Estimating the Spot Covariation of Asset Prices –
Statistical Theory and Empirical Evidence

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Introduction

Covariance estimation is crucial for

- risk management
- portfolio management
- strategic asset allocation
- asset pricing
- hedging
- quantification of systemic risk
- ...

⇒ Benefit from high-frequency data!
• Recent literature shows strong empirical evidence for distinct time variations in daily and long-term correlations between asset prices.

• But: Surprisingly little is known about intraday variations of asset return covariances.

Questions:

• Do covariances, correlations and betas systematically vary within a day
  ⇒ Is there intraday correlation risk?

• How do covariances, correlations and betas behave in extreme market periods?
Why Important?

• Intraday risk management: Assess intraday correlation risks.

• Market microstructure research: Studies on HF trading, impact of market fragmentation, benefits of circuit breakers.

• Analysis of days with distinct information & “Flash Crashes”: Asymmetry of correlation behaviour during bull/bear markets at lower frequencies (e.g., De Santis & Gerard, 1997).
  ⇒ Similar effects during intraday intervals?

• Crucial for co-jump tests (e.g. Bibinger & Winkelmann, 2014).
In a perfect world ...

• Consider a $d$-dimensional continuous martingale price process,

$$X_t = X_0 + \int_0^t \Sigma^{1/2}(s) dB_s , t \in [0, 1],$$

where $B_t$ denotes a standard Brownian motion.

• Objects of interest: $\int_0^t \Sigma(s) ds$ and $\Sigma(s)$.

• If $X_t$ is discretely observed with $X_{i/n}, i = 0, \ldots, n$, a natural estimator for $\int_0^t \Sigma(s) ds$ is

$$RC_n = \sum_{i=1}^n (X_{i/n} - X_{(i-1)/n}) (X_{i/n} - X_{(i-1)/n})^\top$$

with

$$\text{vec} \left( n^{1/2} \left( RC_n - \int_0^1 \Sigma(t) \, dt \right) \right) \xrightarrow{\mathcal{L}} N \left( 0, \int_0^1 (\Sigma(t) \otimes \Sigma(t) dt) \mathcal{Z} \right).$$
Example

- For $d = 1$:

$$n^{1/2} \left( RC_n - \int_0^1 \sigma^2(s) \, ds \right) \xrightarrow{\mathcal{L}} \mathbf{N} \left( 0, 2 \int_0^1 \sigma^4(s) \, ds \right).$$

- For $d = 2$:

$$\Sigma \otimes \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}, \quad \mathcal{Z} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$(\Sigma \otimes \Sigma) \mathcal{Z} = \begin{pmatrix} 2\sigma_1^4 & 2\sigma_1^3\sigma_2 & 2\sigma_1^3\sigma_2 & 2\sigma_1^2\sigma_2^2 \\ 2\rho\sigma_1^3\sigma_2 & (1 + \rho^2)\sigma_1^2\sigma_2^2 & (1 + \rho^2)\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 \\ 2\rho\sigma_1^3\sigma_2 & (1 + \rho^2)\sigma_1^2\sigma_2^2 & (1 + \rho^2)\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 \\ 2\rho^2\sigma_1^2\sigma_2^2 & 2\rho\sigma_1\sigma_2^3 & 2\rho\sigma_1\sigma_2^3 & 2\sigma_2^4 \end{pmatrix}.$$
1. Introduction

Real Intraday Price Path
Realized Covariances in Practice

**Signature Plot**

**Epps Effect**
1. Introduction

• Challenges:
  • Market microstructure noise
  • Asynchronicity of observations
  • Efficiency
  • Positive definiteness

• Approaches:
  • Hayashi/Yoshida (2011)
  • Realized kernels (Barndorff-Nielsen et al, 2011)
  • Pre-averaging (Christensen et al, 2012)
  • QML (Ait-Sahalia et al, 2010)
  • Spectral estimation (Bibinger/Reiss, 2013)

• Open questions:
  • How to optimally deal with asynchronicity and different speeds in observation frequencies?
  • How to construct spot covariance estimators?
Extend and adapt Local Method of Moments (LMM) approach by Bibinger et al. (2014) to spot covariance matrix estimation.
⇒ Build on locally constant approximations of the process
⇒ Robust to microstructure noise and asynchronicity.

• Allow for autocorrelated noise and propose consistent autocorrelation estimators.
  ⇒ Can use tick-by-tick data.
• Derive stable central limit theorem.
  ⇒ Prove rate optimality of estimator.
• Simulation study shows optimal implementation of estimator.
• First empirical evidence on spot covariances & correlations.
Relation to Literature

• Integrated **covariance matrix estimation:**
  - Hayashi/Yoshida (2011);
  - Barndorff-Nielsen et al (2011);
  - Christensen et al (2012);
  - Ait-Sahalia et al (2010);
  - Bibinger et al. (2014).

• **Spot** volatility estimation:
  - Foster & Nelson (1996);
  - Kristensen (2010);
  - Mancini et al. (2012);
  - Bos et al. (2012);
Outline

1. Introduction
2. LMM: Univariate Case
3. Estimation of Spot Covariances
4. Empirical Results
5. Conclusions
2. Local Method of Moments: Univariate Setting
Univariate Setting

- Consider equi-distantly observed (log) price process:

\[ Y_{i/n} = X_{i/n} + \varepsilon_{i/n}, \quad i = 1, \ldots, n, \quad (E_0) \]
\[ dX_t = \sigma(t)dB_t, \quad \varepsilon_{i/n} \overset{iid}{\sim} N(0, \eta^2), \]

where \( \varepsilon_{i/n} \) denotes microstructure noise with variance \( \eta^2 \).

- Experiment \((E_0)\) is asymptotically equivalent to the "continuous-time white noise" process

\[ dY_t = X_t dt + \psi dW_t, \quad (E_1) \]

where \( X_t \perp W_t \) and \( \psi := \eta / \sqrt{n} \).

- Asymptotic equivalence (in the Le Cam sense) for \( n \to \infty \) provided a certain Hölder-regularity of \( \sigma_t \) (Reiss, 2011).
Local Parametric Approximation

• Consider blocks $[kh, (k+1)h], k = 0, \ldots, h^{-1} - 1.

• Assume that block lengths shrink sufficiently fast with increasing $n$: $h^\alpha = o\left(n^{-1/4}\right)$ for $\alpha \in (1/2, 1]$.

• Observing $(\mathcal{E}_0)$ is asymptotically equivalent to observing

$$dY_t = X^h_t dt + \psi dW_t, \quad (\mathcal{E}_2)$$

with the efficient (log-) price process

$$dX^h_t = [\sigma(t)]_h dB_t, \quad [t]_h = [t/h]h,$$

where $[\sigma(t)]_h$ denotes the block $h$-specific constant volatility.
• On block $k$, we have

$$\tilde{Y}_{i^*}^k = \tilde{X}_{i^*}^k + \varepsilon_{i^*}, \quad i^* = i - kn,$$

with

$$d\tilde{X}_{t^*}^k = \sigma_k dB_{t^*}, \quad t^* = t - kh, \quad t \in [kh, (k + 1) h],$$

$\sigma_k$: spot volatility at the beginning of block $k$.

• Observed returns:

$$\Delta \tilde{Y}_{i^*}^k := \tilde{Y}_{i^*}^k - \tilde{Y}_{i^*}^{k-1} = \Delta \tilde{X}_{i^*}^k + \varepsilon_{i^*} - \varepsilon_{i^*-1},$$

with $\Delta \tilde{X}_{i^*}^k \sim N(0, \sigma_k^2/n)$, $\varepsilon_{i^*} \sim N(0, \eta^2)$ and $i^* = 1, \ldots, nh$.

• $\Delta \tilde{Y}_{i^*}^k$ follow MA(1) process with $\mathbb{E} \left[ \Delta \tilde{Y}_{i^*}^k \right] = 0$ and

$$\text{Cov} \left[ \Delta \tilde{Y}_{i^*}^k, \Delta \tilde{Y}_{i^*-l}^k \right] = \begin{cases} 
\sigma_k^2/n + 2\eta^2 & \text{if } l = 0 \\
-\eta^2 & \text{if } l = 1 \\
0 & \text{otherwise.}
\end{cases}$$
Spectral Statistics

- **Idea**: Constructing a statistic in the spectral domain which yields maximal information about $[\sigma(t)]_h$.

- Define a set of block-specific functions $\varphi_{jk}(t)$ which form an othonormal system in $L^2([0, 1])$.

- Defining $\Phi_{jk}(t) := \int \varphi_{jk}(t) dt$ and setting $\Phi_{jk}(kh) = \Phi_{jk}((k + h)h) = 0$, yields

\[
\int_{kh}^{(k+1)h} \varphi_{jk}(t) dY_t = \int_{kh}^{(k+1)h} \varphi_{jk}(t)X_t h dt + \psi \int_{kh}^{(k+1)h} \varphi_{jk}(t) dW_t,
\]

\[
= -\int_{kh}^{(k+1)h} \Phi_{jk}(t) [\sigma(t)]_h dB_t + \psi \int_{kh}^{(k+1)h} \varphi_{jk}(t) dW_t
\]

\[
\overset{d}{=} \left( \int_{kh}^{(k+1)h} \Phi_{jk}^2(t) [\sigma^2(t)]_h dt + \psi^2 \right)^{1/2} \xi_{jk},
\]

where $(\xi_{jk})_{j \geq 1}$ is $N(0, 1)$ and independent across $j$. 
• Maximizing information load of \( \int_{kh}^{(k+1)h} \varphi_{jk}(t) dY_t \) wrt to \( \lfloor \sigma^2(t) \rfloor_h \) yields

\[
\varphi_{jk} = \sqrt{2/h} \cos \left( \frac{t - kh}{h} j \pi \right) \mathbf{1}_{\{kh,(k+1)h\}}
\]

with antiderivative given by

\[
\Phi_{jk} = \frac{\sqrt{2h}}{jh} \sin \left( \frac{t - kh}{h} j \pi \right) \mathbf{1}_{\{kh,(k+1)h\}}.
\]

• Then, for the statistics \( S_{jk} = \int_{kh}^{(k+1)h} \varphi_{jk}(t) dY_t \), we have

\[
S_{jk} \sim N \left( 0, \int_{kh}^{(k+1)h} \varphi_{jk}^2 \lfloor \sigma(t)^2 \rfloor_h dt + \psi^2 \right)
\]

\[
= N \left( 0, \sigma(kh)^2 \int_{kh}^{(k+1)h} \Phi_{jk}^2 dt + \psi^2 \right)
\]

where \( \sigma(kh) = [\lfloor \sigma(t) \rfloor]_k \) for \( t \in [kh,(k+1)h] \).

• Thus: \( S_{jk} \sim N \left( 0, \frac{h^2}{j^2 \pi^2} \sigma^2(kh) + \psi^2 \right) \)
Non-Equidistant Observations

- Consider the process

\[ Y_i = X_{t_i} + \varepsilon_i, \quad (E_0^*) \]

where \( t_i = F^{-1}(i/n) \), where \( F : [0, 1] \to [0, 1] \) is a differentiable cdf with \( F'(\cdot) > 0 \) denoting the local observation density.

- Then, \( (E_0^*) \) is asymptotically equivalent to

\[ dY_t = X_t dt + \psi(t) dW_t, \quad (E_1^*) \]

where \( \psi(t) := \eta/\sqrt{nF'(t)}. \)

- Locally constant approximation:

\[ dY_t = X_t^h dt + [\psi(t)]_h dW_t, \quad (E_2^*) \]

with \( [\psi(t)]_h = \eta/\sqrt{n} \left[ \frac{1}{F'(t)} \right]_h. \)
• Then, under \((E_2^*)\), we have

\[
\int_{kh}^{(k+1)h} \varphi_{jk}(t) dY_t = \int_{kh}^{(k+1)h} \varphi_{jk} X_t^h \, dt + \int_{kh}^{(k+1)h} \varphi_{jk} \mathcal{J}(t) \, dW_t
\]

\[
= \left( \|\Phi_{jk}\|^2 \sigma(kh)^2 + \psi(kh)^2 \right)^{1/2} \xi_{jk},
\]

where \(\xi_{jk} \sim N(0, 1)\).

• Hence:

\[
S_{jk} \sim N \left( 0, \|\Phi_{jk}\|^2 \sigma(kh)^2 + \frac{\eta^2}{nF'(kh)} \right).
\]

with \(\|\Phi_{jk}\|^2 := \int_{kh}^{(k+1)h} \Phi_{jk}^2(t) \, dt = h^2/j^2\pi^2\).
Local Method of Moments Estimation

- $nh - 1$ independent moment estimators of $\sigma^2_k$:
  \[
  \hat{\sigma}^2_{jk} := \| \Phi_{jk} \|^{-2} \left( S_{jk}^2 - \frac{\eta^2}{nF'(kh)} \right), \quad j = 1, \ldots, nh - 1.
  \]

- Combine them to:
  \[
  \hat{\sigma}^2_k = \sum_{j=1}^{nh-1} w_{jk} \hat{\sigma}^2_{jk} \quad \text{with} \quad \sum_{j=1}^{nh-1} w_{jk} = 1.
  \]

- Minimize variance by choosing weights prop. to Fisher inf. of $\hat{\sigma}^2_{jk}$:
  \[
  w_{jk} = \frac{I_{jk}}{\sum_{l=1}^{nh-1} I_{lk}}, \quad I_{jk} = \frac{1}{2} \left( \sigma^2_k + \| \Phi_{jk} \|^{-2} \frac{\eta^2}{nF'(kh)} \right)^{-2}.
  \]
Estimation of Integrated Variance

- Estimator of $\int_0^1 \sigma_t^2 dt$:

$$\hat{\mathbb{V}}_{\text{LMM}} := h \sum_{k=0}^{h^{-1}} I_k^{-1} \sum_{j=1}^{nh-1} I_{jk} \hat{\sigma}_{jk}^2, \quad I_k := \sum_{j=1}^{nh-1} I_{jk}.$$ 

- CLT with $n^{1/4}$ rate and $AVAR = 8\eta \int_0^1 \sigma_t^3 dt$ (Reiss, 2011).
3. Estimation of Spot Covariances
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Setup

- Efficient log-price $X_t$ follows continuous Itô semi-martingale:
  \[
  X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dB_s, \quad t \in [0, 1],
  \]
  where $B_s$ is $d$-dimensional standard Brownian motion.

- $(d \times d)$ spot covariance matrix: $\Sigma_s = \sigma_s \sigma_s^\top$.

- Observations are non-synchronous and noisy:
  \[
  Y_{i}^{(p)} = X_{t_i^{(p)}}^{(p)} + \epsilon_i^{(p)}, \quad i = 0, \ldots, n_p, \quad p = 1, \ldots, d,
  \]
  with observation times $t_i^{(p)}$ and observation errors $\epsilon_i^{(p)}$.

- Let $n = \min_{1 \leq p \leq d} n_p$ denote number of obs. of "slowest" asset.
  \[\Rightarrow\] HF asymptotics with $n/n_p \to \nu_p$ for $0 < \nu_p < 1$. 

Assumption 1

\((b_s)_{s \in [0,1]}\) is a càdlàg process with \(b_s \in C^\nu,R([0,1], \mathbb{R}^d)\) for some \(R < \infty\) and some \(\nu > 0\).

Assumption 2

(i) \((\sigma_s)_{s \in [0,1]}\) follows a càdlàg process with \(\Sigma_s = \sigma_s \sigma_s^\top \geq \Sigma\) uniformly for some strictly positive definite matrix \(\Sigma\).

(ii) For \(\sigma_s \in C^{\alpha,R}([0,1], \mathbb{R}^{d \times d'})\) with \(R < \infty\) and \(\alpha \in (0, 1/2]\), \(\sigma_s = f(\sigma_s^{(1)}, \sigma_s^{(2)})\) with \(f : \mathbb{R}^{2d \times 2d'} \to \mathbb{R}^{d \times d'}\) continuously differentiable, where

- \(\sigma_s^{(1)}\) is a continuous Itô semi-martingale and
- \(\sigma_s^{(2)} \in C^{\alpha,R}([0,1], \mathbb{R}^{d \times d'})\) with \(R < \infty\).

(iii) For \(\sigma_s \in C^{\alpha,R}([0,1], \mathbb{R}^{d \times d'})\) with \(R < \infty\) and \(\alpha \in (1/2, 1]\), \(\sigma^{(1)}\) vanishes.
Assumption 3

(i) $\epsilon = \{\epsilon_i^{(p)}, i = 0, \ldots, n_p, p = 1, \ldots, d\}$ is independent of $X$ and $\epsilon_i^{(p)}$ is independent of $\epsilon_j^{(q)}$ for all $i, j$ and $p \neq q$.

(ii) At least first eight moments of $\epsilon_i^{(p)}, i = 0, \ldots, n_p$, exist for $p = 1, \ldots, d$.

(iii) $\text{Cov}(\epsilon_i^{(p)}, \epsilon_{i+u}^{(p)}) = 0$ for $u > R$, $R < \infty$ and $p = 1, \ldots, d$.

Define:

$$
\eta_p = \eta_0^{(p)} + 2 \sum_{u=1}^R \eta_u^{(p)}, \quad \text{with} \quad \eta_u^{(p)} := \text{Cov}(\epsilon_i^{(p)}, \epsilon_{i+u}^{(p)}), u \leq R,
$$

with $\eta_u^{(p)}, 0 \leq u \leq R$, constant for all $0 \leq i \leq n - u$.

Impose $\eta_p > 0$ for all $p$. 
Assumption 4

There exist differentiable c.d.f.s $F_p$, $p = 1, \ldots, d$, such that observations satisfy

$$t_i^{(p)} = F_p^{-1}(i/n_p), \quad 0 \leq i \leq n_p, p \in \{1, \ldots, d\},$$

where $F'_p \in C^{\alpha,R}([0, 1], [0, 1])$, $p = 1, \ldots, d$, with $\alpha$ being the smoothness exponent in Assumption 2 for $R < \infty$.

Definition 1

In the asymptotic framework with $n/n_p \to \nu_p$, where $0 < \nu_p < \infty$, $p = 1, \ldots, d$, for $n \to \infty$, define the continuous-time noise level matrix

$$H_s = \text{diag} \left( \left( \eta_p \nu_p (F_p^{-1})'(s) \right)^{1/2} \right)_{1 \leq p \leq d}. \quad (3)$$
Local Method of Moments Estimation

- Estimation using LMM approach by Bibinger et al. (2014).
- Partition interval \([0, 1]\) into blocks \([kh_n, (k + 1)h_n]\), \(k = 0, \ldots, h_n^{-1} - 1\) with \(h_n \to 0\) as \(n \to \infty\).
- Approximate original process by process with block-wise constant covariance matrices \(\Sigma_{kh_n}\) and noise levels \(H^n_k\).

\[\Rightarrow\] Estimation error can be asymptotically neglected for sufficient smoothness of \(\Sigma_t\) and \(F_p\) and block sizes \(h_n\) shrinking sufficiently fast.

- Bibinger et al. (2014) propose integrated covariance matrix estimator in simplified setting.

\[\Rightarrow\] Here: estimate spot covariance matrix in generalized setting.
3. Estimation of Spot Covariances

- Local spectral statistics:

\[ S_{jk} = \pi j h_n^{-1} \left( \sum_{i=1}^{n_p} \left( Y_i^{(p)} - Y_{i-1}^{(p)} \right) \Phi_{jk} \left( \frac{t_{i-1}^{(p)} + t_i^{(p)}}{2} \right) \right)_{1 \leq p \leq d}, \]

where

\[ \Phi_{jk}(t) = \sqrt{\frac{2h_n}{j\pi}} \sin \left( j\pi h_n^{-1} (t - k h_n) \right) 1_{[k h_n, (k+1) h_n)}(t), j \geq 1. \]

- Can show that

\[ \text{Cov}(S_{jk}) = (\Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} H_k^n)(1 + o(1)), \]

where \( H_k^n \) has entries

\[ (H_k^n)_{pp} = n_p^{-1} \eta_p (F_p^{-1})'(k h_n), \]

\[ \Rightarrow \text{Estimate } \Sigma_{kh_n} \text{ by } S_{jk} S_{jk}^\top - \pi^2 j^2 h_n^{-2} H_k^n ! \]
An Initial Spot Covariance Matrix Estimator

- Average across frequencies $j = 1, \ldots, J_n^p$ and adjacent blocks:

$$\text{vec} \left( \hat{\Sigma}_{khn}^{\text{pre}} \right) = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} (J_n^p)^{-1} \sum_{j=1}^{J_n^p} \text{vec} \left( S_{jk} S_{jk}^\top - \pi^2 j^2 h_n^{-2} \hat{H}_k \right),$$

where $L_{s,n} = \max\{\lfloor sh_n^{-1} \rfloor - K_n, 0\}$,
$U_{s,n} = \min\{\lfloor sh_n^{-1} \rfloor + K_n, \lceil h_n^{-1} \rceil - 1\}$

- $\hat{H}_k^n$ is a $\sqrt{n}$-consistent estimator of $H_k^n$ with diagonal element

$$\left( \hat{H}_k^n \right)^{(pp)} = \frac{\hat{\eta}_p}{h_n} \sum_{kh_n \leq t_i^{(p)} \leq (k+1)h_n} \left( t_i^{(p)} - t_i^{(p-1)} \right)^2,$$

with $\hat{\eta}_p$ being long-run noise variance estimator.
LMM Spot Covariance Matrix Estimator

- Equal weights for frequencies $j = 1, \ldots, J_n^p$ in general not optimal.
- Increase efficiency: obtain pre-estimated spot covariance matrices using $\text{vec} \left( \hat{\Sigma}^{\text{pre}}_{kh_n} \right)$ and derive estimated optimal weight matrices $\hat{W}_j$.

$LMM$ spot covariance matrix estimator:

$$\text{vec} \left( \hat{\Sigma}_s \right) = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} \sum_{j=1}^{J_n} \hat{W}_j \left( \hat{H}^n_k, \hat{\Sigma}^{\text{pre}}_{kh_n} \right) \times \text{vec} \left( S_{jk} S_{jk}^\top - \pi^2 j^2 h_n^{-2} \hat{H}^n_k \right) .$$
• Optimal weights proportional to local Fisher info matrices:

\[
W_j \left( H_k^n, \Sigma_{kh_n} \right) = \left( \sum_{u=1}^{J_n} \left( \Sigma_{kh_n} + \pi^2 u^2 h_n^{-2} H_k^n \right)^{-\otimes 2} \right)^{-1} \\
\times \left( \Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} H_k^n \right)^{-\otimes 2} \\
= I_k^{-1} I_{jk},
\]

with

\[
I_{jk} = \left( \Sigma_{kh_n} + \pi^2 j^2 h_n^{-2} H_k^n \right)^{-\otimes 2},
\]

and \( I_k = \sum_{j=1}^{J_n} I_{jk} \).

• Note: \( \hat{\Sigma}_s \) symmetric, but not necessarily positive semi-definite.

\[ \Rightarrow \] E.g., project on space of positive semi-definite matrices.
Pointwise Central Limit Theorem

Theorem 1
Assume a setup with observations of type (2), a signal (1) and validity of Assumptions 1-4.

Then, for $h_n = \kappa_1 \log (n)n^{-1/2}$, $K_n = \kappa_2 n^\beta (\log (n))^{-1}$ with constants $\kappa_1, \kappa_2$ and $0 < \beta < \alpha(2\alpha + 1)^{-1}$, for $J_n \to \infty$ and $n/n_p \to \nu_p$ with $0 < \nu_p < \infty, p = 1, \ldots, d$, as $n \to \infty$, $\hat{\Sigma}_s$ satisfies:

$$n^{\beta/2} \text{vec}(\hat{\Sigma}_s - \Sigma_s) \xrightarrow{d-(st)} N\left(0, 2\left(\Sigma \otimes \Sigma_H^{1/2} + \Sigma_H^{1/2} \otimes \Sigma\right) Z\right), s \in [0, 1],$$

where $\Sigma_H = H\left(H^{-1}\Sigma H^{-1}\right)^{1/2} H$ with noise level $H$ from (3) and $Z = \text{COV}(\text{vec}(ZZ^\top))$ for $Z \sim N(0, E_d)$ being a standard normally distributed random vector.
Feasible Central Limit Theorem

Corollary 1

Under the assumptions of Theorem 1, \( \hat{\Sigma}_s \) satisfies

\[
(U_{s,n} - L_{s,n} + 1)^{1/2} (\hat{\nabla}_s^n)^{-1/2} \text{vec} (\hat{\Sigma}_s - \Sigma_s) \xrightarrow{d} N(0, Z), \ s \in [0, 1],
\]

where

\[
\nabla_s^n = (U_{s,n} - L_{s,n} + 1)^{-1} \sum_{k=L_{s,n}}^{U_{s,n}} \left( \sum_{j=1}^{J_n} I_{jk} \right)^{-1}.
\]
Spot Correlations and Betas

- Spot correlation estimator: 
  \[ \hat{\rho}_s^{(pq)} = \frac{\hat{\Sigma}_s^{(pq)}}{\sqrt{\hat{\Sigma}_s^{(pp)} \hat{\Sigma}_s^{(qq)}}}. \]

- Spot beta estimator: 
  \[ \hat{\beta}_s^{(pq)} = \frac{\hat{\Sigma}_s^{(pq)}}{\hat{\Sigma}_s^{(pp)}}. \]

- Delta method yields:
  \[ n^{\beta/2} \text{vec} \left( \hat{\rho}_s^{(pq)} - \rho_s^{(pq)} \right) \xrightarrow{d-(st)} N \left( 0, \mathbb{A} \mathbb{V}_{\rho,s} \right), \ s \in [0, 1], \]
  \[ n^{\beta/2} \text{vec} \left( \hat{\beta}_s^{(pq)} - \beta_s^{(pq)} \right) \xrightarrow{d-(st)} N \left( 0, \mathbb{A} \mathbb{V}_{\beta,s} \right), \ s \in [0, 1]. \]

⇒ Analogously for feasible CLTs.
Estimating Noise Autocovariances

• Estimation of long-run noise variance $\eta_p, p = 1, \ldots, d$, only requires component-wise autocovariance estimates.

  $\Rightarrow$ Restrict analysis to $d = 1$: $n + 1$ observations of $Y_i = X_{t_i} + \epsilon_i, i = 0, \ldots, n$.

• Fix $R \geq 0$ and successively estimate autocovariances by

$$\hat{\eta}_R = (2n)^{-1} \sum_{i=1}^{n} (\Delta_i Y)^2 + n^{-1} \sum_{r=1}^{R} \sum_{i=1}^{n-r} \Delta_i Y \Delta_{i+r} Y,$$

$$\hat{\eta}_r - \hat{\eta}_{r+1} = (2n)^{-1} \sum_{i=1}^{n} (\Delta_i Y)^2 + n^{-1} \sum_{u=1}^{r} \sum_{i=1}^{n-u} \Delta_i Y \Delta_{i+u} Y, \quad 0 \leq r \leq R - 1.$$
• The variance of $\hat{\eta}_r, 0 \leq r \leq R$, is consistently estimated by

$$\widehat{\text{Var}}(\hat{\eta}_r) = n^{-1} \left( V_{r+1}^n + V_r^n + 2C_{r,r+1}^n \right),$$

with

$$C_{r,r+1}^n = \left( \frac{\hat{\Gamma}_0^{00}}{4} + \frac{1}{2} \sum_{u=1}^r \hat{\Gamma}_u^{00} + \sum_{u=0}^r \sum_{u'=1}^{r+1} \left( \hat{\Gamma}_0^{uu'} + 2 \sum_{q=1}^R \hat{\Gamma}_q^{uu'} \right) \right),$$

and $V_r^n = C_{r,r}^n$, where $\hat{\Gamma}_q^{rr'}, q, r, r' \in \{0, \ldots, R\}$ is the fourth sample moment of $\Delta_i Y$.

• In particular, for $r = R$, $\widehat{\text{Var}}(\hat{\eta}_R) = n^{-1} V_R^n$. 
Theorem 2

Under Assumption 3 and \( H_0^Q : \eta_u = 0 \) for all \( u \geq Q \), \( Q = R + 1 \), we have

\[
T^n_Q(Y) = \sqrt{n/V^n_Q} \hat{\eta}_Q \xrightarrow{d} N(0, 1).
\]

Suitable strategy for selecting \( R \):

- Compute \( T^n_Q(Y) \) for \( Q \leq \tilde{Q} = \tilde{R} + 1 \) “large”.
- Incorporate all autocovariances until first hypothesis of zero autocovariance cannot be rejected.

\( \Rightarrow \) Using \( \tilde{R} \), compute long-run noise variance estimate as

\[
\hat{\eta} = \hat{\eta}_0 + 2 \sum_{u=1}^{\tilde{R}} \hat{\eta}_u.
\]
4. Empirical Results
4. Empirical Results

Data

- Mid-quotes and transaction prices for 30 most liquid NASDAQ100 constituents and PowerShares QQQ ETF.
- Sample period from May 2010 to April 2014.
- Data sampled from LOBSTER database: https://lobster.wiwi.hu-berlin.de/
- Handle (few) errors in the trade and mid-quote samples using cleaning procedures by Barndorff-Nielsen et al. (2009).
- Preliminary analysis: huge share of zero returns in quote data.
  ⇒ Focus on quote revisions to reduce computational burden.
Choice of Inputs and Implementation

- Theory requires:
  \[ h_n = \mathcal{O}(\log(n)n^{-1/2}) , \quad J_n = \mathcal{O}(\log(n)) , \]
  \[ J_{p,n} \] fixed at a value not “too large” (e.g., \( J_{p,n} = 5 \)) and \( K_n = \mathcal{O}(n^{1/4-\varepsilon}) \) for \( \varepsilon > 0 \) “small”.

- Introduce proportionality parameters:
  \[ h_n = \theta_h \log(n)n^{-1/2}, \quad J_n = \lfloor \theta_J \log(n) \rfloor \quad \text{and} \quad K_n = \lceil \theta_K n^{1/4-\delta} \rfloor, \]
  where \( \theta_h, \theta_J, \theta_K > 0 \).

  \[ \Rightarrow \] Based on simulations: \( \theta_h = 0.2, \theta_J = 8, \theta_K = 0.4, J_{p,n} = 5 \).

- Estimate
  - \( 30 \times 30 \) spot covariance matrices for NASDAQ100 constituents: spot covariances and correlations, volatilities.
  - \( 31 \times 31 \) spot covariance matrices including QQQ ETF: spot betas with QQQ as market proxy.
# Summary Statistics of Input Values

<table>
<thead>
<tr>
<th>Input</th>
<th>( q_{0.05} )</th>
<th>Mean</th>
<th>( q_{0.95} )</th>
<th>Std.</th>
</tr>
</thead>
<tbody>
<tr>
<td>([ h_n^{-1} ])</td>
<td>18.000</td>
<td>22.516</td>
<td>29.000</td>
<td>3.922</td>
</tr>
<tr>
<td>( J_n )</td>
<td>48.000</td>
<td>53.532</td>
<td>60.000</td>
<td>3.672</td>
</tr>
<tr>
<td>( K_n )</td>
<td>2.000</td>
<td>2.435</td>
<td>3.000</td>
<td>0.300</td>
</tr>
</tbody>
</table>
4. Empirical Results

Cross-Sectional Deciles of Avg. Covariance and Correlation

Spot estimates are averaged across days. Then, cross-sectional sample deciles of across-day averages are computed.
4. Empirical Results

Cross-Sectional Deciles of Avg. Beta and Volatility

(a) Spot Betas

Spot estimates are averaged across days. Then, cross-sectional sample deciles of across-day averages are computed.
Cross-Sectional Deciles of Std. Dev. of Covariance and Correlation

(a) Spot Covariances
(b) Spot Correlations

Sample standard deviations of spot estimates are computed across days. Then, cross-sectional sample deciles of across-day standard deviations are computed.
4. Empirical Results

Cross-Sectional Deciles of Std. Dev. of Beta and Volatility

Sample standard deviations of spot estimates are computed across days. Then, cross-sectional sample deciles of across-day standard deviations are computed.
Cross-Sectional Medians of Intraday Variation Proxy for Covariance and Correlation

Total intraday variation proxy: \( \sum_{i=1}^{ng} |f(t_i) - f(t_{i-1})| \left[ \sum_{i=1}^{ng} |f(t_i)| \Delta t_i \right]^{-1} \).
Cross-Sectional Medians of Intraday Variation Proxy for Beta and Volatility

Total intraday variation proxy: $\sum_{i=1}^{n_g} |f(t_i) - f(t_{i-1})| \left[\sum_{i=1}^{n_g} |f(t_i)| \Delta t_i\right]^{-1}$. 

(a) Spot Betas

(b) Spot Volatilities
Event I: “Flash Crash” (05/06/10)


(2) E-Mini market makers cut back trading.

(3) NASDAQ stops order routing to ARCA.

(4) Rumors suggesting that decline occurred due to “fat-finger” error, and not bad news.

(5) NASDAQ resumes routing to ARCA.
4. Empirical Results

05/06/10: QQQ Transaction Prices

(a) Entire trading day

(b) 1:30 pm – 4:00 pm
4. Empirical Results

05/06/10: Cross-Sectional Deciles of Covariance and Correlation

(a) Spot Covariances

(b) Spot Correlations
05/06/10: Cross-Sectional Deciles of Beta and Volatility

(a) Spot Betas

(b) Spot Volatilities
Event II: “Twitter Flash Crash” (04/23/13)

(1) Fake tweet from the account of AP stating “Breaking: Two Explosions in the White House and Barack Obama is injured”.

(2) Official denial by AP.

(3) AP’s twitter account suspended.
4. Empirical Results

04/23/13: QQQ Transaction Prices

(a) Entire trading day

(b) 1:00 pm – 1:30 pm
04/23/13: Cross-Sectional Deciles of Covariance and Correlation

(a) Spot Covariances

(b) Spot Correlations
04/23/13: Cross-Sectional Deciles of Beta and Volatility

(a) Spot Betas
(b) Spot Volatilities
5. Conclusions
Conclusions

• Introduce spot covariance matrix estimator relying on LMM approach by Bibinger et al. (2014).

• Extend LMM to allow for autocorrelated noise and provide method for choosing order of dependence.

• Derive stable CLT along with feasible version.

• Simulation study demonstrates how to implement estimator.

• Empirical evidence based on NASDAQ100 stocks:
  • Spot covariances, correlations & volatilities exhibit considerable intraday seasonality.
  • Distinct intraday changes of (co-)volatilities in periods of extreme market movements.