Expert Opinions and Dynamic Portfolio Optimization Under Partial Information

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Agenda

1. Dynamic Portfolio Optimization Under Partial Information
2. Expert Opinions
3. Maximizing Log-Utility in a Model with Gaussian Drift
4. Maximizing Power Utility in an HMM Model
Introduction

Initial capital $x_0 > 0$

Horizon $[0, T]$

Aim Maximize expected utility of terminal wealth

Problem Find an optimal investment strategy

How many shares of which asset

have to be held at which time by the portfolio manager?

Market model continuously tradable assets

drift depends on unobservable factor process

investor only observes stock prices and expert opinions
Financial Market with Partial Information

\((\Omega, \mathcal{G} = (\mathcal{G}_t)_{t \in [0,T]}, P)\) filtered probability space

**Money market** with interest rate 0

**Stock market** prices \(S_t = (S^1_t, \ldots, S^n_t)^\top\), returns \(dR^i_t = dS^i_t / S^i_t\)

\(dR_t = \mu_t \, dt + \sigma \, dW^R_t\)

\(W^R = (W^R_t)_{t \in [0,T]}\) \(n\)-dimensional Brownian motion

\(\mu = (\mu_t)_{t \in [0,T]}\) stochastic drift, independent on \(W^R\)

\(\sigma\) volatility, non-singular

**Information**

investor filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]} \subset \mathcal{G}\)

classical situation \(\mathcal{F}_t = \mathcal{F}^R_t = \mathcal{F}^S_t \subset \mathcal{F}^{WR}, \mu \subset \mathcal{G}\)

filtering with observation \(R\) and signal \(\mu\)

we may also consider additional information leading to \(\mathcal{F}^H_t \subset \mathcal{G}\)

optimal strategies depend on filter \(\hat{\mu}^H_t = E[\mu_t | \mathcal{F}^H_t]\)

and its dynamics
Trading and Portfolio Optimization

- $X_t$ wealth (portfolio value) at time $t$
- $\pi = (\pi_t)_{t \in [0, T]}$ trading strategy
  - $\pi^i_t$ is fraction of wealth $X_t$ invested in stock $i$
  - $\pi$ has to be $\mathbb{F}^H$-adapted
- Wealth $X_t = X^\pi_t$ is controlled by $\pi$ and satisfies
  \[ dX_t = X_t \pi_t^T dR_t = X_t \pi_t^T (\mu_t dt + \sigma dW_t^R), \quad X_0 = x_0 \]
- Evaluation of terminal wealth with utility function $U$, e.g.
  \[ U_\theta(x) = \frac{x^\theta}{\theta}, \quad \theta < 1, \quad \theta \neq 0, \quad \text{or} \quad U_0(x) = \log(x) \]
- Stochastic control problem: maximize expected utility
  \[ E[U(X^\pi_T)] \quad \text{over admissible strategies } \pi \quad \text{for } x_0 > 0 \]
Optimal Strategies in Special Cases

- For constant $\mu$: $\pi_t^* = \frac{1}{1-\theta} (\sigma \sigma^\top)^{-1} \mu = \text{const}$  
  Merton strategy

- For stochastic $\mu$, information $\mathbb{F}^H$ and $U = U_0 = \log$ optimal strategy is obtained by substituting filter $\hat{\mu}_t^H$ for $\mu$ (Certainty equivalence principle)
  
  ▶ Proof: (for $n = 1$ and $x_0 = 1$):
  
  $$\log X_t^\pi = \int_0^T \left( \pi_t \mu_t - \frac{1}{2} (\sigma \pi_t)^2 \right) dt + \int_0^T \pi_t \sigma dW_t^R$$

  ▶ For $\mathbb{F}^H$-adapted $\pi$ we obtain
  
  $$E[\log X_T^\pi] = \int_0^T E \left[ \pi_t E \left[ \mu_t \mid \mathcal{F}_t^H \right] - \frac{1}{2} (\sigma \pi_t)^2 \right] dt + 0$$
  
  $$= \int_0^T E \left[ \pi_t \hat{\mu}_t^H - \frac{1}{2} (\sigma \pi_t)^2 \right] dt$$

  ▶ Pointwise maximization yields $\pi_t^* = \sigma^{-2} \hat{\mu}_t^H$

- In general we expect a dependency of $\pi_t^*$ on filter $\hat{\mu}_t^H$ and its dynamics
Drift Models

- **Bayesian Model**: $\mu$ is random but time-independent
  KARATZAS/XUE (1991)

- **Linear Gaussian Model (LGM)** or Kim-Omberg model

  $$d\mu_t = \kappa(\bar{\mu} - \mu_t)dt + \delta dW_t^\mu$$

  leads to Kalman filter

- **Hidden Markov Model (HMM)**

  $\mu$ as a continuous-time Markov chain

  leads to Wonham or HMM filter
Linear Gaussian Model (LGM)

Drift is a Gaussian mean-reversion (Ornstein-Uhlenbeck) process

\[ d\mu_t = \kappa(\bar{\mu} - \mu_t)dt + \delta dW^\mu_t \]

where \( W^\mu_t \) is a Brownian motion (in)dependent of \( W^R_t \)

Closed-form solution available

Stationary distribution for \( t \to \infty \) is \( \mathcal{N}(\bar{\mu}, \frac{\delta^2}{2\kappa}) \)
Hidden Markov Model (HMM)

Drift $\mu_t = \mu(Y_t)$ is a finite-state Markov chain, independent of $W_t^R$

$Y$ has state space $\{e_1, \ldots, e_d\}$, unit vectors in $\mathbb{R}^d$

$\mu(Y_t) = MY_t$ where $M = (\mu_1, \ldots, \mu_d)$ contains states of drift
generator or rate matrix $Q \in \mathbb{R}^{d \times d}$

diagonal: $Q_{kk} = -\lambda_k$ exponential rate of leaving state $k$

conditional transition prob. $P(Y_t = e_l \mid Y_{t-} = k, Y_t \neq Y_{t-}) = Q_{kl}/\lambda_k$

initial distribution $(\rho^1, \ldots, \rho^d)^\top$
Expert Opinions

- Motivation: static Black-Litterman model
  Practitioners apply Bayesian updating to combine subjective views (such as “asset 1 will grow by 5%”) with empirical or implied drift estimates.

- Present paper includes such views or expert opinions into dynamic models with partial observation.

- Investor receives noisy signals about current drift
  - at fixed and known points in time e.g. analysts, company reports
  - at random (unknown) points in time e.g. news, ratings
timing does not carry any useful information
  - jump times of a Poisson process
  - in continuous time (limiting case)
Discrete-Time Expert Opinions

- At times $T_k$ investor observes r.v. $Z_n \in \mathcal{Z}$ (views)
- $Z_k$ depends on current drift $\mu_{T_k}$, density $f(z, \mu_{T_n})$ ($Z_k$) cond. independent given $\mathcal{F}_T^\mu = \sigma(\mu_s : s \in [0, T])$

Examples
- **Absolute view:** $Z_k = \mu_{T_k} + \sqrt{\Gamma_k} \varepsilon_k$, $(\varepsilon_k)$ i.i.d. $N(0, 1)$
  The view “$S$ will grow by 5%” is modelled by $Z_k = 0.05$
  $\Gamma_k$ models confidence of expert

- **Relative view (2 assets):** $Z_k = \mu_{T_k}^1 - \mu_{T_k}^2 + \sqrt{\Gamma_k} \varepsilon_k$
  $T_k = t_k$ fixed or $T_k$ jump times of a Poisson process with intensity $\lambda$
  $\Rightarrow$ marked point process
Literature on Expert Opinions

- Discrete-time expert opinions

- Continuous-time expert opinions
  - LGM: DAVIS/LLEO (2013)
  - HMM: SEIFRIED/SASS/W. (working paper)

- Diffusion approximations
Maximizing Log-Utility in a Model with Gaussian Drift

- We consider one stock \((n = 1)\) with

\[
\begin{align*}
\text{returns} & \quad dR_t = \mu_t dt + \sigma dW_t^R \\
\text{and drift} & \quad d\mu_t = \kappa (\mu - \mu_t) dt + \delta dW_t^\mu, \quad \mu_0 \sim \mathcal{N}(m_0, \eta_0)
\end{align*}
\]

- N expert opinions arrive at fixed times \(0 = t_0 < t_1 < \ldots < t_{N-1} < T\). Views are modeled as Gaussian unbiased estimates \(Z_k\) of the current drift. 

\[
Z_k = \mu_{tk} + \sqrt{\Gamma_k} \varepsilon_k, \quad \text{for i.i.d. } \varepsilon_1, \ldots, \varepsilon_N \sim \mathcal{N}(0, 1).
\]

\(\Gamma_k > 0\) describes the confidence of the expert.

- We distinguish four information regimes for the investor

\[
\begin{align*}
\mathbb{F}_R & \quad \text{observing returns only} \\
\mathbb{F}_E & \quad \text{expert opinions only} \\
\mathbb{F}_C & \quad \text{both returns and expert opinions} \\
\mathbb{F}_F & \quad \text{having full information}
\end{align*}
\]

- We have to compute filters 

\[
\hat{\mu}_t^H = E[\mu_t|\mathcal{F}_t^H]
\]

and conditional variances 

\[
q_t^H = E[(\mu_t - \hat{\mu}_t^H)^2|\mathcal{F}_t^H] \quad \text{for } H = R, E, C, F.
\]
Filtering: Returns Only \((H = R)\)

- For \(H = R\), we are in the classical Kalman filter case and get

\[
d\hat{\mu}_t^R = \kappa(\bar{\mu} - \hat{\mu}_t^R)dt + \sigma^{-2}q_t^R(dR_t - \hat{\mu}_t^R dt), \quad \hat{\mu}_0^R = m_0,
\]

and deterministic conditional variance satisfying the Riccati equation

\[
\frac{dq_t^R}{dt} = \delta^2 - 2\kappa q_t^R - \sigma^{-2}(q_t^R)^2, \quad q_0^R = \eta_0.
\]

For \(n = 1\) we have a closed-form solution for \(q_t^R\).

- For \(t \to \infty\) we have \(q_t^R \to q_\infty^R := \kappa \sigma^2\left(\sqrt{1 + \left(\frac{\delta}{\kappa \sigma}\right)^2} - 1\right)\)

- \(q_t^R\) is decreasing if \(\eta_0 > q_\infty^R\) and increasing if \(\eta_0 < q_\infty^R\).
Filtering: Returns and Expert Opinions ($H = C$)

- Between the information dates the filter and the conditional variance evolve as in regime $H = R$ (Kalman filter).

- At the information dates $t_k$ the expert opinion $Z_k \sim \mathcal{N}(\mu_{t_k}, \Gamma_k)$ leads to a Bayesian update

\[
\begin{align*}
\hat{\mu}_{t_k}^C &= \rho_k \hat{\mu}_{t_k-}^C + (1 - \rho_k) Z_n \\
q_{t_k}^C &= \rho_k q_{t_k-}^C
\end{align*}
\]

with the factor

\[
\rho_k = \frac{\Gamma_k}{\Gamma_k + q_{t_k-}} \in (0, 1)
\]

- For $H = E$ we get corresponding updating formulas and between the information dates we can consider the limiting case $\sigma = \infty$ in regime $H = C$.

- For $H = F$ we have full information and thus $\hat{\mu}_t^F = \mu_t$ and $q_t^F = 0$. 
Example: Filter $\hat{\mu}_t^H$
Example: Filter $\hat{\mu}_t^H$
Example: Conditional Variance $q_t^H$

$$
\mu_0 \sim \mathcal{N}(\mu_\nu, \frac{\delta^2}{2\kappa})
$$

$\mu_0$ known ($\eta_0 = 0$)

$q_0^R = \frac{\delta^2}{2\kappa}$

$q_0^C = q_0^E$

$t_0, t_1, t_2, \ldots, t_{N-1}, T$
Properties of Conditional Variance

- The conditional variances in all cases \( H = R, E, C, F \) are deterministic, thus

\[
q_t^H = E[(\mu_t - \hat{\mu}_t^H)^2 | \mathcal{F}_t^H] = E[(\mu_t - \hat{\mu}_t^H)^2]
\]

\[
= E[\mu_t^2] - E[(\hat{\mu}_t^H)^2] = \text{Var}(\mu_t) - \text{Var}(\hat{\mu}_t^H)
\]

- Small \( q_t^H \) means that the filter is close to the true state.

Therefore \( q_t^H \) may serve as good performance measure.

- \( q_t^C \leq q_t^R \) and \( q_t^C \leq q_t^E \)
Asymptotics of Conditional Variance for \( t \to \infty \)

- Let \( T = \infty \)
- \( q_t^R \to q_R^\infty \) for \( t \to \infty \)
- For equidistant expert opinions with \( \Gamma_k = \Gamma > 0 \) we get for \( H = E, C \)

\[
\limsup_{t \to \infty} q_t^H = U^H \quad \text{and} \quad \liminf_{t \to \infty} q_t^H = L^H,
\]

where \( U^H > L^H > 0 \) can be computed explicitly.
Asymptotics of Conditional Variance for $N \to \infty$

- Let $T < \infty$ fixed
- Expert opinions arrive more frequent and have some minimum confidence:
  For $N \to \infty$ and $H = E, C$ it holds $q_t^{H,N} \to q_t^F = 0$ (full info, LLN)
- Expert opinions arrive more frequent and become ”less confident”:
  Consider equidistant expert opinions with $t_k = k \Delta_N$ with $\Delta_N = T/N$
  $\Gamma_k = \Gamma = \frac{\sigma_J^2}{\Delta_N}$ with $\sigma_J > 0$
- Diffusion process $dJ_t = \mu_t dt + \sigma_J dW_t^J$ models ”continuous-time expert”
  An estimator for the drift in $[t_k, t_k + \Delta_N]$ is $Z_k = \frac{1}{\Delta_N} (J_{t_k + \Delta_N} - J_{t_k})$
  For constant $\mu_t = \mu_{t_k}$ we have $Z_k \sim \mathcal{N}(\mu_{t_k}, \sigma_J^2/\Delta_N)$
- Let $\hat{\mu}_t^J$ and $q_t^J$ denote Kalman filter and cond. variance from observing $J$

**Diffusion approximation**

For $N \to \infty$ it holds

$|q_{t}^{E,N} - q_{t}^{J}| \to 0$ uniformly for all $t \in [0, T]$

$\int_0^T E[ |\hat{\mu}_{t}^{E,N} - \hat{\mu}_{t}^{J}|^2 ]dt \to 0$
Example: Diffusion approximation

Diffusion approximation

For $N \to \infty$ it holds

$$|q_{t}^{E,N} - q_{t}^{J}| \to 0$$

uniformly for all $t \in [0, T]$

$$\int_{0}^{T} E[ |\hat{\mu}_{t}^{E,N} - \hat{\mu}_{t}^{J}|^2 ] dt \to 0$$

Conditional variances $q_{t}^{E,N}$ and $q_{t}^{J}$
Optimal Expected Logarithmic Utility

\[ V(x_0) = \sup_{\pi \in \mathcal{A}^H} E[\log X_T^\pi] \]

Already known: for \( U = U_0 = \log \) and information \( \mathbb{F}^H \) the optimal strategy is

\[ \pi_t^* = \sigma^{-2} \hat{\mu}_t^H \]

Therefore,

\[ V^H(x_0) = E[\log X_T^{\pi_t^*}] = \log x_0 + E \left[ \int_0^T \left( \pi_t^* \hat{\mu}_t^H - \frac{1}{2} (\sigma \pi_t^*)^2 \right) dt \right] \]

\[ = \log x_0 + \frac{1}{2 \sigma^2} \int_0^T E \left[ \left( \hat{\mu}_t^H \right)^2 \right] dt - \int_0^T E[\mu_t^2] - q_t^H \]

Theorem (Gabih/Kondakji/Sass/W. 2014)

\[ V^H(x_0) = \log x_0 + \frac{1}{2 \sigma^2} \left( \int_0^T E[\mu_t^2] dt - \int_0^T q_t^H dt \right) \]

where the integrals (in all four cases) can be computed explicitly.

Properties derived for \( q_t^H \) allow to derive corresponding properties for \( V^H(x_0) \).
Efficiency

We want to quantify the monetary value of the information.

<table>
<thead>
<tr>
<th>Investor</th>
<th>$F$</th>
<th>$H$ (= $R, E, C$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Information (observations)</td>
<td>$F^F$</td>
<td>$F^H$</td>
</tr>
<tr>
<td>Initial capital</td>
<td>$x_0^F = 1$</td>
<td>$x_0^H$</td>
</tr>
<tr>
<td>Opt. terminal wealth</td>
<td>$X_T^F$</td>
<td>$X_T^H$</td>
</tr>
</tbody>
</table>

How much initial capital $x_0^H$ needs $H$ to obtain the same expected utility as $F$?

Solve $V^F(1) = V^H(x_0^H):$

$$x_0^H = \exp \left( \frac{1}{2\sigma^2} \int_0^T q_t^H \, dt \right)$$

Loss of information $x_0^H - 1$

Efficiency $q^H = 1/x_0^H$

for (non-fully informed) $H$-investor.
### Efficiency: Numerical Example

Value $V^H(1)$ and efficiency $\varrho^H$ in % for various numbers $N$

<table>
<thead>
<tr>
<th>$R$</th>
<th>$V^H(1)$</th>
<th>$\varrho^H$</th>
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</thead>
<tbody>
<tr>
<td>0.3213</td>
<td>35.63</td>
<td></td>
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</tbody>
</table>

<table>
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<tr>
<th>$N$</th>
<th>$E$</th>
<th>$C$</th>
<th>$E$</th>
<th>$C$</th>
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</thead>
<tbody>
<tr>
<td>10</td>
<td>0.5208</td>
<td>0.6008</td>
<td>43.49</td>
<td>47.12</td>
</tr>
<tr>
<td>100</td>
<td>0.9957</td>
<td>1.0017</td>
<td>69.94</td>
<td>70.36</td>
</tr>
<tr>
<td>1,000</td>
<td>1.2297</td>
<td>1.2299</td>
<td>88.37</td>
<td>88.39</td>
</tr>
<tr>
<td>10,000</td>
<td>1.3134</td>
<td>1.3134</td>
<td>96.09</td>
<td>96.09</td>
</tr>
<tr>
<td>100,000</td>
<td>1.3407</td>
<td>1.3407</td>
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<td>98.74</td>
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<tr>
<td>10,000,000</td>
<td>1.3521</td>
<td>1.3521</td>
<td>99.87</td>
<td>99.87</td>
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<tr>
<td>1.3533</td>
<td>100.00</td>
<td></td>
</tr>
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</table>
Maximizing Power Utility in an HMM Model

- We consider \( n \) stocks with returns \( dR_t = \mu(Y_t)dt + \sigma dW_t^R \) and drift driven by a finite-state Markov chain \( Y \).
- Expert opinions arrive at jump times of a Poisson process with intensity \( \lambda \) and modeled by a marked point process \((T_k, Z_k)\).
- \( Z_n \) depends on current state \( Y_{T_n} \), density \( f(z, Y_{T_n}) \) \((Z_n)\) cond. independent given \( \mathcal{F}_T^Y = \sigma(Y_s : s \in [0, T]) \).
- We are mainly interested in the information regimes
  \( \mathcal{F}^R \) observing returns only
  \( \mathcal{F}^C \) both returns and expert opinions

and want to maximize \( E[U(X^T_{\pi})] \) for power utility \( U_\theta(x) = \frac{x^\theta}{\theta}, \ \theta < 1, \ \theta \neq 0 \).

- For \( H = F \) (full info) the problem is solved in Rieder & Bäuerle (2004)
HMM Filtering: Returns Only ($H = R$)

Returns: \[ dR_t = \frac{dS_t}{S_t} = \mu(Y_t) \, dt + \sigma \, dW_t \] observations

Drift: \[ \mu(Y_t) = MY_t \] non-observable (hidden) state

Investor filtration: \[ \mathbb{F}^R = (\mathcal{F}^R_t)_{t \in [0,T]} \quad \text{with} \quad \mathcal{F}^R_t = \sigma(R_u : u \leq t) \subset \mathcal{G}_t \]

Filter: \[ p^k_t := P(Y_t = e_k|\mathcal{F}^R_t) \]

\[ \hat{\mu}(Y_t) := E[\mu(Y_t)|\mathcal{F}^R_t] = \mu(p_t) = \sum_{j=1}^{d} p^j_t \mu_j \]

Innovations process: \[ \hat{W}_t^R := \sigma^{-1}(R_t - \int_0^t \hat{\mu}(Y_s)ds) \] is an $\mathbb{F}^R$-BM


\[ p^k_0 = \rho^k \]

\[ dp^k_t = \sum_{j=1}^{d} Q^{jk} p^j_t dt + \beta_k(p_t)^T d\hat{W}_t^R \]

where \[ \beta_k(p) = p^k \sigma^{-1}(\mu_k - \sum_{j=1}^{d} p^j \mu_j) \]
Extra information has no impact on filter \( p_t \) between ‘information dates’ \( T_n \).

Bayesian updating at \( t = T_n \):

\[
p^k_{T_n} \propto p^k_{T_{n-1}} f(Z_n, e_k) \quad \text{recall: } f(\cdot, Y_{T_n}) \text{ is density of } Z_n \text{ given } Y_{T_n}
\]

with normalizer

\[
\sum_{j=1}^d p^j_{T_{n-1}} f(Z_n, e_j) =: \bar{f}(Z_n, p_{T_{n-1}})
\]

HMM filter

\[
p^k_0 = \rho^k
\]

\[
dp^k_t = \sum_{j=1}^d Q^{jk} p^j_t dt + \beta_k(p_t)^\top d\tilde{W}_t^R + p^k_{t-} \int_Z \left( \frac{f(z, e_k)}{f(z, p_{t-})} - 1 \right) \tilde{l}(dt \times dz)
\]

Compensated measure

\[
\tilde{l}(dt \times dz) := l(dt \times dz) - \lambda dt \sum_{k=1}^d p^k_{t-} f(z, e_k) dz
\]

Zakai equation for the unnormalized filter & robust filter

see ELLIOTT, SIU, YANG (2010), KONDAKJI (2012)
Start animation
Filter: Example

Drift

drift

Drift

Wonham Filter

Stock Price

\[ \exp( \int_0^t a(X_s) \, ds ) \]

Stock Price \( S_t \)
Filter: Example

Drift

Stock Price

\( \exp(\int_0^t a(X_s) \, ds) \)

Stock Price \( S_t \)
Optimization Problem Under Partial Information

Wealth
\[ dX_t^{(\pi)} = X_t^{(\pi)} \pi_t^T (\mu(Y_t) \, dt + \sigma dW_t), \quad X_0^{(\pi)} = x_0 \]

Admissible strategies
\[ \mathcal{A}^H = \{ (\pi_t)_{t \in [0,T]} \mid \pi_t \in K \subset \mathbb{R}^n \text{ with } K \text{ compact} \} \]

\[ \pi \text{ is } \mathbb{F}^H\text{-adapted} \]

Reward function
\[ v(t, x, \pi) = E_{t,x}[ U(X_T^{(\pi)}) ] \quad \text{for } \pi \in \mathcal{A}^H \]

Value function
\[ V(t, x) = \sup_{\pi \in \mathcal{A}^H} v(t, x, \pi) \]

Find optimal strategy \( \pi^* \in \mathcal{A}^H \) such that \( V(0, x_0) = v(0, x_0, \pi^*) \)
Reduction to an OP Under Full Information

Consider augmented state process \((X_t, p_t)\)

\[
\text{Wealth} \quad dX_t^{(\pi)} = X_t^{(\pi)} \pi_t^\top \left( \mu(Y_t) \right) dt + \sigma d\tilde{W}_t^R, \quad X_0^{(\pi)} = x_0
\]

\[
\text{Filter} \quad dp_t^k = \sum_{j=1}^d Q^{jk} p_j t dt + \beta_k(p_t)^\top d\tilde{W}_t^R
\]

\[
+ p_{tk}^k \int_{\mathcal{Z}} \left( \frac{f(z,e_k)}{f(z,p_{tk})} - 1 \right) \tilde{l}(dt \times dz), \quad p_0^k = \rho^k
\]

\[
\text{Reward function} \quad v(t, x, p, \pi) = E_{t,x,p}[U(X_T^{(\pi)})] \quad \text{for} \quad \pi \in \mathcal{A}^H
\]

\[
\text{Value function} \quad V(t, x, p) = \sup_{\pi \in \mathcal{A}^H} v(t, x, p, \pi)
\]

Find \(\pi^* \in \mathcal{A}^H(0)\) such that \(V(0, x_0, \rho) = v(0, x_0, \rho, \pi^*)\)
Dynamic Programming Approach

Transformation to risk-sensitive control problem

NAGAI & RUNGGALDIER (2008), DAVIS & LLEO (2012)

Change of measure:

\[ P^{(\pi)}(A) = E[Z^{\pi}1_A] \quad \text{for} \quad A \in \mathcal{F}_T \]

where

\[ Z^{\pi} := \exp \left\{ \theta \int_0^T \pi_s^T \sigma d\tilde{W}_s^R - \frac{\theta^2}{2} \int_0^T \pi_s^T \sigma \sigma^T \pi_s ds \right\} \]

Reward function

\[ E_{t,x,p}[U(X_T^{(\pi)})] = \frac{x_\theta}{\theta} E^{(\pi)}_{t,p} \left[ \exp \left\{ - \int_t^T b(p_s, \pi_s) ds \right\} \right] \]

where

\[ b(p, \pi) := -\theta \left( \pi^T Mp - \frac{1 - \theta}{2} \pi^T \sigma \sigma^T \pi \right) \]

Value function

\[ V(t, p) = \sup_{\pi \in \mathcal{A}^H} v(t, p, \pi) \quad \text{for} \quad 0 < \theta < 1 \]

Find \( \pi^* \in \mathcal{A}^H \) such that \( V(0, \rho) = v(0, \rho, \pi^*) \)
Dynamic Programming Equation (DPE)

State
\[ dp_t = \alpha(p_t, \pi_t)dt + \beta^\top(p_t)dB_t + \int_Z \gamma_I(p_t, z)\tilde{l}(dt \times dz) \]

Generator
\[ \mathcal{L}^a g(p) = \frac{1}{2} \text{tr} \left[ \beta^\top(p) \beta(p) D^2 g \right] + \alpha^\top(p, a) \nabla g \]
\[ + \lambda \int_Z \{ g(p + \gamma_I(p, z)) - g(p) \} \tilde{f}(z, p)dz \]

DPE (Generalized Hamilton-Jacobi-Bellman Equation)

\[ V_t(t, p) + \sup_{a \in K} \left\{ \mathcal{L}^a V(t, p) - b(p, \pi) V(t, p) \right\} = 0 \]

Terminal condition \[ V(T, p) = 1 \]

Candidate for the Optimal Strategy

\[ \pi^* = \pi^*(t, p) = \frac{1}{1 - \theta} (\sigma \sigma^\top)^{-1} \left\{ Mp + \frac{1}{V(t, p)} \sigma \beta(p) \nabla_p V(t, p) \right\} \]

myopic strategy + correction

Certainty equivalence principle does not hold

Justification and regularization of DPE

Computation of the Optimal Strategy

Dynamic Programming Equation

\[ V_t(t, p) + \sup_{a \in K} \left\{ \mathcal{L}^a V(t, p) - b(p, a)V(t, p) \right\} = 0 \]

terminal condition \( V(T, p) = 1 \)

Generator \( \mathcal{L}^a g(p) = \frac{1}{2} tr \left[ \beta^\top(p) \beta(p) D^2 g \right] + \alpha^\top(p, a) \nabla g \)

\[ + \lambda \int_{\mathcal{Z}} \left\{ g(p + \gamma(p, z)) - g(p) \right\} f(z, p) dz \]

Plugging in the optimal strategy

\[ \pi^* = \pi^*(t, p) = \frac{1}{(1 - \theta)} (\sigma \sigma^\top)^{-1} \left\{ M p + \frac{1}{V(t, p)} \sigma \beta(p) \nabla p V(t, p) \right\} \]

yields a nonlinear partial integro-differential equation (PIDE)

Normalization of \( p \): reduction to \( d - 1 \) "spatial" variables

For \( d = 2 \) states: only one "spatial" variable, ellipticity condition is satisfied

Solve PIDE numerically using an explicit finite difference scheme.
Computation of the Optimal Strategy (cont.)

Optimal Strategy $h(t,p)$

Optimal Strategy $h(0,p)$ at Time $t=0$

- Myopic strategy
- Only return
- Return + expert op.
### Monetary Value of Expert Opinions

We want to quantify the value of the extra information.

<table>
<thead>
<tr>
<th>Investor</th>
<th>R</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Information (observations)</td>
<td>only returns</td>
<td>returns + expert opinions</td>
</tr>
<tr>
<td>Initial capital</td>
<td>$x_0^R$</td>
<td>$x_0^C = 1$</td>
</tr>
<tr>
<td>Opt. terminal wealth</td>
<td>$X_T^R$</td>
<td>$X_T^C$</td>
</tr>
</tbody>
</table>

How much initial capital needs R to obtain the same expected utility as C?

$$E[U(X_T^R)] = E[U(X_T^C)]$$

$$\frac{(x_0^R)^\theta}{\theta} V^R(0, p) = \frac{(x_0^C)^\theta}{\theta} V^C(0, p) \quad (x_0^C = 1)$$

$$x_0^R = \left( \frac{V^C(0, p)}{V^R(0, p)} \right)^{\frac{1}{\theta}}$$

Difference $x_0^R - x_0^C$ measures information gain of investor C.
Expert opinions $Z_k \sim \mathcal{N}(\mu(Y_{T_k}), \Gamma)$ with "confidence" $\Gamma = s^2$ arrive with intensity $\lambda$

Limit case $\lambda \to \infty$ full information on the drift state of Markov chain is observable

see RIEDER & BÄUERLE (2004)
Diffusion Approximations

For intensity $\lambda \to \infty$ and expert’s variance $\Gamma$ which is bounded $\Rightarrow$ full information (LLN)

increasing (properly scaled) $\Rightarrow$ ? (CLT)

- $d = 2$ states of the drift $\mu_1$ and $\mu_2$
- conditional distributions of $Z_n$: truncated Gaussian $\mathcal{N}(\mu_j, \Gamma)$, $j = 1, 2$
- expert’s variance $\Gamma = c\lambda$ grows linearly with intensity $\lambda$
- truncate to "$\kappa$–sigma–interval" $[m - \kappa\sqrt{c\lambda}, m + \kappa\sqrt{c\lambda}]$

Updated HMM filter $p = (p_1, p_2)^\top = (\nu, 1 - \nu)^\top$, $\nu = \nu^\lambda$ satisfies

$$
\begin{align*}
\text{d}\nu_t^\lambda &= \alpha(\nu_t^\lambda)\text{d}t + \nu_t^\lambda(1 - \nu_t^\lambda) \frac{\mu_1 - \mu_2}{\sigma} \text{d}\tilde{W}_t^R + \int_{\mathcal{Z}} \gamma_j(\nu_t^\lambda, z) \tilde{l}(\text{d}t \times \text{d}z) \\
\downarrow \quad \lambda \to \infty \\
\text{d}\xi_t &= \alpha(\xi_t)\text{d}t + \xi_t(1 - \xi_t) \frac{\mu_1 - \mu_2}{\sigma} \text{d}\tilde{W}_t^R + \xi_t(1 - \xi_t) \frac{\mu_1 - \mu_2}{\sigma_J} \text{d}\tilde{W}_t^J \\
\end{align*}
$$

$(*$) corresponds to observation of a Markov-modulated Brownian motion

$$
\text{d}J_t = \mu(Y_t)\text{d}t + \sigma_J \text{d}W_t^J \quad \text{where} \quad \sigma_J = c + o(\kappa) \quad \text{for} \quad \kappa \to \infty
$$

continuous-time expert DAVIS & LLEO (2013b)
Diffusion Approximations: Example

Compare value functions $V^\lambda(0, \nu)$ and optimal strategies $\pi^\lambda(0, \nu)$ at time $t = 0$ with corresponding values for the limiting case $\lambda = \infty$ for $\Gamma = c\lambda$, $c = 0.05$.
Conclusion

- Portfolio optimization under partial information on the drift
- Investor observes stock prices and expert opinions
- Closed-form solutions for log-utility and LGM
- For HMM and power utility:
  - non-linear dynamic programming equation with a jump part
- Computation of the optimal strategy
References


