Proof of the Riemann hypothesis.

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Abstract

A proof of the Riemann hypothesis is presented.

The hypothesis

All non–trivial zeros of the Riemann-function ζ are located on the vertical line $\mathrm{re}(s) = -\zeta(0)$ in the complex plane where $s = \sigma + \tau i \in \mathbb{C}$ and $(\sigma, \tau) \in \mathbb{R}^2$. The real and imaginary part of a complex number s are denoted by $\mathrm{re}(s)$ and $\mathrm{im}(s)$ respectively.

The proof

Sub–Lemma **1** : The sequence of trivial zeros of $\zeta(s) = 0$ is infinite.

Proof **1** : According to the functional equation of the zeta function $\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s)$, and $\zeta(0) = -\frac{1}{2}$ with $0 \equiv 0 + 0i$, all trivial zeros of ζ, i.e. the set $\{s \in \mathbb{C} \,|\, \zeta(s) = 0 \wedge \mathrm{im}(s) = 0\} = \{-2n \,|\, n \in \mathbb{N}\} = -2\mathbb{N}$ is infinite. □

Since the Γ–function has no zeros, only poles on non–positive integers, we may interpret the functional equation as showing us that the invariant operation of the zeta function is a combined operation of a parity operator $\mathcal{P} : s \mapsto t \doteq 1 - s$, a complex amplitude rescaling operator $\mathcal{R} : t \mapsto \frac{\Gamma(1-t)}{\pi^{1-t}}$ and a complex temporal signal modulating operator $\mathcal{T} : t \mapsto 2^t \sin(t\,\frac{\pi}{2})$

In other words :

$$\zeta = \mathcal{T} * \mathcal{R} * \zeta \circ \mathcal{P}, \text{ i.e. } \zeta(t) = \mathcal{T}(t) \cdot \mathcal{R}(t) \cdot \zeta(1-t) \qquad (1)$$

With M denoting a subset of $\mathbb{C}$ and $\nu(M)$ being the number of non–trivial zeros of the ζ–function in M we may formulate the following lemma.

Lemma **1 (2)** : $\nu\{\frac{1}{2} + \tau i \,|\, \tau \in \mathbb{R}\} = \infty$

Proof **2** : This follows from early works of Hardy, Bohr and Landau (1914) who not only proved this lemma, but were further able to show, that

$$\forall \varepsilon \in \mathbb{R}_+ : \forall (\sigma, \tau) \in \mathbb{R}^2 : \lim_{T \to \infty} \frac{\nu\{\sigma + \tau i \,|\, 0 \leq \tau \leq T \,\wedge\, \frac{1}{2} - \varepsilon \leq \sigma \leq \frac{1}{2} + \varepsilon\}}{\frac{T}{2\pi}(\ln \frac{T}{2\pi} - 1)} = 1$$

Proof of the Riemann Hypothesis

At infinity, the combined amplitudes of $\mathcal{R}$ and $\mathcal{T}$ are being damped by the the proof of *Lemma* 1. The limiting points $(\frac{1}{2}, +(\infty - \delta)i)$ and $(\frac{1}{2}, -(\infty - \delta)i)$ where δ is an arbitrarily small real number, are both residing on the top and bottom of a line, given by $\{(x, y) \in \mathbb{R}^2 \,|\, x = \frac{1}{2}\}$ We may safely connect the two symmetric points through a unique line – where the clarity of the uniqueness is given by the fact, that these points are converging at infinity. $\square$

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