Approximating Early Exercise Boundaries for American Options

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Abstract

American options are different to European style options in that the contract buyer has the right to exercise the option at any time on or before maturity . The freedom to exercise an American option whenever the holder wishes, introduces a boundary problem to solving the Black-Scholes equation popularly used to price the European options. The contract holder will ideally, of course, only exercise the option prior to the expiry date if the present payoff at time t exceeds the discounted expectation of the possible future values of the option from time t to T. So, only if what the holder of the options gets out of exercising early exceeds the market's view of the expected future return in keeping the option alive, early exercise of the options will take place. Otherwise, he or she will continue to hold on to the option. At every time t there will be a region of values of the underlying whereby it is best to exercise the option (Exercise region) and a complimentary region whereby it is best to keep the option (Free region). There will also be a particular value S(t) of the underlying stock which defines the optimal exercise boundary separating the two regions. In this paper, we evaluate the early exercise boundary using an efficient method developed by **Ait Sahlia** and **Lai (1999)** in R and provide numerical results and graphs.

1 Introduction

The problem of pricing an American option is not as simple as pricing an European one. It has a large literature consisting of relentless researches and propositions to efficiently calculate the price of an American option. Binomial Asset Pricing Model developed for discrete-time pricing of derivatives can be used to evaluate the price of an American option. But in the continuous time frame where the gaps between two successive time points are infinitesimally small, the computation of prices becomes tedious. A significant advancement in the pricing of American option has evolved with the integral representation of the difference between the American option and the European one due to **Kim (1990)**, **Jacka (1991)**, and **Carr**, **Jarrow**, and **Myneni (1992)** . The integral representation (as shown mathematically, later) is accurate but involves finding the boundary function first , before calculating the price of the American option. While **Huang**, **Subrahmanyam**, and **Yu (1996)** proposed to approximate the integrands in both the integral premium representation formula and the integral equation of the boundary function by piece-wise constant functions, **Ju (1998)** proposed to approximate the boundary by a piece-wise exponential function. The method employed by Ju was very significant, as that lead to an increase in efficiency in calculating American option prices as compared to any other method which was existing at that point of time.

Ait Sahlia and **Lai (1999)** went on to bring significant improvements upon the method employed by Ju. Although Sahlia and Lai used the same idea as proposed by Ju, they just introduced ingenious transformations of variables with which the calculation of the early exercise boundary becomes much simpler as the piecewise exponential functions employed by Ju get converted into linear functions after the transformations are applied. Their results also support Ju's findings strongly. Moreover, instead of approximating the boundary between successive time points using an usual step function approximation, Sahlia and Lai proposed to linearly interpolate the boundary between two successive time points. These transformations and linear interpolation technique which have been used by Sahlia and Lai as improvements on Ju's technique lead to significant speed in calculation of results and accurate approximations as well.

The next section describes the basic background of the American option pricing based on which all the researches stand. We, mathematically show what are the conditions required to be satisfied in order to calculate the price of an American option, starting from a insight with the famous Binomial Asset Pricing Model. Later on, we become more specific and explain in details the numerical technique employed by Sahlia and Lai. We provide computation results using their technique in R. Finally, we summarize and conclude.

2 Problem Background

Pricing of American options invariably involves approximating the early exercise boundary for optimal exercise of the options. So, our first important question lies in how to decide when to exercise the American option optimally. At the onset we see a simplistic setting on the famous Binomial Asset Pricing Model given by **Cox, Ross and Rubinstein (1979)**. The significance of this model and hence its enduring popularity is its method of evaluating the underlying over time. Although this uses a discrete-time model of the underlying prices over time, this is nonetheless very important to have a helpful insight in understanding how the price process of an option develops from time to time.

2.1 The Binomial Asset Pricing Model

The Binomial Model is a model adequately solving the problem of pricing derivative securities. In this model we assume that the stock price moves either up or down at any given point of time. Thus, if the stock price is S_o currently, then at the next point of time its price will either be uS_o or dS_o . d and u are two positive numbers with,

$$0 < d < 1 < u$$

The above inequality makes sense . This is because $d < 1$ ensures that the stock price can become less at the next point of time , while $u > 1$ ensures that the stock price can become greater as well. Also, to be more precise, the values d and u satisfy the following inequality,

$$d < 1 + r < u$$

where r is the risk free rate of return (from bank, say) . The above inequality is obvious because if $1 + r$ is greater than u , then it means that the return on the money market is always risklessly greater than that on the stock and hence no one would ever want to invest in the stock. Also $d < 1$ has been justified earlier and hence d cannot be greater than $1 + r$ until and unless r is negative which does not make any sense. With this setting , we proceed to explain how the Binomial Asset Pricing Model evaluates the price of European options. We would then extend it to American options with the supermatingale property as will be explained later.

We denote the upward stock movement by head (H) of a coin toss , while the downward movement by tail (T). Thus, for a stock which is currently priced in the market at S_o, it would become uS_o if the toss results in head or dS_o if the toss results in tail. If the strike price of the European call option is K, then at time point one, the pay off is $(uS_o - K)$ if the coin toss results in head and zero otherwise. Thus the value of the option at time point one can be written as,

$$V_1(w) = max(S_1(w) - K, 0)$$

where $S_1(w)$ is uS_o if w = H and dS_o otherwise. Thus, we see that the payoff of the option at the next time point basically depends on the stock price at that point. The price of the option is formulated by this model very logically with this notion in mind. Logically, the option writer at the time of writing the option must hold an amount of money which would generate an amount required to pay off the contract holder at the expiration date in case favorable situation crops up

($S_T > K$ for call option and $S_T < K$ for put option). However, the option writer is not certain as to what the stock price will be at the expiration date. And thus the concept of expected stock prices comes up. Also, since the position of the option writer now becomes a risky one after writing off the option, he would try to hedge himself against possible losses in the future. In the process of doing so, he buys some shares of the stock so that at the expiration date if the stock prices go above K and the call option is exercised he is able to hedge himself with the stocks that he owns. For put options too, this logic holds correspondingly. The most important thing at this point of time is to decide the price that he would charge for writing off the option or to be more precise, what is the price of the option?

For evaluating the price of the option the binomial model takes up a very simple, yet an ingenious technique. At any given point of time, the option writer should hold such an amount which when invested in the money market would give him a return at the next point of time that would exactly equal the expected stock price at the latter point of time. With this framework, if the option price is taken to be V_o and the number of shares bought to hedge is Δ_o at the initial point of time, then the value of the portfolio initially is $(V_0 - \Delta_0 S_0)$ where this quantity can be negative. Basically this is the amount that he has in the money market. Thus at time point 1 the value of the portfolio would be

$$X_1(w) = \Delta_0 S_1(w) + (1+r)(V_0 - \Delta_0 S_0)$$

where $w = \{H, T\}$. Putting the two different values of w that it can take and equating them to the value of the option accordingly at time point 1, we get two equations as shown below,

$$V_1(H) = \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0)$$

$$V_1(T) = \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0)$$

Solving these two equations gives the values of Δ_0 and V_0 as follows,

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

$$V_0 = \frac{1}{1+r}[\frac{1+r-d}{u-d}V_1(H) + \frac{u-1-r}{u-d}V_1(T)]$$

This is the arbitrage price for the European option with payoff V_1 at time point 1. If we assume $p = \frac{1+r-d}{u-d}$ and $q = \frac{u-1-r}{u-d}$ i.e. $q = 1 - p$, then we can rewrite the solution for V_0 as,

$$V_0 = \frac{1}{1+r}[pV_1(H) + qV_1(T)]$$

where p and q represent the risk-neutral probabilities. The probabilities p and q do not represent the actual probability of the movement of stock prices, but they can be just thought as risk-neutral probability measure giving the expected value of the option at a future point of time. Extending this framework and logic to any point of time k we get the pattern,

$$V_{k-1} = \frac{1}{1+r}[\frac{1+r-d}{u-d}V_k(H) + \frac{u-1-r}{u-d}V_k(T)]$$

where the price of the option at the $(k-1)^{th}$ point of time is given as the discounted value of the expected payoff of the option at time point k. Thus, evaluating the price for European option using Binomial model seems extremely intuitive, logical and clear. The price of the option has to be such so that option writer is able to payoff the holder at time of maturity in case the latter exercises. In order to achieve this position, at any point of time he observes all the stock price movements and hedges his position accordingly maintaining the exact amount which just equals the discounted expected payoff value at the next point of time. Thus he is able to be in a position to payoff at the time of maturity and hedge himself as well.

2.2 Stopping Times and American Options

An American option gives the holder the right to exercise at any point of time on or before the maturity date. Because of this early exercise feature an American option is at least as valuable as its European counterpart. However, the American call option paying no dividends has the same price as the European call option. In other cases the value of the early exercise premium can be substantial.We will primarily be dealing with put options. We saw in case of European options under the Binomial Model that the discounted price process is a martingale (the expected value of the stock prices tends to remain unchanged) where,

$$V_{k-1} = \frac{1}{1+r}[\frac{1+r-d}{u-d}V_k(H) + \frac{u-1-r}{u-d}V_k(T)]$$

and thus the price of the option could be written as,

$$V_0 = E[\frac{V_T}{(1+r)^T}]$$

which means that the price of the option is the discounted expected price of the option at the time of maturity T. However, unlike European options the price process for American options obeys supermartingale property. This is due to the fact that an American option can be exercised at any point of time on or before the maturity date and it is at least as much valuable as the payoff associated with its immediate exercise. This means that if the option holder sees that the expected stock price underlying the option is going to fall (in case of an American put), then he would exercise the option now and would be better off. Previously, we saw that at any point of time the option writer holds that much amount of money which can generate an amount exactly equaling the expected payoff at the immediate next point of time. Here, in case of American options at any point of time he needs to have the maximum of the immediate payoff and the discounted expected payoff for the next point of time. Assuming that the payoff for exercising at any point of time k is equal to $g(x)$where x represents the current price of the underlying, we mathematically express the above statement as follows,

$$V_{k-1}(x) = max\{\frac{1}{1+r}[pV_k(ux) + qV_k(dx)], g_{k-1}(x)\}$$

where $g_{k-1}(x)$ is the value of the option if exercised immediately.

Although the logic and intuition more or less seems similar to the one used for evaluating the prices of European options, there is a subtle difference at the starting point of the two approaches. Unlike European options where we had fixed maturity and exercise date, it was simple to develop such a portfolio which at any point of time is just as worth as the discounted payoff of the option at the next time period and subsequently developed the notion of risk-neutral pricing, here in case of American options we begin with a risk-neutral pricing formula and it can be shown that this price is the smallest initial capital that is required to develop a portfolio in order to hedge one's position from time to time. Since an American option can be exercised at any point of time , the owner would always try to exercise it at such a point of time so that he reaps the maximum benefit. Mathematical formulation of this requires the concept of Stopping Times.

Mathematically, a stopping time Γ is a random variable taking value in $[0, \infty]$ and satisfying

$$\{\Gamma \leq t\} \in F(t) \text{ for } \text{ all } t{\leq}0$$

$F(t)$ is a set of sigma algebra which can be thought of as the set of information available till time point t. Intuitively, this means that the owner of the American option can exercise it at any time point t based on a strategy that takes into account only the information available at time t i.e. the information about the stock price movements till time point t. There can be various ways in which a stopping time is defined. It seems logical that the price of the American option has to be such so that it is the maximum of the present discounted values of all possible future payoffs. To express it mathematically, we write,

$$v_*(x) = \max_{\Gamma \in T} E[e^{-r\Gamma}(K - S(\Gamma))]$$

where T is the set of all stopping times. The equation means that the owner of the option should choose the exercise strategy that maximizes this expected payoff discounted back to time zero and thus we define the price to be the maximum over the discounted expected payoffs at all possible stopping times. It turns out to be that v_* is the initial capital required for an agent to hedge a short position in the American put regardless of the exercise strategy of the owner of the put. Thus this is the price of the American put. It is clear from the above discussion that for pricing an American option it is essential to find out the optimal stopping time or the optimal exercise time which would then enable us to price the American option. We would first see how a perpetual American put (infinite maturity time) is priced.

In a perpetual American put there is no expiration date beyond which it cannot be exercised and hence this makes every date like every other date. What this means is that the price of a perpetual American put cannot depend on the time scale but only on the stock price at any given point of time. To be precise, this means that the owner should exercise the option as soon as the value of the stock reaches a particular level. So our next question lies in how to decide what is that optimum level of the stock price to which when it falls, the option should be exercised. Thus, here our stopping time is of the form,

$$\Gamma_L \;=\; min[t\geq 0;\; S(t) \;=\; L] \tag{1}$$

where Γ_L is set equal to infinity if the stock price never reaches the level L. We have to consider only $L \;<\; K$ (since, we are considering an American put). Thus, if the stock level reaches L then the option is exercised and the owner receives $(K \;-\; L)$. Thus, the price of the option or the discounted value of the payoff of the option from the exercise point of time to the time point zero is given by,

$$v_L(S(0)) \;=\; (K - L)E(e^{-r\Gamma_L}) \text{ for all } S(0)\geq L. \tag{2}$$

It can be shown that the if at the time of signing the contract the stock price is x then the price of the option or rather, the solution to the above equation is given by,

$$v_L(x) = \begin{cases} K \;-\; x, & \text{if } 0\leq x\leq L, \\ (K \;-\; L)(\frac{x}{L})^{\frac{2r}{\sigma^2}} & \text{if } x \geq L \end{cases} \tag{3}$$

It can be shown graphically, that the optimum level, L at which, when the stock price reaches the option should be exercised is

$$L_* \;=\; \frac{2r}{2r \;+\; \sigma^2}K \tag{4}$$

where K is the strike price.

Analytical characterization shows that the price of the option in the two regions $0\leq x\leq L$ and $x\geq L$ match with each other at $x = L$ and the price functions in the two regions are differentiable at L as well. The second derivative of $v_L(x)$, however, has a jump at $x = L$. In particular the function $v_L(x)$ is found to satisfy the following conditions,

$$v(x)\geq (K \;-\; x)^+ \text{ for all } x\geq 0, \tag{5}$$

$$rv(x) \;-\; rxv'(x) \;-\frac{1}{2}\sigma^2x^2v''(x) \geq 0 \text{ for all } x\geq 0, \tag{6}$$

and

$$\text{for each } x\geq 0, \text{ equality holds in either of the above two equations.} \tag{7}$$

Equation (7) is particularly important as the equations (5) and (6) do not necessarily imply the existence of such an early exercise boundary. The equation (7) ensures that there exist two distinct regions, S or the stopping set and C or the continuation set given by,

$$S = [x \geq 0; v_{L_*}(x) = (K - x)^+] \tag{8}$$

and

$$C = [x \geq 0; v_{L_*}(x) > (K - x)^+] \tag{9}$$

respectively. If the stock price $x \in C$ then equality holds in equation (6). This means that the price of the option at that point of time which is the maximum of the immediate payoff and the discounted expected payoff at the next point of time equals the latter. Thus it is beneficial for the owner to wait and hold the option. When $x \in S$ then inequality holds in equation (6) which technically means that the discounted expected payoff of the next period of time is lesser than the immediate payoff. Thus, in this case the option is said to have reached its intrinsic value and its optimal for the owner to exercise the option. If, however he fails to exercise the option, then the option writer can consume the extra amount.

Extending the case of the perpetual American put to a finite expiration American put the equations (5) , (6) and (7) or the complementarity conditions get modified to,

$$v(t,x) \geq (K - x)^+ \text{ for all } t \in [0,T], x \geq 0, \tag{10}$$

$$rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0 \text{ for all } t\ [0,T), x \geq 0, \tag{11}$$

and

$$\text{for each } t \in [0,T) x \geq 0, \text{ equality holds in either of the above two equations.} \tag{12}$$

Just like the perpetual American put, here too the owner has to wait until the stock price reaches some pre-determined level. However, now this is a function of time $(T - t)$, unlike perpetual American put where it was independent of time. This is intuitively clear because at time point T the owner will exercise only if the stock price is below K (the strike price). Thus we see that at each point of time over the life of the option there exists a particular value of the stock price i.e. $x = L(T - t)$ such that if the stock price is below that level then the owner should exercise , otherwise he should continue holding the option. So, our concern finally boils down to evaluating this optimum level at each point of time which acts as a boundary between the stopping set and the continuation set.

Extensive literature exists on calculating efficiently this early exercise boundary of American options. However, the evaluation of boundary using the an integral representation of the difference between the American option and its European counterpart as given by **Kim (1990)** ,**Jacka (1991)**,and **Carr**, **Jarrow**, and **Myneni (1992)** has led to considerable research modifying and improving upon their proposed technique. One such efficient modification and transformation has been carried out by **Ait Sahlia (1999)** and **Lai (1999)** which significantly simplifies the problem of approximating the early exercise boundary and hence an easy evaluation of the American option prices.

The next section presents the technique, rather the ingenious transformations made by **Ait Sahlia (1999)** and **Lai (1999)** in the Ju's exponential piecewise approximation technique for the early exercise boundary. It is followed by our numerical computations (using their technique) in R which give close approximations to the results obtained by them.

3 Numerical Method to Evaluate the Early Exercise Boundary

The main motivation towards solving the early exercise boundary using advanced and developed techniques lie behind the idea of the proposition of the integral difference between the American option and its European counterpart made by **Kim (1990)** ,**Jacka (1991)**,and **Carr**,**Jarrow**, and **Myneni (1992)** . Specifically the integral representation formula is,

$$U(t,P) = U_E(t,P) + \int_t^T [rKe^{-r(\Gamma-t)}N(-d_2(P,B_\Gamma,\Gamma-t)) \; - \; \mu P e^{-\mu(\Gamma-t)}N(-d_1(P,B_\Gamma,\Gamma-t))]d\Gamma \tag{13}$$

where,

$$U_E(t,P) = Ke^{-r(T-t)}N(-d_2(P,K,T \; - \; t)) \; - \; Pe^{-\mu(T-t)}N(-d_1(P,K,T \; - \; t))$$

where,

$$d_1(x,y,\Gamma) \;=\; \frac{ln(\frac{x}{y}) \; + \; (r \; - \; \mu \; + \; \frac{1}{2}\sigma^2)\Gamma}{\sigma\sqrt{\Gamma}}$$

and

$$d_2(x,y,\Gamma) \;=\; d_1(x,y,\Gamma) \; - \; \sigma\sqrt{\Gamma}$$

Since $U(t,B_t) \;=\; K \; - \; B_t$, it follows from integral 13 that the boundary B_t satisfies the integral equation,

$$K - B_t = U_E(t,P) + \int_t^T [rKe^{-r(\Gamma-t)}N(-d_2(B_t,B_\Gamma,\Gamma-t)) \; - \; \mu B_t e^{-\mu(\Gamma-t)}N(-d_1(B_t,B_\Gamma,\Gamma-t))]d\Gamma \tag{14}$$

Kim (1990) proposed to solve this integral by approximating it with step function. The method presented by **Ait Sahlia (1999)** and **Lai (1999)** is a numerical improvement upon

Ju's method of solving the integral (14) (by piecewise exponential function) using suitable transformations under which the piecewise exponential boundary B_Γ becomes a piecewise linear function. The transformations introduced by them after assuming $\rho = \frac{r}{\sigma^2}$ and $\alpha = \frac{\mu}{r}$ are,

$$s = \sigma^2(t-T) \tag{15}$$

and

$$z = log(\frac{P}{K}) - (\rho - \alpha\rho - \frac{1}{2})s \tag{16}$$

Here, r is the risk free rate of return, while σ and μ are the volatility and dividend rate of the underlying stock respectively. With these transformations, what we now have to find is the function $\bar{z}(s)$ at various values of s (which is in turn a function of t) and linearly interpolate the function $\bar{z}(s)$ between the successive values of s. We substitute B_t by $Ke^{\bar{z}(s) + (\rho - \alpha\rho - \frac{1}{2})s}$. Under these transformations and substitutions equation (14) gets expressed in a simpler form. For details about the integral equation and its rearrangement for numerical computations, one can refer to the paper by **Ait Sahlia** and **Lai (1999)**. The mathematical formulation and working out of the expressions can be seen in the above mentioned paper. We, however, will present here only the brief outline of the idea behind solving the integral equations with the transformations mentioned.

The time period of the option (say T) is divided into a set of discrete time points. The idea is to evaluate the $\bar{z}(s)$ at the various points of s (which is a function of t). Having calculated the values at discrete time points we linearly interpolate the value of $\bar{z}(s)$ between successive time points to get the value in a continuous time set. After evaluating the value of $\bar{z}(s)$ at various points of s we can get the value of B_t only by substituting the value of $\bar{z}(s)$ in $Ke^{\bar{z}(s) + (\rho - \alpha\rho - \frac{1}{2})s}$ over the various intervals of s where the function $\bar{z}(s)$ has been linearly interpolated accordingly. We solve the integral equation on a grid of time points or the s values. For calculating the value of the function $\bar{z}(s)$ at the m^{th} value of s , we take into account all the $\bar{z}(s)$ values at the previous s values. The value of $\bar{z}(s)$ at each time point s is obtained by solving a non-linear equation as shown in the paper by **Ait Sahlia** and **Lai (1999)**. The solution at each time point is obtained recursively, having declared a candidate value for the solution at the start of recursion and then obtaining the value of $\bar{z}(s)$ for which the non-linear equations attain zero value or is beyond some pre-specified tolerance level. For details about the non-linear equation, its evaluation and solution, the paper by **Ait Sahlia** and **Lai (1999)** can be referred to as mentioned earlier.

The next section shows our evaluation of the early exercise boundary by the technique given by **Ait Sahlia** and **Lai (1999)**. We provide numerical results and graphs obtained in R which seem quite acceptable with the results cited in the paper by **Ait Sahlia** and **Lai** (1999).

4 Results and Discussion

We use even spacing in the choice of grid points i.e we divide the time interval $[0\ ,\ T]$ or the range of s $[-\sigma^2 T\ ,\ 0]$ (since, $s = \sigma^2(t - T)$) into intervals of equal length. We can choose δ, a small value, which is the gap between any two successive s values. The number of intervals in which the range of s gets divided is given by $n = \frac{\sigma^2 T}{\delta}$. The tolerance level used by us for solving the boundary (using a non-linear equation) is 10^{-4}.

The following tables and the corresponding graphs show our computation results for the early exercise boundary i.e. $\bar{z}(s)$ at various values of s using the method given by **Ait Sahlia** and **Lai** (1999).

Table 1: $\alpha = 1.2,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ **and** $\delta = 10^{-2}$ **:**

$-s$	$\bar{z}(s)$
0.03	-0.3915193
0.06	-0.4979153
0.09	-0.5706395
0.12	-0.6277348
0.15	-0.6756809
0.18	-0.7184308
0.21	-0.7565337
0.24	-0.7919107
0.27	-0.824343
0.30	-0.8547986

The above table shows the value of $\bar{z}(s)$ at only ten different values of s although the computation has been done for thirty ($n = \frac{\sigma^2 T}{\delta} = 30$, in this case) different values of s. Figure (1), the graph corresponding to the inputs for the above computed table is provided.

We change the value of δ from 10^{-2} to 5×10^{-3} and obtain the following table.

Table 2: $\alpha = 1.2,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ **and** $\delta = 5\times10^{-3}$ **:**

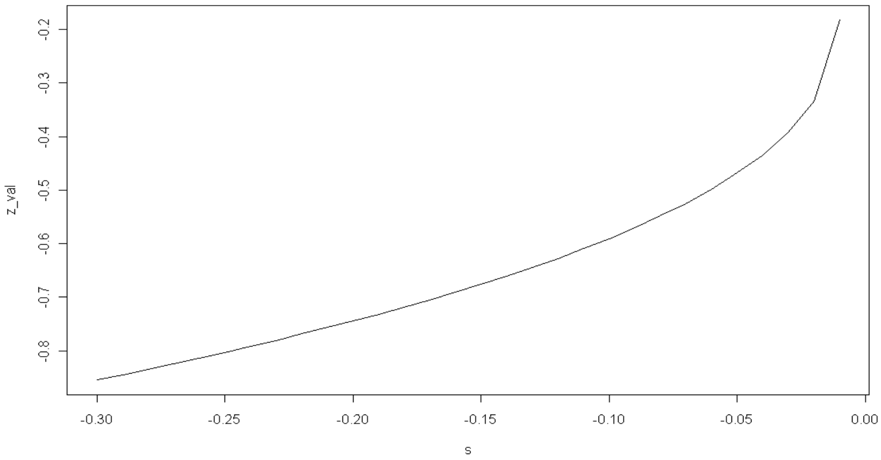

Figure 1: $\alpha = 1.2,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ and $\delta = 10^{-2}$

$-s$	$\bar{z}(s)$
0.03	-0.3974470
0.06	-0.4987005
0.09	-0.569923
0.12	-0.6269421
0.15	-0.675415
0.18	-0.7176445
0.21	-0.7561635
0.24	-0.7910882
0.27	-0.8235218
0.30	-0.854013

The above table shows the value of $\bar{z}(s)$ at only ten different values of s (as earlier) although the computation has been done for sixty ($n = \frac{\sigma^2 T}{\delta} = 60$, in this case) different values of s. Figure (2), the graph corresponding to the inputs for the above computed table is provided.

Figure (2) attempts to show that as δ become smaller i.e. the distance between the successive time points decreases ($\delta = 5{\times}10^{-3}$ here), the exercise boundary becomes smoother. Nonetheless, the boundary is already quite accurate with $\delta = 10^{-2}$. Thus, in our case it suffices to take $\delta = 10^{-2}$ without making it any smaller.

We use $\delta = 5{\times}10^{-3}$ here and change the value of α to 0.8 to obtain the following table.

Table 3: $\alpha = 0.8,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ **and** $\delta = 5{\times}10^{-3}$ **:**

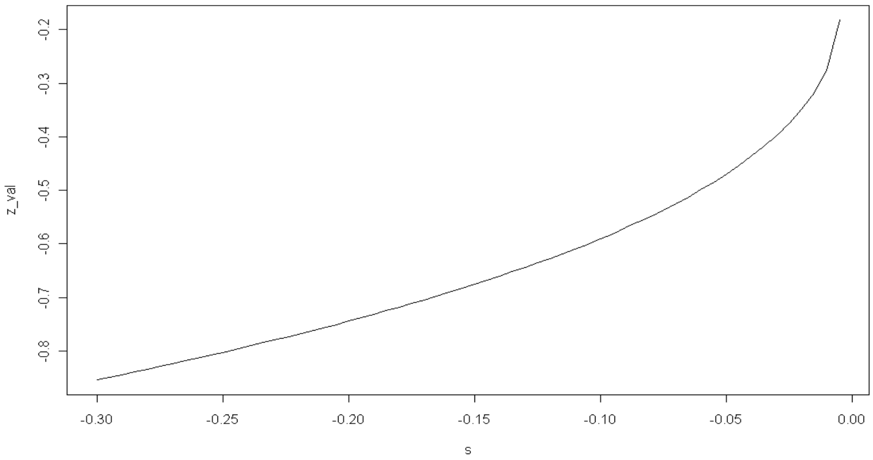

Figure 2: Plot of $\bar{z}(s)$ against s with $\alpha = 1.2,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ and $\delta = 5{\times}10^{-3}$

$-s$	$\bar{z}(s)$
0.03	-0.2659134
0.06	-0.4210514
0.09	-0.481041
0.12	-0.5261339
0.15	-0.5650311
0.18	-0.5994227
0.21	-0.6304025
0.24	-0.6587024
0.27	-0.6848577
0.30	-0.7091025

The above table shows the value of $\bar{z}(s)$ at only ten different values of s, as earlier, although the computation has been done for sixty ($n = \frac{\sigma^2 T}{\delta} = 60$, in this case) different values of s. Here, there is a minor difference with the previous two tables. In the previous two cases the value of α was 1.2 and for $\alpha > 1$ we initialize $\bar{z}(0)$ as $ln(\alpha)$, whereas for $\alpha{\le}1$, we initialize $\bar{z}(0)$ as zero. This matter has just been highlighted and is of no major significance. The exercise boundary for the previous cases as well as this case appear to be of the same shape, with only difference being that, with decreasing values of α the boundary moves up, though it looks almost same as far as its shape is concerned. This basically means that, as α becomes smaller , or the dividend becomes smaller ($\alpha = \mu/r$), the stock price level at which the option should be exercised at any point of time increases i.e. the early exercise boundary moves up. Figure (3) is the graph corresponding to the inputs for the above computed table.

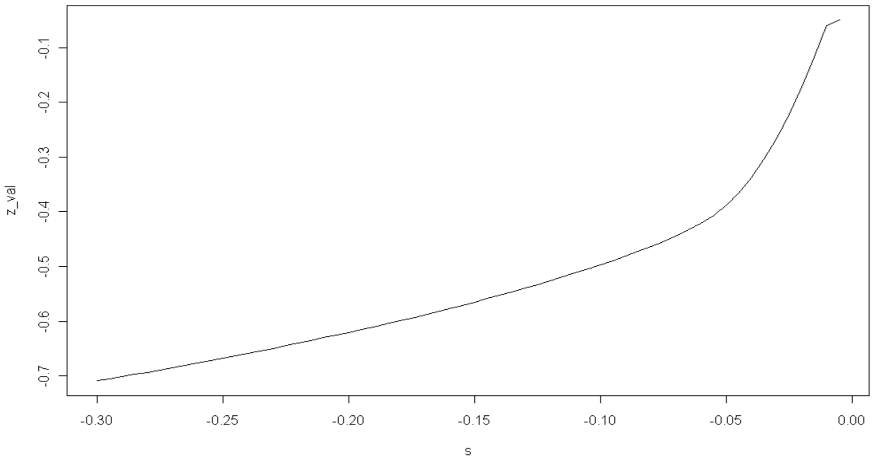

Figure 3: Plot of $\bar{z}(s)$ against s with $\alpha = 0.8,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ and $\delta = 5{\times}10^{-3}$

The following table produces the output for $\alpha = 0.001$ with the remaining parameters left unchanged.

Table 4: $\alpha = 0.001,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ **and** $\delta = 5{\times}10^{-3}$ **:**

$-s$	$\bar{z}(s)$
0.03	-0.2288506
0.06	-0.3186725
0.09	-0.3529692
0.12	-0.3792697
0.15	-0.4013473
0.18	-0.4202262
0.21	-0.4365501
0.24	-0.4508725
0.27	-0.4636028
0.30	-0.47503

The above table again shows the value of $\bar{z}(s)$ at only ten different values of s, as earlier, although the computation has been done for sixty ($n = \frac{\sigma^2 T}{\delta} = 60$, in this case) different values of s. However, this result is of significance and hence has been provided. Previously we presented tables with $\alpha = 1.2$ and $\alpha = 0.8$, both of which means there is a finite rate of dividend ($\alpha = \frac{\mu}{r}$). However, we might be interested in computing the boundary when there is no dividend at all. There is a bit of problem with the computation here, when $\alpha = 0$ as explanations follow.

The lower bound for $\bar{z}(s)$ as shown by Sahlia and Lai is given by,

$$\bar{z}_l(s) \;=\; -\,[\rho(1-\alpha)-\frac{1}{2}]s \;-\; ln[\frac{\theta}{(\theta-1)}] \tag{17}$$

where,

$$\theta \;=\; -\,[\rho(1-\alpha)-\frac{1}{2}] \;-\; ([\rho(1-\alpha)-\frac{1}{2}]^2+2\rho)^{0.5} \tag{18}$$

From equations (17) and (18) it can be seen that if ρ equals 0.5 and α equals 0 then the lower bound of $\bar{z}(s)$ for any s attains $-\infty$ (For θ, we take the negative root in equation (18). This is because the upper bound is given by $\bar{z}_u(s) \;=\; -\,[\rho(1-\alpha)-\frac{1}{2}]s \;-\; [ln\alpha]^+$ and the lower bound has to be smaller than the upper bound). This results in a difficulty in computation of the exercise boundary. Nonetheless, it can be calculated with some modifications in our code which we will not carry out for our computation. However, we, in the above table, showed the values of $\bar{z}(s)$ for $\alpha = 0.001$. Our objective in calculating the boundary for $\alpha = 0.001$ is to show that without the modifications required for the particular case with the lower bound attaining infinity,we can leave that as a special case and continue employing our code for small value of α and get satisfactory results. Below, the graph corresponding to the inputs for the above computed table is provided.

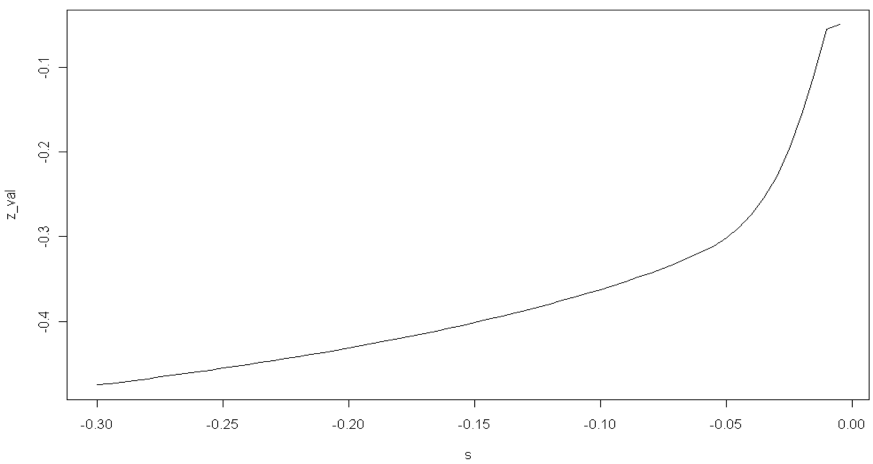

Figure 4: Plot of $\bar{z}(s)$ against s with $\alpha = 0.001,\ \rho = 0.5,\ \sigma^2 = 0.01,\ T = 30$ and $\delta = 5{\times}10^{-3}$

5 Extensions and Generalizations

In this section we will consider some extensions and generalizations of what we have done. We have carried out the computation using the technique given by Sahlia and Lai, in a simplistic manner. We have calculated the value of the function $\bar{z}(s)$ for different values of s i.e. we have calculated in the transformed co-ordinates. One can easily substitute the value of $\bar{z}(s)$ in B_t by $Ke^{\bar{z}(s)\ +\ (\rho\ -\ \alpha\rho\ -\ \frac{1}{2})s}$, to get the exercise boundary in the original co-ordinates. Also, having calculated the boundary (in original co-ordinates), one can find out the actual price of the American option which is infact the motivation behind all the research. We have only shown that our computation results for $\bar{z}(s)$ agree with the findings by Sahlia and Lai approximately, as that is sufficient to verify our computation results with those of Sahlia and Lai. Knowing the exercise boundary and the price of the option, one can find the hedge parameters as well. Nonetheless, the details of these are clearly stated in the paper by **Ait Sahlia** and **Lai (1999)**, which one can refer to.

6 Conclusion

The transformations applied by Sahlia and Lai to linearize the exponential piecewise functions for approximating the early exercise boundary is of particular significance. Although the method of solving the integral equation by piecewise exponential function, as shown by **Ju (1998)** is ingenious in its own way, improving the accuracy and computation speed over any other existing methods till then, the transformations proposed by **Ait Sahlia** and **Lai (1999)** simplify the computation all the more, thus improving the technique further and supporting Ju's findings as well at the same time. The idea to linearly interpolate the boundary, instead of using typical step-wise functions between successive time points is also of major significance. Our computation uses a simple, logical technique of using recursions to solve the non-linear equation. All these lead to an efficient approximation to the early exercise boundary that involves finding its values at few time points and linearly interpolating between them.

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