Volatility, Information Feedback and Market Microstructure Noise: A Tale of Two Regimes

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Introduction

- Major topic in financial econometrics over the last decade: 

  *How can we optimally use financial high-frequency data to construct efficient volatility estimators for aggregated periods (intraday, day, week, ...)?*

- Typical starting point:

  \[ p_{t_i} = p_{t_i}^* + \epsilon_i, \quad \epsilon_i \sim (0, \sigma^2), \quad i = 1, \ldots, n, \]

  \[ p_t^* = p_0^* + \int_0^t \sigma^*^2(s) dB_s, \quad t \in [0, T], \]

  where \( p_t \) denotes the (observed) log price, \( B_t \) is a standard Brownian motion \( B_t \), and \( \epsilon_i \) is microstructure "noise".

- Object of interest: \( \int_0^T \sigma^*^2(s) ds \), corresponding to the variance of a \( T \)-period return.
If $p_t$ are discretely observed with $p_{i/n}, i = 0, \ldots, n$, a natural estimator for $\int_0^T \sigma^2(s) \, ds$ is given by the realized variance,

$$RV^n = \sum_{i=1}^n \left( p_{i/n} - p_{(i-1)/n} \right)^2,$$

which is consistent and efficient with

$$n^{1/2} \left( RV^n - \int_0^1 \sigma^* s^2 \, ds \right) \xrightarrow{\mathcal{L}} N \left( 0, 2 \int_0^1 \sigma^* s^4 \, ds \right).$$

Suggestion: Sampling on highest possible frequency!
Real Intraday Price Path
Problem: HF prices are subject to noise, i.e., we only observe
\[ p_i = p_i^* + \epsilon_i, \quad i = 1, \ldots, n, \]
where \( \epsilon_i \) is associated with market microstructure "noise" (MMN).

⇒ If we let \( n \to \infty \), RV becomes biased and inconsistent.
⇒ If MMN is i.i.d., (log) returns \((p_i - p_{i-1})\) are negatively autocorrelated.
1-sec and 2-sec autocorrelations over 10min windows

⇒ MMN cannot be i.i.d.!
⇒ Noise properties change locally!
Huge literature on efficiently estimating $\sigma_{\varepsilon^*}^2$

- kernel estimators (Barndorff-Nielsen et al, 2008), pre-averaging (Jacod et al, 2009), MLE (Ait-Sahalia et al 2005), multi-scale estimators (Zhang 2006), spectral estimators (Reiss, 2011), ...

But:

- Assumptions on noise statistically motivated!
- Missing link to microstructure theory and trading behavior
- All approaches rely on the classical RW+Noise decomposition

$$p_i = p_i^* + \varepsilon_i, \quad i = 1, \ldots, n.$$
A Model With Information/Trading Feedback

- **Idea:** Model with mis-pricing component:
  \[ p_i = p_{i-1} - \alpha(p_{i-1} - p_{i-1}^*) + \epsilon_i \]

- Changes in observed prices are caused by two sources:
  - Market microstructure noise
  - Mis-pricing component due to deviations between observed prices and efficient prices ("error correction")

- **Reasoning:**
  - "Non-informational" shocks cause "mis-pricing"
  - Prices are permanently in dis-equilibrium

- Speed by which observed prices react to inherent mis-pricing governed by \( \alpha \) ⇒ Measuring "market efficiency"
Model captures two fundamental market regimes:

- Mis-pricing removed by "contrarian behavior"
  \[ \Rightarrow \text{negative autocorrelations in observed returns} \]
- Mis-pricing enforced by "momentum behavior"
  \[ \Rightarrow \text{positive autocorrelations in observed returns} \]

State of market driven by relationship between

- speed of price reversion \( \alpha \),
- noise-to-signal ratio.

Why important?

- Model opens up channels for market microstructure foundations of HF-based volatility estimation.
- HF-based assessment of market efficiency.
- Statistical implications!
Outline

1. Introduction
2. A Model with Information Feedback
3. Two Market Regimes
4. Estimation
5. Empirical Evidence
6. Model Generalization
7. Conclusions
2. A Model with Information Feedback
Setup

- Model in discrete time, i.e., \( i \in \{0, 1, 2, \ldots, n\} \) with \( n = T/\Delta \).
- Observed log prices \( p_i \) are assumed to be driven by

\[
p_{i+1} = p_i - \alpha (p_i - p_i^*) + \epsilon_{i+1}, \quad 0 < \alpha < 2, \quad \epsilon_{i+1} \sim \text{iid } N(0, \sigma^2_{\epsilon}),
\]

\[
p_{i+1}^* = p_i^* + \epsilon_{i+1}^*, \quad \epsilon_{i+1}^* \sim \text{iid } N(0, \sigma^2_{\epsilon^*}), \quad \mathbb{E}[\epsilon_{i+1}\epsilon_{i+1}^*] = 0.
\]

- \( \mu_i := p_i - p_i^* \) mis-pricing component.
- \( \alpha \): speed of price reversion
2. A Model with Information Feedback

Simulations of $p_i$ and $p_i^*$ for different $\alpha$
Alternative Representation

- Model can be written as

\[
\begin{align*}
    p_i &= p_i^* + \mu_i, \\
    p_i^* &= p_{i-1}^* + \varepsilon_i^*, \quad \varepsilon_i^* \sim i.i.d. \ N(0, \sigma_{\varepsilon^*}^2), \\
    \mu_i &= (1 - \alpha) \mu_{i-1} + \epsilon_i^\mu,
\end{align*}
\]

where \( \epsilon_i^\mu := \epsilon_i - \varepsilon_i^* \sim i.i.d. \ W N(0, \sigma_\mu^2) \) with \( \sigma_\mu^2 := \sigma_{\varepsilon^*}^2 + \sigma_\varepsilon^2. \)

- \( \mu_i \) follows mean zero AR(1) process with

\[
\mathbb{V}[\mu_i] = \frac{\sigma_\mu^2}{\alpha(2 - \alpha)}
\]
2. A Model with Information Feedback

- The error covariance matrix $\Sigma$ is given by

$$\Sigma : = \begin{bmatrix}
E[(\varepsilon_i^\mu)^2] & E[\varepsilon_i^\mu \varepsilon_i^*] \\
E[\varepsilon_i^\mu \varepsilon_i^*] & E[(\varepsilon_i^*)^2]
\end{bmatrix} = \begin{bmatrix}
E[\mu_i^2] & E[\mu_i \varepsilon_i^*] \\
E[\mu_i \varepsilon_i^*] & E[(\varepsilon_i^*)^2]
\end{bmatrix}$$

$$= \begin{bmatrix}
\sigma_{\varepsilon}^2 + \sigma_{\varepsilon^*}^2 & -\sigma_{\varepsilon^*}^2 \\
-\sigma_{\varepsilon^*}^2 & \sigma_{\varepsilon^*}^2
\end{bmatrix}$$

with $E[\varepsilon_i^* \varepsilon_i^\mu_{i-h}] = 0 \quad \forall \; h$.

- Observed returns $r_i = p_i - p_{i-1}$ are then given by

$$r_i = -\alpha \mu_{i-1} + \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} N(0, \sigma_{\varepsilon}^2),$$

with

$$E[\varepsilon_i \mu_{i+h}] = (1 - \alpha)^h \sigma_{\varepsilon}^2 \quad \forall h \geq 0$$
$$E[\varepsilon_i \mu_{i-h}] = 0 \quad \forall h > 0$$
$$E[\varepsilon_i \varepsilon_i^\mu] = \sigma_{\varepsilon}^2$$
$$E[\varepsilon_i \varepsilon_i^\mu_{i-h}] = 0 \quad \forall h \neq 0$$
Illustration of $\nabla [\mu_i] / \sigma^2_{\mu}$ depending on $\alpha$ for $\alpha \in (0, 2)$. 
Special Case $\alpha = 1$ (Perfect "Efficiency")

- For $\alpha = 1$ we obtain

\[ p_{i+1} = p_i^* + \epsilon_{i+1} = p_{i+1}^* + \epsilon_{i+1}^\mu, \]

where $\epsilon_{i}^\mu := \epsilon_i - \epsilon_i^* = p_i - p_i^* = \mu_i$ is iid with

\[ \nabla[\epsilon_{i}^\mu] = \sigma_{\mu_i}^2 = \sigma_{\epsilon_i}^2 + \sigma_{\epsilon_i^*}^2 \]

\[ \mathbb{E}[\epsilon_i^* \epsilon_i^\mu] = \mathbb{E}[(p_i^* - p_i^*_{i-1})\epsilon_i^\mu] = -\sigma_{\epsilon_i^*}^2 \]

$\Rightarrow$ RW plus endogenous iid noise!

$\Rightarrow$ $\mathbb{E}[r_i, r_{i-1}] = -\sigma_{\epsilon_i}^2$

$\Rightarrow$ Endogeneity structurally built into the model!
3. Two Market Regimes
Return Variances

The return variance is given by

\[
\mathbb{V}[r_i] = \sigma^2_\epsilon + \alpha^2 \mathbb{V}[\mu_i] = \frac{1}{2 - \alpha} (2\sigma^2_\epsilon + \alpha \sigma^2_\epsilon^*) .
\]

implying \( \mathbb{V}[r_i] \geq \sigma^2_\epsilon \).

We define the *noise-to-signal ratio* \( \lambda \) as

\[
\lambda = \frac{\sigma^2_\epsilon}{\sigma^2_\epsilon^*} .
\]

Then, the unconditional return variance is given by

\[
\mathbb{V}[r_i] = \sigma^2_\epsilon^* \frac{2\lambda + \alpha}{2 - \alpha}.
\]
It follows that

\[ \mathbb{V}[r_i] \leq \mathbb{V}[r_i^*] \quad \text{if} \quad \lambda \leq 1 - \alpha, \]
\[ \mathbb{V}[r_i] > \mathbb{V}[r_i^*] \quad \text{otherwise}. \]

\( \Rightarrow \) If proportion of ”informational variance” (\( \lambda \)) is high, and \( p_i \) sluggishly follows \( p_i^* \), changes in efficient price are passed over to the observed price in mitigated way.
Regimes in Return Autocovariances

- **Lemma.** Assume $\sigma^2_\epsilon > 0$, $0 < \alpha < 2$, and $h \geq 1$. Then,

$$\text{Cov}[r_i, r_{i-h}] = \psi(h-1) \sigma^2_\epsilon \frac{(1 - \alpha - \lambda)}{2 - \alpha},$$

with $\psi(h-1) = \alpha (1 - \alpha)^{h-1}$, and $\psi(0) = 1$, if $\alpha = 1$.

- **Corollary.** Assume $\sigma^2_\epsilon > 0$, $0 < \alpha < 2$, and $h \geq 1$.

  (i) If $0 < \alpha < 1$, then
  $$\text{sgn}\{\text{Cov}[r_i, r_{i-h}]\} = \text{sgn}\{(1 - \alpha) - \lambda\}.$$  
  
  (ii) If $\alpha = 1$, then $\text{Cov}[r_i, r_{i-1}] = -\sigma^2_\epsilon < 0$, and $\text{Cov}[r_i, r_{i-h}] = 0$, for $h > 1$.

  (iii) If $1 < \alpha < 2$, then $\text{sgn}\{\text{Cov}[r_i, r_{i-h}]\} = \text{sgn}\{(-1)^h\}$. 

3. Two Market Regimes

\[ \text{Cov}[r_i, r_{i-h}] = 0 \] holds as long as

\[ \lambda = 1 - \alpha \]

**Implications:**

- If \( \alpha = 1 \) and there is noise (\( \lambda > 0 \)) returns cannot be uncorrelated!

- As long there is noise (\( \lambda > 0 \)), price updating must be sluggish (\( \alpha < 1 \)) to ensure \( \text{Cov}[r_i, r_{i-h}] = 0 \)!
Implications for the Realized Variance

- Consider log prices, $p_0, p_\Delta, \ldots, p_{i\Delta}, \ldots, p_T$ at equidistant points $i = 1, \ldots, T/\Delta - 1, T/\Delta$, with grid size $\Delta$ and $n = T/\Delta$ and $r_{i\Delta} = p_{i\Delta} - p_{(i-1)\Delta}$.

- The realized return variance measure at time $T$ is given as

$$RV_T^\Delta = \sum_{i=1}^{T/\Delta} r_{i\Delta}^2.$$
**Theorem.** For $0 < \alpha < 1$, the expected time-$T$ realized variance sampled at calendar time grid size $\Delta$ equals,

$$\langle p \rangle_T^\Delta = T \cdot \sigma_{\varepsilon^*}^2 + T \cdot \sigma_{\varepsilon^*}^2 \cdot \phi(\Delta) \frac{\lambda - (1 - \alpha)}{(2 - \alpha)},$$

with

$$\phi(\Delta) = \frac{2}{\alpha \Delta} \left( 1 - (1 - \alpha)^\Delta \right).$$

The mapping $\Delta \mapsto \phi(\Delta)$, from $\mathbb{R}_+$ into $(0, -\frac{2}{\alpha} \ln(1 - \alpha))$ is strictly decreasing with

(i) $\lim_{\Delta \to 0} \phi(\Delta) = -\frac{2}{\alpha} \ln(1 - \alpha)$,

(ii) $\lim_{\Delta \to \infty} \phi(\Delta) = 0$. 
Volatility Signature Plots

(i) if $\lambda > (1 - \alpha)$, then $T \cdot \sigma^2 * \epsilon < \langle p \rangle_T^\Delta$,
(ii) if $\lambda < (1 - \alpha)$, then $T \cdot \sigma^2 * \epsilon > \langle p \rangle_T^\Delta$. 

\begin{center}
\begin{tikzpicture}
\begin{axis}[
width=\textwidth,
height=10cm,
axis lines=left,
xtick={0,2,4,6,8,10},
ytick={0.6,0.8,1.0,1.2,1.4},
xticklabels={$0$,$2$,$4$,$6$,$8$,$10$},
yticklabels={$0.6$,$0.8$,$1.0$,$1.2$,$1.4$},
xlabel=$\Delta$
]
\addplot[green,thick] coordinates { (0,1.4) (10,0.6) };
\addplot[blue,thick] coordinates { (0,1.4) (10,0.6) };
\addplot[cyan,thick] coordinates { (0,1.4) (10,0.6) };
\addplot[teal,thick] coordinates { (0,1.4) (10,0.6) };
\addplot[olive,thick] coordinates { (0,1.4) (10,0.6) };
\addplot[gray,thick] coordinates { (0,1.4) (10,0.6) };
\addplot[black,thick] coordinates { (0,1.4) (10,0.6) };
\end{axis}
\end{tikzpicture}
\end{center}
4. Estimation
State-Space Representation

- Denote $X_i$ as a state vector at $i$ with $X_i := (\mu_i \mu_{i-1} \epsilon_i)$.
- Then, $r_i$ can be written as
  \[ r_i = FX_i, \]
  \[ X_i = GX_{i-1} + w_i \]
  with $F = (0 \ -\alpha \ 1)$ and
  \[
  G = \begin{pmatrix}
  (1 - \alpha) & 0 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 0
  \end{pmatrix}, \quad w_i = \begin{pmatrix}
  \epsilon_i^\mu \\
  0 \\
  \epsilon_i
  \end{pmatrix}, \quad \Sigma_w = \begin{pmatrix}
  \sigma^2_\epsilon + \sigma^2_{\epsilon^*} & 0 & \sigma^2_\epsilon \\
  0 & 0 & 0 \\
  \sigma^2_\epsilon & 0 & \sigma^2_\epsilon
  \end{pmatrix}.
  \]

- Parameters can be estimated by maximum likelihood using the Kalman filter.
Alternative: Moment Estimation

- We can employ the unconditional moment restrictions

\[
\phi_1(r_i; \alpha; \sigma^2_\varepsilon, \sigma^2_{\varepsilon*}) = \sigma^2_\varepsilon n - \sigma^2_\mu n \phi(\Delta) \frac{1 - \alpha - \lambda}{(2 - \alpha)(\lambda + 1)} - \sum_{i=1}^{n} r_i^2
\]

\[
\phi_2(r_i; \alpha; \sigma^2_\varepsilon, \sigma^2_{\varepsilon*}) = r_i^2 - \frac{1}{2 - \alpha} (2\sigma^2_\varepsilon + \alpha \sigma^2_{\varepsilon*}),
\]

\[
\phi_{2,h}(r_i; \alpha; \sigma^2_\varepsilon, \sigma^2_{\varepsilon*}) = r_i r_{i-h} - \psi(h - 1)\sigma^2_{\varepsilon*} \frac{1 - \alpha - \lambda}{2 - \alpha},
\]

with \( \psi(h) = \alpha(1 - \alpha)^h \geq 0 \) and \( h = 1, 2, \ldots \).
A GMM estimator can be formulated as

\[ \hat{\theta}(\mathcal{W}) = \arg \min_\theta \left[ \frac{1}{n} \sum_{i=1}^{n} \tilde{m}(r_i; \theta) \right]' \mathcal{W}_n \left[ \frac{1}{n} \sum_{i=1}^{n} \tilde{m}(r_i; \theta) \right], \]

where \( \theta = (\alpha, \sigma_{\varepsilon}^2, \sigma_{\varepsilon^*}^2)' \), while
\( \tilde{m}(r_i; \theta) = (\phi_1(r_i; \theta), \phi_2(r_i; \theta), \phi_{2,1}(r_i; \theta), \phi_{2,2}(r_i; \theta), \ldots) \) represents a set of model implied moment conditions, and \( \mathcal{W}_n \) is a conforming positive definite weighting matrix.
5. Empirical Evidence
5. Empirical Evidence

Data

- Data sampled from LOBSTER (https://lobsterdata.com/)
- Mid-quote returns from NASDAQ 100 constituents, first 40 trading days of 2014

Empirical distribution of per-stock averages of daily mid-quote revisions
## Significant first-order return autocorrelations

<table>
<thead>
<tr>
<th></th>
<th>Two-Sided</th>
<th>One-Sided $\text{Cor}(r_i, r_{i-1}) &lt; 0$</th>
<th>One-Sided $\text{Cor}(r_i, r_{i-1}) &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta$</td>
<td>1%</td>
<td>5%</td>
</tr>
<tr>
<td>$T = 30\text{min}$</td>
<td>$N = 41,600$</td>
<td>25.8</td>
<td>37.2</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>1sec</td>
<td>15.7</td>
<td>27.3</td>
</tr>
<tr>
<td></td>
<td>2sec</td>
<td>9.4</td>
<td>19.7</td>
</tr>
<tr>
<td></td>
<td>5sec</td>
<td>6.4</td>
<td>15.4</td>
</tr>
<tr>
<td>$T = 10\text{min}$</td>
<td>$N = 124,800$</td>
<td>17.9</td>
<td>28.7</td>
</tr>
<tr>
<td></td>
<td>1sec</td>
<td>10.3</td>
<td>19.9</td>
</tr>
<tr>
<td></td>
<td>2sec</td>
<td>5.0</td>
<td>13.0</td>
</tr>
<tr>
<td></td>
<td>5sec</td>
<td>3.0</td>
<td>9.3</td>
</tr>
<tr>
<td>$T = 5\text{min}$</td>
<td>$N = 249,600$</td>
<td>14.1</td>
<td>23.9</td>
</tr>
<tr>
<td></td>
<td>1sec</td>
<td>7.9</td>
<td>15.9</td>
</tr>
<tr>
<td></td>
<td>2sec</td>
<td>3.5</td>
<td>9.6</td>
</tr>
<tr>
<td></td>
<td>5sec</td>
<td>1.5</td>
<td>6.0</td>
</tr>
</tbody>
</table>

---

5. Empirical Evidence
Prop. of sign. ACFs (10% level), $T \in \{5, 10, 30\}$ min
Proportion of stocks with significant window-to-window ACF, $T \in \{5, 10, 30\}$ min
5. Empirical Evidence

Distribution of Parameter Estimates

- \( \alpha \)
- \( \lambda \)
- \( \lambda - (1 - \alpha) \)
- \( \text{Cor}(r_i, r_{i-1}) \)
- \( \sigma^2_{\epsilon} \)
- \( \sigma^2_z \)

- Group 1
- Group 2
- Group 3
- Group 4
- Group 5
- All
## Summary Statistics of Estimates

### $T = 10\text{min}, \Delta = 1\text{sec}$

<table>
<thead>
<tr>
<th></th>
<th>$q5$</th>
<th>$q25$</th>
<th>Median</th>
<th>Mean</th>
<th>$q75$</th>
<th>$q95$</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>0.778</td>
<td>0.864</td>
<td>0.909</td>
<td>0.909</td>
<td>0.963</td>
<td>1.052</td>
<td>0.105</td>
</tr>
<tr>
<td>$\hat{\lambda}$</td>
<td>0.077</td>
<td>0.088</td>
<td>0.096</td>
<td>0.102</td>
<td>0.106</td>
<td>0.129</td>
<td>0.073</td>
</tr>
<tr>
<td>$\hat{\lambda} - (1 - \hat{\alpha})$</td>
<td>-0.150</td>
<td>-0.048</td>
<td>0.005</td>
<td>0.011</td>
<td>0.069</td>
<td>0.181</td>
<td>0.124</td>
</tr>
<tr>
<td>$\bar{\text{Cor}}(r_i, r_{i-1})$</td>
<td>-0.144</td>
<td>-0.056</td>
<td>-0.000</td>
<td>-0.008</td>
<td>0.041</td>
<td>0.121</td>
<td>0.081</td>
</tr>
<tr>
<td>$\hat{\sigma}_{\varepsilon}^2 \cdot (10^{-8})$</td>
<td>0.060</td>
<td>0.140</td>
<td>0.254</td>
<td>0.563</td>
<td>0.517</td>
<td>1.850</td>
<td>1.219</td>
</tr>
<tr>
<td>$\hat{\sigma}_{\epsilon}^2 \cdot (10^{-8})$</td>
<td>0.005</td>
<td>0.013</td>
<td>0.025</td>
<td>0.055</td>
<td>0.050</td>
<td>0.180</td>
<td>0.125</td>
</tr>
</tbody>
</table>

### $T = 10\text{min}, \Delta = 2\text{sec}$

<table>
<thead>
<tr>
<th></th>
<th>$q5$</th>
<th>$q25$</th>
<th>Median</th>
<th>Mean</th>
<th>$q75$</th>
<th>$q95$</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>0.726</td>
<td>0.846</td>
<td>0.905</td>
<td>0.891</td>
<td>0.961</td>
<td>1.056</td>
<td>0.138</td>
</tr>
<tr>
<td>$\hat{\lambda}$</td>
<td>0.072</td>
<td>0.086</td>
<td>0.095</td>
<td>0.104</td>
<td>0.106</td>
<td>0.138</td>
<td>0.101</td>
</tr>
<tr>
<td>$\hat{\lambda} - (1 - \hat{\alpha})$</td>
<td>-0.210</td>
<td>-0.070</td>
<td>-0.001</td>
<td>-0.004</td>
<td>0.066</td>
<td>0.188</td>
<td>0.158</td>
</tr>
<tr>
<td>$\bar{\text{Cor}}(r_i, r_{i-1})$</td>
<td>-0.149</td>
<td>-0.053</td>
<td>0.000</td>
<td>0.002</td>
<td>0.058</td>
<td>0.151</td>
<td>0.091</td>
</tr>
<tr>
<td>$\hat{\sigma}_{\varepsilon}^2 \cdot (10^{-8})$</td>
<td>0.116</td>
<td>0.279</td>
<td>0.516</td>
<td>1.177</td>
<td>1.058</td>
<td>3.875</td>
<td>2.670</td>
</tr>
<tr>
<td>$\hat{\sigma}_{\epsilon}^2 \cdot (10^{-8})$</td>
<td>0.009</td>
<td>0.026</td>
<td>0.048</td>
<td>0.118</td>
<td>0.101</td>
<td>0.394</td>
<td>0.287</td>
</tr>
</tbody>
</table>
Volatility Signature Plots
TS Plots of Estimates of Yahoo and Microsoft
Intraday Seasonalities

5. Empirical Evidence
Temporal Aggregation

Let $p_i$, sampled at step size $\Delta \geq 1$, with observations for $i = 0, \Delta, 2 \cdot \Delta, \ldots T - \Delta$, governed by

$$p_{i+\Delta} = p_i - \alpha \Delta (p_i - p_i^*) + \varepsilon_{i+\Delta,\Delta}, \quad p_{i+\Delta}^* = p_i^* + \hat{\varepsilon}_{i+\Delta,\Delta}^*$$

where $\varepsilon_{i+\Delta,\Delta} \sim \mathcal{N}(0, \sigma_{\varepsilon,\Delta}^2)$ and $\hat{\varepsilon}_{i+\Delta,\Delta}^* \sim \mathcal{N}(0, \sigma_{\hat{\varepsilon}^*,\Delta}^2)$ for all $i \in \{k \cdot \Delta, k = 0, 1, 2, 3, \ldots\}$.

Then, for $\Delta \geq 1$ and $0 < \alpha < 1$, we have,

$$\alpha_{\Delta} = 1 - (1 - \alpha)^{\Delta}, \quad \sigma_{\varepsilon,\Delta}^2 = g_\varepsilon \sigma_{\varepsilon}^2 + g_{\varepsilon^*} \sigma_{\varepsilon^*}^2, \quad \sigma_{\varepsilon^*,\Delta}^2 = \Delta \cdot \sigma_{\varepsilon^*},$$

with $g_\varepsilon$ and $g_{\varepsilon^*}$ denoting two functions depending on $\alpha$ and $\Delta$. 
5. Empirical Evidence

- $\alpha_\Delta$ is strictly increasing in $\Delta$, with $\lim_{\Delta \to \infty} \alpha_\Delta = 1$

- For the noise-to-signal ratio $\lambda_\Delta$ for models estimated at lower frequencies we have

$$\lambda_\Delta = \frac{\sigma^2_{\varepsilon,\Delta}}{\sigma^2_{\varepsilon^*,\Delta}} = \frac{1}{\Delta} \left( g_{\varepsilon} \lambda + g_{\varepsilon^*} \right)$$

with $\lim_{\Delta \to \infty} \lambda_\Delta = 1$. 
Temporal Aggregation
6. Model Generalization
6. Model Generalization

- Assume the model

\[ p_{i+1} = p_i - \alpha (p_i - p_i^*) + \epsilon_{i+1}, \quad \epsilon_{i+1} \sim \text{iid } N(0, \sigma_\epsilon^2), \]
\[ p_{i+1}^* = p_i^* + \epsilon_{i+1}^*, \quad \epsilon_{i+1}^* \sim \text{iid } N(0, \sigma_{\epsilon^*}^2), \]

with \( \mathbb{E}[\epsilon_{i+1}\epsilon_{i+1}^*] = \gamma \neq 0. \)

- For \( \gamma = \alpha \sigma_{\epsilon^*}^2 \): Model by Amihud & Mendelson (1987):

\[ p_i = p_{i-1} - \alpha (p_{i-1} - p_i^*) + \tilde{\epsilon}_i, \]

with \( \tilde{\epsilon}_i := \epsilon_i - \alpha \epsilon_i^* \) and \( \mathbb{E}[\tilde{\epsilon}_i\epsilon_i^*] = 0. \)

- For \( \gamma = \sigma_{\epsilon^*}^2 \): RW-plus-iid-noise model

\[ p_i = p_i^* + \epsilon_i \quad \text{with } \mathbb{E}[\epsilon_i\epsilon_i^*] = 0. \]

- For \( \gamma = 0 \), we obtain the original model.
Estimates for 2 days for AAPL
LR Tests for $H_0 : \gamma = \alpha \sigma_\varepsilon^2$ (left) and $H_0 : \gamma = 0$ (right)

Top: $T = 10\text{min}, \Delta = 2\text{secs}$; Bottom: $T = 10\text{min}, \Delta = 5\text{secs}$
8. Conclusions
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Conclusions

▶ Evidence for a model with information feedback
▶ Show mostly sluggish price updating due to mis-pricing
▶ Extent of market efficiency varies over time ⇒ Identification of local states of "contrarian trading" and "momentum trading"
▶ Strong intraday and cross-sectional variation

Implications

▶ Channels for bridging the gap between high-frequency statistics and market microstructure theory
▶ New implications for volatility estimation
▶ Can be extended in various directions