Matrix and Tensor Factorization from a Machine Learning Perspective

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Matrix Factorization - SVD

- **SVD**: Decompose $p_1 \times p_2$ matrix $Y := V_1 D V_2^T$
  - $V_1$ are $k$ eigenvectors of $YY^T$
  - $V_2$ are $k$ eigenvectors of $Y^T Y$
  - $D := \sqrt{\text{eig}(\text{diag}(YY^T))}$
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Properties:
- Method from linear algebra for arbitrary $Y$
- Tool for descriptive and exploratory data analysis
- Decomposition optimizes the Frobenius norm with orthogonality constraints
Tensor Factorization - Tucker Decomposition

- **Tucker Decomposition:**
  Decompose $p_1 \times p_2 \times p_3$ tensor $Y := D \times_1 V_1 \times_2 V_2 \times_3 V_3$
  - $V_1$ are $k_1$ eigenvectors of mode-1 unfolded $Y$
  - $V_2$ are $k_2$ eigenvectors of mode-2 unfolded $Y$
  - $V_3$ are $k_3$ eigenvectors of mode-3 unfolded $Y$
  - $D \in \mathbb{R}^{k_1 \times k_2 \times k_3}$ non-diagonal core tensor
Tensor Factorization - Tucker Decomposition

- **Tucker Decomposition:**
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  - $D \in \mathbb{R}^{k_1 \times k_2 \times k_3}$ non-diagonal core tensor

**Properties:**
- Extension of SVD method for arbitrary tensor $Y \rightarrow$ orthogonal representation
- Tool for descriptive and exploratory data analysis
- Decomposition optimizes the Frobenius norm
- Expensive inference: sequences of SVD + core tensor $D$
Tensor Factorization - Canonical Decomposition

- **Canonical Decomposition (CD):**
  Decompose $p_1 \times p_2 \times p_3$ tensor $Y := D \times_1 V_1 \times_2 V_2 \times_3 V_3$
  - with diagonal, identity tensor $D$
  - as a sum of $k$ rank-one tensors $Y = \sum_{f=1}^{k} v_{1,f} \circ v_{2,f} \circ v_{3,f}$

Properties:
- Extension of SVD method for arbitrary tensor $Y$
- Tool for descriptive and exploratory data analysis
- Decomposition optimizes the Frobenius norm
- Fast inference due to less parameters
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Machine Learning Perspective

Machine Learning:

...is the task of

- learning from (noisy) experience $E$ with respect to some class of tasks $T$ and performance measure $P$
  - Experience $E$: data
  - Tasks $T$: predictions
  - Performance $P$: root mean square error (RMSE), misclassification,...
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- to **generalize** from noisy data to accurately **predict** new cases
Machine Learning Perspective

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- to generalize from noisy data to accurately predict new cases

In other words:

- Prediction accuracy on new cases $=$ Machine Learning
- No hypothesis generation/testing $=$ Data Mining
Applications of the Machine Learning Perspective

Many different applications:
- Social Network Analysis
- Recommender Systems
- Graph Analysis
- Image/Video Analysis
- …
Applications of the Machine Learning Perspective

Overall Task:

- **Prediction** of missing friendships, interesting items, corrupted pixels, missing edges, etc.
- **Data representation**: matrix/tensor with missing entries

\[ Y = \begin{array}{cccc}
  u_1 & 3 & & 5 \\
  u_2 & 3 & 4 & 4 \\
  & & & \\
  u_{p_1} & 5 & 2 & 2 \\
\end{array} \]
Applications of the Machine Learning Perspective

**Overall Task:**

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- **Data representation:** matrix/tensor with missing entries

Further Properties:

- **Unobserved Heterogeneity**, e.g. different consumption preferences of different users and items
- **Large-scale**: millions of interactions
- **Poor Data Quality**: high noise due to indirect data collection

→ Factorization Models
Factorization Models from a Machine Learning Perspective

Common usage of matrix (and tensor) factorization:

- Identification/Interpretation of unobserved heterogeneity, i.e. latent dependencies between instances of a mode (rows, columns, ...)

- Data compression, e.g., instead of $p_1p_2$ values, store only $(p_1 + p_2)k$ values

- Data preprocessing: uncorrelate $p$ predictor variables of a design matrix $X$
Factorization Models from a Machine Learning Perspective

Machine Learning perspective on factorization models:

- Factorization models seen as predictive models
  - No probabilistic embedding, e.g. for Bayesian Analysis
  - Gaussian likelihood:
    \[ Y = V_1 D V_2^T + E, \quad \forall e_\ell \in E : e_\ell \sim N(0, \sigma^2 = 1) \]
  - No missing value treatment; often treated as 0

- Scalable Learning for large scale datasets
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  - Non-informative prior:
    \[ p(V_1, D, V_2) \propto 1 \]
  - Does not distinguish between signal and noise
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  ► Does not distinguish between signal and noise
► No interest in latent representation, e.g. interpretation
  ► Elimination of orthonormality constraint \( Y = V_1, V_2^T + E \)
  ► → for general tensors:
    \[ Y = \sum_{f=1}^{k} v_{1,f} \circ v_{2,f} \circ \ldots \circ v_{m,f} + E \]
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  ▶ \( \rightarrow \) for general tensors:
    \[ Y = \sum_{f=1}^{k} v_{1,f} \circ v_{2,f} \circ \ldots \circ v_{m,f} + E \]
Related Work

Existing Extensions of Factorization Models:

- More general likelihood for multinomial, count data, etc.
  → Exponential Family PCA\(^1\)

- Inference only on observed tensor entries

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- More general likelihood for multinomial, count data, etc.
  - Exponential Family PCA\(^1\)

- Inference only on observed tensor entries

- Introduction of prior distributions\(^2\)

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Related Work

**Existing Extensions of Factorization Models:**

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- Scalable Learning algorithms for large scale data: Gradient Descent Models\(^3\)

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**Missing Extensions:**

- Extensions of the predictive models, i.e. SVD, CD

- Comparison to standard predictive models

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Outline

Generalized Factorization Model

Relations to standard Models

Empirical Evaluation

Conclusion and Discussion
Outline

Generalized Factorization Model

Relations to standard Models

Empirical Evaluation

Conclusion and Discussion
Probabilistic SVD

Frobenius norm optimal:

\[
\argmin_{\theta = \{V_1, V_2\}} \|Y - V_1 V_2^T\|_F = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (y_{i,j} - v_{1,i} v_{2,j})^2
\]
Probabilistic SVD

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\[
\arg\min_{\theta=\{V_1, V_2\}} \|Y - V_1 V_2^T\|_F = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (y_{i,j} - v_{1,i}^T v_{2,j})^2
\]

\[
\begin{align*}
\begin{array}{c}
\bar{Y} \\
Y_{i,j}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
V_1 \\
V_{1,i} \cdot V_{2,j}
\end{array}
\end{align*}
\]
Probabilistic SVD

Frobenius norm optimal:

$$\arg\min_{\theta=\{V_1, V_2\}} \|Y - V_1 V_2^T\|_F = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (y_{i,j} - v_{1,i}^T v_{2,j})^2$$

$$y_{i,j} = v_{1,i}^T v_{2,j}$$

$$v_{1,i}^T v_{2,j} = \sum_{f=1}^{k} v_{1,i,f} v_{2,j,f}$$
Probabilistic SVD

**Frobenius norm optimal:**

$$\arg\min_{\theta=\{V_1, V_2\}} \| Y - V_1 V_2^T \|_F = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (y_{i,j} - v_{1,i}^T v_{2,j})^2$$

\[ \propto \text{Gaussian maximum likelihood:} \]

$$\arg\max_{\theta=\{V_1, V_2\}} \prod_{i=1}^{p_1} \prod_{j=1}^{p_2} \exp \left( -\frac{1}{2} (y_{i,j} - v_{1,i}^T v_{2,j})^2 \right)$$
Probabilistic CD - Extend SVD to arbitrary $m$

**Frobenius norm optimal:**

$$\arg\min_{\theta=\{V_1,V_2,\ldots,V_m\}} \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} \cdots \sum_{i_m=1}^{p_m} (y_{i_1,i_2,\ldots,i_m} - \sum_{f=1}^{k} v_{1,i_1,f} v_{2,i_2,f} \cdots v_{m,i_m,f})^2 \quad y^C_{i_1,i_2,\ldots,i_m}$$
Probabilistic CD - Extend SVD to arbitrary $m$

Frobenius norm optimal:

$$\arg\min_{\theta=\{V_1, V_2, \ldots, V_m\}} \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} \cdots \sum_{i_m=1}^{p_m} (y_{i_1, i_2, \ldots, i_m} - \sum_{f=1}^{k} v_{1, i_1, f} v_{2, i_2, f} \cdots v_{m, i_m, f})^2$$
Probabilistic CD - Extend SVD to arbitrary \( m \)

**Frobenius norm optimal:**

\[
\begin{align*}
\argmin_{\theta=\{V_1, V_2, \ldots, V_m\}} & \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} \cdots \sum_{i_m=1}^{p_m} (y_{i_1,i_2,\ldots,i_m} - \sum_{f=1}^{k} v_{1,i_1,f} v_{2,i_2,f} \cdots v_{m,i_m,f})^2 \\
\end{align*}
\]
Probabilistic CD - Extend SVD to arbitrary $m$

**Frobenius norm optimal:**

$$\arg\min_{\theta=\{V_1, V_2, \ldots, V_m\}} \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} \ldots \sum_{i_m=1}^{p_m} (y_{i_1, i_2, \ldots, i_m} - k \sum_{f=1}^{k} V_{i_1, f} V_{i_2, f} \cdots V_{i_m, f})^2$$

$\propto$ **Gaussian maximum likelihood:**

$$\arg\max_{\theta=\{V_1, V_2, \ldots\}} \prod_{i_1=1}^{p_1} \prod_{i_2=1}^{p_2} \ldots \prod_{i_m=1}^{p_m} \exp \left(-\frac{1}{2} (y_{i_1, i_2, \ldots, i_m} - y_{i_1, i_2, \ldots, i_m}^{CD})^2 \right)$$
GFM I: Predictive Model Interpretation

Adapt Notation:

- Introduce for each tensor element $\ell = 1, \ldots, n$ vector-valued indicator vectors $x_{\ell,j}$, $j = 1, \ldots, m$ of length $p_j$. 
GFM I: Predictive Model Interpretation

Adapt Notation:

- Introduce for each tensor element $\ell = 1, \ldots, n$ vector-valued indicator vectors $x_{\ell,j}, j = 1, \ldots, m$ of length $p_j$

- Form for each tensor element $y_{i_1, \ldots, i_m}$ a predictor vector $x_{\ell} \in \mathbb{R}^{p_1 + p_2 + \ldots + p_m}$ by concatenating $x_j$:

$$ x_{\ell}^{CD} = \begin{pmatrix} 0, \ldots, 1, \ldots, 0, 0, \ldots, 1, \ldots, 0, \ldots \end{pmatrix} $$

$$ = \begin{pmatrix} \overbrace{x_{\ell,1}}^{x_{\ell,1}}, \overbrace{x_{\ell,2}}^{x_{\ell,2}} \end{pmatrix} $$
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$$x_{\ell}^{CD} = \left( \begin{array}{c} 0, \ldots, 1, \ldots, 0, 0, \ldots, 1, \ldots, 0, \ldots \end{array} \right)$$

$$\quad = \left( \begin{array}{c} x_{\ell,1} \quad x_{\ell,2} \end{array} \right)$$

- Denote with $V = \{V_1, \ldots, V_m\}$ the set of all predictive model parameters
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- Denote with $V = \{V_1, \ldots, V_m\}$ the set of all predictive model parameters

- Rewrite $y_{i_1, \ldots, i_m}^{CD}$ as $y_{i_1, \ldots, i_m}^{CD} = y_{\ell}^{CD} = f(x_{\ell}^{CD} | V)$

- with general

$$f(x_{\ell} | V) = \sum_{f=1}^{k} \prod_{j=1}^{m} x_{\ell,j}^{T} v_{j,f} = \sum_{f=1}^{k} v_{1,i_1,f} v_{2,i_2,f} \cdots v_{m,i_m,f}$$
GFM I: Predictive Model Interpretation

More intuitive interpretation: 2-level hierarchical representation

\[ y_\ell \sim N(\mu_\ell, \alpha = 1) \]

\[ f(x_\ell | V) = \mu_\ell = \sum_{f=1}^{k} \prod_{j=1}^{m} x_{\ell,j}^T \mathbf{v}_{j,f} \mu_{j,f}(x_\ell) \]
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More intuitive interpretation: 2-level hierarchical representation

\[ y_\ell \sim N(\mu_\ell, \alpha = 1) \]

► each pair of mode \( j \) and latent dimension \( f \) has a different linear model
   \[ \mu_{j,f}(x_\ell) = x_{\ell,j}^T v_{j,f} \]
   with predictor vector \( x_{\ell,j} \) describing each mode \( j \) and \( p_j \) different model parameters \( v_{j,f} \) per mode \( j \) and latent dimension \( f \)
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- → factorization models are 2-level hierarchical (multi-)linear models with
  - \( \mu_{j,f}(x_\ell) \) per latent dimension \( f \) and mode \( j \) in the upper level
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- \( \rightarrow \) factorization models are 2-level hierarchical (multi-)linear models with
  - \( \mu_{j,f}(x_\ell) \) per latent dimension \( f \) and mode \( j \) in the upper level
  - modeled as linear functions of mode-dependent predictors \( x_{\ell,j} \)
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  - modeled as linear functions of mode-dependent predictors \( x_{\ell,j} \)
  - merged by the dot product \( \sum_{f=1}^k \prod_{j=1}^m \mu_{j,f}(x_\ell) \) to get \( f(x_\ell|V) \)
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- \( \rightarrow \) factorization models are 2-level hierarchical (multi-)linear models with
  - \( \mu_{j,f}(x_\ell) \) per latent dimension \( f \) and mode \( j \) in the upper level
  - modeled as linear functions of mode-dependent predictors \( x_{\ell,j} \)
  - merged by the dot product \( \sum_{f=1}^{k} \prod_{j=1}^{m} \mu_{j,f}(x_\ell) \)
  - and \( y_{\ell}^{CD} = f(x_{\ell}^{CD}|V) \) using \( x_{\ell}^{CD} \)
Important example: Matrix Factorization

Level 1:

- $\mu_{1,f}(x_{\ell}) = x_{\ell,1}^T v_{1,f}$ (k row means)
- $\mu_{2,f}(x_{\ell}) = x_{\ell,2}^T v_{2,f}$ (k column means)
Important example: Matrix Factorization

Level 1:
- $\mu_1,f(x_l) = x^T_{l,1} v_{1,f}$ (k row means)
- $\mu_2,f(x_l) = x^T_{l,2} v_{2,f}$ (k column means)

Level 2:
- $f(x_{CD}^l | V) = \sum_{f=1}^k \mu_1,f(x_l) \mu_2,f(x_l) = v^T_{1,f} v_{2,f}$
GFM II: Predictive Model Extension

From $x_{\ell}^{CD}$ to arbitrary $x_{\ell}$:

- $x_{\ell} = (x_{\ell,1}, \ldots, x_{\ell,m})$ still consists of $m$ subvectors for each mode
GFM II: Predictive Model Extension

From $x_{\ell}^{CD}$ to arbitrary $x_{\ell}$:

- $x_{\ell} = (x_{\ell,1}, \ldots, x_{\ell,m})$ still consists of $m$ subvectors for each mode
- While $x_{\ell,j} \in x_{\ell}^{CD}$ has exactly one active position, thus $p_j - 1$ zero values, e.g.,

$$
x_{\ell,1} = (1, 0, \ldots, 0) \quad \text{1st row}
$$

$$
x_{\ell,2} = (0, 1, 0, \ldots, 0) \quad \text{2nd column}
$$
GFM II: Predictive Model Extension

From $x^{CD}_\ell$ to arbitrary $x_\ell$:

- $x_\ell = (x_{\ell,1}, \ldots, x_{\ell,m})$ still consists of $m$ subvectors for each mode
- While $x_{\ell,j} \in x^{CD}_\ell$ has exactly one active position, thus $p_j - 1$ zero values, e.g.,

  $x_{\ell,1} = (1, 0, \ldots, 0)$  \quad $x_{\ell,2} = (0, 1, 0, \ldots, 0)$

- Mode-vectors $x_{\ell,j}$ may be defined arbitrarily, e.g., to take into account more complex conditional dependencies
Understanding induced dependencies:

**Overall independence assumption:** \( y_\ell | x_\ell \sim N(\mu_\ell = f(x_\ell | V), 1) \)

▶ **Simplest case SVD:** Correlation only on identical mode-indices, e.g. row or column index ↔ rows/columns are independent from other rows/columns
Understanding induced dependencies:

**Overall independence assumption:** \( y_{\ell} | x_{\ell} \sim N(\mu_{\ell} = f(x_{\ell}| V), 1) \)

- **Simplest case SVD:** Correlation only on identical mode-indices, e.g. row or column index \( \leftrightarrow \) rows/columns are independent from other rows/columns

\[
y_{i,j} = \mu_{1,f} x_{i,1} + \mu_{2,f} x_{i,2} + \epsilon_{\ell}
\]

- e.g. \( p_1 = 3 \) rows, \( p_2 = 8 \) columns:

\[
x_{\ell,1} = (1, 0, 0), \quad x_{\ell,2} = (0, 1, 0, 0, 0, 0, 0, 0)
\]
Creating new dependencies:

**Overall independence assumption:** \( y_\ell | x_\ell \sim N(\mu_\ell = f(x_\ell | V), 1) \)

- Dependencies **between** different rows/columns for SVD:
  More than one active indicator per mode, with \( p_1 = 3 \) rows, \( p_2 = 8 \) columns:

\[
\begin{align*}
  x_{\ell,1} &= (1, 0, 0), \\
  x_{\ell,2} &= (0, 1, 1, 0, 1, 0, 0, 0)
\end{align*}
\]

---

Creating new dependencies:

**Overall independence assumption:** \( y_\ell | x_\ell \sim N(\mu_\ell = f(x_\ell | V), 1) \)

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\[
x_{\ell,1} = (1, 0, 0), \quad x_{\ell,2} = (0, 1, 1, 0, 1, 0, 0, 0)
\]

\[
f(x_\ell | V) = \sum_{f=1}^{k} (x_{\ell,1}^T v_{1,f})(x_{\ell,2}^T v_{2,f}) = \sum_{f=1}^{k} v_{1,1,f} (v_{2,2,f} + v_{2,3,f} + v_{2,5,f})
\]

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4 Koren, Y.: Factorization meets the neighborhood: a multifaceted collaborative filtering model, KDD08.
Creating new dependencies:

**Overall independence assumption:** $y_{\ell} | x_{\ell} \sim N(\mu_{\ell} = f(x_{\ell} | V), 1)$

- **Dependencies between different rows/columns for SVD:**
  More than one active indicator per mode, with $p_1 = 3$ rows, $p_2 = 8$ columns:

  $x_{\ell,1} = (1, 0, 0), \ x_{\ell,2} = (0, 1, 1, 0, 1, 0, 0, 0)$

  $f(x_{\ell} | V) = \sum_{f=1}^{k} (x_{\ell,1}^T v_{1,f}) (x_{\ell,2}^T v_{2,f}) = \sum_{f=1}^{k} v_{1,1,f} (v_{2,2,f} + v_{2,3,f} + v_{2,5,f})$

- **Example from recommender systems:** SVD++$^4$

---

$^4$Koren, Y.: Factorization meets the neighborhood: a multifaceted collaborative filtering model, KDD08.
Creating new dependencies:

**Overall independence assumption:** \( y_{\ell | x_{\ell}} \sim N(\mu_{\ell} = f(x_{\ell}|V), 1) \)

- **Dependencies beyond rows and/or columns for SVD:**
  
  \( p_1 = 3 \) rows and \( p_2 = 8 \) columns + additional rating information:

  \[
  x_{\ell,1} = (1, 0, 0), \quad x_{\ell,2} = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
  \]

  **Example from recommender systems:** Factorized Transition Tensors

\[
f(x_{\ell}|V) = \sum_{f=1}^{k} (x_{\ell,1}^T v_{1,f})(x_{\ell,2}^T v_{2,f}) = \sum_{f=1}^{k} v_{1,1,f}(3 \cdot v_{2,3,f} + 4 \cdot v_{2,5,f})
\]

---

Creating new dependencies:

Overall independence assumption: \( y_\ell | x_\ell \sim N(\mu_\ell = f(x_\ell | V), 1) \)

- Dependencies beyond rows and/or columns for SVD:
  \( p_1 = 3 \) rows and \( p_2 = 8 \) columns + additional rating information:

  \[
  x_{\ell,1} = (1, 0, 0), \quad x_{\ell,2} = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 3, 0, 4, 0, 0, 0)
  \]

\[
\begin{array}{c}
  i \\
  \downarrow \\
  y_{i,j} \\
  \downarrow \\
  j
\end{array}
= \begin{array}{c}
  i \\
  \downarrow \\
  V_1 \\
  \downarrow \\
  j
\end{array}
\]

\[
f(x_\ell | V) = \sum_{f=1}^{k} (x_{\ell,1}^T v_{1,f})(x_{\ell,2}^T v_{2,f}) = \sum_{f=1}^{k} v_{1,1,f} (3 \cdot v_{2,3,f} + 4 \cdot v_{2,5,f})
\]

- Example from recommender systems: Factorized Transition Tensors\(^5\)

---

\(^5\) Rendle, S., Freudenthaler, C., Schmidt-Thieme, L.: Factorizing personalized Markov chains for next-basket
GFM II: Predictive Model Extension II

\[ f(x_\ell | V) = \mu_\ell = \sum_{f=1}^{k} \prod_{j=1}^{m} x_{\ell,j}^T v_{j,f} = \sum_{f=1}^{k} \prod_{j=1}^{m} \sum_{i=1}^{p_j} x_{\ell,j,i} v_{j,i,f} \]

- Only one predictive model per mode \( m \)
- Increase predictive accuracy: combine several reasonable predictive models per mode
GFM II: Predictive Model Extension II

\[
f(x_\ell | V) = \mu_\ell = \sum_{f=1}^{k} \prod_{j=1}^{m} x_{\ell,j}^T v_{j,f} = \sum_{f=1}^{k} \prod_{j=1}^{m} \sum_{i=1}^{p_j} x_{\ell,j,i} \cdot 1 \cdot v_{j,i,f}
\]

- Only one predictive model per mode \( m \)
- Increase predictive accuracy: combine several reasonable predictive models per mode
- Introduce mode-defining selection vectors \( d_j \in \{0, 1\}^p \) on \( x_\ell = (x_1, \ldots, x_m) \in \mathbb{R}^p \), \( j = 1, \ldots, m \)
- One set of selection vectors \( \{d_1, \ldots, d_m\} \) defines one model

\[
f(x_\ell | V) = \mu_\ell = \sum_{f=1}^{k} \prod_{j=1}^{m} \sum_{i=1}^{p} x_{\ell,i} d_{j,i} v_{i,f}
\]
GFM II: Predictive Model Extension II

\[ f(x_\ell|V) = \mu_\ell = \sum_{f=1}^{k} \prod_{j=1}^{m} x_{\ell,j}^T v_{j,f} = \sum_{f=1}^{k} \prod_{j=1}^{m} \sum_{i_{j}=1}^{p_{j}} x_{\ell,j,i_{j}} \cdot 1 \cdot v_{j,i_{j},f} \]

- Only one predictive model per mode \( m \)
- Increase predictive accuracy: combine several reasonable predictive models per mode

- Introduce mode-defining selection vectors \( d_{j} \in \{0, 1\}^{p} \) on \( x_{\ell} = (x_{1}, \ldots, x_{m}) \in \mathbb{R}^{p} \), \( j = 1, \ldots, m \)

- One set of selection vectors \( \{d_{1}, \ldots, d_{m}\} \) defines one model

\[ f(x_\ell|V) = \mu_\ell = \sum_{f=1}^{k} \prod_{j=1}^{m} \sum_{i=1}^{p} x_{\ell,i} d_{j,i} v_{i,f} \]

- Collect several selection sets \( \{d_{1}, \ldots, d_{m}\} \in \mathcal{D} \):

\[ f^{GFM}(x_\ell|V) = \mu_\ell = \sum_{f=1}^{k} \sum_{\{d_{1}, \ldots, d_{m}\} \in \mathcal{D}} \prod_{j=1}^{m} \sum_{i=1}^{p} x_{\ell,i} d_{j,i} v_{i,f} \]
GFM II: Predictive Model Learning?

\[ f^{GFM}(x_\ell | V) = \sum_{i_1=1}^{p} \cdots \sum_{i_m=1}^{p} x_\ell, i_1 \cdots x_\ell, i_m \sum_{D \in \mathcal{D}} \sum_{f=1}^{k} d_1, i_1 \cdots d_m, i_m \sum_{f=1}^{k} v_{i_1, f} \cdots v_{i_m, f} \]

- Infer model selection \( \mathcal{D} \rightarrow \) Bayesian model averaging
  + Also negative interaction effects of even order
    - Redundant parameterization
    - Expensive model prediction \( O(|D|kmp) \)
GFM II: Predictive Model Selection

- Specify $D$, i.e. select a specific model
GFM II: Predictive Model Selection

- Specify $D$, i.e. select a specific model

Extended Matrix Factorization

- $m = 2$ different modes, $p$ different mode-models $\rightarrow |D| = mp = 2p$

$$D = \left\{ \left\{ (d_{1,1}, \ldots, d_{1,p}), (d_{2,1}, \ldots, d_{2,p}) \right\} : d_{1,i_1} = 1, d_{2,i_2} = 1 \forall i_2 > i_1, 0 \text{ else} \right\},$$
GFM II: Predictive Model Selection

- Specify $\mathcal{D}$, i.e. select a specific model

Extended Matrix Factorization

- $m = 2$ different modes, $p$ different mode-models $\rightarrow |\mathcal{D}| = mp = 2p$

\[
\mathcal{D} = \left\{ \left\{ (d_{1,1}, \ldots, d_{1,p}), (d_{2,1}, \ldots, d_{2,p}) \right\} : d_{1,i_1} = 1, d_{2,i_2} = 1 \forall i_2 > i_1, 0 \text{ else} \right\},
\]

- ex.1: $d_1 = (1, 0, 0, \ldots, 0)$, $d_2 = (0, 1, 1, 1, \ldots, 1)$
- ex.2: $d_1 = (0, 1, 0, \ldots, 0)$, $d_2 = (0, 0, 1, 1, \ldots, 1)$

\[
f(x_\ell | V) = \mu_\ell = \sum_{i_1=1}^{p} \sum_{i_2>i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f}
\]
Selecting a prior distribution:

So far:

- $y_\ell \mid x_\ell \sim N(\mu_\ell = f(x_\ell \mid V), 1)$
- More general $x_\ell$ unifies recent factorization model enhancements
Selecting a prior distribution:

So far:

- $y_\ell|{x_\ell} \sim N(\mu_\ell = f({x_\ell}|V), 1)$
- More general $x_\ell$ unifies recent factorization model enhancements

Next step:

- Selecting a prior distribution for model parameters $V$, using prior knowledge:
  - If all tensor elements $\in \mathbb{R}^k \rightarrow k$ linearly independent real valued latent dimensions
Selecting a prior distribution:

So far:

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Next step:

- Selecting a prior distribution for model parameters \( V \), using prior knowledge:
  - If all tensor elements \( \in \mathbb{R}^k \rightarrow k \) linearly independent real valued latent dimensions
  - Each tensor element is a sum over \( k \) linear functions of the same \( x_\ell \)
  - Different variances \( \sigma_f^2 \) along \( k \) latent dimensions
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So far:

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  - Different variances $\sigma_f^2$ along $k$ latent dimensions
  - Centered at some dimension dependent mean $\mu_f$
Selecting a prior distribution:

So far:

- \( y_\ell | x_\ell \sim N(\mu_\ell = f(x_\ell | V), 1) \)
- More general \( x_\ell \) unifies recent factorization model enhancements

Next step:

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  - Each tensor element is a sum over \( k \) linear functions of the same \( x_\ell \)
  - Different variances \( \sigma_f^2 \) along \( k \) latent dimensions
  - Centered at some dimension dependent mean \( \mu_f \)
  - Center \( \mu_f \) and variance \( \sigma_f^2 \) may differ for different modes
Selecting a prior distribution:

→ **Conjugate Gaussian prior** distribution:
Each latent $k$-dim representation $\mathbf{v}_i \in \mathbb{R}^k$, $i = 1, \ldots, p$ is an independent draw

$$
\mathbf{v}_i \sim N((\mu_1, \ldots, \mu_k), \text{diag}(\sigma_1^2, \ldots, \sigma_k^2))
$$
Selecting a prior distribution:

→ **Conjugate Gaussian prior** distribution:
Each latent $k$-dim representation $\mathbf{v}_i \in \mathbb{R}^k$, $i = 1, \ldots, p$ is an independent draw

$$\mathbf{v}_i \sim N(\mathbf{\mu}, \text{diag}(\sigma_1^2, \ldots, \sigma_k^2))$$

→ **Conjugate Gaussian hyperprior**:
Each prior mean $\mu_f$ is considered an independent realization of

$$\mu_f \sim N(\mu_0, \frac{c}{\lambda_f})$$

→ **Conjugate Gamma hyperprior**:
Each precision $\lambda_f = \sigma_f^{-2}$ is considered an independent realization of

$$\lambda_f \sim G(\alpha_0, \beta_0)$$
Learning Generalized Factorization Models

▶ Stochastic Gradient Descent: scalable, simple
Learning Generalized Factorization Models

- Stochastic Gradient Descent: scalable, simple
- Alternating Least Squares
Learning Generalized Factorization Models

- Stochastic Gradient Descent: scalable, simple
- Alternating Least Squares
- Bayesian Analysis, i.e. standard Gibbs sampling:
  - Easy to derive for multi-linear models like

\[
f(x_\ell | V) = \sum_{i_1=1}^{p} \sum_{i_2>i_1} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f}
\]
Learning Generalized Factorization Models

- Stochastic Gradient Descent: scalable, simple
- Alternating Least Squares
- Bayesian Analysis, i.e. standard Gibbs sampling:
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    \[
    f(x_\ell|V) = \sum_{i_1=1}^{p} \sum_{i_2 > i_1} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f}
    \]
  - Learning on large datasets: block size = 1 → \( O(N_z k) \)
Outline

Generalized Factorization Model

Relations to standard Models

Empirical Evaluation

Conclusion and Discussion
## Polynomial Regression

\[
f(x_\ell | V) = \sum_{i_1=1}^{p} x_{\ell, i_1} \beta_{i_1} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell, i_1} x_{\ell, i_2} \beta_{i_1, i_2} + \ldots + \sum_{i_1=1}^{p} \ldots \sum_{i_o \geq i_{o-1}}^{p} x_{\ell, i_1} \ldots x_{\ell, i_1} \beta_{i_1, \ldots, i_o}
\]

- Order-\(o\) polynomial regression models
Polynomial Regression

\[ f(x_\ell|V) = \sum_{i_1=1}^{p} x_{\ell,i_1} \beta_{i_1} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \beta_{i_1,i_2} + \ldots + \sum_{i_1=1}^{p} \ldots \sum_{i_o \geq i_o-1}^{p} x_{\ell,i_1} \ldots x_{\ell,i_1} \beta_{i_1, \ldots, i_o} \]

- Order-\(o\) polynomial regression models

\[ f^{GFM}(x_\ell|V) = \sum_{f=1}^{k} \sum_{\{d_1, \ldots, d_m\} \in \mathcal{D}} \prod_{j=1}^{m} \sum_{i_1}^{p} x_{\ell,i} d_{j,i} v_{i,f} \]

- Generalized Factorization Model of \(m\)-mode tensor
Polynomial Regression

\[
f(x_\ell | V) = \sum_{i_1=1}^{p} x_{\ell, i_1} \beta_{i_1} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell, i_1} x_{\ell, i_2} \beta_{i_1, i_2} + \ldots + \sum_{i_1=1}^{p} \ldots \sum_{i_o \geq i_{o-1}}^{p} x_{\ell, i_1} \ldots x_{\ell, i_o} \beta_{i_1, \ldots, i_o}
\]

- Order-\( o \) polynomial regression models

\[
f^{GFM}(x_\ell | V) = \sum_{f=1}^{k} \sum_{\{d_1, \ldots, d_m\} \in D} \prod_{j=1}^{m} \sum_{i=1}^{p} x_{\ell, i} d_{j, i} v_{i, f}
\]

- Generalized Factorization Model of \( m \)-mode tensor
  - Define one of the \( p \) predictor variables as constant, e.g. \( x_{\ell, 1} = 1 \) \( \forall \ell = 1, \ldots, n \)
  - Fix corresponding parameter vector \( v_1 \in V = (1, \ldots, 1) \)

\[ k \]
Polynomial Regression

\[ f(x_\ell|V) = \sum_{i_1=1}^{p} x_{\ell,i_1} \beta_{i_1} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \beta_{i_1,i_2} + \ldots + \sum_{i_1=1}^{p} \ldots \sum_{i_o \geq i_{o-1}}^{p} x_{\ell,i_1} \ldots x_{\ell,i_1} \beta_{i_1,\ldots,i_o} \]

- Order-\(o\) polynomial regression models

\[ f^{GFM}(x_\ell|V) = \sum_{f=1}^{k} \sum_{\{d_1,\ldots,d_m\} \in \mathcal{D}} \prod_{j=1}^{m} \sum_{i=1}^{p} x_{\ell,i} d_{j,i} v_{i,f} \]

- Generalized Factorization Model of \(m\)-mode tensor
  - Define one of the \(p\) predictor variables as constant, e.g. \(x_{\ell,1} = 1, \forall \ell = 1,\ldots,n\)
  - Fix corresponding parameter vector \(v_1 \in V = (1,\ldots,1)_k\)
  - Selecting \(\mathcal{D}\) accordingly gives

\[ f^{GFM}(x_\ell|V) = \sum_{i_1=1}^{p} x_{\ell,i_1} \sum_{f=1}^{k} v_{i_1,f} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f} + \ldots + \sum_{i_1=1}^{p} \ldots \sum_{i_m \geq i_{m-1}}^{p} x_{\ell,i_1} \ldots x_{\ell,i_m} \sum_{f=1}^{k} v_{i_1,f} \ldots v_{i_m,f} \beta_{i_1,\ldots,i_m} \]

Christoph Freudenthaler

ISMLL, University of Hildesheim

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Polynomial Regression vs. GFM

Generalized Factorization Model include Polynomial Regression

- For factorized parameters, e.g. $\beta_{i_1, i_2} = \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f}$
- If number of modes $m$ equals order $o$
Factorization Machines\textsuperscript{6} vs. GFM

- Very similar to previous factorized polynomial regression model

\[
f^{GFM}(x_\ell | V) = \sum_{i_1=1}^{p} x_{\ell,i_1} \sum_{f=1}^{k} v_{i_1,f} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f} + \ldots
\]

\[
f^{FM}(x_\ell | V) = \sum_{i_1=1}^{p} x_{\ell,i_1} \beta_{i_1} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f} + \ldots
\]

\textsuperscript{6}Rendle, S.: Factorization Machines. ICDM10.
Factorization Machines\(^6\) vs. GFM

- Very similar to previous factorized polynomial regression model

\[
f^{GFM}(x_\ell | V) = \sum_{i_1=1}^{p} x_{\ell,i_1} \sum_{f=1}^{k} v_{i_1,f} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f} + \ldots
\]

\[
f^{FM}(x_\ell | V) = \sum_{i_1=1}^{p} x_{\ell,i_1} \beta_{i_1} + \sum_{i_1=1}^{p} \sum_{i_2 \geq i_1}^{p} x_{\ell,i_1} x_{\ell,i_2} \sum_{f=1}^{k} v_{i_1,f} v_{i_2,f} + \ldots
\]

- Factorization Machines using simple Gibbs are successfully applied in several challenges

\(^6\)Rendle, S.: Factorization Machines. ICDM10.
Neural Networks vs. GFM

\[ f(x_l, v_1) \]

\[ f(x_l, v_2) \]

\[ y_l \]

\[ w_1 \]

\[ w_2 \]

\[ x_{l,1}, x_{l,2}, x_{l,3}, x_{l,4} \]

\[ v_{1,1}, v_{1,2}, v_{3,1}, v_{4,1}, v_{4,2} \]
Neural Networks vs. GFM

- **Input**: predictor vector $\mathbf{x}_\ell \in \mathbb{R}^P$, $\ell = 1, \ldots, n$
Neural Networks vs. GFM

Input: predictor vector $\mathbf{x}_\ell \in \mathbb{R}^p$, $\ell = 1, \ldots, n$

Hidden: Hidden nodes correspond to latent dimensions $\rightarrow k$ nodes
Neural Networks vs. GFM

▶ **Input**: predictor vector $x_\ell \in \mathbb{R}^p$, $\ell = 1, \ldots, n$

▶ **Hidden**: Hidden nodes correspond to latent dimensions $\rightarrow k$ nodes

▶ **Activation Function**: $f(x_\ell, v_f) := \sum_{\{d_1, \ldots, d_m\} \in D} \prod_{j=1}^m \sum_{i=1}^p x_{\ell,i} d_{j,i} v_{i,f}$
Neural Networks vs. GFM

- **Input**: predictor vector $x_{\ell} \in \mathbb{R}^p$, $\ell = 1, \ldots, n$
- **Hidden**: Hidden nodes correspond to latent dimensions $\rightarrow k$ nodes
- **Activation Function**: $f(x_{\ell}, v_f) := \sum_{\{d_1, \ldots, d_m\} \in \mathcal{D}} \prod_{j=1}^{m} \sum_{i=1}^{p} x_{\ell,i} d_{j,i} v_{i,f}$
- **Output**: Weights $w_f = 1$
Outline

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Empirical Evaluation on Recommender System Data

- **Task**: movie rating prediction
- **Data**: 100M ratings of 480k users on 18k items

![Graph showing empirical evaluation results](image)

**Netflix**

**k (latent Dimensions)**

**RMSE**

- SGD KNN256
- SGD KNN256++
- BFM KNN256
- BFM KNN256++
- SGD PMF
- BPMF
- BFM (u,i)
- BFM (u,t,f)
- BFM (u,i,u,t,f,i,f)
- BFM Ensemble
Bayesian Factorization Machines - Kaggle

▶ **Task:** student performance prediction on GMAT (Graduate Management Admission Test), SAT and ACT (college admission test)

▶ **118** contestants

▶ **Data**

<table>
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<th>Capped Binomial Deviance</th>
<th>Entries</th>
<th>Last Submission UTC (Best Submission - Last)</th>
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Bayesian Factorization Machines - KDDCup’11

▶ **Task:** music rating prediction
▶ **≈ 1000** contestants
▶ **Data**\(^9\): 250M ratings of 1M users on 625k *items* (songs, tracks, album or artists)

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<tr>
<th>Rank</th>
<th>Team Name</th>
<th>Best Score (RMSE)</th>
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\(^9\)http://kddcup.yahoo.com/
Outline

Generalized Factorization Model

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Empirical Evaluation

Conclusion and Discussion
In a nutshell:

- GFM, thus matrix and tensor Factorization are types of regression models
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- GFM, thus matrix and tensor Factorization are types of regression models
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It's now safe to turn off your computer.