Proof of the Riemann hypothesis.

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Abstract

A proof of the Riemann hypothesis is presented.

The hypothesis

All non–trivial zeros of the Riemann-function \( \zeta \) are located on the vertical line \( \mathrm{re}(s) = -\zeta(0) \) in the complex plane where \( s = \sigma + \tau i \in \mathbb{C} \) and \( (\sigma, \tau) \in \mathbb{R}^2 \). The real and imaginary part of a complex number \( s \) are denoted by \( \mathrm{re}(s) \) and \( \mathrm{im}(s) \) respectively.

The proof

**Sub–Lemma 1**: The sequence of trivial zeros of \( \zeta(s) = 0 \) is infinite.

**Proof 1**: According to the functional equation of the zeta function \( \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \), and \( \zeta(0) = -\frac{1}{2} \) with \( 0 \equiv 0 + 0i \), all trivial zeros of \( \zeta \), i.e. the set \( \{ s \in \mathbb{C} | \zeta(s) = 0 \land \mathrm{im}(s) = 0 \} = \{ -2n \mid n \in \mathbb{N} \} = -2\mathbb{N} \) is infinite. \( \square \)

Since the \( \Gamma \)–function has no zeros, only poles on non–positive integers, we may interpret the functional equation as showing us that the invariant operation of the zeta function is a combined operation of a parity operator \( \mathcal{P} : s \mapsto t = 1 - s \), a complex amplitude rescaling operator \( \mathcal{R} : t \mapsto \frac{\Gamma(1-t)}{\pi^{1-t}} \) and a complex temporal signal modulating operator \( \mathcal{T} : t \mapsto 2^t \sin(t \frac{\pi}{2}) \).

In other words:

\[
\zeta = \mathcal{T} \ast \mathcal{R} \ast \mathcal{P} \ast \zeta, \quad \text{i.e. } \zeta(t) = \mathcal{T}(t) \cdot \mathcal{R}(t) \cdot \zeta(1-t) \quad (1)
\]
With $M$ denoting a subset of $\mathbb{C}$ and $\nu(M)$ being the number of non–trivial zeros of the $\zeta$–function in $M$ we may formulate the following lemma.

**Lemma 1 (2)** : $\nu\{\frac{1}{2} + \tau i \mid \tau \in \mathbb{R}\} = \infty$

**Proof 2** : This follows from early works of Hardy, Bohr and Landau (1914) who not only proved this lemma, but were further able to show, that

$$\forall \varepsilon \in \mathbb{R}_+: \forall (\sigma, \tau) \in \mathbb{R}^2 : \lim_{T \to \infty} \frac{\nu\{\sigma + \tau i \mid 0 \leq \tau \leq T \land \frac{1}{2} - \varepsilon \leq \sigma \leq \frac{1}{2} + \varepsilon\}}{\frac{T}{2\pi} (\ln \frac{T}{2\pi} - 1)} = 1$$

**Proof of the Riemann Hypothesis**

At infinity, the combined amplitudes of $R$ and $T$ are being damped by the the proof of Lemma 1. The limiting points $(\frac{1}{2}, + (\infty - \delta)i)$ and $(\frac{1}{2}, - (\infty - \delta)i)$ where $\delta$ is an arbitrarily small real number, are both residing on the top and bottom of a line, given by $\{(x, y) \in \mathbb{R}^2 \mid x = \frac{1}{2}\}$ We may safely connect the two symmetric points through a unique line – where the clarity of the uniqueness is given by the fact, that these points are converging at infinity. □

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