \delta_n V_{T_n}^\tau$ *the cumulative dividend process corresponding to* V^τ*. Then we have for* $g \in L^\infty\big([K,\infty)\big)$

$$1_{\{\tau>t\}} E^Q\big(g(V_t) \mid \mathcal{F}_t^{\mathbb{M}}\big) = 1_{\{\tau>t\}} \frac{E^Q\big(g(V_t^\tau) 1_{\{V_t^\tau > K\}} \mid \mathcal{F}_t^{\tilde{Z}} \vee \mathcal{F}_t^{\tilde{D}}\big)}{Q\big(V_t^\tau > K \mid \mathcal{F}_t^{\tilde{Z}} \vee \mathcal{F}_t^{\tilde{D}}\big)}. \tag{4.2}$$

Proof. For notational simplicity we ignore the dividend observation in the proof so that $\mathbb{F}^{\mathbb{M}} = \mathbb{F}^Z \vee \mathbb{F}^N$. In the first step we show that $\mathbb{F}^Z$ can be replaced with the filtration $\mathbb{F}^{\tilde{Z}}$ that is generated by the stopped asset value process. This will follow from the relation

$$E^Q\big(g(V_t) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{\mathbb{M}}\big) = E^Q\big(g(V_t^\tau) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{Z^\tau} \vee \mathcal{F}_t^N\big), \tag{4.3}$$

where $\mathbb{F}^{Z^\tau}$ is the filtration generated by the stopped process Z^τ. To this, note first that $\mathbb{F}^{Z^\tau} \vee \mathbb{F}^N$ is a subfiltration of $\mathbb{F}^{\mathbb{M}}$ (as τ is an $\mathbb{F}^{\mathbb{M}}$ stopping time), so that the right hand side of (4.3) is $\mathcal{F}_t^{\mathbb{M}}$-measurable. Moreover, for $\tau > t$ one has $V_t^\tau = V_t$ and $(Z_s^\tau)_{s=0}^t = (Z_s)_{s=0}^t$. Hence we get for any bounded measurable functional $h\colon \mathcal{C}\big([0,T]\big) \to \mathbb{R}$ that

$$\begin{aligned} E^Q\big(g(V_t) 1_{\{\tau>t\}} h((Z_s)_{s=0}^t)\big) &= E^Q\big(g(V_t^\tau) 1_{\{\tau>t\}} h((Z_s^\tau)_{s=0}^t)\big) \\ &= E^Q\Big(E^Q\big(g(V_t^\tau) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{Z^\tau} \vee \mathcal{F}_t^N\big) h((Z_s^\tau)_{s=0}^t)\Big). \end{aligned} \tag{4.4}$$

Due to the presence of the indicator $1_{\{\tau>t\}}$ in (4.4) we may replace $h((Z_s^\tau)_{s=0}^t)$ with $h((Z_s)_{s=0}^t)$ in that equation, so that we obtain (4.3) by the definition of conditional expectations. A similar argument shows that $E^Q\big(g(V_t) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{\tilde{Z}} \vee \mathcal{F}_t^N\big) = E^Q\big(g(V_t) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{Z^\tau} \vee \mathcal{F}_t^N\big)$, which gives the equality

$$E^Q\big(g(V_t) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{\mathbb{M}}\big) = E^Q\big(g(V_t^\tau) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{\tilde{Z}} \vee \mathcal{F}_t^N\big). \tag{4.5}$$

Using the Dellacherie formula (see for instance Lemma 3.1 in Elliott, Jeanblanc and Yor (2000)) and the relation $\{\tau > t\} = \{V_t^\tau > K\}$, we finally get

$$\begin{aligned} E^Q\big(g(V_t^\tau) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{\tilde{Z}} \vee \mathcal{F}_t^N\big) &= 1_{\{\tau>t\}} \frac{E^Q\big(g(V_t^\tau) 1_{\{\tau>t\}} \mid \mathcal{F}_t^{\tilde{Z}}\big)}{Q\big(\tau > t \mid \mathcal{F}_t^{\tilde{Z}}\big)} \\ &= 1_{\{\tau>t\}} \frac{E^Q\big(g(V_t^\tau) 1_{\{V_t^\tau > K\}} \mid \mathcal{F}_t^{\tilde{Z}}\big)}{Q\big(V_t^\tau > K \mid \mathcal{F}_t^{\tilde{Z}}\big)}, \end{aligned}$$

as claimed. □

With the notation $f(v) := g(v) 1_{\{v>K\}}$ Proposition 4.1 shows that in order to evaluate the right side of (4.2) one needs to compute for generic $f \in L^\infty\big([K,\infty)\big)$ conditional expectations of the form

$$E^Q\big(f(V_t^\tau) \mid \mathcal{F}_t^{\tilde{Z}} \vee \mathcal{F}_t^{\tilde{D}}\big) \tag{4.6}$$

This is a stochastic filtering problem with signal process given by V^τ (the asset value process stopped at the first exit time of the halfspace (K,∞)) and with standard diffusion and point process information.

In the sequel we study this problem using results of Pardoux (1978) on the filtering of diffusions stopped at the first exit time of some bounded domain. In order to be in a situation where the results of Pardoux (1978) are applicable we choose some large number N and replace the unbounded halfspace (K, ∞) with the bounded domain (K, N). For this we define the stopping time $\sigma_N = \inf\{t \geq 0\colon V_t \geq N\}$ and we replace the original asset value process V with the stopped process $V^N := (V_{t\wedge\sigma_N})_{t\geq 0}$. Applying Proposition 4.1 to the process V^N leads to a filtering problem with signal process $X := (V^N)^\tau$. More precisely, one has to compute conditional expectations of the form

$$E^Q\big(f(X_t) \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big) \tag{4.7}$$

where, with a slight abuse of notation, $Z_t = \int_0^t a(X_s)ds + W_t$ and $D_t = \sum_{\{n\colon T_n \leq t\}} \delta_n X_{T_n}$. Note that $\tau \wedge \sigma_N$ is the first exit time of V from the domain (K, N). Moreover, it holds by definition that $X_t = V_{t\wedge\tau\wedge\sigma_N}$, i.e. X is equal to the asset value process V stopped at the boundary of the bounded domain (K, N). Hence the state space of X is given by $S^X := [K, N]$ and the analysis of Pardoux (1978) applies to the problem (4.7).

The next proposition shows that replacing V with the stopped process V^N does not affect the financial implications of the analysis, provided that N is sufficiently large.

Proposition 4.2. *1. Fix some horizon date $T > 0$ and let $\mathbb{F}$ be an arbitrary subfiltration of $\mathbb{G}$. Then for $\epsilon > 0$, it holds that*

$$Q\Big(\sup_{0\leq t\leq T} Q(\sigma_N \leq t \mid \mathcal{F}_t) > \epsilon\Big) \leq \frac{1}{\epsilon} Q(\sigma_N \leq T) \to 0 \text{ as } N \to \infty,$$

i.e. the conditional probability that V reaches the upper boundary N can be made arbitrarily small by making N sufficiently large, uniformly for all subfiltrations $\mathbb{F}$ of $\mathbb{G}$.

2. Consider a function $f\colon [K, \infty) \to \mathbb{R})$ so that $Y_t = \sup_{s\leq t}\big|f(V_s^\tau)\big|$ is an integrable process. Then, as $N \to \infty$, the solution of the filtering problem (4.7) (the problem with domain (K, N)) converges in probability to the solution of the filtering problem (4.6) (the problem with domain (K, ∞).)

The proof of the proposition is given in Appendix A. We mention that Statement 2 applies to the full-information value of the corporate securities discussed in Section 3.

Zakai equations. As in Pardoux (1978) we adopt the reference probability approach to solve the filtering problem (4.7). Under this approach one considers the model under an equivalent measure Q^* such that Z and X are independent and reverts to the original dynamics via a change of measure. In a first step we consider the filtering problem without the additional dividend information; dividends will be included in Section 4.3.

It will be convenient to model the pair (X, Z) on a product space $(\Omega, \mathcal{G}, \mathbb{G}, Q^*)$. Denote by $(\Omega_2, \mathcal{G}^2, \mathbb{G}^2, Q_2)$ some filtered probability that supports an l-dimensional Wiener process $Z = (Z_t(\omega_2))_{0\leq t\leq T}$. Given some probability space $(\Omega_1, \mathcal{G}^1, \mathbb{G}^1, Q_1)$ supporting the process X we let $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{G} = \mathcal{G}^1 \otimes \mathcal{G}^2$, $\mathbb{G} = \mathbb{G}^1 \otimes \mathbb{G}^2$ and $Q^* = Q_1 \otimes Q_2$, and we extend all processes to the product space in the obvious way. Note that this construction implies that under Q^*, Z is an l-dimensional Brownian motion independent of X. Consider a Girsanov-type measure transform of the form $L_t = (dQ/dQ^*)|_{\mathcal{F}_t}$ with

$$L_t = L_t(\omega_1, \omega_2) = \exp\Big(\int_0^t a\big(X_s(\omega_1)\big)^\top dZ_s(\omega_2) - \frac{1}{2}\int_0^t |a\big(X_s(\omega_1)\big)|^2\, ds\Big). \tag{4.8}$$

Girsanov's theorem for Brownian motion implies that under Q the pair (X, Z) has the correct joint law. Using the abstract Bayes formula, one has for $f \in L^\infty(S^X)$ that

$$E^Q\big(f(X_t) \mid \mathcal{F}_t^Z\big) = \frac{E^{Q^*}\big(f(X_t)L_t \mid \mathcal{F}_t^Z\big)}{E^{Q^*}\big(L_t \mid \mathcal{F}_t^Z\big)}\,. \tag{4.9}$$

We concentrate on the numerator. Using the product structure of the underlying probability space we get that

$$E^{Q^*}\big(f(X_t)L_t \mid \mathcal{F}_t^Z\big)(\omega) = E^{Q_1}\big(f(X_t)L_t(\cdot, \omega_2)\big) =: \Sigma_t f(\omega) \tag{4.10}$$

In Theorem 1.3 and 1.4 of Pardoux (1978) the following characterization of Σ_t is derived.

Proposition 4.3. *Denote by $(T_t)_{t\geq 0}$ the transition semigroup of the Markov process X, that is for $f \in L^\infty(S^X)$ and $x \in S^X$, $T_t f(x) = E_x^Q(f(X_t))$. Then the following holds*

1. *$\Sigma_t f$ as defined in (4.10) satisfies the equation*

$$\Sigma_t f = \Sigma_0(T_t f) + \sum_{i=1}^{l} \int_0^t \Sigma_s(a_i T_{t-s} f)\, dZ_s^i \tag{4.11}$$

2. *Let $\tilde{\Sigma}$ be an $\mathbb{F}^Z$ adapted process taking values in the set of bounded and positive measures on S^X. Suppose that for $f \in \mathcal{C}^0(S^X)$ $\tilde{\Sigma}_t f := \int_{S^X} f(x)\tilde{\Sigma}_t(dx)$ satisfies equation (4.11) and that moreover $\Sigma_0 = \tilde{\Sigma}_0$. Then for all $0 \leq t \leq T$, $\Sigma_t = \tilde{\Sigma}_t$ a.s.*

Comments. In the sequel we will mostly use the vector notation $\int_0^t \Sigma_s(a^\top T_{t-s} f)\, dZ_s$ to denote the stochastic integral in (4.11). Equation (4.11) can be viewed as mild form of the classical Zakai equation. In fact, it is easily seen that for f in the domain of the generator $\mathcal{L}_X$ of X, (4.11) is equivalent to the equation

$$\Sigma_t f = \Sigma_0 f + \int_0^t \Sigma_s(\mathcal{L}_X f) ds + \int_0^t \Sigma_s(a^\top f) dZ_s.$$

However, in the sequel we need to determine $\Sigma_t f$ also for non-smooth functions such as $f(x) = 1_{\{K\}}(x)$, so that we prefer to work with (4.11).

4.2 An SPDE for the Density of Σ_t

In this section we derive an SPDE for the density $u = u(t, \cdot)$ of the solution Σ_t of the Zakai equation (4.11). We begin with the necessary notation. First, we introduce the Sobolev spaces

$$H^k(S^X) = \big\{u \in L^2(S^X)\colon \frac{d^\alpha u}{dx^\alpha} \in L^2(S^X) \text{ for } \alpha \leq k\big\},$$

where the derivatives are assumed to exist in the weak sense. Moreover, we let $H_0^1(S^X) = \{u \in H^1(S^X)\colon u = 0 \text{ on the boundary } \partial S^X\}$ (the trace on ∂S^X exists by standard results on Sobolev spaces). The scalar product in $L^2(S^X)$ is denoted by $(\cdot\,,\cdot)_{S^X}$.

Consider for $f \in H^2(S^X)$ the differential operator $\mathcal{L}^*$ with

$$\mathcal{L}^* f(x) = \frac{1}{2}\frac{d^2}{dx^2}\big(\sigma^2 x^2 f\big)(x) - \frac{d}{dx}\big((r - \bar{\delta}\lambda^D)xf\big)(x). \tag{4.12}$$

$\mathcal{L}^*$ is adjoint to $\mathcal{L}_V$ in the following sense: one has $\big(f, \mathcal{L}_V g\big)_{S^X} = \big(\mathcal{L}^* f, g\big)_{S^X}$ whenever $f, g \in H^2(S^X) \cap H^1_0(S^X)$. Next we define an extension of $-\mathcal{L}^*$ to the entire space $H^1_0(S^X)$. For this we denote by $H^1_0(S^X)'$ the dual space of $H^1_0(S^X)$ and by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1_0(S^X)'$ and $H^1_0(S^X)$. Then we may define a bounded linear bounded operator $\mathcal{A}^*$ from $H^1_0(S^X)$ to $H^1_0(S^X)'$ by

$$\langle \mathcal{A}^* f, g \rangle = \frac{1}{2}\big(\sigma^2 x^2 \frac{df}{dx}, \frac{dg}{dx}\big)_{S^X} + \big((\sigma^2 - r + \bar{\delta}\lambda^D) x f\, ,\, \frac{dg}{dx}\,\big)_{S^X} . \tag{4.13}$$

Partial integration shows that for $f \in H^2(S^X) \cap H^1_0(S^X)$ and $g \in H^1_0(S^X)$ one has $\langle \mathcal{A}^* f, g\rangle = -\big(\mathcal{L}^* f, g\big)_{S^X}$, so that $\mathcal{A}^*$ is in fact an extension of $-\mathcal{L}^*$.

We will show that the density of Σ_t can be described in terms of the SPDE

$$du(t) = -\mathcal{A}^* u(t) dt + a^\top u(t) dZ_t, \quad u(0) = \pi_0, \tag{4.14}$$

This equation is to be understood as an equation in the dual space $H^1_0(S^X)'$, that is for every $v \in H^1_0(S^X)$ one has the relation

$$\big(u(t), v\big)_{S^X} = \big(u(0), v\big)_{S^X} - \int_0^t \langle \mathcal{A}^* u(s), v\rangle ds + \sum_{i=1}^{l} \int_0^t \big(a_i u(s), v\big)_{S^X} dZ_s\,. \tag{4.15}$$

In the sequel we will mostly denote the stochastic integral with respect to the vector process Z by $\int_0^t \big(a^\top u(s), v\big)_{S^X} dZ_s$.

Theorem 4.4. *Suppose that Assumptions 2.1 and 2.2 hold and that the initial density π_0 belongs to $H^1_0(S^X)$. Then the following holds.*

1. *There is a unique $\mathbb{F}^Z$-adapted solution $u \in L^2\big(\Omega \times [0,T], Q^* \otimes dt; H^1_0(S^X)\big)$ of equation* (4.14).
2. *u has additional regularity: it holds that $u(t) \in H^2(S^X)$ a.s. and that the trajectories of u belong to $C\big([0,T], H^1_0(S^X)\big)$, the space of $H^1_0(S^X)$-valued continuous functions with the supremum norm. Moreover, $u(t,\cdot) \geq 0$ Q^* a.s.*
3. *The process $u(t)$ (essentially) describes the solution of the measure-valued Zakai equation* (4.11)*: for $f \in L^\infty(S^X)$ one has*

$$\Sigma_t f = \big(u(t), f\big)_{S^X} + \nu_K(t) f(K) + \nu_N(t) f(N), \quad \textit{where} \tag{4.16}$$

$$0 \leq \nu_K(t) = \int_0^t \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s, K) ds, \tag{4.17}$$

$$0 \leq \nu_N(t) = -\int_0^t \frac{1}{2}\sigma^2 N^2 \frac{du}{dx}(s, N) ds + \int_0^t a^\top(N) \nu_N(s) dZ_s\,. \tag{4.18}$$

Comments. Since $u(t)$ belongs to $H^2(S^X) \cap H^1_0(S^X)$, (4.15) can be written as

$$\big(u(t), v\big)_{S^X} = \big(u(0), v\big)_{S^X} + \int_0^t \big(\mathcal{L}^* u(s), v\big)_{S^X} ds + \int_0^t \big(a^\top u(s), v\big)_{S^X}\, dZ_s; \tag{4.19}$$

moreover, an approximation argument shows that (4.19) holds for $v \in L^2(S^X)$ (and not only for $v \in H^1_0(S^X)$).

Statement 3 shows that the measure Σ_t has a Lebesgue-density on the interior of S^X and a point mass on the boundary points K and N. In view of Proposition 4.2, the point mass $\nu_N(t)$ is largely irrelevant; the point mass $\nu_K(t)$ on the other hand will be important in the analysis of the default intensity in Section 5.

The assumption that S^X is a bounded domain is needed in the proof of Statement 2; given the existence of a sufficiently regular nonnegative solution of equation (4.14) the proof of Statement 3 is valid for an unbounded domain as-well.

Proof. Statements 1 and 2 follow directly from Theorems 2.1, 2.3 and 2.6 of Pardoux (1978). We give a sketch of the proof of 3, as this explains why (4.14) is the appropriate SPDE to consider; moreover our arguments justify the form of ν_K and ν_N.

Note first that by standard results for Sobolev spaces, the derivatives of u at the boundary points of S^X exist, as $u(t) \in H^2(S^X)$. Moreover, as $u(t,x) \geq 0$ on S^X, we have $\frac{du}{dx}(t,K) \geq 0$ and $\frac{du}{dx}(t,N) \leq 0$, so that $\nu_K(t) \geq 0$ and $\nu_N(t) \geq 0$. Denote by $\tilde{\Sigma}_t$ the measure-valued process that is defined by the right side of (4.16). In order to show that $\tilde{\Sigma}_t$ solves the mild-form Zakai equation (4.11) fix some continuous function $f\colon S^X \to \mathbb{R}$ and some $t \leq T$, and denote by $\bar{u}(s,x)$ the solution of the terminal and boundary value problem

$$\bar{u}_s + \mathcal{L}_V \bar{u} = 0, \quad (t,x) \in (0,t) \times (K,N),$$

with terminal condition $\bar{u}(t,x) = f(x)$, $x \in S^X$, and boundary conditions $u(s,K) = f(K)$, $u(s,N) = f(N)$, $s \leq t$. It is well-known that $\bar{u}$ describes the transition semigroup of X, that is $\bar{u}(s,x) = T_{t-s}f(x)$, $0 \leq s \leq t$. As $\bar{u}(t) = f$ we obtain from the definition of $\tilde{\Sigma}_t$ and the dynamics of $\nu_K(t)$ and $\nu_N(t)$ that

$$\begin{aligned}\tilde{\Sigma}_t f = \big(u(t), \bar{u}(t)\big)_{S^X} &+ \int_0^t \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s,K) f(K) ds \\ &- \int_0^t \frac{1}{2}\sigma^2 N^2 \frac{du}{dx}(s,N) f(N) ds + \int_0^t a^\top(N) \nu_N(s) f(N)\, dZ_s\,.\end{aligned}$$

Next we compute the differential of $\big(u(t), \bar{u}(t)\big)_{S^X}$. We get, using the Ito product formula, (4.19) and the relation $d\bar{u}(s) = -\mathcal{L}_V \bar{u}(s) ds$, that

$$\begin{aligned}\big(u(t), \bar{u}(t)\big)_{S^X} = \big(u(0), \bar{u}(0)\big)_{S^X} &+ \int_0^t \big(\mathcal{L}^* u(s), \bar{u}(s)\big)_{S^X} ds + \int_0^t \big(a^\top u(s), \bar{u}(s)\big)_{S^X} dZ_s \\ &+ \int_0^t \big(u(t), -\mathcal{L}_V \bar{u}(s)\big)_{S^X} ds\,.\end{aligned}$$

Partial integration gives, using the boundary conditions satisfied by $\bar{u}$,

$$\int_0^t \big(u(t), -\mathcal{L}_V \bar{u}(s)\big)_{S^X} ds = -\int_0^t \big(\mathcal{L}^* u(s), \bar{u}(s)\big)_{S^X} ds + \int_0^t \Big[\frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s,x) f(x)\Big]_K^N ds\,.$$

Hence we get

$$\tilde{\Sigma}_t f = \big(u(0), \bar{u}(0)\big)_{S^X} + \int_0^t \big(a^\top u(s), \bar{u}(s)\big)_{S^X} + a^\top(N) \nu_N(s) f(N)\, dZ_s\,.$$

Now note that for $x \in [K,N]$, $\bar u(s)(x) = T_{t-s}f(x)$. Using that $a(K)=0$ by Assumption 2.2, we obtain that the stochastic integral with respect to Z equals

$$\int_0^t \Big\{ \big(a^\top u(s), T_{t-s}f\big)_{S^X} + a^\top(K)\nu_K(s)T_{t-s}f(K) + a^\top(N)\nu_N(s)T_{t-s}f\,(N) \Big\}\, dZ_s\,.$$

Hence it holds that $\tilde\Sigma_t f = \tilde\Sigma_0(T_t f) + \int_0^t \tilde\Sigma_s(a^\top T_{t-s}f)\, dZ_s$. Moreover, $\Sigma_0 f = \big(\pi_0, f\big)_{S^X} = \tilde\Sigma_0 f$, so that $\Sigma_t = \tilde\Sigma_t$ by Proposition 4.3. □

4.3 Conditional Distribution with respect to $\mathbb{F}^{\mathbb{M}}$

In this subsection we compute the conditional distribution of X with respect to the market filtration $\mathbb{F}^{\mathbb{M}} = \mathbb{F}^Z \vee \mathbb{F}^D \vee \mathbb{F}^N$. We begin by including the dividend information $\mathbb{F}^D$ in the analysis. For this we use an extension of the reference probability argument from Section 4.1. Recall that we denote the dividend dates by T_n, $n \ge 1$, that d_n denotes the dividend paid at T_n and that the conditional density of d_n given $X_{T_n} = x$ is denoted by $\varphi(y,x)$. In the sequel we let $T_0 =$ for notational convenience.

We consider the case of Poissonian dividend dates which is notationally easier. Fix some strictly positive reference density $\varphi^*(y)$ on $\mathbb{R}^+$ and suppose that the space $(\Omega_2, \mathcal{G}^2, \mathbb{G}^2, Q_2)$ supports a random measure $\mu^D(dy,dt)$ with compensating measure equal to $\gamma^{D,*}(dy,dt) = \varphi^*(y)dy\lambda^D dt$ and that μ^D is independent of the Brownian motion Z. In order to revert to the original model dynamics we introduce the density martingale $L_t = L_t^1 L_t^2$ where L_t^1 is as in (4.8) and where $L_t^2 = L_t^2(\omega_1,\omega_2)$ is given the solution of the SDE

$$L_t^2 = 1 + \int_0^t \int_{\mathbb{R}^+} L_{s-}^2 \Big(\frac{\varphi(y,X_s)}{\varphi^*(y)} - 1\Big)\, (\mu^D - \gamma^{D,*})(dy,ds)\,. \tag{4.20}$$

Since $\varphi(\cdot,x)$ and φ are probability densities we get

$$\int_{\mathbb{R}^+} \Big(\frac{\varphi(y,x)}{\varphi^*(y)} - 1\Big)\varphi^*(y)dy = \int_{\mathbb{R}^+} (\varphi(y,x) - \varphi^*(y))dy = 1 - 1 = 0\,. \tag{4.21}$$

Hence $\int_0^t \int_{\mathbb{R}^+} \big(\frac{\varphi(y,X_s)}{\varphi^*(y)} - 1\big)\, \gamma^{D,*}(dy,ds) \equiv 0$ and we obtain that

$$L_t^2 = 1 + \int_0^t \int_{\mathbb{R}^+} L_{s-}^2 \Big(\frac{\varphi(y,X_s)}{\varphi^*(y)} - 1\Big)\mu^D(dy,ds) = \prod_{T_n \le T} \frac{\varphi(d_n, X_{T_n})}{\varphi^*(d_n)}\,. \tag{4.22}$$

Since L^1 and L^2 are orthogonal it holds that

$$dL_t = L_{t-}a(X_t)^\top dZ_t + \int_{\mathbb{R}^+} L_{t-}\big(\frac{\varphi(y,X_t)}{\varphi^*(y)} - 1\big)\, (\mu^D - \gamma^{D,*})(dy,dt)\,.$$

The next lemma shows that $(L_t)_{0\le t\le T}$ is in fact the appropriate density martingale to consider.

Lemma 4.5. *It holds that $E^{Q^*}(L_T) = 1$. Define the measure Q by $(dQ/dQ^*)|_{\mathcal{G}_T} = L_T$. Then under Q the random measure μ^D has $\mathbb{G}$-compensator $\gamma^D(dy,dt) = \varphi(y,X_t)dy\lambda^D dt$. Moreover, the triple (X,Z,D) with $D_t = \int_0^t y\,\mu^D(dy,ds)$ has the joint law postulated in Assumption 2.1.*

The proof is given in Appendix A. In analogy with (4.10) we let $\Sigma_t f\,(\omega) = E^{Q^*}\big(f(X_t)L_t(\cdot,\omega_2)\big)$. With dividends the SPDE for the density of the absolutely continuous part of the measure-valued process Σ_t becomes

$$du(t) = -\mathcal{A}^* u(t)dt + a^\top u(t)dZ_t + \int_{\mathbb{R}^+} u(t-)\,\big(\frac{\varphi(y,\cdot)}{\varphi^*(y)} - 1\big)\,(\mu^D - \gamma^{D,*})(dy,dt)\,, \tag{4.23}$$

with initial condition $u(0) = \pi_0$. The interpretation of (4.23) is analogous to the previous section: for $v \in H_0^1(S^X)$ it holds that

$$\begin{aligned}\big(u(t),v\big)_{S^X} = \big(u(0),v\big)_{S^X} - \int_0^t \langle \mathcal{A}^* u(s), v\rangle ds + \int_0^t \big(a^\top u(s), v\big)_{S^X} dZ_s \\ + \int_0^t \int_{\mathbb{R}^+} \Big(u(s-)\Big(\frac{\varphi(y,\cdot)}{\varphi^*(y)} - 1\Big), v\Big)_{S^X} (\mu^D - \gamma^{D,*})(dy,ds)\,.\end{aligned} \tag{4.24}$$

Note that (4.21) implies that the integral wrt $\gamma^{D,*}(dy,ds)$ can be dropped in (4.24). Hence we can give a simple description of the dynamics (4.23): between dividend dates, that is on (T_{n-1},T_n), $n \geq 1$, $u(t)$ solves the SPDE (4.14) with initial value $u(T_{n-1})$; at $t = T_n$ one has

$$u(T_n,x) = u(T_n-,x)\frac{\varphi(d_n,x)}{\varphi^*(d_n)}\,. \tag{4.25}$$

Since the mapping $x \mapsto \varphi(y,x)$ is smooth by Assumption 2.1, the function $u(T_n)$ defined in (4.25) is an element of $H_0^1(S^X)$. Hence we may apply Theorem 4.4 iteratively on each of the intervals (T_{n-1},T_n), yielding the existence of a unique positive solution $u(t) \in H_0^1(S^X) \cap H^2(S^X)$ of the SPDE (4.23).

The next result extends Theorem 4.4 to the case with dividends.

Proposition 4.6. *Denote by $u(t)$ the solution of the SPDE* (4.23) *and define*

$$\nu_K(t) = \int_0^t \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s,K)ds + \int_0^t \int_{\mathbb{R}^+} \nu_K(s-)\big(\frac{\varphi(y,K)}{\varphi^*(y)} - 1\big)\,(\mu^D - \gamma^{D,*})(dy,ds) \tag{4.26}$$

$$\begin{aligned}\nu_N(t) = -\int_0^t \frac{1}{2}\sigma^2 N^2 \frac{du}{dx}(s,N)ds + \int_0^t a^\top(N)\nu_N(s)dZ_s \\ + \int_0^t \int_{\mathbb{R}^+} \nu_N(s-)\big(\frac{\varphi(y,N)}{\varphi^*(y)} - 1\big)\,(\mu^D - \gamma^{D,*})(dy,ds)\,.\end{aligned} \tag{4.27}$$

Then it holds that $\Sigma_t f = \big(u(t),f\big)_{S^X} + \nu_K(t)f(K) + \nu_N(t)f(N)$.

Proof. We proceed via induction over the dividend dates. For $t \in [0,T_1)$ there is no dividend information and the claim follows from Theorem 4.4. Suppose now that the claim of the Proposition holds for $t \in [0,T_n)$. From (4.22) we have that $L_{T_n} = L_{T_n-}\frac{\varphi(d_n,X_{T_n})}{\varphi^*(d_n)}$ and hence, using the induction hypothesis,

$$\begin{aligned}\Sigma_{T_n} f\,(\omega) &= E^{Q^*}\Big(L_{T_n-}(\cdot,\omega_2)f(X_{T_n})\frac{\varphi(d_n(\omega_2),X_{T_n})}{\varphi^*(d_n(\omega_2))}\Big) \\ &= \big(u(T_n-), f\frac{\varphi(d_n,\cdot)}{\varphi^*(d_n)}\big)_{S^X} + \nu_K(T_n-)f(K)\frac{\varphi(d_n,K)}{\varphi^*(d_n)} + \nu_N(T_n-)f(N)\frac{\varphi(d_n,K)}{\varphi^*(d_n)}\,.\end{aligned} \tag{4.28}$$

The dynamics of $u(t)$, $\nu_K(t)$ and $\nu_N(t)$ imply that

$$u(T_n,x) = u(T_n-,x)\frac{\varphi(d_n,x)}{\varphi^*(d_n)};\ \nu_K(T_n) = \nu_K(T_n-)\frac{\varphi(d_n,K)}{\varphi^*(d_n)};\ \nu_N(T_n) = \nu_N(T_n-)\frac{\varphi(d_n,N)}{\varphi^*(d_n)}\,.$$

Hence (4.28) equals $\big(u(T_n),f\big)_{S^X} + \nu_K(T_n)f(K) + \nu_N(T_n)f(N)$, which proves the claim. $\square$

Remark 4.7. 1. In order to derive the SPDE for u with deterministic dividend dates $t_1, t_2, \dots$ one applies the previous arguments assuming that μ^D has Q^*-compensator $\tilde\gamma^{D,*}(dy,dt) = \sum_{n=1}^\infty \varphi^*(y)dy\,\delta_{t_n}(dt)$; the whole derivation goes through with only notational changes.

2. For the filtering results it does not matter that the d_n are dividend payments, so that our analysis applies also to other types of noisy asset information arriving discretely in time such as rating changes.

Finally we return to the filtering problem with respect to the market filtration $\mathbb{F}^{\mathbb{M}}$. Combining Propositions 4.1 and 4.6 we immediately obtain the following result.

Corollary 4.8. *One has for $f \in L^\infty(S^X)$*

$$1_{\{\tau>t\}} E^Q\big(f(X_t) \mid \mathcal{F}_t^{\mathbb{M}}\big) = 1_{\{\tau>t\}}\big((\pi(t,\cdot),f)_{S^X} + \pi_N(t)f(N)\big), \tag{4.29}$$

with $\pi(t,x) = u(t,x)/C(t)$ and $\pi_N(t) = \nu_N(t)/C(t)$ and with norming constant $C(t)$ given by $C(t) = \big(u(t),1\big)_{S^X} + \nu_N(t)$.

Remark 4.9. In view of Proposition 4.2, for practical purposes the ν_N-term that corresponds to the conditional probability of reaching the upper boundary of S^X prior to the horizon date can be dropped. With this simplification we get for $t \in [0,\tau)$

$$E^Q(f(V_t) \mid \mathcal{F}_t^{\mathbb{M}}) \approx \big(\tilde\pi(t),f\big)_{S^X} \text{ with } \tilde\pi(t,x) = 1_{(K,N)}(x)\frac{u(t,x)}{\big(u(t),1\big)_{S^X}}. \tag{4.30}$$

4.4 Finite-dimensional approximation of the filter equation

The SPDE (4.14) is a stochastic partial differential equation and thus an infinite-dimensional object. In order to solve the filtering problem numerically and to generate price trajectories of for basic corporate securities one needs to approximate (4.14) by a finite-dimensional equation. A natural way to achieve this is the Galerkin approximation method. We first explain the method for the case without dividend payments. Consider m linearly independent basis functions $e_1,\dots,e_m \in H_0^1(S^X)\cap H^2(S^X)$ generating the subspace $\mathcal{H}^{(m)} \subset H_0^1(S^X)$, and denote by $\mathrm{pr}^{(m)}\colon H_0^1(S^X) \to \mathcal{H}^{(m)}$ the projection on this subspace with respect to $(\cdot,\cdot)_{S^X}$. In the Galerkin method the solution $\tilde u$ of the equation

$$d\tilde u(t) = \mathrm{pr}^{(m)} \circ \mathcal{L}^* \circ \mathrm{pr}^{(m)}\,\tilde u(t)dt + \mathrm{pr}^{(m)}(a^\top \mathrm{pr}^{(m)}\,\tilde u(t))\,dZ_t, \quad \tilde u(0) = \mathrm{pr}^{(m)}\,\pi_0 \tag{4.31}$$

is used as an approximation to the solution u of (4.14). Since projections are self-adjoint, we get that for $v \in H_0^1(S^X)$

$$d\big(\tilde u(t),v\big)_{S^X} = \big(\mathcal{L}^*\circ \mathrm{pr}^{(m)}\,\tilde u(t), \mathrm{pr}^{(m)}\,v\big)_{S^X}dt + \big(a^\top \mathrm{pr}^{(m)}\,\tilde u(t), \mathrm{pr}^{(m)}\,v\big)_{S^X}dZ_t. \tag{4.32}$$

Hence $d(\tilde u(t),v)_{S^X} = 0$ if v belongs to $(\mathcal{H}^{(m)})^\perp$. Since moreover $\tilde u(0) = \mathrm{pr}^{(m)}\,\pi_0 \in \mathcal{H}^{(m)}$ we conclude that $\tilde u(t) \in \mathcal{H}^{(m)}$ for all t. Hence $\tilde u$ is of the form $\tilde u(t) = \sum_{i=1}^m \psi_i(t)e_i$, and we now determine an SDE system for the m dimensional process $\Psi^{(m)}(t) = (\psi_1(t), \dots, \psi_m(t))'$. Using (4.32) we get for $j \in \{1, \dots, m\}$

$$d\big(\tilde{u}(t), e_j\big)_{S^X} = \sum_{i=1}^{m} \psi_i(t)\big(\mathcal{L}^* e_i, e_j\big)_{S^X} dt + \sum_{k=1}^{l}\sum_{i=1}^{m} \big(a_k e_i, e_j\big)_{S^X} \psi_i(t) dZ_t^k. \tag{4.33}$$

On the other hand,

$$d(\tilde{u}(t), e_j)_{S^X} = \sum_{i=1}^{m} (e_i, e_j) d\psi_i(t). \tag{4.34}$$

Define now the $m \times m$ matrices A, B and $C^1, \dots, C^l$ with $a_{ij} = (e_i, e_j)_{S^X}$, $b_{ij} = (\mathcal{L}^* e_i, e_j)_{S^X}$ and $c_{ij}^k = (a_k e_i, e_j)_{S^X}$. Equating (4.33) and (4.34), we get the following system of SDEs for $\Psi^{(m)}$

$$d\Psi^{(m)}(t) = A^{-1} B^\top \Psi^{(m)}(t) dt + \sum_{k=1}^{l} A^{-1} C^k \Psi^{(m)}(t) dZ_t^k \tag{4.35}$$

with initial condition $\Psi^{(m)}(0) = A^{-1}\big((\pi_0, e_1)_{S^X}, \dots, (\pi_0, e_m)_{S^X}\big)'$. Equation (4.35) can be solved with numerical methods for SDEs such as a simple Euler scheme or the more advanced splitting up method proposed by Le Gland (1992). Further details regarding the numerical implementation of the Galerkin method are given among others in Frey, Schmidt and Xu (2013). Conditions for the convergence $\tilde{u} \to u$ are well-understood, see for instance Germani and Piccioni (1987): the Galerkin approximation for the filter density converges for $m \to \infty$ if and only if the Galerkin approximation for the deterministic forward PDE $\frac{du}{dt}(t) = \mathcal{L}^* u(t)$ converges.

In the case with dividend information the Galerkin method is applied successively on each interval (T_{n-1}, T_n), $n = 1, 2, \dots$. Denote by $\tilde{u}^{(n)}$ the approximating density over the interval (T_{n-1}, T_n). In line with relation (4.25), the initial condition for the interval (T_n, T_{n+1}) is then given by

$$\tilde{u}(T_n) = \mathrm{pr}^{(m)} \left(\tilde{u}^{(n)}(T_n, \cdot) \frac{\varphi(d_n, \cdot)}{\varphi^*(d_n)} \right),$$

that is by projecting the updated density $\tilde{u}^{(n)}(T_n, \cdot)\frac{\varphi_d(d_n, \cdot)}{\varphi^*(d_n)}$ onto $\mathcal{H}^{(m)}$.

5 Dynamics of Corporate Security Prices

In this section we study the dynamics of corporate security prices in the market filtration. It will be shown that the the price processes of corporate securities are of jump-diffusion type, driven by the noisy asset information Z, by the compensated random measure corresponding to the dividend payments and by the compensated default indicator process.

As a first step we derive the $\mathbb{F}^{\mathbb{M}}$-semimartingale decomposition of the default indicator process $N_t = 1_{\{\tau \le t\}}$ and show that N admits an $\mathbb{F}^{\mathbb{M}}$-intensity.

5.1 Default intensity

Theorem 5.1. *The $\mathbb{F}^{\mathbb{M}}$-compensator of N_t is given by the process $(\Lambda_{t\wedge\tau})_{t\geq 0}$ where $\Lambda_t = \int_0^t \lambda_s ds$ and where the default intensity λ_t is given by*

$$\lambda_t = \frac{1}{2}\sigma^2 K^2 \frac{d\pi}{dx}(t,K)\,. \tag{5.1}$$

Here $\pi(t,x)$ is conditional density of X_t given $\mathcal{F}_t^{\mathbb{M}}$ introduced in Corollary 4.8.

We mention that a similar result was obtained in Duffie and Lando (2001) for the case where the noisy observation of the asset value process arrives only at deterministic time points.

Proof. We use the following well-known result to determine the compensator of N (see for instance Blanchet-Scalliet and Jeanblanc (2004)).

Proposition 5.2. *Let $F_t = Q(\tau \leq t \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D)$ and suppose that $F_t < 1$ for all t. Denote the Doob-Meyer decomposition of the bounded $\mathbb{F}^Z \vee \mathbb{F}^D$-submartingale F by $F_t = M_t^F + A_t^F$. Define the process Λ via*

$$\Lambda_t = \int_0^t (1 - F_{s-})^{-1} dA_s^F\,, \quad t \geq 0.$$

Then $N_t - \Lambda_{t\wedge\tau}$ is an $\mathbb{F}^{\mathbb{M}}$-martingale. In particular, if A^F is absolutely continuous, that is if $dA_t^F = \gamma_t^A dt$, τ has the default intensity $\lambda_t = \gamma_t^A/(1-F_t)$.

In order to apply the proposition we need to compute the Doob-Meyer decomposition of the submartingale F. Here we get

$$F_t = Q(\tau \leq t \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D) = Q(X_t = K \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D) = \frac{\Sigma_t 1_{\{K\}}}{\Sigma_t 1}.$$

Proposition 4.6 gives $\Sigma_t 1_{\{K\}} = \nu_K(t)$. In order to simplify the exposition we assume that the densities $\varphi(\cdot, x)$, $x \in S^X$, have common support. In that case we may choose the reference density φ^* in Proposition 4.6 as $\varphi^*(y) := \varphi(y,K)$, and it follows that $d\nu_K(t) = \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(t,K)dt$.

Next we consider the term $(\Sigma_t 1)^{-1}$. By definition it holds that $\Sigma_t 1 = E^{Q^*}(L_t \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D) = (dQ/dQ^*)|_{\mathcal{F}_t^Z \vee \mathcal{F}_t^D}$. Hence we get that

$$(\Sigma_t 1)^{-1} = (dQ^*/dQ)|_{\mathcal{F}_t^Z \vee \mathcal{F}_t^D}; \tag{5.2}$$

in particular, $(\Sigma_t 1)^{-1}_{t\geq 0}$ is a Q martingale. Ito's product rule therefore gives that

$$A_t^F = \int_0^t \frac{1}{\Sigma_s 1} \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s,K)\, ds.$$

Furthermore we have

$$1 - F_t = Q(X_t > K \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D) = \frac{1}{\Sigma_t 1}\big(\big(u(t),1\big)_{S^X} + \nu_N(t)\big). \tag{5.3}$$

The claim thus follows from Proposition 5.2 and from the definition of $\pi(t,x)$ in Corollary 4.8. ∎

5.2 Filter Equations and Asset Price Dynamics

In this section we derive the nonlinear filtering equations for the market filtration. As a corollary we obtain the dynamics of corporate security prices.

In line with standard notation we denote for $f \in L^\infty([0,T] \times S^X)$ the optional projection of the process $f(t, X_t)$ on the market filtration by $\widehat{f}_t = E^Q(f(t,X_t) \mid \mathcal{F}_t^{\mathbb{M}})$. We denote the generator of the Markov process X by $\mathcal{L}_X$; note that for smooth functions f on S^X one has $\mathcal{L}_X f(x) = 1_{(K,N)}(x)\mathcal{L}_V f(x)$. Next we introduce the processes that drive the filtering equations and hence asset price dynamics. First we let

$$M_t^Z = M_t^{Z,\mathbb{F}^{\mathbb{M}}} = Z_t - \int_0^t E^Q\left(a(X_s) \mid \mathcal{F}_s\right) ds\,, \quad t \geq 0. \tag{5.4}$$

It is well known that M^Z is a $(Q, \mathbb{F}^{\mathbb{M}})$ Brownian motion and hence the martingale part in the $\mathbb{F}^{\mathbb{M}}$-semimartingale decomposition of Z. Moreover, since with Poissonian dividend dates the $\mathbb{G}$-compensator of the random measure μ^D is given by $\varphi(y, X_t)dy\, \lambda^D dt$, the $\mathbb{F}^{\mathbb{M}}$-compensator of μ^D equals

$$\gamma^{D,\mathbb{F}^{\mathbb{M}}}(dt, dy) = (\widehat{\varphi(y)})_t dy\, \lambda^D dt\,, \tag{5.5}$$

where $(\widehat{\varphi(y)})_t$ is short for $E^Q\big(\varphi(y, X_t) \mid \mathcal{F}_t^{\mathbb{M}}\big)$. Similarly, with deterministic dividend dates the $\mathbb{F}^{\mathbb{M}}$-compensator of μ^D is given by

$$\gamma^{D,\mathbb{F}^{\mathbb{M}}}(dt, dy) = \sum_n = 1^\infty (\widehat{\varphi(y)})_t dy\, \delta_{t_n}(dt)\,. \tag{5.6}$$

With this notation at hand we give the nonlinear filtering equations in the following

Theorem 5.3. *For $f \in C^{1,2}([0,T] \times S^X)$ the optional projection $\widehat{f}_t$ has dynamics*

$$\begin{aligned} \widehat{f}_t = \widehat{f}_0 &+ \int_0^t (\widehat{\frac{df}{dt}})_s + (1 - N_{s-})(\widehat{\mathcal{L}_X f})_s \, ds + \int_0^{t\wedge\tau} (\widehat{fa})_s^\top - \widehat{f}_s \widehat{a}_s^\top \, dM_s^{Z,\mathbb{F}^{\mathbb{M}}} \\ &+ \int_0^{t\wedge\tau} (f(s,K) - \widehat{f}_{s-})\, d(N_s - \lambda_s^D ds) \\ &+ \int_0^{t\wedge\tau} \int_{\mathbb{R}^+} \frac{(\widehat{f\varphi(y)})_{s-} - \widehat{f}_{s-}(\widehat{\varphi(y)})_{s-}}{(\widehat{\varphi(y)})_{s-}} \, (\mu^D - \gamma^{D,\mathbb{F}^{\mathbb{M}}})(dy, ds)\,. \end{aligned} \tag{5.7}$$

Proof. Using Corollary 4.8 and the fact that $X_t = K$ on $\{\tau \leq t\}$ one obviously has

$$\widehat{f}_t = 1_{\{\tau \leq t\}} f(t,K) + 1_{\{\tau > t\}} \left\{ \big(\pi(t), f(t,\cdot)\big)_{S^X} + \pi_N(t) f(t,N) \right\}. \tag{5.8}$$

The main part of the proof is thus to compute the dynamics of $\pi_t f := \big(\pi(t), f(t,\cdot)\big)_{S^X} + \pi_N(t) f(t,N)$. In order to keep the notation simple we first consider the case without dividend information. We have

Lemma 5.4. *It holds that*

$$d\pi_t f = \Big(\pi_t(\frac{df}{dt} + \mathcal{L}_X f) - \frac{1}{2}\sigma^2 K^2 \frac{d\pi(t,K)}{dx}(f(t,K) - \pi_t f)\Big)\, dt + \big(\pi_t(a^\top f) - \pi_t a^\top \pi_t f\big)\, d(Z_t - \pi_t a\, dt). \tag{5.9}$$

Proof of Lemma 5.4. We start with the case where f is time-independent. Recall from Corollary 4.8 that $\pi_t f = \frac{1}{C(t)}\left((u(t), f)_{S^X} + \nu_N(t) f(N)\right)$ with $C(t) = (u(t), 1)_{S^X} + \nu_N(t)$. Using (4.19) and (4.18) we get

$$dC(t) = \Big(\big(\mathcal{L}^* u(t), 1\big)_{S^X} - \frac{1}{2}\sigma^2 N^2 \frac{du}{dx}(t, N)\Big)\, dt + \Big(\big(u(t), a^\top\big)_{S^X} + \nu_N(t) a^\top(N)\Big)\, dZ_t. \quad (5.10)$$

Partial integration shows that the drift term in (5.10) equals $-\frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(t, K)$. Hence,

$$\begin{aligned} d\frac{1}{C(t)} = &\frac{1}{2C(t)^2}\sigma^2 K^2 \frac{du}{dx}(t, K)\, dt - \frac{1}{C(t)^2}\Big(\big(u(t), a^\top\big)_{S^X} + \nu_N(t) a^\top(N)\Big)\, dZ_t \\ &+ \frac{1}{C(t)^3}\sum_{j=1}^{l}\big(\big(u(t), a_j\big)_{S^X} + \nu_N(t) a_j\big)^2\, dt\,. \end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} d\Big(\big(u(t), f\big)_{S^X} + \nu_N(t) f(N)\Big) = &\Big(\big(u(t), \mathcal{L}_X f\big)_{S^X} - \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(t, K) f(K)\Big)\, dt \\ &+ \Big(\big(u(t), a^\top f\big)_{S^X} + \nu_N(t) a^\top(N) f(N)\Big)\, dZ_t. \end{aligned} \quad (5.11)$$

Hence we get, using the Ito product formula and the fact that $\pi(t, v) = u(t, v)/C(t)$

$$\begin{aligned} d\pi_t f =& \frac{1}{C(t)} d\left(\big(u(t), f\big)_{S^X} + \nu_N(t) f(N)\right) + \left(\big(u(t), f\big)_{S^X} + \nu_N(t) f(N)\right) d\frac{1}{C(t)} \\ &\qquad + d\left[\frac{1}{C}, \big(u, f\big)_{S^X} + \nu_N f(N)\right]_t \\ =& \Big(\big(\pi(t), \mathcal{L}_X f\big)_{S^X} - \frac{1}{2}\sigma^2 K^2 \frac{d\pi}{dx}(t, K) f(K)\Big)\, dt + \pi_t(a^\top f)\, dZ_t \\ &+ \Big(\frac{1}{2}\sigma^2 K^2 \frac{d\pi}{dx}(t, K)\, \pi_t f + \sum_{j=1}^{l}(\pi_t a_j)^2 \pi_t f\Big)\, dt \;\; - (\pi_t f)(\pi_t a^\top)\, dZ_t - \Big(\sum_{j=1}^{l}(\pi_t a_j)(\pi_t f)\Big)\, dt\,. \end{aligned}$$

Rearranging terms and using that $\pi_t(\mathcal{L}_X f) = (\pi(t), \mathcal{L}_X f)_{S^X}$ gives (5.9). For time-dependent f we have $d\pi_t f = \pi_t(\frac{df}{dt}(t, \cdot))\, dt + d\pi_t f(t, \cdot)$ so that we obtain the additional term $\pi_t(\frac{df}{dt}(t, \cdot))$ in the drift of the dynamics of $\pi_t f$. □

Now we return to the proof of the theorem. We get from (5.8) that

$$d\widehat{f_t} = N_{t-}\frac{df}{dt}(t, K) dt + (f(t, K) - \pi_t f)\, dN_t + (1 - N_{t-}) d\pi_t f$$

Substituting the dynamics of $\pi_t f$ in this equation gives the filter equation (5.7) for the case without dividends.

Finally, we consider the case with dividend payments. Between dividend dates the dynamics of $\widehat{f_t}$ can be derived by similar arguments as before. At a dividend date we have, using Bayesian updating,

$$\pi_{T_n} f - \pi_{T_n -} f = \int_{\mathbb{R}^+} \frac{\pi_{T_n -}(f\varphi(y)) - (\pi_{T_n -} f)(\pi_{T_n -}\varphi(y))}{\pi_{T_n -}(\varphi(y))}\, \mu^D(dy, \{T_n\})$$

Finally we show that the filter equation (5.8) the measure $\mu^D(dy,dt)$ can be replaced with the compensated measure $(\mu^D - \gamma^{D,\mathbb{F}^{\mathbb{M}}})(dy,dt)$, as this helps to identify the semimartingale decomposition of asset prices. To this, note that for any $x \in S^X$ it holds that $\int_{\mathbb{R}^+} \varphi(y,x)dy = 1$, as $\varphi(\cdot,x)$ is a probability density. Hence

$$\int_{\mathbb{R}^+} \pi_t(f\varphi(y))dy = \int_{S^X} \pi(t,x)f(x)\int_{\mathbb{R}^+}\varphi(y,x)dy\,dx + \pi_N(t)f(N)\int_{\mathbb{R}^+}\varphi(y,N)dy = \pi_t f,$$

and we obtain that

$$\int_{\mathbb{R}^+} \frac{\pi_t(f\varphi(y)) - (\pi_t f)(\pi_t\varphi(y))}{\pi_t\varphi(y)}\pi_t\varphi(y) = \pi_t f - \pi_t f = 0\,,$$

and therefore $\int_0^t \int_{\mathbb{R}^+} \frac{\pi_s(f\varphi(y))-(\pi_s f)(\pi_s\varphi(y))}{\pi_s\varphi(y)}\,\gamma^{D,\mathbb{F}^{\mathbb{M}}}(dy,ds) = 0$ for all $t \ge 0$. □

Remark 5.5. Alternatively, one can derive the filter equations using the innovations approach to nonlinear filtering. For this one has to show first that every $\mathbb{F}^{\mathbb{M}}$ martingale can be represented as a sum of stochastic integrals with respect to the processes $N_t - \Lambda_{t\wedge\tau}$ and $M_t^{Z,\mathbb{F}^{\mathbb{M}}}$, and with respect to the random measure measure $\mu^D - \gamma^{D,\mathbb{F}^{\mathbb{M}}}$. Standard arguments can then be used to identify the integrands in the martingale representation of $\widehat{f}_t - \int_0^t(\widehat{\mathcal{L}_X f})_s ds$. This is the route taken in Cetin (2012) for the case without dividend payments.

Finally we may use Theorem 5.3 to derive the dynamics of the corporate securities introduced in Section 3. We concentrate on the equity value; analogous formulas can be written down for the survival claim and the payment-at-default claim. Denote by $S_t = (\widehat{h^{\mathrm{eq}}})_t$ the equity value of the firm at time t where h^{eq} is given in (3.8). Since $h^{\mathrm{eq}}(K) = 0$ and since moreover $\mathcal{L}_X h^{\mathrm{eq}}(v) = rh^{\mathrm{eq}}(v) - \bar{\delta}\lambda^D v$, we get from Theorem 5.3 that

$$\begin{aligned} dS_t = (1-N_{t-})\Big\{&(r\widehat{S}_t - \bar{\delta}\lambda^D\widehat{V}_t)dt + (\widehat{h^{\mathrm{eq}}a^\top})_t - S_t\widehat{a}_t^\top\, dM_t^{Z,\mathbb{F}^{\mathbb{M}}} - \widehat{S}_{t-}\,d(N_t-\lambda_t dt) \\ &+ \int_{\mathbb{R}^+} \frac{(\widehat{h^{\mathrm{eq}}\varphi(y)})_{t-} - S_{t-}(\widehat{\varphi(y)})_{t-}}{(\widehat{\varphi(y)})_{t-}}\,(\mu^D - \gamma^{D,\mathbb{F}^{\mathbb{M}}})(dy,dt)\Big\}\,. \end{aligned} \tag{5.12}$$

There are a number of interesting observations to make from equation (5.12). First, the stock price dynamics are of jump-diffusion type, and markedly different from geometric Brownian motion. Moreover, the processes driving the stock price dynamics (the Brownian motion $M^{Z,\mathbb{F}^{\mathbb{M}}}$, the compensated random measure $(\mu^D - \gamma^{D,\mathbb{F}^{\mathbb{M}}})(dy,dt)$ and the compensated default indicator process $N_t - \int_0^{t\wedge\tau}\lambda_s\,ds)$ are directly related to the arrival of information on the market. Hence the equation formalizes the idea that stock prices are driven by the arrival of new information on the value of the underlying firm.

6 Derivative Pricing

In this section we discuss the pricing of derivative securities in our setup. The pricing of basic corporate securities with $\mathbb{F}^N \vee \mathbb{F}^D$-adapted payoff stream such as equity and debt has been discussed in Section 3. Here we therefore give some general results on the structure of prices of *options on basic corporate securities*, that is securities whose payoff depends on the price of traded basic securities. Examples for such products include equity and bond options or convertible bonds.

We begin with a general result on the pricing of a survival claim with payoff $1_{\{\tau>T\}}H$ for some $\mathcal{F}_T^Z \vee \mathcal{F}_T^D$ measurable random variable H. The result shows that the pricing of this claim can be reduced to the problem of computing a conditional expectation with respect to the reference measure Q^* and the background filtration $\mathcal{F}_t^Z \vee \mathcal{F}_t^D$.

Proposition 6.1. *Consider some integrable, $\mathcal{F}_T^Z \vee \mathcal{F}_T^D$ measurable random variable H. Then it holds for $t \le T$ that*

$$E^Q\big(1_{\{\tau>T\}}H \mid \mathcal{F}_t^{\mathbb{M}}\big) = 1_{\{\tau>t\}} \frac{E^{Q^*}\big(H\left((u(T),1)_{S^X} + \nu_N(T)\right) \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{(u(t),1)_{S^X} + \nu_N(t)}. \tag{6.1}$$

Proof. As in the proof of Theorem 5.1 we let $F_t = Q(\tau \le t \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D)$. Then the Dellacherie-formula gives

$$E^Q\big(1_{\{\tau>T\}}H \mid \mathcal{F}_t^{\mathbb{M}}\big) = 1_{\{\tau>t\}} \frac{E^Q\big((1-F_T)H \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{1-F_t}. \tag{6.2}$$

Moreover, using (5.2) and (5.3) we have for $s \in \{t, T\}$ that

$$1 - F_s = \big((u(s),1\big)_{S^X} + \nu_N(s)\big) \frac{dQ^*}{dQ}|_{\mathcal{F}_s^Z \vee \mathcal{F}_s^D}.$$

Substituting this relation into (6.2) gives

$$E^Q\big(1_{\{\tau>T\}}H \mid \mathcal{F}_t^{\mathbb{M}}\big) = 1_{\{\tau>t\}} \frac{E^Q\Big(\big((u(T),1)_{S^X} + \nu_N(T)\big)\frac{dQ^*}{dQ}|_{\mathcal{F}_T^Z \vee \mathcal{F}_T^D} H \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\Big)}{\big((u(t),1)_{S^X} + \nu_N(t)\big)\frac{dQ^*}{dQ}|_{\mathcal{F}_t^Z \vee \mathcal{F}_t^D}},$$

and this is equal to (6.1) by the abstract Bayes formula. □

Next we specialize this general result to options on traded basic corporate securities. From now on we ignore the point mass $\nu_N(t)$ at the upper boundary of S^X. Consider for concreteness an option on the stock price of the firm with maturity T and payoff $H = g(S_T)$. We get for the price of this claim that

$$\Pi_t^H = E^Q\big(e^{-r(T-t)}1_{\{\tau>T\}}g(S_T) \mid \mathcal{F}_t^{\mathbb{M}}\big) + e^{-r(T-t)}g(0)Q(\tau \le T).$$

The computation of the default probability $Q(\tau \le T)$ has been discussed in detail in Section 3, so that we concentrate on the first term. Using Proposition 6.1 and the fact that $S_T = (u(T), h^{\mathrm{eq}})_{S^X}/(u(T),1)_{S^X}$ we get that this term equals

$$1_{\{\tau>t\}} \frac{1}{(u(t),1)_{S^X}} E^{Q^*}\Big(g\Big(\frac{(u(T),h^{\mathrm{eq}})_{S^X}}{(u(T),1)_{S^X}}\Big)(u(T),1)_{S^X} \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\Big). \tag{6.3}$$

Now standard results on the Markov property of solutions of SPDEs such as Theorem 9.30 of Peszat and Zabczyk (2007) imply that under Q^* the solution $u(t)$ of the SPDE (4.23) is a Markov process. Hence

$$E^{Q^*}\Big(g\Big(\frac{(u(T),h^{\mathrm{eq}})_{S^X}}{(u(T),1)_{S^X}}\Big)(u(T),1)_{S^X} \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\Big) = \tilde{C}(t,u(t)) \tag{6.4}$$

for some function $\tilde{C}(t,u(t))$ of time and of the current value of the unnormalized filter density. Moreover, since the the SPDE (4.23) is linear, $\tilde{C}$ is homogeneous of degree zero in u. Hence we may without loss of generality replace $u(t)$ by the current filter density $\pi(t) = u(t)/(u(t),1)_{S^X}$, and we get $E^Q\big(e^{-r(T-t)}1_{\{\tau>T\}}g(S_T) \mid \mathcal{F}_t^{\mathbb{M}}\big) = 1_{\{\tau>t\}}C(t,\pi(t))$ where

$$C(t,\pi) = E^{Q^*}\Big(g\big(\frac{(u(T),h^{\mathrm{eq}}))_{S^X}}{(u(T),1)_{S^X}}\big)(u(T),1)_{S^X} \mid u(t)=\pi\Big). \tag{6.5}$$

The actual computation of C is best done using Monte Carlo simulation, using a numerical method to solve the SPDE (4.23). The Galerkin approximation described in Section 4.4 is particularly well-suited for this purpose since most of the time-consuming computational steps can be done off-line. Note that (6.5) is an expectation with respect to the reference measure Q^*. Hence one needs to sample from the SDE (4.23) under Q^*, that is the driving process Z is a Brownian motion and the random measure μ^D has compensator $\gamma^{D,*}$. Alternatively, one might evaluate directly the expected value $E^Q\big(e^{-r(T-t)}1_{\{\tau>T\}}g(S_T) \mid \mathcal{F}_t^{\mathbb{M}}\big)$, using a simulation approach sketched in Section 7 below.

Remark 6.2 (Factor structure)**.** Equation 6.5 shows that the price of all securities at time t is given by a function of the current filter density $\pi(t) = u(t)/(u(t,1)_{S^X}$. Since $u(t)$ is moreover a Markov process the model has factor structure with infinite-dimensional factor process $u(t)$.

Remark 6.3 (Model calibration)**.** The fact that pricing formulas depend on the current filter density $\pi(t)$ raises the issue of model calibration. As explained earlier, we view the process Z generating $\mathbb{F}^{\mathbb{M}}$ as abstract source of information so that the density process $\pi(t)$ is not directly observable for investors. On the other hand, pricing formulas need to be evaluated using only publicly available information. Hence we have to back out (an estimate of) $\pi(t)$ from prices of traded basic corporate securities at time t. A crucial observation in this context is the fact that the prices of traded basic corporate securities are linear functions of $\pi(t)$, so that model calibration leads to optimization problems with linear constraints. In order to make this more explicit, we assume that a Galerkin approximation of the form $\pi^{(m)}(t) = \sum_{i=1}^m \psi_i e_i$ with nonnegative basis functions $e_1,\dots,e_m$ is used to approximate the filter density $\pi(t)$. Assume that we observe prices $\Pi_1^*,\dots,\Pi_\ell^*$ of ℓ basic corporate securities with full information value $h_l(t,v)$, $1 \le l \le \ell$. In order to match the observed prices perfectly, the Fourier coefficients $\psi_1,\dots,\psi_m$ need to satisfy the following $\ell+1$ linear constraints

$$\sum_{i=1}^m \psi_i(e_i,1)_{S^X} = 1, \text{ and } \sum_{i=1}^m \psi_i\big(e_i,h_l(t,\cdot)\big)_{S^X} = \Pi_l^*,\ 1 \le l \le \ell\,.$$

Calibration problems of this type have been analyzed in detail in the literature on implied copula models for CDO pricing, and we refer to Hull and White (2006) and Frey and Schmidt (2012) for further information.

There are good theoretical reasons to expect that the model calibrates well to a given term structure of defaultable bond spreads, or, equivalently to a given term structure of survival probabilities $Q(\tau > s)$, $0 \le s \le T$. In fact, Davis and Pistorius (2013) have shown analytically that the initial distribution π_0 can be chosen in such a way that τ has an exponential distribution. In Davis and Pistorius (2013) the exponential distribution has been chosen mainly for analytical convenience, so that one should have good calibration properties for other survival distributions as well. A detailed numerical analysis of model calibration is deferred to future research.

K	r	λ^D	σ (vol of GBM)	initial filter distribution π_0
20	0.02	1	0.2	displaced lognormal, $V - K \sim LN(\ln 15, 0.2)$

Table 1: Parameters used in simulation study.

7 Numerical Experiments

In this section we illustrate the model with a number of numerical experiments. We are particularly interested in the impact of the noisy asset information $\mathbb{F}^Z$ on the asset price dynamics under incomplete information.

We use the following setup for our analysis: Dividends are paid annually; the dividend size is modelled as $d_n = \delta_n V_{t_n}$ where δ_n is lognormal with mean $\bar{\delta} = 1\%$. The noisy asset information Z is two-dimensional with $a_1(v) = c_1 \ln v$ and $a_2(v) = c_2\big(\ln K + \sigma - \ln v)^+$; for $c_2 > 0$ this choice of a_2 models the idea that the market obtains additional information on the asset value of the firm as soon as the asset value is less than one standard deviation away from the default boundary, perhaps because of the firm is monitored particularly closely in that case. The remaining parameters are given in in Table 1.

In order to generate a trajectory of the filter density $\pi(t)$ with initial value π_0 and related quantities such as the stock price S_t we proceed according to the following steps.

1. Generate a random variable $V \sim \pi_0$, a trajectory $(V_s)_{s=0}^T$ of the asset value process with initial value $V_0 = V$ and the associated trajectory $(N_s)_{s=0}^T$ of the default indicator process.

2. Generate realizations $(D_s)_{s=0}^T$ and $(Z_s)_{s=0}^T$ of the cumulative dividend process and of the noisy asset information, using the trajectory of the asset value process generated in Step 1 as input.

3. Compute for the observation generated in Step 2 a trajectory $(u(s))_{s=0}^T$ of the unnormalized filter density with initial value $u(0) = \pi_0$, using the Galerkin approximation described in Section 4.4. Return $\pi(s) = (1 - N_s)\big(u(s)/(u(s), 1)_{S^X}\big)$ and $S_s = (1 - N_s)(\pi(s), h^{\mathrm{eq}})_{S^X}$, $0 \le s \le T$.

Details on the numerical methodology are available on request.

Next we describe the results of our numerical experiments. The trajectory of V and of the corresponding dividend process that is used as input to all filtering experiments is shown in Figure 1. In Figure 2 we plot a trajectory of the stock price S_t and of the corresponding full information value $h^{\mathrm{eq}}(V_t)$ for the case where the market has only dividend information ($c_1 = c_2 = 0$). This can be viewed as an example of the discrete noisy accounting information considered in Duffie and Lando (2001). We see that S_t has very unusual dynamics; in particular it evolves deterministically between dividend dates.

The next graphs use the parameter values $c_1 = 4$ and $c_2 = 0$. They show in particular that more realistic asset price dynamics can be obtained by introducing the continuous noisy asset information $Z_t = \int_0^t c_1 \ln(V_s)ds + W_t$. We begin by plotting the evolution of the filter density in Figure 3. It can be seen that $\pi(t)$ evolves continuously between dividend dates, and that the density jumps at a dividend date. Moreover, the mode of $\pi(t)$ is close to the

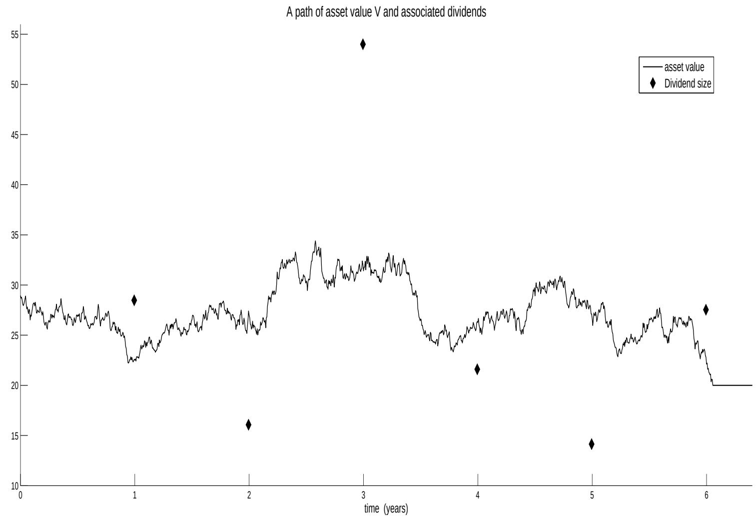

Figure 1: A simulated path of the asset value leading to a default at $\tau \approx 6.05$(years) and a dividend realisation for that path. Dividend payments have been multiplied by 100 to make them comparable in size to the asset value.

default boundary K immediately prior to default. Next we graph the resulting stock price trajectory S_t together with the full informatiuon value $h^{\text{eq}}(V_t)$, see Figure 4. Clearly, for $c_1 > 0$ S_t has nonzero volatility between dividend dates. A comparison of the two trajectories moreover shows that S_t jumps to zero at the default time τ; this reflects the fact that the default time has an intensity under incomplete information so that default comes as a surprise to the market. The corresponding trajectory of the default intensity $\lambda_t = \frac{d\pi}{dx}(t, K)$ is shown in Figure 5. Note that λ_t is quite large when V is close to the default boundary K as one would expect intuitively. Finally we consider the parameter set $c_1 = 4, c_2 = 50$; this corresponds to the situation where the market receives a lot of information on the asset value whenever V is close to the default boundary. In this scenario default is "almost predictable" and the model behaves similar to a structural model close to default. This can be seen from Figure 6 where we plot S_t together with the full information value $h^{\text{eq}}(V_t)$. Note that now S_t and the full information value $h^{\text{eq}}(V_t)$ are very close immediately before the default; in particular, the stock price jump at $t = \tau$ is very small.

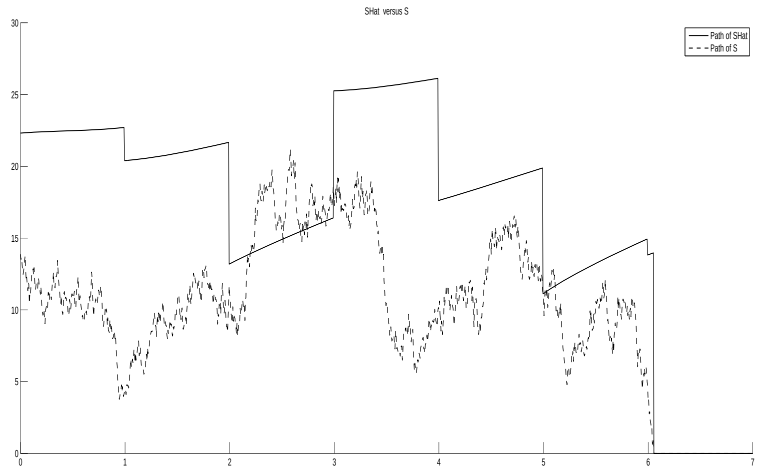

Figure 2: A simulated path of the full information value $h^{\mathrm{eq}}(V_t)$ of the stock (dashed line) and of the stock price S_t (normal line, label Shat) for $c_1 = c_2 = 0$ (only dividend information).

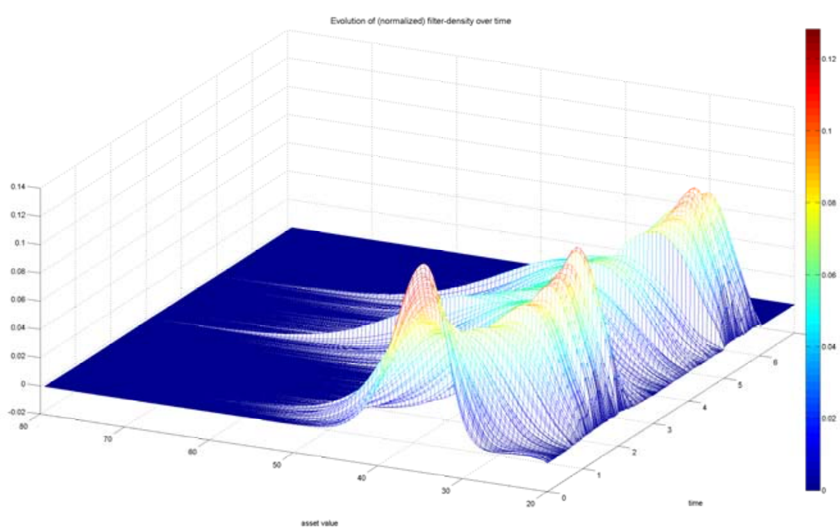

Figure 3: A simulated realisation of the conditional density $\pi(t)$.

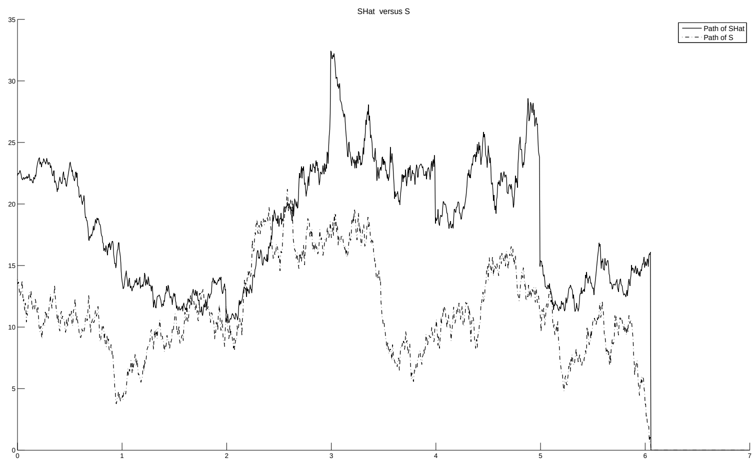

Figure 4: A simulated path of the full information value $h^{\mathrm{eq}}(V_t)$ of the stock (dashed line) and of the stock price S_t (normal line, label Shat) for $c_1 = 4, c_2 = 0$ (noisy asset information).

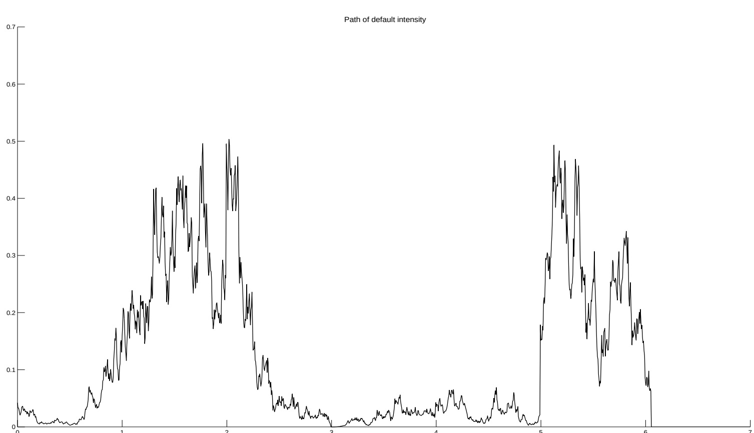

Figure 5: A simulated path of the default intensity for for $c_1 = 4, c_2 = 0$.

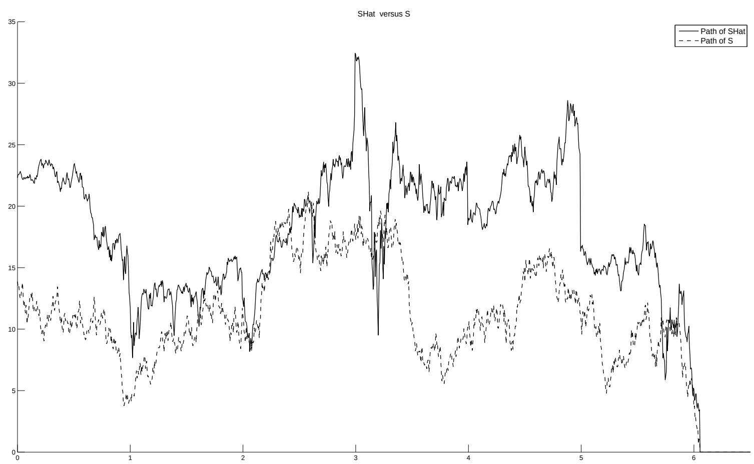

Figure 6: A simulated path of the full information value $h^{\mathrm{eq}}(V_t)$ of the stock (dashed line) and of the stock price S_t (normal line, label Shat) for $c_1 = 4, c_2 = 50$ (close monitoring of V close to the default boundary).

A Additional Proofs

Proof of Proposition 4.2. The process $F_t = Q(\sigma_N \le t \mid \mathcal{F}_t)$, $0 \le t \le T$, is an $\mathbb{F}$-submartingale. Hence we get, using the first submartingale inequality

$$Q\big(\sup_{s\le T} F_s > \epsilon\big) \le \frac{1}{\epsilon} E(F_T^+) = \frac{1}{\epsilon} Q(\sigma_N \le T),$$

which gives Statement 1.

The difficulty in the proof of Statement 2 is the fact that we have to compare conditional expectations with respect to different filtrations. Similarly as in robust filtering, we address this problem using the Kallianpur Striebel formula (4.9) and the product representation (4.10). For simplicity we ignore the dividend observation. Let $X^N = (V^N)^\tau$, and put

$$L_t^N(\omega_1,\omega_2) = \exp\Big(\int_0^t a(X_s^N(\omega_1))^\top dZ_s(\omega_2) - \frac{1}{2}\int_0^t \big|a((X_s^N)(\omega_1))\big|^2\, ds\Big)$$

and define $L_t(\omega_1,\omega_2)$ in the same way, but with V^τ instead of X^N. In view of (4.9) and (4.10) we need to show that $E^{Q_1}\Big(f(X_t^N)L_t^N(\cdot,\omega_2)\Big)$ converges in probability to $E^{Q_1}\Big(f(V_t^\tau)L_t(\cdot,\omega_2)\Big)$ on Ω_2. Now for $t < \sigma_N$ we have $X_t^N = V_t^\tau$ and hence also $L_t^N = L_t$. This gives

$$\Big|E^{Q_1}\Big(f(X_t^N)L_t^N(\cdot,\omega_2) - f(V_t^\tau)L_t(\cdot,\omega_2)\Big)\Big| \le E^{Q_1}\Big(1_{\{\sigma_N\le t\}}(\big|f(X_t^N)\big|L_t^N(\cdot,\omega_2) + \big|f(V_t^\tau)\big|L_t(\cdot,\omega_2))\Big).$$

Hence

$$\begin{aligned} E^{Q^*}\Big(\Big|E^{Q_1}\Big(f(X_t^N)L_t^N(\cdot,\omega_2) - f(V_t^\tau)L_t(\cdot,\omega_2)\Big)\Big|\Big) &\le E^{Q^*}\Big(1_{\{\sigma_N\le t\}}L_t^N\big|f(X_t^N)\big|\Big) + E^{Q^*}\Big(1_{\{\sigma_N\le t\}}L_t\big|f(V_t^\tau)\big|\Big) \\ &\le E^{Q^*}\Big(1_{\{\sigma_N\le t\}}Y_tL_t^N\Big) + E^{Q^*}\Big(1_{\{\sigma_N\le t\}}Y_tL_t\Big). \end{aligned}$$

Since the law of the asset value process is the same under Q and Q^* the last line equals $E^Q\big(1_{\{\sigma_N\le t\}}Y_t\big)+E^Q\big(1_{\{\sigma_N\le t\}}Y_t\big)$ and this converges to zero for $N\to\infty$ by dominated convergence and the assumed integrability of Y. This shows that $E^{Q_1}\big(f(X_t^N)L_t^N(\cdot,\omega_2)\big)$ converges to $E^{Q_1}\big(f(V_t^\tau)L_t(\cdot,\omega_2)\big)$ in $L_1(\Omega_2,\mathcal{G}^2,Q_2)$ and hence also in probability.

$\square$

Proof of Lemma 4.5. The easiest way to show that $E(L_T)=1$ is to condition on the trajectory of X or, more formally, on ω_1. Given ω_1, L^1 and L^2 are independent and hence $E^{Q^*}(L_T(\omega_1,\cdot)) = E^{Q^*}(L_T^1(\omega_1,\cdot))E^{Q^*}(L_T^2(\omega_1,\cdot))$. Now $E^{Q^*}(L_T^1(\omega_1,\cdot)) = 1$ by a standard application of the Novikov-criterion, so that it suffices to show that $E^{Q^*}(L_T^2(\omega_1,\cdot)) = 1$. Here we get by conditioning on $N_T^D = \int_0^T\int_{\mathbb{R}^+} 1\mu^D(dy,ds)$ (the number of dividend dates up to time T) that

$$\begin{aligned} E^{Q^*}(L_T^2(\omega_1,\cdot)) &= \sum_{m=0}^\infty P(N_T^D = m)E^{Q^*}\Big(\prod_{n=1}^m \frac{\varphi(d_n, X_{T_n}(\omega_1))}{\varphi^*(d_n)}\Big) \\ &= \sum_{m=0}^\infty P(N_T^D = m)\prod_{n=1}^m \int_{\mathbb{R}^+} \frac{\varphi(y, X_{T_n}(\omega_1))}{\varphi^*(y)}\varphi^*(y)\,dy\,. \end{aligned}$$

Note that $\int_{\mathbb{R}^+}\frac{\varphi(y,_{T_n}(\omega_1))}{\varphi^*(y)}\varphi^*(y)dy = \int_{\mathbb{R}^+}\varphi(y,X_{T_n}(\omega_1)dy = 1$. Hence we get that $E^{Q^*}(L_T^2(\omega_1,\cdot)) = \sum_{m=0}^\infty P(N_T^D = m) = 1$.

In order to show that $\mu^D(dy,dt)$ has Q-compensator $\gamma^D(dy,dt)$ we use the general Girsanov theorem (see for instance Protter (2005), Theorem 3.40): it holds that a process M is a Q^*-local martingale if and only if $\widetilde{M}_t = M_t - \int_0^t \frac{1}{L_{s-}}d\langle L,M\rangle_s$ is a Q-local martingale. Consider now some bounded predictable function $\beta\colon [0.T] \times \mathbb{R}^+ \to \mathbb{R}$ and define the Q^*-local martingale $M_t = \int_0^t \int_{\mathbb{R}^+} \beta(s,y)\,(\mu^d - \gamma^{D,*})(dy,ds)$. As M is of finite variation, we get that

$$[M,L]_t = \sum_{s\le t} \Delta M_s \Delta L_s = \int_0^t \int_{\mathbb{R}^+} L_{s-}\big(\frac{\varphi(y,X_s)}{\varphi^*(y)} - 1\big)\beta(s,y)\mu^D(dy,ds).$$

It follows that

$$\langle M,L\rangle_t = \int_0^t \int_{\mathbb{R}^+} L_{s-}\big(\frac{\varphi(y,X_s)}{\varphi^*(y)} - 1\big)\beta(s,y)\gamma^{D,*}(ds,dy) = \int_0^t \int_{\mathbb{R}^+} L_{s-}\big(\varphi(y,X_s) - \varphi^*(y)\big)\beta(s,y)\,dy\lambda^D ds.$$

Recall that $\gamma^D(dy,dt) = \varphi(y,X_t)dy\,\lambda^D dt$. Hence we get that

$$\widetilde{M}_t := M_t - \int_0^t \frac{1}{L_s-} d\langle L,M\rangle_s = \int_0^t \int_{\mathbb{R}^+} \beta(s,y)(\mu^D - \gamma^D)(dy,ds)\,.$$

Now $\widetilde{M}$ is a local martingale by the general Girsanov theorem, which shows that $\gamma^D(dy,dt)$ is in fact the Q-compensator of μ^D. The other claims are clear. $\square$

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