Worst-Case Value-at-Risk of Non-Linear Portfolios

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Consider a market consisting of \( m \) assets.

**Optimal Asset Allocation Problem**

Choose the weights vector \( w \in \mathbb{R}^m \) to make the portfolio return high, whilst keeping the associated risk \( \rho(w) \) low.

**Portfolio optimization problem:**

\[
\begin{align*}
\text{minimize} & \quad \rho(w) \\
\text{subject to} & \quad w \in \mathcal{W}.
\end{align*}
\]

**Popular risk measures \( \rho \):**

- **Variance** \( \rightarrow \) Markowitz model
- **Value-at-Risk** \( \rightarrow \) Focus of this talk
Let \( \tilde{r} \) denote the random returns of the \( m \) assets.

The portfolio return is therefore \( w^T \tilde{r} \).

**Value-at-Risk (VaR)**

The minimal level \( \gamma \in \mathbb{R} \) such that the probability of \( -w^T \tilde{r} \)
exceeding \( \gamma \) is smaller than \( \epsilon \).

\[
\text{VaR}_\epsilon(w) = \min \left\{ \gamma : \mathbb{P} \left\{ \gamma \leq -w^T \tilde{r} \right\} \leq \epsilon \right\}
\]
VaR lacks some desirable theoretical properties:

- **Not a coherent** risk measure.
- **Needs precise knowledge** of the distribution of $\tilde{r}$.
- **Non-convex** function of $w$ → VaR minimization **intractable**.

To optimize VaR: resort to VaR approximations.

Example: assume $\tilde{r} \sim \mathcal{N}(\mu_r, \Sigma_r)$, then

$$\text{VaR}_\epsilon(w) = -\mu_r^T w - \Phi^{-1}(\epsilon) \sqrt{w^T \Sigma_r w},$$

Normality assumption **unrealistic** → may **underestimate** the actual VaR.
Worst-Case Value-at-Risk (WCVaR)

- Only know means \( \mu_r \) and covariance matrix \( \Sigma_r \succ 0 \) of \( \tilde{r} \).
- Let \( \mathcal{P}_r \) be the set of all distributions of \( \tilde{r} \) with mean \( \mu_r \) and covariance matrix \( \Sigma_r \).

\[
\text{WCVaR}_\epsilon(w) = \min \left\{ \gamma : \sup_{\mathbb{P} \in \mathcal{P}_r} \mathbb{P} \left\{ \gamma \leq -w^T \tilde{r} \right\} \leq \epsilon \right\}
\]

- WCVaR is immunized against uncertainty in \( \mathbb{P} \): distributionally robust.
- Unless the most pessimistic distribution in \( \mathcal{P}_r \) is the true distribution, actual VaR will be lower than WCVaR.
El Ghaoui *et al.* have shown that

$$\text{WCVaR}_\varepsilon(w) = -\mu^T w + \kappa(\varepsilon)\sqrt{w^T\Sigma w},$$

where $\kappa(\varepsilon) = \sqrt{(1 - \varepsilon)/\varepsilon}$.

Connection to robust optimization:

$$\text{WCVaR}_\varepsilon(w) = \max_{r \in U_\varepsilon} \varepsilon - w^T r,$$

where the ellipsoidal uncertainty set $U_\varepsilon$ is defined as

$$U_\varepsilon = \left\{ r : (r - \mu_r)^T\Sigma_r^{-1}(r - \mu_r) \leq \kappa(\varepsilon)^2 \right\}.$$

Therefore,

$$\min_{w \in \mathcal{W}} \text{WCVaR}_\varepsilon(w) \equiv \min_{w \in \mathcal{W}} \max_{r \in U_\varepsilon} -w^T r.$$
Assume that the market consists of:

- \( n \leq m \) basic assets with returns \( \tilde{\xi} \), and
- \( m - n \) derivatives with returns \( \tilde{\eta} \).

\( \tilde{\xi} \) are only risk factors.

We partition asset returns as \( \tilde{r} = (\tilde{\xi}, \tilde{\eta}) \).

Derivative returns \( \tilde{\eta} \) are **uniquely** determined by basic asset returns \( \tilde{\xi} \). There exists \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) with \( \tilde{r} = f(\tilde{\xi}) \).

\( f \) is highly **non-linear** and can be inferred from:

- Contractual specifications (option payoffs)
- Derivative pricing models
WCVaR is applicable but not suitable for portfolios containing derivatives:

- Moments of $\tilde{\eta}$ are difficult to estimate accurately.
- Disregards perfect dependencies between $\tilde{\eta}$ and $\tilde{\xi}$.

WCVaR severely overestimates the actual VaR, because:

- $\Sigma_r$ only accounts for linear dependencies
- $\mathcal{U}_\epsilon$ is symmetric but derivative returns are skewed
We develop two new Worst-Case VaR models that:

- Use first- and second-order moments of $\tilde{\xi}$ but not $\tilde{\eta}$.
- Incorporate the non-linear dependencies $f$

**Generalized Worst-Case VaR**

Let $\mathcal{P}$ denote set of all distributions of $\tilde{\xi}$ with mean $\mu$ and covariance matrix $\Sigma$.

$$\min \left\{ \gamma : \sup_{\mathcal{P} \in \mathcal{P}} \left\{ \gamma \leq -w^T f(\tilde{\xi}) \right\} \leq \epsilon \right\}$$

- When $f(\tilde{\xi})$ is:
  - convex polyhedral $\rightarrow$ Worst-Case Polyhedral VaR (SOCP)
  - nonconvex quadratic $\rightarrow$ Worst-Case Quadratic VaR (SDP)
Assume that the \( m - n \) derivatives are European put/call options maturing at the end of the investment horizon \( T \).

Basic asset returns: \( \tilde{r}_j = f_j(\tilde{\xi}) = \tilde{\xi}_j \) for \( j = 1, \ldots, n \).

Assume option \( j \) is a call with strike \( k_j \) and premium \( c_j \) on basic asset \( i \) with initial price \( s_i \), then \( \tilde{r}_j \) is

\[
f_j(\tilde{\xi}) = \frac{1}{c_j} \max \left\{ 0, s_i(1 + \tilde{\xi}_i) - k_j \right\} - 1
\]

\[
= \max \left\{ -1, a_j + b_j \tilde{\xi}_i - 1 \right\}, \text{ where } a_j = \frac{s_i - k_j}{c_j}, \quad b_j = \frac{s_i}{c_j}.
\]

Likewise, if option \( j \) is a put with premium \( p_j \), then \( \tilde{r}_j \) is

\[
f_j(\tilde{\xi}) = \max \left\{ -1, a_j + b_j \tilde{\xi}_i - 1 \right\}, \text{ where } a_j = \frac{k_j - s_i}{p_j}, \quad b_j = -\frac{s_i}{p_j}.
\]
In compact notation, we can write $\tilde{r}$ as

$$\tilde{r} = f(\tilde{\xi}) = \left( \max \left\{ -e, a + B\tilde{\xi} - e \right\} \right).$$

Partition weights vector as $w = (w^\xi, w^\eta)$.

No derivative short-sales: $w \in \mathcal{W} \implies w^\eta \geq 0$.

Portfolio return of $w \in \mathcal{W}$ can be expressed as

$$w^T\tilde{r} = w^T f(\tilde{\xi})$$

$$= (w^\xi)^T \tilde{\xi} + (w^\eta)^T \max \left\{ -e, a + B\tilde{\xi} - e \right\}.$$
Use the piecewise linear portfolio model:

$$w^T f(\tilde{\xi}) = (w^x)^T \tilde{\xi} + (w^n)^T \max \left\{ -e, a + B \tilde{\xi} - e \right\}.$$
Theorem: SDP Reformulation of WCPVaR

WCPVaR of $w$ can be computed as an SDP:

$$\text{WCPVaR}_\epsilon(w) = \min \gamma$$

s.t.  

- $M \in S^{n+1}$, $y \in \mathbb{R}^{m-n}$, $\tau \in \mathbb{R}$, $\gamma \in \mathbb{R}$
- $\langle \Omega, M \rangle \leq \tau \epsilon$, $M \succeq 0$, $\tau \geq 0$, $0 \leq y \leq w^n$

$$M + \begin{bmatrix} 0 & w^\xi + B^T y \\ (w^\xi + B^T y)^T & -\tau + 2(\gamma + y^T a - e^T w^n) \end{bmatrix} \succeq 0$$

Where we use the second-order moment matrix $\Omega$:

$$\Omega = \begin{bmatrix} \Sigma + \mu \mu^T & \mu \\ \mu^T & 1 \end{bmatrix}$$
Worst-Case Polyhedral VaR: Convex Reformulations

Theorem: SOCP Reformulation of WCPVaR

WCPVaR of \( w \) can be computed as an SOCP:

\[
WCPVaR_\epsilon(w) = \min_{0 \leq g \leq w^\eta} -\mu^T(w^\xi + B^Tg) + \kappa(\epsilon) \left\| \Sigma^{1/2}(w^\xi + B^Tg) \right\|_2^2 \ldots - a^Tg + e^Tw^\eta
\]

- SOCP has better scalability properties than SDP.
Robust Optimization Perspective on WCPVaR

WCPVaR minimization is equivalent to:

\[
\min_{w \in \mathcal{W}} \max_{r \in U^p_\epsilon} -w^T r.
\]

where the uncertainty set \( U^p_\epsilon \subseteq \mathbb{R}^m \) is defined as

\[
U^p_\epsilon = \left\{ r \in \mathbb{R}^m : \exists \xi \in \mathbb{R}^n \text{ such that } \left( \xi - \mu \right)^T \Sigma^{-1} (\xi - \mu) \leq \kappa(\epsilon)^2 \text{ and } r = f(\xi) \right\}
\]

Unlike \( U_\epsilon \), the set \( U^p_\epsilon \) is not symmetric!
Robust Optimization Perspective on WCPVaR

Zymler, Kuhn and Rustem

Worst-Case Value-at-Risk of Non-Linear Portfolios
Consider Black-Scholes Economy containing:
- Stocks A and B, a call on stock A, and a put on stock B.
- Stocks have drifts of 12% and 8%, and volatilities of 30% and 20%, with instantaneous correlation of 20%.
- Stocks are both $100.
- Options mature in 21 days and have strike prices $100.

Assume we hold equally weighted portfolio.

Goal: calculate VaR of portfolio in 21 days.
- Generate 5,000,000 end-of-period stock and option prices.
- Calculate first- and second-order moments from returns.
- Estimate VaR using: Monte-Carlo VaR, WCVaR, and WCPVaR.
At confidence level $\epsilon = 1\%$:

- WCVaR unrealistically high: 497%.
- WCVaR is 7 times larger than WCPVaR.
- WCPVaR is much closer to actual VaR.
\( m - n \) derivatives can be exotic with arbitrary maturity time. Value of asset \( i = 1 \ldots m \) is representable as \( v_i(\tilde{\xi}, t) \).

For short horizon time \( T \), second-order Taylor expansion is accurate approximation of \( \tilde{r}_i \):

\[
\tilde{r}_i = f_i(\tilde{\xi}) \approx \theta_i + \Delta_i^T \tilde{\xi} + \frac{1}{2} \tilde{\xi}^T \Gamma_i \tilde{\xi} \quad \forall i = 1, \ldots, m.
\]

Portfolio return approximated by (possibly non-convex):

\[
w^T \tilde{r} = w^T f(\xi) \approx \theta(w) + \Delta(w)^T \tilde{\xi} + \frac{1}{2} \tilde{\xi}^T \Gamma(w) \tilde{\xi},
\]

where we use the auxiliary functions

\[
\theta(w) = \sum_{i=1}^{m} w_i \theta_i, \quad \Delta(w) = \sum_{i=1}^{m} w_i \Delta_i, \quad \Gamma(w) = \sum_{i=1}^{m} w_i \Gamma_i.
\]

We now allow short-sales of options in \( w \)
Worst-Case Quadratic VaR

For any \( w \in \mathcal{W} \), we define WCQVaR as

\[
\min \left\{ \gamma : \sup_{\mathcal{P} \in \mathcal{P}} \left\{ \gamma \leq -\theta(w) - \Delta(w)^T \bar{\xi} - \frac{1}{2} \bar{\xi}^T \Gamma(w) \bar{\xi} \right\} \leq \epsilon \right\}
\]

Theorem: SDP Reformulation of WCQVaR

WCQVaR can be found by solving an SDP:

\[
\text{WCQVaR}_\epsilon(w) = \min \gamma \\
\text{s.t.} \quad M \in \mathbb{S}^{n+1}, \quad \tau \in \mathbb{R}, \quad \gamma \in \mathbb{R} \\
\quad \left\langle \Omega, M \right\rangle \leq \tau \epsilon, \quad M \succeq 0, \quad \tau \geq 0, \\
\quad M + \begin{bmatrix} \Gamma(w) & \Delta(w) \\ \Delta(w)^T & -\tau + 2(\gamma + \theta(w)) \end{bmatrix} \succeq 0
\]

There seems to be no SOCP reformulation of WCQVaR.
Robust Optimization Perspective on WCQVaR

- WCQVaR minimization is equivalent to:

\[
\min_{\mathbf{w} \in \mathcal{W}} \max_{\mathbf{Z} \in \mathcal{U}_q} -\langle \mathbf{Q}(\mathbf{w}), \mathbf{Z} \rangle
\]

where

\[
\mathbf{Q}(\mathbf{w}) = \begin{bmatrix}
\frac{1}{2} \Gamma(\mathbf{w}) & \frac{1}{2} \Delta(\mathbf{w}) \\
\frac{1}{2} \Delta(\mathbf{w})^T & \theta(\mathbf{w})
\end{bmatrix},
\]

and the uncertainty set \( \mathcal{U}_q \subseteq \mathbb{S}^{n+1} \) is defined as

\[
\mathcal{U}_q = \left\{ \begin{bmatrix} \mathbf{X} & \xi \\ \xi^T & 1 \end{bmatrix} \in \mathbb{S}^{n+1} : \Omega - \varepsilon \mathbf{Z} \gtrless 0, \mathbf{Z} \gtrless 0 \right\}
\]

- \( \mathcal{U}_q \) is lifted into \( \mathbb{S}^{n+1} \) to compensate for non-convexity.
There is a connection between $\mathcal{U}_\epsilon \subseteq \mathbb{R}^m$ and $\mathcal{U}_\epsilon^q \subseteq \mathbb{S}^{n+1}$.

If we impose: $w \in \mathcal{W} \implies \Gamma(w) \succ 0$ then robust optimization problem reduces to:

$$
\min_{w \in \mathcal{W}} \max_{r \in \mathcal{U}_\epsilon^q} - w^T r
$$

where the uncertainty set $\mathcal{U}_\epsilon^q \subseteq \mathbb{R}^m$ is defined as

$$
\mathcal{U}_\epsilon^q = \left\{ r \in \mathbb{R}^m : \exists \xi \in \mathbb{R}^n \text{ such that } (\xi - \mu)^T \Sigma^{-1} (\xi - \mu) \leq \kappa(\epsilon)^2 \text{ and } r_i = \theta_i + \xi^T \Delta_i + \frac{1}{2} \xi^T \Gamma_i \xi \quad \forall i = 1, \ldots, m \right\}
$$

Unlike $\mathcal{U}_\epsilon$, the set $\mathcal{U}_\epsilon^q$ is not symmetric!
Robust Optimization Perspective on WCQVaR

Zymler, Kuhn and Rustem
Worst-Case Value-at-Risk of Non-Linear Portfolios
Now we want to estimate VaR after 2 days (not 21 days).

- VaR not evaluated at option maturity times → use WCQVaR (not WCPVaR).
- Use Black-Scholes to calculate prices and greeks.

At $\epsilon = 1\%$: WCVaR still 3 times larger than WCQVaR.

Outperformance: option strat 56%, stock-only strat 12%.
Sharpe Ratio: option strat 0.97, stock-only strat 0.13.
Allocation option strategy: 89% stocks, 11% options.
Paper available on optimization-online.