New Control Variates for Lévy Processes and Asian Options

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Outline

- Control variates for Lévy process models
  - Control variate framework
  - Option pricing examples

- Variance reduction for Asian options
  - A unified framework for non-Gaussian models
  - The proposed method is a combination of
    - Control Variate (CV)
    - Conditional Monte Carlo (CMC)
Monte Carlo (MC) Method: General Principles

- Estimation of an unknown parameter: $\mu = \mathbb{E}[Y]$
- Generation of iid sample $Y_1, Y_2, \ldots, Y_n$
- The estimator: $\hat{\mu} = \frac{\sum_{i=1}^{n} Y_i}{n}$
- To quantify the error $\hat{\mu} - \mu$:
  - Central Limit Theorem: $\frac{\hat{\mu} - \mu}{s_n/\sqrt{n}} \Rightarrow N(0, 1)$ as $n \to \infty$.
  - Probabilistic error bound: $\Phi^{-1}(1 - \alpha/2) \cdot s/\sqrt{n}$
  - To get smaller error bound:
    - Increase the sample size $n$ (Larger computational time) $O(1/\sqrt{n})$
    - Decrease the variance $s^2$ (Variance reduction techniques)
Problem definition

- Lévy process $\{L(t), t \geq 0\}$
  - Stationary and independent increments, and $L(0) = 0$

- Functional of $L$:
  - $q(L(t_1), \ldots, L(t_d))$
  - Time grid $0 = t_0 < t_1 < t_2 < \ldots < t_d$ with $t_j = j \Delta t$
  - $t_j = j \Delta t$ ⇒ increments $L(t_i) - L(t_{i-1})$ are iid

- Estimation of $E[q(L(t_1), \ldots, L(t_d))]$ by simulation.

- A new variance reduction method
Problem definition

- In the literature, there exist variance reduction methods suggested for Lévy processes
  - They are often special to the 'process type' or 'problem type'

- A new control variate (CV) method

- It can be applied for any Lévy process for which the probability density function (PDF) of the increments is available in closed form

- Numerical examples: path-dependent options
Control Variate Method

- Estimator: \( Y = q(L) - c^T (V - \mathbb{E}[V]) \)
  - \( V = (V_1, \ldots, V_m)^T \) set of CVs with known \( \mathbb{E}[V] \)
  - \( c = (c_1, \ldots, c_m)^T \) the coefficient vector (optimal \( c^* \) by linear regression)

- Successful if strong linear dependence: \( VRF = 1/(1 - R^2) \).

- Our CV framework:
  - Special CV, tailored to \( q() \)
  - General CVs, selected from a basket of CVs (not tailored to \( q() \))
Functional of a Brownian Motion (BM).

Brownian motion \( \{W(t), t \geq 0\} \) with parameters \( \{\mu, \sigma\} \):

- \( W(t) = \mu t + \sigma B(t) \)
- \( B(t) \) is a standard BM

Functional \( \zeta(W(t_1), \ldots, W(t_d)) \)

- Similar to the original function: \( \zeta \sim q \).

**Known expectation:** \( E[\zeta(W)] \) is available in closed form
similarity of paths: \((W(t_1), \ldots, W(t_d)) \sim (L(t_1), \ldots, L(t_d))\) and similarity of functions: \(\zeta \sim q\)

\[ \Rightarrow \text{Large correlation between } q(L) \text{ and } \zeta(W) \]

For similar paths,

- \(\mu = E[L(1)]\) and \(\sigma = \sqrt{\text{Var}(L(1))}\)

- Using CRN (common random numbers) for path simulation

Comonotonic increments lead to maximal correlation

- \(U \sim U(0, 1)\)

- \(L(t_i) - L(t_{i-1}) \leftarrow F_L^{-1}(U)\)

- \(W(t_i) - W(t_{i-1}) \leftarrow F_{BM}^{-1}(U)\)
Inverse CDFs:

- $F_{BM}^{-1}(U)$, Inverse CDF of normal distribution.
- $F_{L}^{-1}(U)$, non-tractable.

Approximation of $F_{L}^{-1}(U)$ by numerical inversion algorithm of Derflinger et al. (2010)

It requires **only** PDF (probability density function)

For many Lévy processes, PDF is available in closed form (while CDF and the inverse CDF are not).
General CVs

- Simple path characteristics of $L$ and $W$ (e.g. average, maximum)

- They are not tailored to $q()$

- We call them as 'general CVs' since they are applicable to any $q()$, whereas $\zeta(W)$ is called 'special CV' as it is designed considering the special properties of $q()$.

- Let $\gamma(W,L)$ be a function of the paths of $W$ and $L$ that evaluates the set of path characteristics.
Algorithm

Require: special CV function $\zeta()$, general CV function $\gamma()$

1: for $i = 1$ to $n$ do
2:     for $j = 1$ to $d$ do
3:         Generate uniform variate $U \sim U(0, 1)$.
4:         Set $X_j \leftarrow F^{-1}_L(U)$ and $Z_j \leftarrow F^{-1}_{BM}(U)$.
5:         Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
6:     end for
7:     Set $Y_i \leftarrow q(L) - c_1 (\zeta(W) - E[\zeta(W)]) - c_2^T (\gamma(W, L) - E[\gamma(W, L)])$.
8: end for
Algorithm

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5:     Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
6:   end for
7:   Set $Y_i \leftarrow q(L) - c_1 \left( \zeta(W) - E[\zeta(W)] \right) - c_2^T \gamma(W, L) - E[\gamma(W, L)]$
8: end for
Algorithm

**Require:** special CV function $\zeta()$, general CV function $\gamma()$

1: for $i = 1$ to $n$ do  
2: \hspace{1em} for $j = 1$ to $d$ do  
3: \hspace{2em} Generate uniform variate $U \sim U(0, 1)$.  
4: \hspace{2em} Set $X_j \leftarrow F_L^{-1}(U)$ and $Z_j \leftarrow F_{BM}^{-1}(U)$.  
5: \hspace{2em} Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$  
6: \hspace{1em} end for  
7: Set $Y_i \leftarrow q(L) - c_1 (\zeta(W) - E[\zeta(W)]) - c_2^T (\gamma(W, L) - E[\gamma(W, L)])$.  
8: end for
Algorithm

Require: special CV function $\zeta()$, general CV function $\gamma()$

1: for $i = 1$ to $n$ do
2:   for $j = 1$ to $d$ do
3:       Generate uniform variate $U \sim U(0, 1)$.
4:       Set $X_j \leftarrow F_L^{-1}(U)$ and $Z_j \leftarrow F_{BM}^{-1}(U)$.
5:       Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
6:     end for
7:     Set $Y_i \leftarrow q(L) - c_1 (\zeta(W) - \mathbb{E} [\zeta(W)]) - c_2^T (\gamma(W,L) - \mathbb{E} [\gamma(W,L)])$.
8: end for
CV Selection

- In algorithm, the user has to provide the CV functions $\zeta()$ and $\gamma()$.

- The selection of special CV $\zeta()$ depends on the problem type, as it is tailored to $q()$.

- Our approach for the selection of general CVs:
  - A large *basket* of CV candidates
  - Stepwise backward linear regression.
    - The $t$-statistics of regression coefficients: $t = \frac{\hat{\beta}}{s.e.(\hat{\beta})}$
    - Check the significance: $t \in (-5, 5)$?

- pilot simulation run
CV Selection

- Stepwise backward regression
  1. Start with a full regression model
  2. Remove the CV with the smallest absolute $t$ statistic from the model, if its value is smaller than 5
  3. Recompute the $t$-statistics of the remaining CVs for the new regression model
  4. Steps 2-3 are repeated until all absolute $t$ values $> 5$
  5. Use the remaining CVs for the main simulation

- Why not use all CVs in the basket?
  - Simulation or evaluation of expectation of some CVs can be expensive.
  - Backward regression automatically eliminates the CV if it is not useful.
Complexity

- A single regression with \( k \) covariates requires \( O(n_p k^2) \) operations
  - \( k \) number of CVs
  - \( n_p \) sample size of pilot simulation

- The worst case: All CVs are useless \( O(n_p k^3) \)

- Since \( n_p < n \), no substantial increase in the computational time
Basket of CVs

- Path characteristics of which the expectation is available in closed form.
- Not exhaustive and depends on our knowledge of the closed form solutions.
- No CV that require a numerical method to evaluate the expectation.

Our notation
- CVL: path characteristics of $L$ (internal CVs)
- CVW: path characteristics of $W$ (external CVs)
### Table: Basket of CVs.

<table>
<thead>
<tr>
<th>Label</th>
<th>CV</th>
<th>Label</th>
<th>CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVL1</td>
<td>$L(t_d)$</td>
<td>CVW1</td>
<td>$W(t_d)$</td>
</tr>
<tr>
<td>CVL2</td>
<td>$\exp(L(t_d))$</td>
<td>CVW2</td>
<td>$\exp(W(t_d))$</td>
</tr>
<tr>
<td>CVL3</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} L(t_i)$</td>
<td>CVW3</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} W(t_i)$</td>
</tr>
<tr>
<td>CVL4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} L(t_i)\right)$</td>
<td>CVW4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} W(t_i)\right)$</td>
</tr>
<tr>
<td>CVL5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(L(t_i))$</td>
<td>CVW5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(W(t_i))$</td>
</tr>
<tr>
<td></td>
<td>CVW6: $\max_{0 \leq i \leq d} W(t_i)$</td>
<td></td>
<td>CVW7: $\exp(\max_{0 \leq i \leq d} W(t_i))$</td>
</tr>
<tr>
<td></td>
<td>CVW8: $\sup_{0 \leq u \leq t_d} W(u)$</td>
<td></td>
<td>CVW9: $\exp(\sup_{0 \leq u \leq t_d} W(u))$</td>
</tr>
</tbody>
</table>
Basket of CVs

- All CVs in the basket are easy to simulate

- A bit more difficult CVs: \( \sup_{0\leq u\leq t_d} W(u) \) and \( e^{\sup_{0\leq u\leq t_d} W(u)} \)

- Simulation of \( \sup_{0\leq u\leq t_d} W(u) \) conditional on \((W(t_1), \ldots, W(t_d))\)

\[
\sup_{0\leq u\leq t_d} W(u) = \max_{1\leq i\leq d} \left( \sup_{t_{i-1}\leq u\leq t_i} W(u) \right).
\]

- generate the maxima of \( d \) Brownian bridges
Basket of CVs

- CDF of the maximum of a Brownian bridge

\[
P \left( \sup_{0 \leq u \leq t} W(u) \leq x \mid W(t) = y \right) = 1 - \exp \left( -\frac{2x(x-y)}{\sigma^2 t} \right),
\]

- Inversion

\[
x = 0.5 \left( y + \sqrt{y^2 - 2\sigma^2 t \log U} \right),
\]

where \( U \sim U(0,1) \) is a uniform random number

- \( \mathbb{E} \left[ \sup_{0 \leq u \leq t_d} W(u) \mid W(t_1), \ldots, W(t_d) \right] \) as alternative to \( \sup_{0 \leq u \leq t_d} W(u) \)

- requires numerical integration, not efficient
Basket of CVs

- CVs in the basket are strongly correlated with each other.

- *Multicollinearity:* It inflates the standard errors of the estimates of the regression coefficients.

- It can be a problem for the accuracy of the estimates of the $t$ statistics, when the sample size is too small.

- $n_p = 10^4$ is generally sufficient to get relatively stable estimates of the $t$ values.
## Basket of CVs: Expectation formulas

**Table:** Expectation formulas for the CVs depending on the terminal value and the averages.

<table>
<thead>
<tr>
<th>Label</th>
<th>CV</th>
<th>Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVL1</td>
<td>$L(t_d)$</td>
<td>$d E[X]$</td>
</tr>
<tr>
<td>CVL2</td>
<td>$\exp(L(t_d))$</td>
<td>$M_{\Delta t}(1)^d$</td>
</tr>
<tr>
<td>CVL3</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} L(t_i)$</td>
<td>$E[X](d + 1)/2$</td>
</tr>
<tr>
<td>CVL4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} L(t_i)\right)$</td>
<td>$\prod_{i=1}^{d} M_{\Delta t}(i/d)$</td>
</tr>
<tr>
<td>CVL5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(L(t_i))$</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} M_{\Delta t}(1)^i$</td>
</tr>
<tr>
<td>CVW1</td>
<td>$W(t_d)$</td>
<td>$d \mu \Delta t$</td>
</tr>
<tr>
<td>CVW2</td>
<td>$\exp(W(t_d))$</td>
<td>$e^{d(\mu \Delta t + \sigma^2 \Delta t/2)}$</td>
</tr>
<tr>
<td>CVW3</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} W(t_i)$</td>
<td>$\mu \Delta t(d + 1)/2$</td>
</tr>
<tr>
<td>CVW4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} W(t_i)\right)$</td>
<td>$\exp(\tilde{\mu} + \tilde{\sigma}^2/2)$</td>
</tr>
<tr>
<td>CVW5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(W(t_i))$</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} e^{i(\mu \Delta t + \sigma^2 \Delta t/2)}$</td>
</tr>
</tbody>
</table>
Basket of CVs: Expectation formulas

Table: Expectation formulas for the CVs depending on maximum.

<table>
<thead>
<tr>
<th>Label</th>
<th>CV</th>
<th>Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVW6</td>
<td>$\max_{0 \leq i \leq d} W(t_i)$</td>
<td>by Spitzer’s identity</td>
</tr>
<tr>
<td>CVW7</td>
<td>$\exp(\max_{0 \leq i \leq d} W(t_i))$</td>
<td>by Öhgren (2001)</td>
</tr>
<tr>
<td>CVW8</td>
<td>$\sup_{0 \leq u \leq t_d} W(u)$</td>
<td>e.g. Shreve (2004)</td>
</tr>
<tr>
<td>CVW9</td>
<td>$\exp(\sup_{0 \leq u \leq t_d} W(u))$</td>
<td>e.g. Shreve (2004)</td>
</tr>
</tbody>
</table>
A Simple Example

- We use only general CVs in the basket \textit{without} a special CV

\[ q(L) = \exp \left( \max_{0 \leq i \leq d} L(t_i) \right) \]

- \( L \) is a generalized hyperbolic (GH) process
  \begin{itemize}
  
  \item \( \Delta t = 1/250 \)
  
  \item \( \lambda = 1.5, \alpha = 189.3, \beta = -5.71, \delta = 0.0062, \mu = 0.001 \)
  
  \item the increment distribution is close to normal but has a higher kurtosis
  
  \end{itemize}

- Variance Reduction Factors (VRFs)
  \begin{itemize}
  
  \item For \( d = 5 \), VRF = 560
  
  \item For \( d = 50 \), VRF = 395
  
  \end{itemize}
Examples from Option Pricing

- Underlying stock: $S(t) = S(0)e^{L(t)}$, 
  - Non-normal logreturns with high kurtosis.

- Payoff of path dependent options: $\psi(S(t_1), \ldots, S(t_d))$

- Price
  
  \[ e^{-rt_d}E[\psi(S(t_1), \ldots, S(t_d))], \]

- $q(L) = \psi(S(0)e^{L})$
Examples from Option Pricing

- **Special CV:** a similar option with analytically available price under geometric Brownian Motion (GBM)
  - \( \zeta() \) corresponds to \( \psi_{CV}() \) payoff function of new option.
  - \( \{\tilde{S}(t), t \geq 0\} \) stock price under GBM:
    \[
    \tilde{S}(t) = \tilde{S}(0)e^{W(t)} = \tilde{S}(0)\exp((r - \sigma^2/2)t + \sigma B(t)).
    \]
  - We set \( \sigma = \sqrt{\text{Var}(L(1))} \) and \( \tilde{S}(0) = S(0) \)

- **General CVs:** Use the basket
Option Examples

- **Asian Option:** \( \psi_A(S) = \left( \frac{\sum_{i=1}^{d} S(t_i)}{d} - K \right)^+ \)

  Special CV: \( \psi_G(\tilde{S}) = \left( \left( \prod_{i=1}^{d} \tilde{S}(t_i) \right)^{1/d} - K \right)^+ \).

- **Lookback Option:** \( \psi_L(S) = \left( \max_{0 \leq i \leq d} S(t_i) - K \right)^+ \),

  Special CV: \( \psi_{LC}(\tilde{S}) = \left( \sup_{0 \leq u \leq t_d} \tilde{S}(u) - K \right)^+ \).
Numerical Results

Table: Results for Asian and lookback options under GH process with \( T = 1, \Delta t = 1/250, r = 0.05, S(0) = 100, n = 10^4 \). Error: 95% error bound.

<table>
<thead>
<tr>
<th>Option</th>
<th>( K )</th>
<th>Price</th>
<th>Error</th>
<th>VRF-A</th>
<th>VRF-G</th>
<th>VRF-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asian</td>
<td>90</td>
<td>12.239</td>
<td>0.004</td>
<td>1,743</td>
<td>185</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>4.912</td>
<td>0.005</td>
<td>530</td>
<td>51</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>1.240</td>
<td>0.006</td>
<td>121</td>
<td>13</td>
<td>40</td>
</tr>
<tr>
<td>Lookback</td>
<td>110</td>
<td>7.534</td>
<td>0.012</td>
<td>294</td>
<td>57</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>3.297</td>
<td>0.012</td>
<td>160</td>
<td>35</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>130</td>
<td>1.266</td>
<td>0.011</td>
<td>79</td>
<td>17</td>
<td>32</td>
</tr>
</tbody>
</table>

- VRF-A: VRF obtained by using all (significant) CVs,
- VRF-G: VRF obtained by using only general CVs (CVLs and CVWs),
- VRF-S: VRF obtained by using only special CV
Numerical Results

- Efficiency factor: \( EF = \frac{\sigma_N^2 t_N}{\sigma_{CV}^2 t_{CV}} \)
  - \( t_N \) and \( t_{CV} \) are the CPU times of naive simulation and CV method.

- In naive simulation, we used the subordination (the standard method in the literature).

- Asian option: \( t_N/t_{CV} = 1 \)

- Lookback option: \( t_N/t_{CV} = 0.7 \)

- time of the pilot simulation run is between 30% and 50% of the main simulation
Success of the method

- Proximity of increment (log-return) distribution to the normal distribution.

- Shape depends on $\Delta t$
  - $\Delta t \to \infty$, gets close to normal
  - $\Delta t \to 0$, very high kurtosis

- In option pricing,
  - $\Delta t = 1/4$, quarterly monitoring
  - $\Delta t = 1/12$, monthly monitoring
  - $\Delta t = 1/50$, weekly monitoring
  - $\Delta t = 1/250$, bold daily monitoring
  - $\Delta t \to 0$, continuous monitoring (not possible in practice)
### Asian option example for variance gamma (VG) process

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\Delta t$</th>
<th>Price</th>
<th>Error</th>
<th>VRF</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>1/4</td>
<td>31.562</td>
<td>0.004</td>
<td>2,966</td>
</tr>
<tr>
<td></td>
<td>1/12</td>
<td>31.156</td>
<td>0.004</td>
<td>2,582</td>
</tr>
<tr>
<td></td>
<td>1/50</td>
<td>31.002</td>
<td>0.006</td>
<td>894</td>
</tr>
<tr>
<td></td>
<td>1/250</td>
<td>30.975</td>
<td>0.019</td>
<td>87</td>
</tr>
<tr>
<td>100</td>
<td>1/4</td>
<td>5.903</td>
<td>0.003</td>
<td>1,949</td>
</tr>
<tr>
<td></td>
<td>1/12</td>
<td>5.229</td>
<td>0.003</td>
<td>1,815</td>
</tr>
<tr>
<td></td>
<td>1/50</td>
<td>4.972</td>
<td>0.005</td>
<td>795</td>
</tr>
<tr>
<td></td>
<td>1/250</td>
<td>4.912</td>
<td>0.015</td>
<td>72</td>
</tr>
<tr>
<td>130</td>
<td>1/4</td>
<td>0.082</td>
<td>0.002</td>
<td>114</td>
</tr>
<tr>
<td></td>
<td>1/12</td>
<td>0.034</td>
<td>0.001</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>1/50</td>
<td>0.025</td>
<td>0.001</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td>1/250</td>
<td>0.020</td>
<td>0.002</td>
<td>15</td>
</tr>
</tbody>
</table>

**Table:** Using 'special CV' for Asian VG options with $T = 1$ and different $\Delta t$'s; $n = 10,000$; Error: 95% error bound; VRF: variance reduction factor.
Conclusions

- A general control variate framework for the functionals of Lévy processes.

- The method exploits the strong correlation between the original Lévy process and an auxiliary Brownian motion
  - Numerical inversion of CDFs

- In the CV framework,
  - special control variates tailored to the functionals
  - general control variates selected from a large basket of control variate candidates

- In the application to path dependent options, we observe moderate to large variance reductions
Asian options

- Stock price process \( \{ S(t), t \geq 0 \} \)

Arithmetic average call option

\[
P_A(S) = \left( \frac{1}{d} \sum_{i=1}^{d} S(t_i) - K \right)^+
\]

- Time grid \( 0 = t_0 < t_1 < t_2 < \ldots < t_d = T \) with \( t_j = j \Delta t \)
- Option price: \( e^{-rT} \mathbb{E}[P_A(S)] \)

Geometric Brownian motion (GBM)

\[
S(t) = S(0) \exp \left\{ (r - \sigma^2/2) t + \sigma B(t) \right\}, \quad t \geq 0
\]

- No closed form solution for the price
Simulation of Asian options

- Efficient numerical methods under GBM
  - PDE based finite difference methods, e.g. Večeř (2001)
  - Approximations, e.g. Curran (1994); Lord (2006)

- Monte Carlo simulation
  - Advantage: Probabilistic error bound
  - Disadvantage: Slow convergence rate

- Variance reduction method
CVs for Asian options

- Classical CV method of Kemna and Vorst (1990)
  - Arithmetic and geometric averages:
    \[ A = \frac{1}{d} \sum_{i=1}^{d} S(t_i) \quad \text{and} \quad G = \left( \prod_{i=1}^{d} S(t_i) \right)^{1/d} \]
  - If \( S(t_i)'s \) are close to each other, then \( A \sim G \)
  - \( P_A = (A - K)^+ \sim (G - K)^+ = P_G \)
- \( E[P_G] \) is available in closed form under GBM
- Very successful, if \( \sigma \) and \( T \) are small
CVs for Asian options

- Lower bound $\mathbb{E}[(A - K) \mathbf{1}_{\{G > K\}}]$ suggested by Curran (1994):

$$
(A - K)^+ = (A - K)^+ \mathbf{1}_{\{G \leq K\}} + (A - K)^+ \mathbf{1}_{\{G > K\}}
$$

$$
= (A - K)^+ \mathbf{1}_{\{G \leq K\}} + (A - K) \mathbf{1}_{\{G > K\}},
$$

- New CV by Dingeç and Hörmann (2013)

$$
Y_{CV} = P_A - c (W - \mathbb{E}[W]),
$$

where $W = (A - K) \mathbf{1}_{\{G > K\}}$.

- $\mathbb{E}[W]$ is available in closed form under GBM
CVs for Asian options

- If we set $c = 1$, then $Y_{CV} = (A - K)^+ 1_{\{G \leq K\}} + E[W]$

- Conditional Monte Carlo (CMC) for $Y = (A - K)^+ 1_{\{G \leq K\}}$
  - New estimator as conditional expectation: $E[Y|Z] = \int Y \, dF(G)$
  - All variance due to $G$ is removed

- Algorithm in Dingeç and Hörmann (2013)
  - New CV + CMC + additional CVs
  - Larger VRF than the classical CV
  - Special to GBM
Non-Gaussian models

- Under GBM, log-returns are iid normals

- Observed facts
  - Non-normality of log-returns, Higher kurtosis, heavier tails than normal
  - Volatility clustering
    - Large absolute log-returns are followed by large absolute log-returns
    - Non-linear dependency between log-returns

- Alternative models to GBM
  - Lévy process, (i.i.d. log-returns)
  - Stochastic volatility models
  - Regime switching models
A unified framework for non-Gaussian models

- Three models
  - Generalized hyperbolic (GH) Lévy process (Prause, 1999)
  - Heston stochastic volatility (SV) model (Heston, 1993)
  - Regime switching (RS) model (Hardy, 2001)

- A unified framework
  - Stock price process \( S(t) = S(0)e^{X(t)} \)
  - Log-returns: \( \Delta X_i = X(t_i) - X(t_{i-1}) \)
Unified Framework

- The unified representation

\[ \Delta X_i = \Gamma_i + \Lambda_i Z_i, \quad i = 1, \ldots, d, \]

- \( \Gamma_i, \Lambda_i \)'s are modulated by stochastic process \( \{V(t), t \geq 0\} \)
- \( \Gamma = f_m(V) \) and \( \Lambda = f_v(V) \).
- \( Z_i \)'s are i.i.d. standard normal variables independent of \( V(t) \).
- \( (\Delta X_1, \ldots, \Delta X_d | V) \) is multivariate normal

- The variance process \( V(t) \)
  - GH Lévy: GIG process (subordinator)
  - Heston: CIR process
  - Regime switching: Discrete Time Markov Chain (DTMC)
Typical Control Variate Methods for the Unified Framework

- The standard CV approach mentioned in Glasserman (2004)

\[ Y_{CV} = P_A - c(\tilde{P}_G - E[\tilde{P}_G]). \]

where \( \tilde{P}_G \) denote the payoff of the geometric average option under GBM

- Using common \( Z \) to introduce correlation

- A more elaborate CV approach by Zhang (2011)

\[ Y_{CV} = P_A - c(P_G - E[P_G|V]). \]

- They do not reduce the variance due to \( V \)
New Control Variate Method

- CV of Dingeç and Hörmann (2013): \( W = (A - K) \mathbf{1}_{\{G > K\}} \)

- It will reduce the variance coming from both random variables \( Z \) and \( V \)

- Evaluation of the expectation \( \mu_W = E[W] \)
  - Lemmens et al. (2010) use \( \mu_W \) as lower bound for the price under Lévy processes
  - Our new observation:
    Formulas of Lemmens et al. (2010) can be used for any model allowing the computation of joint characteristic function (JCF) of the log-return vector \( \Delta X = (\Delta X_1, \ldots, \Delta X_d) \).
Expectation of the CV

- Formula of Lemmens et al. (2010) (after simplifications)

\[
\mu_W = \frac{g(0)}{2} - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega L} g(\omega) - g(0)}{\omega} d\omega,
\]

- \(g(\omega) = \frac{S(0)}{d} \sum_{j=1}^{d} \phi_{\bar{X},X_j}(\omega, -i) - K \phi_{\bar{X}}(\omega)\).

\(\phi_{\bar{X}}(\omega)\): CF of \(\bar{X} = \sum_{j=1}^{d} X(t_j) / d\)

\(\phi_{\bar{X},X_j}(\omega_1, \omega_2)\): bivariate CF of \(\bar{X}\) and \(X(t_j)\)

- Both CFs can be evaluated, if \(\phi_{\Delta X}(u) = E[e^{iu^T\Delta X}]\) is available
Expectation of the CV

- Formula of Lemmens et al. (2010) (after simplifications)

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- \( g(\omega) = \frac{S(0)}{d} \sum_{j=1}^{d} \varphi_{\bar{X},X_j}(\omega, -i) - K \varphi_{\bar{X}}(\omega). \)

\( \varphi_{\bar{X}}(\omega) \): CF of \( \bar{X} = \sum_{j=1}^{d} X(t_j)/d \)

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- Both CFs can be evaluated, if \( \varphi_{\Delta X}(u) = E[e^{iu^T \Delta X}] \) is available
Expectation of the CV

- Formulas for JCF: $\varphi_{\Delta X}(u) = \mathbb{E}[e^{iu^T \Delta X}]$

- Lévy process: only requires CF of i.i.d. increment

- Heston model: given by Rockinger and Semenova (2005) for affine jump diffusion models

- Regime switching model: it is possible to derive a simple recursion
Improving the CV Method by CMC

- By setting \( c = 1 \), we get
  \[
  Y_{CV} = Y + \mu_W \quad \text{with} \quad Y = (A - K)^+ 1_{\{G \leq K\}}.
  \]

- Conditional Monte Carlo (CMC) for \( Y = (A - K)^+ 1_{\{G \leq K\}} \)
  
  - Simulation output as a function of two random inputs \( Y = q(V, Z) \)
  
  - The idea: Simulation of standard multinormal vector \( Z \) in a specific direction \( \vartheta \in \mathbb{R}^d, ||\vartheta|| = 1 \) by the formula
    \[
    Z = \vartheta \Xi + (I_d - \vartheta \vartheta^T)Z', \quad \Xi \sim N(0, 1), \quad Z' \sim N(0, I_d),
    \]
    where \( I_d \) is \( d \times d \) identity matrix.
  
  - Select the direction depending on \( V \),
    \[
    \vartheta_i(V) = \frac{(d - i + 1)\Lambda_i}{\sqrt{\sum_{j=1}^{d}(d - j + 1)^2\Lambda_j^2}}, \quad i = 1, \ldots, d,
    \]
    where \( \Lambda = f_V(V) \).
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$$\vartheta_i(V) = \frac{(d - i + 1)\Lambda_i}{\sqrt{\sum_{j=1}^{d} (d - j + 1)^2 \Lambda_i^2}}, \quad i = 1, \ldots, d, \quad (2)$$

  where $\Lambda = f_v(V)$. 

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Improving the CV Method by CMC

By setting $c = 1$, we get

$$ Y_{CV} = Y + \mu_W $$

with

$$ Y = (A - K)^+ \mathbf{1}_{\{G \leq K\}}. $$

Conditional Monte Carlo (CMC) for $Y = (A - K)^+ \mathbf{1}_{\{G \leq K\}}$

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Dingec, Hoermann, Sak (IE-BOUN) New CVs for Levy processes June 20, 2013 43 / 50
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Conditional Monte Carlo

- $Y = q(V, \Xi, Z')$

- Use $E[Y|Z', V]$ as an estimator.

$$E[Y|Z' = z, V = v] = \frac{1}{d} \sum_{i=1}^{d} s_i(z, v) e^{a_i(v)^2/2} \left[ \Phi(k(v) - a_i(v)) - \Phi(b(z, v) - a_i(v)) \right] - K \left[ \Phi(k(v)) - \Phi(b(z, v)) \right].$$

- $\Phi()$: CDF of std. normal dist.

- $b(z, v)$ is the root of equation, $A(x) - K = 0$, which is found by Newton’s method.
Algorithm

1: Compute $\mu_W$
2: for $i = 1$ to $n$ do
3:     Simulate a variance path $V$
4:     Simulate $Z' \sim N(0, I_d)$
5:     Compute $E[Y|Z', V]$
6:     Set $Y_i \leftarrow e^{-rT}(E[Y|Z', V] + \mu_W)$
7: end for
8: return $\bar{Y}$ and the error bound $\Phi^{-1}(1 - \alpha/2) \cdot s/\sqrt{n}$.

- Up to 10 times slower than naive simulation
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## Numerical Results

<table>
<thead>
<tr>
<th>Model</th>
<th>$T$</th>
<th>$K$</th>
<th>Price</th>
<th>Error</th>
<th>VRF</th>
</tr>
</thead>
<tbody>
<tr>
<td>GH ($\Delta t = 1/250$)</td>
<td>1</td>
<td>90</td>
<td>12.23708</td>
<td>0.00002</td>
<td>$8.8 \times 10^7$</td>
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<tr>
<td></td>
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<tr>
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<td>110</td>
<td>1.24135</td>
<td>0.00004</td>
<td>$2.8 \times 10^6$</td>
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<tr>
<td></td>
<td>2</td>
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<td>14.38544</td>
<td>0.00004</td>
<td>$3.0 \times 10^7$</td>
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<tr>
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<tr>
<td>Heston SV ($\Delta t = 1/12$)</td>
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<td>90</td>
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<td>110</td>
<td>2.19650</td>
<td>0.00005</td>
<td>$2.5 \times 10^6$</td>
</tr>
<tr>
<td>RS ($\Delta t = 1/12$)</td>
<td>1</td>
<td>90</td>
<td>12.46569</td>
<td>0.00004</td>
<td>$1.9 \times 10^7$</td>
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<td>3.11583</td>
<td>0.00010</td>
<td>$1.7 \times 10^6$</td>
</tr>
</tbody>
</table>

**Table:** Variance reduction factors (VRF) compared to naive simulation. $S(0) = 100, r = 0.05, n = 10^4$
Conclusions

- A new efficient simulation method developed for Asian option pricing under a general model framework
  - GH Lévy process
  - Heston stochastic volatility model
  - Discrete-time regime switching model

- Combination of CV and CMC
  - CV is applicable to all models in which the numerical computation of JCF of the log-return vector is possible
  - CMC is applicable to all models having normal mean-variance mixture representation

- Numerical results show significant variance reduction compared to naive simulation
References I


Thank You