SHORT UNIVERSAL GENERATORS VIA GENERALIZED RATIO-OF-UNIFORMS METHOD

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ABSTRACT. We use inequalities to design short universal algorithms that can be used to generate random variates from large classes of univariate continuous or discrete distributions (including all log-concave distributions). The expected time is uniformly bounded over all these distributions for a particular generator. The algorithms can be implemented in a few lines of high level language code.

1. INTRODUCTION

In the last decade several approaches have been introduced for so called universal (or black box) methods for generating non-uniform random variates. Recent papers propose methods where a hat function that approximates the respective probability density function or probability vector is constructed (see e.g. Ahrens (1993), Hörmann (1995), Ahrens (1995), Evans and Swartz (1998), Leydold (2000a), Leydold (2000b); Hörmann and Derflinger (1996), Hörmann and Derflinger (1997)). These methods have (extremely) fast marginal generation time, but require a setup step, which is expensive compared to the average cost of generating one random variate. Although this setup step can be made short at the price of a much higher marginal generation time (e.g. Gilks and Wild 1992) the resulting algorithms are rather complex.

Another approach by Devroye (1984b) on the other hand uses universal inequalities that hold for every log-concave distribution. The algorithm for continuous random variates is based on the following proposition.

Theorem 1 (Devroye (1986), §VII.2.5, Theorem 2.4)**.** *If f is a log-concave density with mode $\mu = 0$ and $f(0) = 1$, then writing q for $F(0)$, where F denotes the c.d.f. of the distribution, we have*

$$f(x) \le \begin{cases} \min(1, e^{1-x/(1-q)}), & (x \ge 0) \\ \min(1, e^{1+x/q}), & (x < 0) \end{cases} \tag{1}$$

The area under the bounding curve is 2.

If $F(\mu)$ is not known, a modified universal hat exists with area 4 (see Devroye (1986), §VII.2.3). In both cases these universal hats are not optimal. Devroye (1984b) derives the properties of the optimal hat and provides a (rather expensive) generator for the corresponding distribution. The areas below the optimal bounding curves are $\pi^2/6$ and $\pi^2/3$, respectively, i.e., about 18% better. Algorithms that

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utilize this theorem can be found in Devroye (1986). Devroye (1987) gives an analogous result for discrete log-concave distributions.

In Leydold (2001) the ratio-of-uniforms method is used to derive more general inequalities that were used to compile faster and simpler algorithms that work for a larger class of distributions, including all log-concave distributions. As in Devroye's algorithm and in opposition to other black-box algorithms, hardly any setup step is required. Thus it is superior in the changing parameter case.

In this paper this approach is extended. We introduce universal bounding curves based on the generalized ratio-of-uniforms method by Wakefield, Gelfand, and Smith (1991). The new algorithms are still universal and the expected numbers of uniform random numbers are uniformly bounded for each of these algorithms. They are applicable to a large class of so called T-concave distributions (Hörmann 1995) at the expense of higher marginal generation times. For the subclass of heavy-tailed T-concave distributions these inequalities can be used to compile even faster algorithms. Therefore they complement the universal algorithm introduced by Devroye (1984a) (see also Devroye (1986, §VII.3.2)) that uses moments of monotone distributions.

The new bounding curves can also be used to derive general upper bounds for probability density functions of T-concave distributions. These are optimal in the sense that constructing any better majorizing function requires more information about such a density function.

2. Ratio-of-uniforms and T-concave distributions

2.1. **Ratio-of-uniforms.** The ratio-of-uniforms method introduced by Kinderman and Monahan (1977) is a flexible method that can be adjusted to a large variety of distributions. It has become a popular transformation method to generate non-uniform random variates, since it results in exact, efficient, fast and easy to implement algorithms. It is based on the following (slightly modified) theorem.

Theorem 2 (Kinderman and Monahan 1977)**.** *Let $f(x)$ be a positive integrable function with support (x_0, x_1) not necessarily finite. If (V, U) is uniformly distributed in*

$$\mathcal{A} = \mathcal{A}(f) = \{(v,u)\colon 0 < u \le \sqrt{f(v/u+\mu)},\ x_0 < v/u + \mu < x_1\}, \tag{2}$$

then $X = V/U + \mu$ has probability density function $f(x)/\int f$.

For sampling random points uniformly distributed in $\mathcal{A}$, rejection from a convenient enveloping region is used, usually from the minimal bounding rectangle. Its boundaries have to be calculated analytically for each distribution. If $\mathcal{A}$ is convex, however, it is easy to construct a universal bounding rectangle without computing these boundaries (see below).

2.2. ***T*-concave distributions.** A probability density function $f(x)$ is called *T-concave* if there exists a monotonically increasing, differentiable transformation $T(x)$ such that $T(f(x))$ is concave. A distribution is called T-concave if its probability density function is T-concave. The *transformed density rejection* method is an acceptance/rejection technique that uses T-concavity of a large class of distributions to construct hat function and squeeze for the density automatically (see Hörmann (1995) or Evans and Swartz (1998) for details). Hörmann (1995) suggests the family T_c of transformations with $T_0(x) = \log(x)$ and $T_c(x) = -x^c$ for $-1 < c < 0$. ($c \le -1$ is possible for densities with compact domain.) If a function

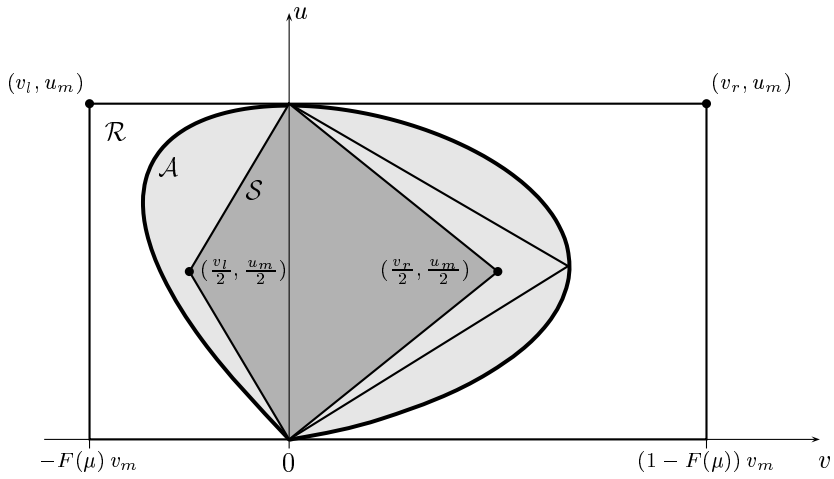

FIGURE 1. $\mathcal{A}$ and universal bounding rectangle $\mathcal{R}$ and squeeze $\mathcal{S}$ for gamma(3) distribution.

f is T_{c_1}-concave then it is also T_c-concave for every $c < c_1$ (Hörmann 1995). An equivalent transformation is $\tilde{T}_c(x) = (x^c - 1)/c$, for $c \neq 0$, and $\tilde{T}_0(x) = \log(x)$, also known as Box-Cox-transformation. It is continuous in c for $c = 0$.

Stadlober (1989) and Dieter (1989) have clarified the relationship of the ratio-of-uniforms method to the ordinary acceptance/rejection method. It can be viewed as rejection from a table-mountain shaped density (see Figure 2). Leydold (2000a) has shown a deeper connection to transformed density rejection. Moreover, a full characterization of all distributions with convex region $\mathcal{A}$ is derived.

Theorem 3 (Leydold 2000a)**.** *$\mathcal{A}(f)$ is convex if and only if $f(x)$ is T-concave with transformation $T(x) = -1/\sqrt{x}$, i.e., if and only if $-1/\sqrt{f(x)}$ is a concave function.*

Notice that this class of T-concave distributions includes all log-concave distributions.

2.3. **Universal inequalities.** Using the convexity of $\mathcal{A}(f)$ for this class of distributions we arrive at the following proposition (see Figure 1).

Theorem 4 (Leydold 2001, Theorems 4 and 5)**.** *For a distribution with density f, c.d.f. F and mode μ let*

$$\begin{aligned} \mathcal{R} &= \{(v,u)\colon v_l \le v \le v_r, 0 \le u \le u_m\}, \\ \mathcal{Q} &= \{(v,u)\colon -v_m \le v \le v_m, 0 \le u \le u_m\}, \end{aligned} \tag{3}$$

and

$$\mathcal{S} = \{(v,u)\colon v_l\, u \le u_m\, v \le v_r\, u,\ v_l\,(u_m - u) \le u_m\, v \le v_r\,(u_m - u)\}, \tag{4}$$

where

$$\begin{aligned} u_m &= \sqrt{f(\mu)} & v_l &= -F(\mu)\, v_m \\ v_m &= (\textstyle\int f)/\sqrt{f(\mu)} & v_r &= (1 - F(\mu))\, v_m. \end{aligned} \tag{5}$$

Then for any T-concave distribution with $T(x) = -1/\sqrt{x}$ we find

$$\mathcal{S} \subset \mathcal{A} \subset \mathcal{R} \subset \mathcal{Q}. \tag{6}$$

Moreover

$$2\,|\mathcal{S}| = |\mathcal{A}| = \frac{1}{2}|\mathcal{R}| = \frac{1}{4}|\mathcal{Q}| \tag{7}$$

where $|\mathcal{A}|$ *denotes the area of* $\mathcal{A}$.

Notice that only the knowledge of $F(\mu)$ is required, e.g., $F(\mu) = \frac{1}{2}$ for symmetric distributions, or $F(\mu) = 0$ for monotonically decreasing densities. Moreover, $\mathcal{Q}$ does not depend on $F(\mu)$ at all. By equation (7) the rejection constant of an algorithm that uses Theorem 4 is 2 when $F(\mu)$ is known and 4 otherwise.

Theorem 3 is proved by means of an appropriate transformation (see Theorem 6 below). This transformation can be used to derive universal upper and lower bounds for T-concave distributions (Leydold 2001, Theorem 7), see Figure 2.

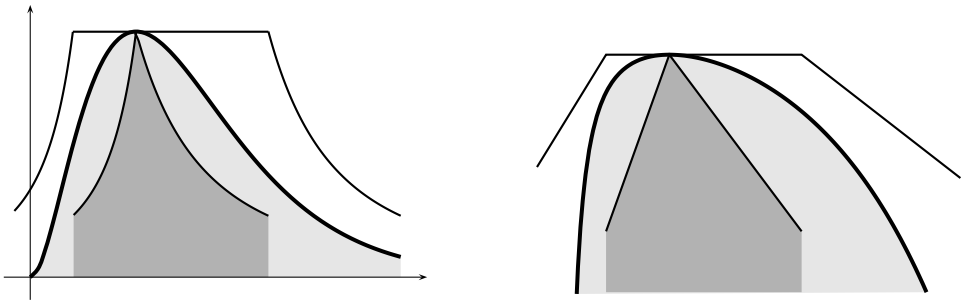

FIGURE 2. Universal hat and squeeze for gamma(3) distribution. Original scale (l.h.s.) and transformed scale (r.h.s.)

3. A GENERALIZATION

Theorem 5 (Wakefield, Gelfand, and Smith 1991)**.** *Let* $f(x)$ *be a positive integrable function with support* (x_0, x_1) *not necessarily finite. Let* g *be a strictly increasing differentiable function on* $[0, \infty)$ *such that* $g(0) = 0$*, and let* k *and* μ *be constants. Suppose the pair of variables* (V, U) *is uniformly distributed over the region*

$$\mathcal{A}_g = \mathcal{A}_g(f) = \{(v,u)\colon 0 < u \le g^{-1}[k\,f(v/g'(u)+\mu)],\ x_0 < v/g'(u)+\mu < x_1\}, \tag{8}$$

then $X = V/g'(U) + \mu$ *has probability density function* $f(x)/\int f$.

Wakefield et al. (1991) have suggested power functions $g_r(u) = u^{r+1}/(r+1)$, $r > 0$ and $k = 1/(r+1)$. Then equation (8) becomes

$$\mathcal{A}_r = \mathcal{A}_r(f) = \{(v,u)\colon 0 < u \le \sqrt[r+1]{f(v/u^r+\mu)},\ x_0 < v/u^r + \mu < x_1\}. \tag{9}$$

The *minimal bounding rectangle* for $\mathcal{A}_r$ is given by (see Wakefield et al. (1991))

$$\mathcal{R}_{\text{mbr}} = \{(v,u)\colon v^- \le v \le v^+, 0 \le u \le u_m\} \tag{10}$$

where

$$\begin{aligned} u_m &= \sup_{x_0<x<x_1} \sqrt[r+1]{f(x)} \\ v^- &= \inf_{x_0<x<x_1} (x-\mu)[f(x)]^{r/(r+1)} \\ v^+ &= \sup_{x_0<x<x_1} (x-\mu)[f(x)]^{r/(r+1)} \end{aligned} \tag{11}$$

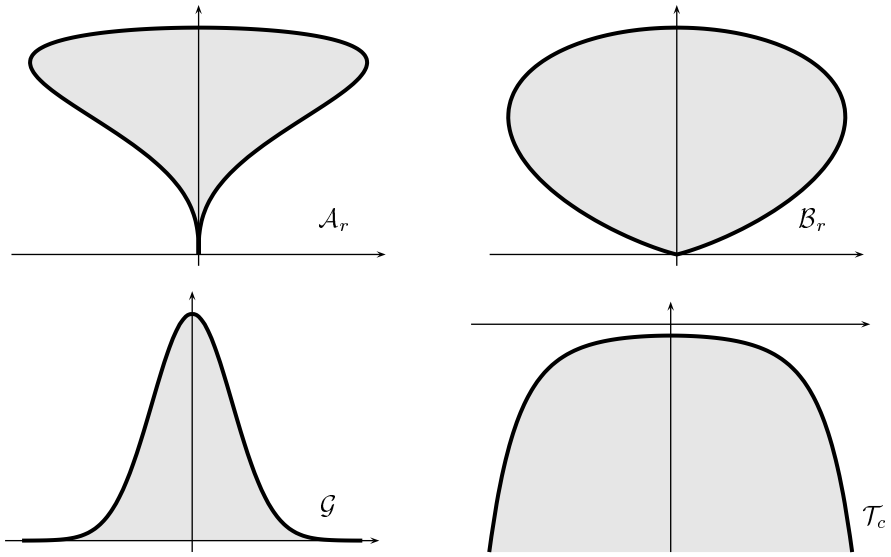

FIGURE 3. $\mathcal{A}_r$, $\mathcal{B}_r$, $\mathcal{G}$, and $\mathcal{T}_c$ for the normal distribution with $r = 3$ and $c = -3/4$. Notice that $\mathcal{B}_r$ is convex if and only if $\mathcal{T}_c$ is convex (Theorem 6).

There also exists a generalization of Theorem 3. Consider the following regions

$$\begin{aligned}\mathcal{B}_r &= \mathcal{B}_r(f) = \{(v,w)\colon 0 < w \le [f(v/w+\mu)]^{r/(r+1)},\ x_0 < v/w+\mu < x_1\}\\ \mathcal{G} &= \mathcal{G}(f) = \{(x,y)\colon 0 < y \le f(x), x_0 < x < x_1\}\\ \mathcal{T}_c &= \mathcal{T}_c(f) = \{(x,y)\colon 0 < y \le -(f(x))^c, x_0 < x < x_1\}\end{aligned} \tag{12}$$

$\mathcal{G}(f)$ is the region between the graph of f and the x-axis. $\mathcal{T}_c(f)$ is the region below the graph of the transformed density $T_c(f(x))$.

Theorem 6. *$\mathcal{B}_r(f)$ is convex if and only if $f(x)$ is T_c-concave with $c = -\frac{r}{r+1}$.*

Transformations between the above regions play a crucial role for the proof of this theorem as well as for the further developement of the theory of this type of generation methods. Consider the transformations

$$\begin{aligned}\Phi_{\mathcal{AB}}\colon &\quad \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}\times\mathbb{R}^+, \quad (V,U) \mapsto (V,W) = (V, U^r)\\ \Phi_{\mathcal{BG}}\colon &\quad \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}\times\mathbb{R}^+, \quad (V,W) \mapsto (X,Y) = (V/W+\mu, W^{(r+1)/r})\\ \Phi_{\mathcal{GT}}\colon &\quad \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}\times\mathbb{R}^-, \quad (X,Y) \mapsto (X,Z) = (X, -Y^c)\end{aligned} \tag{13}$$

Notice that $\Phi_{\mathcal{AB}}$ maps $\mathcal{A}_r$ one-to-one onto $\mathcal{B}_r$. Analogously for $\Phi_{\mathcal{BG}}$ and $\Phi_{\mathcal{GT}}$.

Proof of the Theorem. Let $c = -\frac{r}{r+1}$. The transformation

$$\begin{aligned}\Phi_{\mathcal{BT}} = \Phi_{\mathcal{GT}} \circ \Phi_{\mathcal{BG}}\colon\ &\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}\times\mathbb{R}^-,\\ &(V,W) \mapsto (\tfrac{V}{W}+\mu, -W^c\tfrac{r+1}{r}) = (\tfrac{V}{W}+\mu, -\tfrac{1}{W})\end{aligned}$$

maps $\mathcal{B}_r$ one-to-one onto $\mathcal{T}_c$. Notice that f is T_c-concave if and only if $\mathcal{T}_c$ is convex. Thus it remains to show that $\mathcal{B}_r$ is convex if and only if $\mathcal{T}_c$ is convex, and consequently that straight lines remain straight lines under transformation $\Phi_{\mathcal{BT}}$.

Let $a\,x + b\,y = d$ be a straight line in $\mathcal{T}_c$. Then $a\,(v/w+\mu) - b/w = d$ and $a\,v+(1-d)\,w = b$, i.e., a straight line in $\mathcal{B}_r$. Analogously a straight line $a\,v+b\,w = d$ in $\mathcal{B}_r$ is mapped onto the line $a\,x + d\,y = a\,\mu - b$ in $\mathcal{T}_c$. □

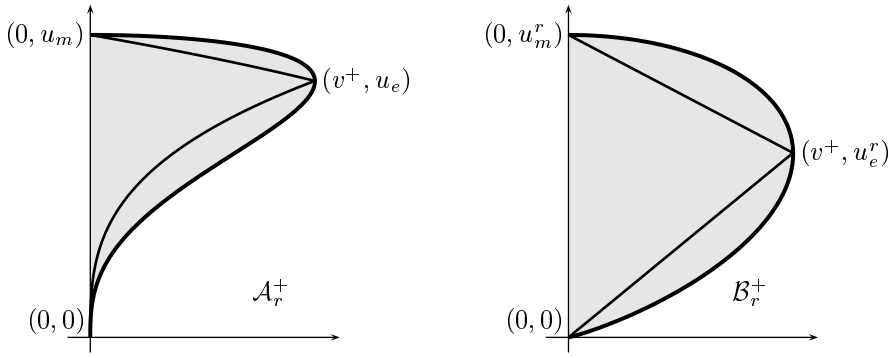

FIGURE 4. The region $\mathcal{B}_r^+$ with enclosed triangle with vertices at $(0,0)$, $(0, u_m^r)$ and extremal point (v^+, u_e^r) (r.h.s.). $\mathcal{A}_r^+$ and the corresponding enclosed region on the l.h.s.

The transformation $\Phi_{\mathcal{AG}} = \Phi_{\mathcal{BG}} \circ \Phi_{\mathcal{AB}} \colon (V,U) \mapsto (X,Y) = (V/U^r + \mu, U^{r+1})$ has Jacobian $r+1$ and thus Theorem 5 follows for the power function $g(u) = u^{r+1}/(r+1)$. (For an arbitrary function $g(u)$ the proof is completely analogously.) Moreover

$$|\mathcal{A}_r(f)| = \int f/(r+1). \tag{14}$$

Remark. For $r = 1$ we have $\mathcal{B}_1 = \mathcal{A}_1 = \mathcal{A}$ (equations (12), (9) and (2)). Thus Theorems 2 and 3 are corollaries of the respective Theorems 5 and 6.

3.1. **A universal envelope.** Now let $f(x)$ be a T_c-concave density function with $c = -\frac{r}{r+1}$ and mode μ. Define $\mathcal{A}_r^+ = \{(v,u) \in \mathcal{A}_r \colon v \geq 0\}$, $\mathcal{A}_r^- = \{(v,u) \in \mathcal{A}_r \colon v \leq 0\}$, and analogously $\mathcal{B}_r^+$ and $\mathcal{B}_r^-$. By Theorem 6, $\mathcal{B}_r^+$ is convex and thus contains the triangle with vertices at $(0,0)$, $(0, u_m^r)$ and the extremal point with coordinates (v^+, u_e^r), where $u_e \geq 0$ (see Figure 4). Then transformation $\Phi_{\mathcal{AB}}^{-1}$ maps this triangle into a three-side region in $\mathcal{A}_r$ of area

$$\int_0^{v^+} \left([u_m^r + v\,(u_e^r - u_m^r)/v^+]^{1/r} - [v\,u_e^r/v^+]^{1/r} \right) dv = v^+ \, u_m^r \, \frac{r}{r+1} \, \frac{u_e - u_m}{u_e^r - u_m^r}. \tag{15}$$

By the convexity of $\mathcal{B}_r^+$ this cannot exceed $|\mathcal{A}^+|$ and consequently

$$v^+ \leq |\mathcal{A}^+| \, \frac{1}{u_m^r} \, \frac{r+1}{r} \, \frac{u_e^r - u_m^r}{u_e - u_m} = |\mathcal{A}^+| \, \frac{1}{u_m} \, \frac{r+1}{r} \, \frac{(u_e/u_m)^r - 1}{(u_e/u_m) - 1}. \tag{16}$$

Using (14) and the fact that $|\mathcal{A}^+| = (1 - F(\mu))\,|\mathcal{A}|$ we arrive at

$$v^+ \leq (1 - F(\mu)) \, (\textstyle\int f) \, \dfrac{1}{r\,u_m} \, \dfrac{(u_e/u_m)^r - 1}{(u_e/u_m) - 1}. \tag{17}$$

An analogous (lower) bound can be derived for v^-. Thus we have the following universal envelopes.

Theorem 7. *Let $f(x)$ be a T_c-concave density function with $c = -\frac{r}{r+1}$, $r > 0$, and mode μ. Let F denote the c.d.f. of the distribution and let*

$$\begin{aligned} \mathcal{R}_r &= \{(v,u) \colon v_l(u) \leq v \leq v_r(u), 0 \leq u \leq u_m\} \\ \mathcal{Q}_r &= \{(v,u) \colon -v_m(u) \leq v \leq v_m(u), 0 \leq u \leq u_m\} \end{aligned} \tag{18}$$

where

$$\begin{aligned} u_m &= \sqrt[r+1]{f(\mu)} & v_m(u) &= (\int f)/(r\,u_m)\,\tfrac{(u/u_m)^r-1}{(u/u_m)-1} \\ v_l(u) &= -F(\mu)\,v_m(u) & v_r(u) &= (1-F(\mu))\,v_m(u) \end{aligned} \tag{19}$$

Then $\mathcal{A}_r \subset \mathcal{R}_r \subset \mathcal{Q}_r$ *and*

$$|\mathcal{R}_r| = \frac{r+1}{r}(\gamma + \psi(r+1))\,|\mathcal{A}_r| \qquad \textit{and} \qquad |\mathcal{Q}_r| = 2\,|\mathcal{R}_r| \tag{20}$$

where $\gamma = 0.577216\ldots$ *denotes Euler's (gamma) constant and* $\psi(z) = \Gamma'(z)/\Gamma(z)$ *the Digamma function (also known as Euler's psi function). The envelopes* $\mathcal{R}_r$ *and* $\mathcal{Q}_r$ *are optimal.*

Proof. By equation (17) every point $(v,u) \in \mathcal{A}_r$ satisfies the respective conditions for $\mathcal{R}_r$ and $\mathcal{Q}_r$ in equation (18). Consequently $\mathcal{A}_r \subset \mathcal{R}_r \subset \mathcal{Q}_r$. It remains to verify (20). From (19) and (14) we find $|\mathcal{R}_r| = |\mathcal{R}_r^+| + |\mathcal{R}_r^-| = \int_0^{u_m} (\int f)/(r\,u_m)\,\frac{(u/u_m)^r-1}{(u/u_m)-1}\,du = \int_0^{u_m} |\mathcal{A}_r|\,\frac{r+1}{r}\,\frac{(u/u_m)^r-1}{(u/u_m)-1}\,\frac{1}{u_m}\,du = |\mathcal{A}_r|\,\frac{r+1}{r}\int_0^1 \frac{z^r-1}{z-1}\,dz = |\mathcal{A}_r|\,\frac{r+1}{r}\,(\gamma + \psi(r+1))$, where the last equality follows from formula 8.361(7) in Gradshteyn and Ryzhnik (1994, p.952). □

Figure 5 illustrates the situation for the Cauchy distribution and $r = 3$. Figure 6 plots the ratio of $|\mathcal{R}_r|/|\mathcal{A}_r|$ against r. Generating points uniformly distributed over $\mathcal{R}_r$ requires an appropriate enveloping region. Transformed density rejection can be used to construct such a region. We need the following two lemmata.

Lemma 8 (Derflinger 2000)**.** $\frac{x-1}{x^r-1}$ *is convex for* $r \geq 1$ *and* $x \in [0,1)$.

Proof. The proposition is trivial for $r = 1$. Thus assume $r > 1$. By a straight forward computation we find $((x-1)/(x^r-1))'' = \frac{r\,x^{r-2}}{(x^r-1)^3} \cdot [(r-1)(x^{r+1}-1) - (r+1)(x^r-x)]$, which is greater than or equal to 0 if and only if the second factor is not greater than 0, i.e., if and only if $A(x) = \frac{x^{r+1}-1}{r+1} - \frac{x^r-x}{r-1} \leq 0$. Notice that $A'(x) = x^r - \frac{r\,x^{r-1}-1}{r-1}$ and $A''(x) = r\,(x-1)\,x^{r-2} \leq 0$. Consequently $A'(x)$ is monotonically decreasing and, since $A'(1) = 0$, $A'(x) \geq 0$. Hence $A(x)$ is monotonically increasing and, since $A(1) = 0$, $A(x) \leq 0$ and the proposition follows. □

Lemma 9. *Let* $\tau(x) = -\frac{x-1}{x^r-1}$*, and let* $a = \tau(x_0) - x_0\,\tau'(x_0)$ *and* $b = \tau'(x_0)$ *for an* $x_0 \in (0,1)$*. Assume* $r \geq 1$*, then* $-\frac{1}{a+b\,x} \geq \frac{x^r-1}{x-1}$ *for all* $x \in [0,1]$.

Proof. By Lemma 8, $\tau(x)$ is concave. Thus we find for the tangent in x_0, $a + b\,x \geq \tau(x)$, for all $x \in [0,1)$. Now notice that $\tau'(x) = (x^r-1)^{-2} \cdot (1 + r\,(x-1)\,x^{r-1} - x^r)$, and $\lim_{x\to 1} \tau'(x) = (r-1)/(2\,r) \geq 0$, by l'Hospital's rule. Thus, by the concavity of $\tau(x)$, $\tau'(x) > 0$ for all $x \in (0,1)$, i.e., $b > 0$. Hence $a + b\,x \leq a + b$. Notice that $a + b$ is the tangent of τ with construction point x_0 evaluated at $x = 1$. By the concavity of τ, $a + b$ is maximized in $x_0 = 0$. Hence $\tau(x) \leq a + b\,x < a + b \leq \tau(0) + \tau'(0) = 0$. Consequently $-1/(a + b\,x) \geq -1/\tau(x)$ and the proposition follows. □

The following enveloping region for $\mathcal{R}_r$ has been suggested by Derflinger (2000).

Theorem 10. *Let* $f(x)$ *be a* T_c*-concave probability density function with* $c = -\frac{r}{r+1}$*,* $r \geq 1$*, and mode* μ*. Let* F *denote the c.d.f. of the distribution. Let*

$$x_r = 1 - 2.187/(r + 5 - 1.28/r)^{0.9460} \tag{21}$$

and denote $a_r = \tau(x_r) - x_r\,\tau'(x_r)$ *and* $b_r = \tau'(x_r)$, *where* $\tau(x) = -(x-1)/(x^r-1)$. *Let*

$$\begin{aligned}\tilde{\mathcal{R}}_r &= \{(v,u)\colon \tilde{v}_l(u) \le v \le \tilde{v}_r(u), 0 \le u \le u_m\}\\ \tilde{\mathcal{Q}}_r &= \{(v,u)\colon -\tilde{v}_m(u) \le v \le \tilde{v}_m(u), 0 \le u \le u_m\}\end{aligned} \tag{22}$$

where

$$\begin{aligned}u_m &= \sqrt[r+1]{f(\mu)} & \tilde{v}_m(u) &= (\int f)/(r\,u_m)^{\frac{-1}{a_r + b_r\,(u/u_m)}}\\ \tilde{v}_l(u) &= -F(\mu)\,\tilde{v}_m(u) & \tilde{v}_r(u) &= (1-F(\mu))\,\tilde{v}_m(u)\end{aligned} \tag{23}$$

Then $\mathcal{A}_r \subset \tilde{\mathcal{R}}_r \subset \tilde{\mathcal{Q}}_r$ *and*

$$|\tilde{\mathcal{R}}_r| = \frac{r+1}{r}\,\frac{1}{b_r}\,\log\left(\frac{a_r}{a_r+b_r}\right)\,|\mathcal{A}_r| \qquad \textit{and} \qquad |\tilde{\mathcal{Q}}_r| = 2\,|\tilde{\mathcal{R}}_r|. \tag{24}$$

Proof. $\mathcal{A}_r(f) \subset \mathcal{R}_r \subseteq \tilde{\mathcal{R}}_r$ follows immediately from Theorem 7 and Lemma 9. To verify (24) we find $|\tilde{\mathcal{R}}_r| = |\tilde{\mathcal{R}}_r^+| + |\tilde{\mathcal{R}}_r^-| = \int_0^{u_m} |\mathcal{A}_r|\,\frac{r+1}{r}\frac{1}{u_m}\,\frac{-1}{a_r+b_r\,(u/u_m)}\,du = |\mathcal{A}_r|\,\frac{r+1}{r}\int_0^1 \frac{1}{a_r+b_r\,z}\,dz = |\mathcal{A}_r|\,\frac{r+1}{r}\,\frac{1}{b_r}\,\log(a_r/(a_r+b_r))$. □

Remark. The parameters for x_r in equation (21) have been found by minimizing the area of $\tilde{\mathcal{R}}_r$. Numerical computations show that $|\tilde{\mathcal{R}}_r| < 1.13\,|\mathcal{R}_r|$.

To generate points uniformly in $\tilde{\mathcal{R}}_r$ we have to generate random variates with density proportional to $h(x) = -1/(a+b\,x)$ on $[0,1]$. This can be done by inversion. Notice that $a + b\,x < 0$ for all $x \in [0,1]$. The c.d.f. is given by

$$H(x) = \frac{\log(a/(a+b\,x))}{\log(a/(a+b))}$$

and thus we find for the inverse of the c.d.f.

$$H^{-1}(u) = \tfrac{a}{b}(\exp(-u\,\log(a/(a+b))) - 1).$$

Using the above results we can compile algorithm GSROUC.

Algorithm GSROUC

Require: p.d.f. $f(x)$, parameter $r > 1$, area $\int f$, mode μ;
c.d.f. at mode $F(\mu)$ (optional)

/* Setup */
$p \leftarrow 1 - 2.187/(r+5-1.28/r)^{0.9460}$.
$b \leftarrow (1 - r\,p^{r-1} + (r-1)\,p^r)/(p^r-1)^2$.
$a \leftarrow -(p-1)/(p^r-1) - p\,b$.
$u_m \leftarrow \sqrt[r+1]{f(\mu)}, \quad v_m \leftarrow (\int f)/(r\,u_m)$.
if $F(\mu)$ is provided **then**
  $v_l \leftarrow -F(\mu)\,v_m, \quad v_r \leftarrow v_m + v_l$.
else
  $v_l \leftarrow -v_m, \quad v_r \leftarrow v_m$.
/* Generator */
repeat
  Generate W uniformly on $(0, \log(a/(a+b)))$.
  Generate Z uniformly on (v_l, v_r).
  $U \leftarrow (\exp(-W) - 1)\,a/b$.
  $V \leftarrow -Z/(a+b\,U), \quad U \leftarrow U\,u_m$.
  $X \leftarrow V/U^r + \mu$.
until $U^{r+1} \le f(X)$.
return X.

Remark. There is no need for an algorithm for $r < 1$ since then the much simpler and faster algorithm that applies Theorem 4 can be used (see algorithm `SROU` in Leydold (2001)).

Remark. It should be noted that for $r \to 0$ we have $c \to 0$ and $T_c \to \log$. Moreover, we find $\lim_{r\to 0} \frac{r+1}{r}(\gamma+\psi(r+1)) = \pi^2/6$. The optimal algorithm by Devroye (1984b) (see remark after Theorem 1) is thus a limit case of the new method. However, this is of purely theoretical interest, since the regions $\mathcal{A}_r$ become long and skinny when r tends to 0.

3.2. **A universal squeeze.** Analogously to Theorem 4 there exists an (optimal) squeeze for $\mathcal{A}_r$. Figure 5 illustrates the situation for the Cauchy distribution and $r = 3$.

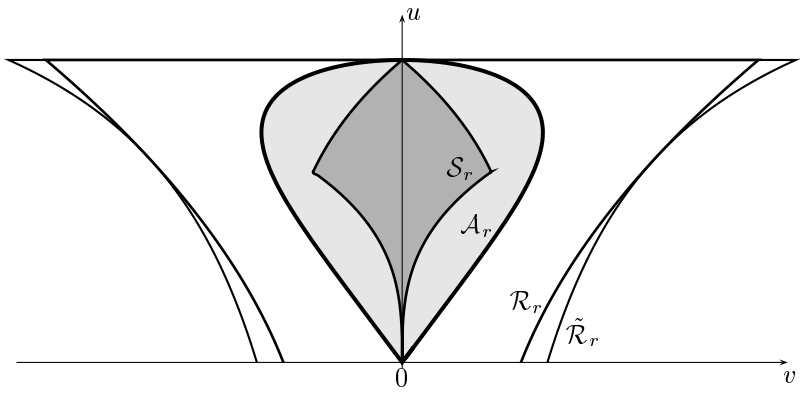

FIGURE 5. $\mathcal{A}_r$, universal bounding envelopes $\mathcal{R}_r$ and $\tilde{\mathcal{R}}_r$, and universal squeeze $\mathcal{S}_r$ for the Cauchy distribution. ($r = 3$)

Theorem 11. *Let $f(x)$ be a T_c-concave probability density function with $c = -\frac{r}{r+1}$, $r > 0$, and mode μ. If $F(\mu)$ is given, where F denotes the c.d.f. of the distribution, then there exists a set $\mathcal{S}_r = \mathcal{S}_r(f)$, such that $\mathcal{S}_r \subset \mathcal{A}_r(f)$. We have $(V, U) \in \mathcal{S}_r$ if and only if either*

$$0 \le \frac{V}{U^r} \le \frac{\bar{v}_r}{u_m^r} \qquad \text{and} \qquad U^r\,\bar{v}_r + V\,r\,u_m^r \le \bar{v}_r\,u_m^r \tag{25}$$

or

$$0 \ge \frac{V}{U^r} \ge \frac{\bar{v}_l}{u_m^r} \qquad \text{and} \qquad U^r\,\bar{v}_l + V\,r\,u_m^r \ge \bar{v}_l\,u_m^r. \tag{26}$$

where u_m is as defined in Theorem 7 and $\bar{v}_r = (1 - F(\mu))\,(\int f)/u_m$ and $\bar{v}_l = -F(\mu)\,(\int f)/u_m$. Moreover

$$|\mathcal{S}_r| = |\mathcal{A}_r| \left(1 - 1/(r+1)^{1/r}\right). \tag{27}$$

Proof. By Theorem 6, $\mathcal{B}_r(f)$ is convex. Let $\mathcal{S}_r$ denote the universal squeeze region (which might be empty) and let $\mathcal{C} = \Phi_{\mathcal{AB}}(\mathcal{S}_r)$ denote the corresponding region in $\mathcal{B}_r$. Assume that $\mathcal{A}_r^+ \neq \emptyset$. Let Δ be the triangle defined by the set of all points (v, u^r) that satisfy the inequalities $v \ge 0$, $v/u^r \le \bar{v}_r/u_m^r$ and $u^r\,\bar{v}_r + v\,r\,u_m^r \le \bar{v}_r\,u_m^r$. Its vertices are $(0,0)$, $(0, u_m^r)$ and $(\bar{v}_r/(r+1), u_m^r/(r+1))$. Define $\mathcal{C}^+ = \{(v,u) \in \mathcal{C}\colon v \ge 0\}$ and $\mathcal{R}_r^+ = \{(v,u) \in \mathcal{R}_r\colon v \ge 0\}$. Every straight line through a point

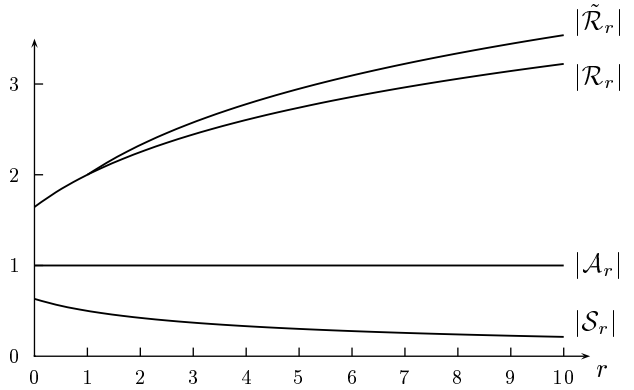

FIGURE 6. Respective ratios of $|\tilde{\mathcal{R}}_r|$, $|\mathcal{R}_r|$ and $|\mathcal{S}_r|$ to $|\mathcal{A}_r|$ as functions of r. (Notice that $\tilde{\mathcal{R}}_r$ is not defined for $r < 1$.)

(V, U^r) with $(V, U) \in \mathcal{R}_r^+ \setminus \mathcal{A}_r^+$ that does not intersect $\mathcal{B}_r^+$, is transformed by $\Phi_{\mathcal{AB}}^{-1}$ into a curve that splits $\mathcal{R}_r^+$ into two parts such that (i) $\mathcal{A}_r^+$ and the edge $(0,0)(0,u_m)$ are completely contained in the left hand part, and (ii) the area of the left hand part must be at least $|\mathcal{A}_r^+|$. $\mathcal{S}_r^+$ is then the intersection of the left hand parts of all such curves. Consequently $\mathcal{C}^+$ must be contained in the triangles with respective vertices at $(0,0)$, $(0,u_m^r)$ and $(v_r(0),0)$, and $(0,0)$, $(0,u_m^r)$ and $(v_r(u_m),u_m^r)$, where $v_r(u)$ is the bounding curve for $\mathcal{R}$ given in equation (19). Notice that $v_r(u_m) = \bar{v}_r$ and $v_r(0) = \bar{v}_r/r$, and thus the respective right hand vertices are $(\bar{v}_r/r, 0)$ and $(\bar{v}_r, u_m^r)$. Since the intersection of these triangles is given by Δ, we have $\Delta \supseteq \mathcal{C}^+$. Now notice that $\mathcal{C}^+$ is convex, because it is the intersection of triangles. It remains to show that the vertex $(\bar{v}_r/(r+1), u_m^r/(r+1)) \in \mathcal{C}^+$, since then $\Delta \subseteq \mathcal{C}^+$. A straight line through $(a,0)$ and (b,u_m^r) is given by

$$v = w/u_m^r\,(b-a) + a, \tag{28}$$

where (v,w) is point on that line. Hence the transformed curved (using $\Phi_{\mathcal{AB}}^{-1}$) is given by $v = u^r/u_m^r\,(b-a) + a$ and for the area of the left hand side region in $\mathcal{R}_r^+$ cut off by this curve we find $\int_0^{u_m}(u^r/u_m^r\,(b-a)+a)\,du = (b + r\,a)\,u_m/(r+1)$. By constraint (ii) this must be $\geq |\mathcal{A}_r^+| = \bar{v}_r\,u_m/(r+1)$. The latter equality follows from (14) and the definition of $\bar{v}_r$. Hence $b + r\,a \geq \bar{v}_r$. Now suppose $(\bar{v}_r/(r+1), u_m^r/(r+1)) \notin \mathcal{C}^+$. Then by (28) $\bar{v}_r/(r+1) > (u_m^r/(r+1))/u_m^r\,(b-a) + a$ and thus $\bar{v}_r > b + r\,a$, a contradiction. Therefore $\Delta \subseteq \mathcal{C}^+$. Consequently $\mathcal{C}^+ = \Delta$ and the inequality (25) follows. Analogously we find a set $\mathcal{C}^-$ and inequality (26) for the left hand rectangle $\mathcal{R}_r^-$.

To verify (27) we can compute $|\mathcal{S}_r^+|$ in the same way as equation (15). We only have to replace the vertex (v^+, u_e^r) by the vertex $(\bar{v}_r/(r+1), u_m^r/(r+1))$, i.e., we have to replace v^+ by $(1 - F(\mu)\,(\int f)/(u_m\,(r+1)) = (1 - F(\mu)\,|\mathcal{A}_r|/u_m$ and u_e by $u_m/(r+1)^{1/r}$ in equation (15). Since $|\mathcal{S}_r| = |\mathcal{S}_r^-| + |\mathcal{S}_r^+|$, (27) follows by a straight forward computation. □

4. T_c-CONCAVE DISTRIBUTIONS

The envelopes and squeezes introduced in the previous section can be used to derive general upper and lower bounds that hold for all T_c-concave distributions. It is a straight forward computation. Universal majorizing and minimizing functions for the density function are obtained by applying the transformation $\Phi_{\mathcal{BG}} \circ \Phi_{\mathcal{AB}}$ (see eq.(13)) on the respective boundaries of $\mathcal{R}_r$ and $\mathcal{Q}_r$ (Theorem 7), and $\mathcal{S}_r$ (Theorem 11). We only give the upper bounds that do not require the evaluation of the c.d.f. at the mode. All other bounds can be derived in completely the same way.

Theorem 12. *Let $f(x)$ be a T_c-concave density function with $c = -\frac{r}{r+1}$, $r > 0$, and mode μ. Let $u_m = \sqrt[r+1]{f(\mu)}$ (as in Theorem 7) and $x_m = \int f / f(\mu)$. Define $\theta\colon (0,\infty) \to (0,\infty)$ and $\psi\colon (0,\infty) \to (0,\infty)$ by*

$$\theta^{-1}(x) = x\,\frac{1-x^r}{1-x} \qquad \textit{and} \qquad \psi(x) = \theta^{r+1}(x\,r\,f(\mu)/\textstyle\int f). \tag{29}$$

Let

$$h(x) = \begin{cases} f(\mu) & \textit{for } -x_m \le x-\mu \le x_m \\ f(\mu)/\psi(|x-\mu|) & \textit{otherwise} \end{cases} \tag{30}$$

Then $h(x) \ge f(x)$ for all x.

Proof. First notice that $\theta^{-1}\colon (0,\infty) \to (0,\infty)$ is strictly monotonically increasing and $\theta^{-1}(0) = 0$. Hence its inverse map θ exists and is well defined. This can easily be seen, since $(\theta^{-1})'(x) = (1-(r+1)\,x^r + r\,x^{r+1})/(x-1)^2$ is positive for all $x > 0$, because the denominator is alway non-negative and the numerator $1-(r+1)\,x^r + r\,x^{r+1}$ is 0 for $x = 1$, strictly monotonically decreasing for $x \in (0,1)$ and strictly monotonically increasing for $x > 1$ (use its derivative).

Now $\Phi_{\mathcal{BG}} \circ \Phi_{\mathcal{AB}}\colon (V,U) \mapsto (V/U^r + \mu, U^{r+1})$ maps the bounding curve of $\mathcal{Q}_r$ into the graph of a hat function for the density f. The straight line through the upper edge of $\mathcal{Q}_r$ in Theorem 7 is mapped into $y = u_m^{r+1} = f(\mu)$. The bounding curve $v(u) = (\int f)/(r\,u_m)\frac{(u/u_m)^r-1}{(u/u_m)-1}$ is mapped into the curve given by $y = u^{r+1}$ and $x(y)-\mu = v(u)/u^r = (\int f)/(r\,u_m^{r+1})\,\frac{1-(u_m/u)^r}{1-(u_m/u)}\,(u_m/u) = (\int f)/(r\,u_m^{r+1})\,\theta^{-1}(u_m/u) = (\int f)/(r\,f(\mu))\,\theta^{-1}\left(\sqrt[r+1]{f(\mu)/y}\right)$.

Hence $\sqrt[r+1]{y(x)} = \sqrt[r+1]{f(\mu)}/\theta((x-\mu)\,r\,f(\mu)/\int f)$ and $y(x) = f(\mu)/\psi(x-\mu)$ for $x \ge \mu$. Since the right upper vertex of $\mathcal{Q}_r$ is given by $((\int f)/u_m, u_m)$ the corresponding vertex in the graph of the hat function is at $x = v/u^r = (\int f)/f(\mu) = x_m$. We get an analogously result for the left hand bounding curve. Thus the result follows. □

This upper bound can be applied to sample from the T_c-concave distribution. For sampling from the hat distribution the inverse-of-f method (Devroye 1986, §IV.6.3, p.178) can be used. However this requires an appropriate hat function to $\sqrt[r+1]{\theta^{-1}(x)}$ similar to those in Theorem 10.

Remark. The bound of Theorem 12 is optimal in the sense that any improvement requires more information about the distribution besides the location of the mode and the c.d.f. at the mode.

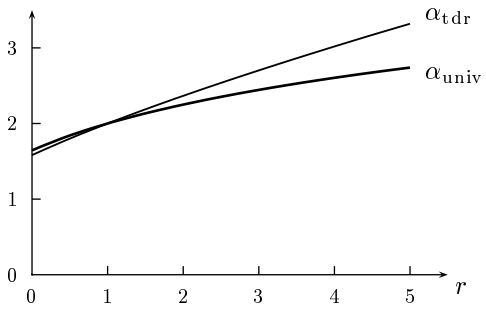

FIGURE 7. α_{univ} and α_{tdr}.

Remark. By our construction of the hat function in Theorem 12 the rejection constant α follows immediately from equation (20), i.e.,

$$\alpha = \alpha_{\mathrm{univ}}(r) = \frac{r+1}{r}(\gamma + \psi(r+1)). \tag{31}$$

(To be more precise: this is the rejections constant when we use $F(\mu)$ for constructing the hat.) Another way to construct a mountain-table shaped hat for a T_c-concave density is by means of transformed density rejection with the mode of the density and two points on either side of it as construction points. By the theory developed by Deflinger it is possible to optimize such a hat function for a given density (Derflinger et al. 2002). The rejection constant α is then bounded from above for a given $c = -r/(r+1)$ by

$$\alpha \leq \alpha_{\mathrm{tdr}}(r) = \frac{1}{1-(1/(r+1))^{1/r}}. \tag{32}$$

It is also shown that this bound is sharp, i.e. for any $r \geq 0$ there exists a T_c-concave density where equality holds in (32). It is interesting to note that $\alpha_{\mathrm{univ}}(1) = \alpha_{\mathrm{tdr}}(1)$, $\alpha_{\mathrm{univ}} > \alpha_{\mathrm{tdr}}$ for $r < 1$, and $\alpha_{\mathrm{univ}} < \alpha_{\mathrm{tdr}}$ for $r > 1$, see Figure 7. Consequently, the performance of the simple generator that is based on Theorem 7 is better than the worst-case performance of a more specialized generator based on TDR with three points of contact.

5. Heavy-tailed distributions

We call a distribution a heavy-tailed T_c-concave distribution, $c < 0$, if its probability density function $f(x)$ is T_c-concave and $(x-\mu)\,f(x)^{-c}$ is monotonically increasing on $\mathbb{R}$. Notice that for densities with support not equal to $\mathbb{R}$ this only may hold if it is monotone with support $(-\infty, \mu)$ or (μ, ∞).

Again let μ be the mode of the density f and define $\mathcal{A}_r^+$ and $\mathcal{B}_r^+$ as in §3.1. If we have a heavy-tailed T_c-concave distribution, $\mathcal{B}_r^+$ is convex and contains the triangle with vertices at $(0,0)$, $(0, u_m^r)$ and $(v^+, 0)$, where u_m and v^+ are defined by (11). Moreover if $\mathcal{B}_r^+$ contains a boundary point (v_b, u_b^r), $v_b > 0$, then $\mathcal{B}_r^+$ also contains the quadrangle with vertices at $(0,0)$, $(0, u_m^r)$, (v_b, u_b^r) and $(v_b, 0)$. By

transformation $\Phi_{\mathcal{AB}}^{-1}$ it is mapped into a four-sided region in $\mathcal{A}_r$ of area

$$\int_0^{v_b} (u_m^r + v\,(u_b^r - u_m^r)/v_b)^{1/r}\,dv = v_b\,\frac{r}{r+1}\,\frac{u_b^{r+1} - u_m^{r+1}}{u_b^r - u_m^r} \tag{33}$$

which cannot exceed $|\mathcal{A}_r^+|$. Thus we find analogously to §3.1

$$v_b \le (1 - F(\mu))\,(\int\!\!f)\,\frac{1}{r\,u_m}\,\frac{(u_b/u_m)^r - 1}{(u_b/u_m)^{r+1} - 1}. \tag{34}$$

An analogous (lower) bound can be deduced for a boundary point (v_b, u_b^r) with $v_b < 0$. Thus we have the following universal envelopes.

Theorem 13. *Let $f(x)$ be a heavy-tailed T_c-concave density function with $c = -\frac{r}{r+1}$, $r > 0$, and mode μ. Let F denote the c.d.f. of the distribution and let*

$$\begin{aligned} \mathcal{R}_r &= \{(v,u)\colon v_l(u) \le v \le v_r(u), 0 \le u \le u_m\} \\ \mathcal{Q}_r &= \{(v,u)\colon -v_m(u) \le v \le v_m(u), 0 \le u \le u_m\} \end{aligned} \tag{35}$$

where

$$\begin{aligned} u_m &= \sqrt[r+1]{f(\mu)} & v_m(u) &= (\int\!\!f)/(r\,u_m)\,\tfrac{(u/u_m)^r - 1}{(u/u_m)^{r+1} - 1} \\ v_l(u) &= -F(\mu)\,v_m(u) & v_r(u) &= (1 - F(\mu))\,v_m(u) \end{aligned} \tag{36}$$

Then $\mathcal{A}_r \subset \mathcal{R}_r \subset \mathcal{Q}_r$ and

$$|\mathcal{R}_r| = -\frac{1}{r}\left(\gamma + \psi\left(\frac{1}{r+1}\right)\right)|\mathcal{A}_r| \qquad \textit{and} \qquad |\mathcal{Q}_r| = 2\,|\mathcal{R}_r| \tag{37}$$

where $\gamma = 0.577216\ldots$ denotes Euler's (gamma) constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ the Digamma function (also known as Euler's psi function).

Proof. By the above considerations it remains to verify (37). From (36) and (14) we find $|\mathcal{R}_r| = |\mathcal{R}_r^+| + |\mathcal{R}_r^-| = \int_0^{u_m} (\int\!\!f)/(r\,u_m)\,\frac{(u/u_m)^r - 1}{(u/u_m)^{r+1} - 1}\,du = \int_0^{u_m} |\mathcal{A}_r|\,\frac{r+1}{r}\,\frac{(u/u_m)^r - 1}{(u/u_m)^{r+1} - 1}\,\frac{1}{u_m}\,du = |\mathcal{A}_r|\,\frac{r+1}{r}\int_0^1 \frac{z^r - 1}{z^{r+1} - 1}\,dz$. Substitute t for z^{r+1} yields $-\frac{1}{r+1}\int_0^1 \frac{t^{-r/(r+1)} - 1}{t - 1}\,dt = -\frac{1}{r+1}\,(\gamma + \psi(\frac{1}{r+1})$ for the integral, where the last equality follows from formula 8.361(7) in Gradshteyn and Ryzhnik (1994, p.952). Thus the result follows. □

The envelopes $\mathcal{R}_r$ and $\mathcal{Q}_r$ in Theorem 13 are optimal. Figure 8 illustrates the situation for Student's distributions with $\nu = 1/3$ degrees of freedom. Notice that $|\mathcal{R}_r|/|\mathcal{A}_r|$ converges to 1 when $r \to \infty$ (see Figure 9).

Generating points uniformly over $\mathcal{R}_r$ requires an appropriate enveloping region. Using a rectangle is the most convenient one. Notice that $\frac{(u/u_m)^r - 1}{(u/u_m)^{r+1} - 1}$ is maximized at $u = 0$ in $[0, u_m]$. Thus we find the following (minimal) bounding rectangle for $\mathcal{R}_r$.

Theorem 14. *Let $f(x)$ be a heavy-tailed T_c-concave density function with $c = -\frac{r}{r+1}$, $r > 0$, and mode μ. Let F denote the c.d.f. of the distribution and let*

$$\begin{aligned} \tilde{\mathcal{R}}_r &= \{(v,u)\colon \tilde{v}_l \le v \le \tilde{v}_r, 0 \le u \le u_m\} \\ \tilde{\mathcal{Q}}_r &= \{(v,u)\colon -\tilde{v}_m \le v \le \tilde{v}_m, 0 \le u \le u_m\} \end{aligned} \tag{38}$$

where

$$\begin{aligned} u_m &= \sqrt[r+1]{f(\mu)} & \tilde{v}_m &= (\int\!\!f)/(r\,u_m) \\ \tilde{v}_l &= -F(\mu)\,\tilde{v}_m & \tilde{v}_r &= (1 - F(\mu))\,\tilde{v}_m \end{aligned} \tag{39}$$

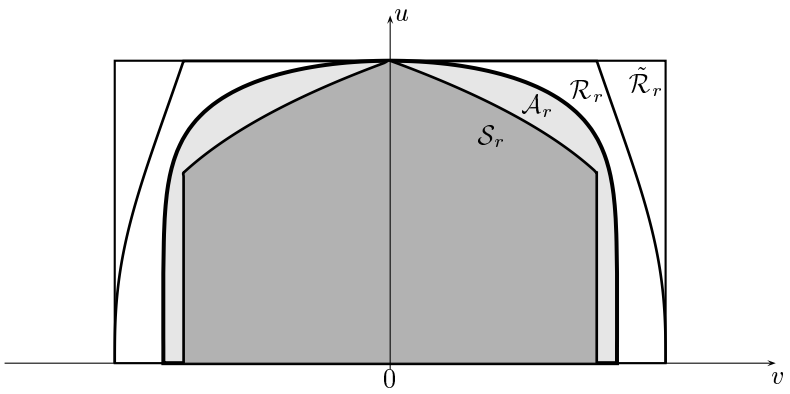

FIGURE 8. $\mathcal{A}_r$ and universal bounding envelopes $\mathcal{R}_r$ and $\tilde{\mathcal{R}}_r$, and universal squeeze $\mathcal{S}_r$ for Student's distribution with $\nu = 1/3$ degree of freedom. $(r = 3)$

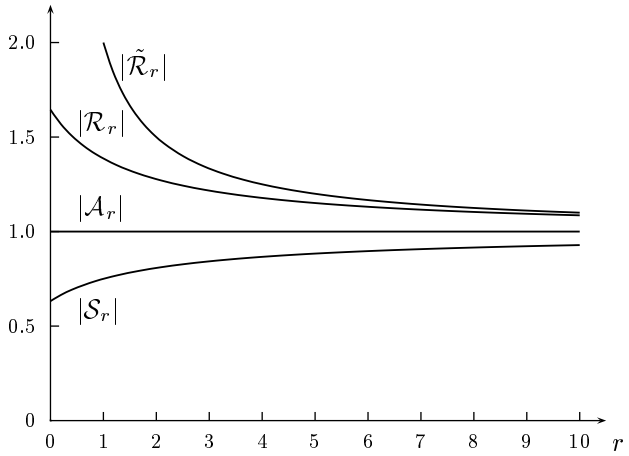

FIGURE 9. Respective ratios of $|\tilde{\mathcal{R}}_r|$, $|\mathcal{R}_r|$ and $|\mathcal{S}_r|$ to $|\mathcal{A}_r|$ as functions of r for heavy-tailed distributions

Then $\mathcal{A}_r \subset \tilde{\mathcal{R}}_r \subset \tilde{\mathcal{Q}}_r$ *and*

$$|\tilde{\mathcal{R}}_r| = \frac{r+1}{r}\,|\mathcal{A}_r| \qquad \textit{and} \qquad |\tilde{\mathcal{Q}}_r| = 2\,|\tilde{\mathcal{R}}_r|. \tag{40}$$

Although the envelope $\tilde{\mathcal{R}}_r$ is not optimal for small r, we find that $|\tilde{\mathcal{R}}_r|$ converges to $|\mathcal{R}_r|$ for $r \to \infty$ (see Figure 9). In particular we have $|\tilde{\mathcal{R}}_r| < 1.1\,|\mathcal{R}_r|$ for $r \geq 4$. So it is only necessary to find sophisticated generators for the distribution with density proportional to $\frac{z^r-1}{z^{r+1}-1}$ if r is small. For $r = 1$, $\frac{z^r-1}{z^{r+1}-1}$ reduces to $1/(z+1)$.

We also can find a universal squeeze for heavy-tailed distributions.

Theorem 15. *Let $f(x)$ be a heavy-tailed T_c-concave probability density function with $c = -\frac{r}{r+1}$, $r > 0$, and mode μ. If $F(\mu)$ is given, where F denotes the c.d.f. of the distribution, then there exists a set $\mathcal{S}_r = \mathcal{S}_r(f)$, such that $\mathcal{S}_r \subset \mathcal{A}_r(f)$. We*

have $(V,U) \in \mathcal{S}_r$ *if and only if*

$$\bar{v}_l\,(u_m^r - U^r) \le r\,u_m^r\,V \le \bar{v}_r\,(u_m^r - U^r) \tag{41}$$

and

$$\bar{v}_l/(r+1) \le V \le \bar{v}_r/(r+1) \tag{42}$$

where u_m *is as defined in Theorem 13 and* $\bar{v}_r = (1-F(\mu))\,(\int\!\!f)/u_m$ *and* $\bar{v}_l = -F(\mu)\,(\int\!\!f)/u_m$. *Moreover*

$$|\mathcal{S}_r| = |\mathcal{A}_r|\left(1 - 1/(r+1)^{(r+1)/r}\right). \tag{43}$$

Proof. Since every heavy-tailed T_c-concave distribution is T_c-concave, inequality (41) follows immediately from Theorem 11. Furthermore notice that $(\bar{v}_r/(r+1), u_m^r/(r+1))$ is contained in the transformed squeeze region (in $\mathcal{B}_r^+$, see proof of Theorem 11). Moreover if $\mathcal{B}_r^+$ contains a boundary point (v_b, u_b^r), then $\mathcal{B}_r^+$ also contains the quadrangle with vertices at $(0,0)$, $(0,u_m^r)$, (v_b,u_b^r) and $(v_b,0)$ (as show above). Thus (42) follows.

To verify (43) we compute

$$\int_0^{v_b} (u_m^r + w/v_b\,(u_b^r - u_m^r))^{1/r}\,dw = \frac{r}{r+1}\,u_m\,v_b\,\frac{(u_b/u_m)^{r+1}-1}{(u_b/u_m)^r - 1}. \tag{44}$$

Using the vertex $(\bar{v}_r/(r+1), u_m^r/(r+1))$ for (v_b,u_b^r) gives $v_b = \bar{v}_r/(r+1) = (1-F(\mu))\int\!\!f\,/(u_m\,(r+1)) = (1-F(\mu))\,|\mathcal{A}_r|/u_m$ and $u_b = u_m/(r+1)^{1/r}$. Then (44) reduces to $(1-F(\mu))\,|\mathcal{A}_r|\,\left(1-1/(r+1)^{(r+1)/r}\right)$, and the result follows from the analogous considerations for $\mathcal{B}_r^-$. □

Remark. If f is the probability density function of a heavy-tailed T_c-concave distribution for a particular c, then this does not hold for any other $c' \neq c$. Indeed $(x-\mu)\,f(x)^{-c}$ is increasing and bounded only if f behaves asymptotically as $x^{1/c}$ for $x \to \infty$. Thus for any $c' \neq c$, $(x-\mu)\,f(x)^{-c}$ either converges to 0 or is unbounded.

Now let f be a differentiable function. Derflinger (2000) calls the largest $c \le 0$ with $(T_c(f(x)))'' \le 0$ the *T_c-concavity* of f at x, denoted by $\mathrm{tc}_f(x)$. (We set $\mathrm{tconv}(f(x)) = -\infty$, if no such c exists.) Then a monotonically decreasing density is heavy-tailed T_c-concave if and only if f is T_c-concave and $c = \liminf_{x\to\infty} \mathrm{tc}_f(x) > -1$. An analogous characterization holds for monotonically increasing and symmetric densities. Some standard distributions (e.g. t-distributions) have the property, that $\mathrm{tc}_f(x)$ is monotonically decreasing on (μ,∞) and increasing in $(-\infty,\mu)$.

6. Discrete distributions

Devroye (1987) has introduced a universal hat for discrete log-concave distributions that can be applied for the design of a black box algorithm. In Leydold (2001) a faster and more generally applicable algorithm based on the ratio-of-uniforms method has been introduced. Therefore it is not amazing that a version of Theorem 10 can be found for discrete distributions. However some modifications are necessary.

A discrete distribution with probability vector p_i, with support $I \subseteq \mathbb{Z}$, is called *T-concave* if

$$T(p_i) \ge \tfrac{1}{2}(T(p_{i-1}) + T(p_{i+1})) \qquad \text{for all } i \in I. \tag{45}$$

For log-concave distributions we have $T(x) = \log(x)$ and $p_i^2 \geq p_{i-1}\, p_{i+1}$. Obviously p_i is unimodal. Denote its mode by μ. For the following assume that p_i is T_c-concave with $c = -r/(r+1)$ for an $r > 0$. Let

$$f_p(x) = \begin{cases} p_{\lfloor x \rfloor} & \text{for } \lfloor x \rfloor \in I \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Since f_p is a step function, $\mathcal{B}_r(f_p)$ cannot be convex, where $\mathcal{B}_r$ is defined by (12). The outer vertices of $\mathcal{B}_r$ (its "spikes") are given by $((i+1-\mu)p_i^{r/(r+1)}, p_i^{r/(r+1)})$ for all $i \geq \mu$, and $((i-\mu)p_i^{r/(r+1)}, p_i^{r/(r+1)})$ for all $i \leq \mu$, with $i \in I \subseteq \mathbb{Z}$ (use transformation $\Phi_{\mathcal{BG}}$ from eq. (13)). Let (v^+, u_e^r) be the right extremal point of $\mathcal{B}_r$ and let $\Delta_{\mathcal{B}}$ denote the triangle with vertices at $(0,0)$, $(0, u_m^r)$ and (v^+, u_e^r) in $\mathcal{B}_r^+ = \{(v, u^r) \in \mathcal{B}_r \colon v \geq 0\}$. Notice that by inequality (45) there are no outer vertices of $\mathcal{B}_r$ in the interior of $\Delta_{\mathcal{B}}$. Let $\Delta_{\mathcal{A}}$ be the image of $\Delta_{\mathcal{B}}$ under transformation $\Phi_{\mathcal{AB}}^{-1}$.

Lemma 16. *$|\Delta_{\mathcal{A}}| \leq |\mathcal{A}_r^+|$, where $\mathcal{A}_r^+ = \{(v,u) \in \mathcal{A}_r \colon v \geq 0\}$.*

Proof. For the proof we again need the transformation (13). Denote the images of $\Delta_{\mathcal{B}}$ under the respective transformations $\Phi_{\mathcal{BG}}$ and $\Phi_{\mathcal{GT}} \circ \Phi_{\mathcal{BG}}$ by $\Delta_{\mathcal{G}}$ and $\Delta_{\mathcal{T}}$. The latter transformation $\Phi_{\mathcal{GT}} \circ \Phi_{\mathcal{BG}}$ maps straight lines into straight lines (see proof of Theorem 6). Thus $\Delta_{\mathcal{T}}$ is a convex (unbounded) polygon. Moreover the edge $(0,0)(0,u_m^r)$ is mapped into the y-axis.

Now assume $u_e = 0$. Then $\Delta_{\mathcal{T}}$ is bounded only by the images of the edges $(0,0)(0,u_m^r)$ and $(0,u_m^r)(v^+,u_e^r)$. Since both are straight lines, they are subsets of $\mathcal{T}_c$, $c = -r/(r+1)$, by inequality (45). Hence $\Delta_{\mathcal{T}} \subseteq \mathcal{T}_c$ and consequently $\Delta_{\mathcal{A}} \subseteq \mathcal{A}_r^+$ and the proposition follows.

If $u_e > 0$ then edge $(0,u_m)(v^+,u_e^r)$ of $\Delta_{\mathcal{B}}$ is contained in the quadrangle $Q_{\mathcal{B}}$ with vertices in $(0,u_m^r)$, (u_m^r,u_m^r), (v^+,u_e^r) and $(v^+-u_e^r,u_e^r)$, see Figure 10 (l.h.s.). Using $\Phi_{\mathcal{GT}} \circ \Phi_{\mathcal{BG}}$, Q is mapped into a set $Q_{\mathcal{T}}$ which again is a quadrangle by Theorem 6, see Figure 10 (r.h.s.). Notice that we find for the areas $|Q_{\mathcal{T}} \cap \Delta_{\mathcal{T}}| = |Q_{\mathcal{T}}|/2$ and $|Q_{\mathcal{T}} \cap \mathcal{T}_c| \geq |Q_{\mathcal{T}}|/2$. Figure 10 shows the "worst case" where equality holds in (45) for all i between the mode and the point that corresponds to the extremal point (v^+, u_e^r). Now we can use transformation $\Phi_{\mathcal{GT}}^{-1}$ to map $Q_{\mathcal{T}}$ into a set $Q_{\mathcal{G}}$. We have to show that $|\Delta_{\mathcal{G}}| \leq |\mathcal{G}^+|$, where $\mathcal{G}^+ = \{(x,y) \in \mathcal{G} \colon y \leq f(x)\}$. Then the result follows due to the fact that that the Jacobian of $\Phi_{\mathcal{BG}} \circ \Phi_{\mathcal{AB}}$ is constant and thus $|\Delta_{\mathcal{A}}| \leq |\mathcal{A}^+|$ follows as proposed.

Now let $Q_{\mathcal{G}}$ denote the image of $Q_{\mathcal{T}}$ under transformation $\Phi_{\mathcal{GT}}^{-1}$. To verify $|\Delta_{\mathcal{G}}| \leq |\mathcal{G}^+|$ it is sufficient to show that the area of the image of $Q_{\mathcal{T}} \cap \mathcal{T}_c$ under transformation $\Phi_{\mathcal{GT}}^{-1}$, i.e. $Q_{\mathcal{G}} \cap \mathcal{G}$, is larger than the area of the image of the triangle $Q_{\mathcal{T}} \cap \Delta_{\mathcal{T}}$, i.e. $Q_{\mathcal{G}} \cap \Delta_{\mathcal{G}}$. The absolute value of the Jabobian of $\Phi_{\mathcal{GT}}^{-1}$ is given by $1/c\,(-y)^{1/c-1}$ which is decreasing for decreasing y, since $y < 0$. Hence we immediately find (see r.h.s. of Figure 10) $|Q_{\mathcal{G}} \cap \Delta_{\mathcal{G}}| \leq |Q_{\mathcal{G}}|/2$. To see that $|Q_{\mathcal{G}} \cap \mathcal{G}| > |Q_{\mathcal{G}}|/2$ notice that we can split $Q_{\mathcal{T}}$ horizontally along the upper edges of the bars in Figure 10. In each of these parts the shaded region that belongs to $\mathcal{T}_c$ has at least the same area as its complement outside. Moreover the shaded part is more concentrated towards the x-axis than the its complement. Thus if we look at the images of this region and of its complement under transformation $\Phi_{\mathcal{GT}}^{-1}$ the area of the former is always greater than the area of the latter and thus $|Q_{\mathcal{G}} \cap \mathcal{G}| > |Q_{\mathcal{G}}|/2$. Hence $|Q_{\mathcal{G}} \cap \mathcal{G}| > |Q_{\mathcal{G}} \cap \Delta_{\mathcal{G}}|$ and the result follows. □

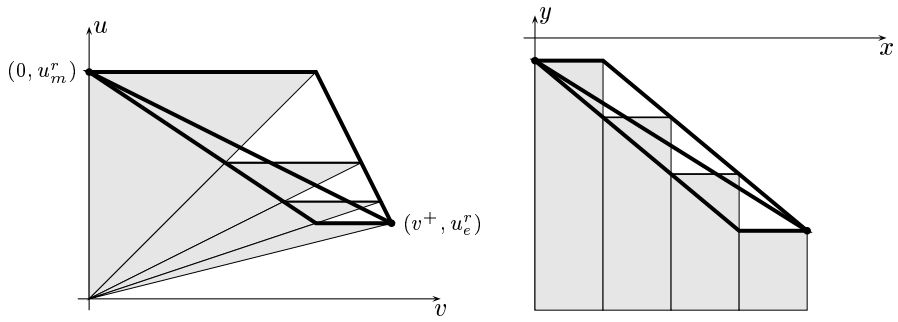

FIGURE 10. Quadrangle $Q_{\mathcal{B}}$ for $\mathcal{B}_r$ (l.h.s.) and the corresponding quadrangle $Q_{\mathcal{T}}$ for $\mathcal{T}_c$ (r.h.s.)

It is obvious that an analogous result holds for $\mathcal{A}_r^-$ and we arrive at the following proposition analogously to our considerations of Theorem 7. Notice that $\sup_{i<\mu} p_i = p_{\mu-1}$.

Theorem 17. *Let p_i, $i \in \mathbb{Z}$, be a T_c-concave probability vector with $c = -r/(r+1)$, $r > 0$ and mode μ. Let F denote the c.d.f. of the distribution and let*

$$\begin{aligned} u_l &= \sqrt[r+1]{p_{\mu-1}} \quad & v_l(u) &= (\textstyle\sum p_i)/(r\,u_l)\,\frac{(u/u_l)^r-1}{(u/u_l)-1} \\ u_r &= \sqrt[r+1]{p_{\mu}} \quad & v_r(u) &= (\textstyle\sum p_i)/(r\,u_r)\,\frac{(u/u_r)^r-1}{(u/u_r)-1} \end{aligned} \tag{47}$$

where we set $v_l(u) = 0$ if $u_l = 0$. Let $\mathcal{R}_r = \mathcal{R}_r^- \cup \mathcal{R}_r^+$ with

$$\begin{aligned} \mathcal{R}_r^- &= \{(v,u)\colon -F(\mu)\,v_l(u) \le v \le 0,\, 0 \le u \le u_l\} \\ \mathcal{R}_r^+ &= \{(v,u)\colon 0 \le v \le (1-F(\mu))\,v_r(u), 0 \le u \le u_r\} \end{aligned} \tag{48}$$

and $\mathcal{Q}_r = \mathcal{Q}_r^- \cup \mathcal{Q}_r^+$ with

$$\begin{aligned} \mathcal{Q}_r^- &= \{(v,u)\colon v_l(u) \le v \le 0,\, 0 \le u \le u_l\} \\ \mathcal{Q}_r^+ &= \{(v,u)\colon 0 \le v \le v_r(u), 0 \le u \le u_r\} \end{aligned} \tag{49}$$

Then $\mathcal{A}_r \subset \mathcal{R}_r \subset \mathcal{Q}_r$ and

$$|\mathcal{R}_r| = \frac{r+1}{r}(\gamma + \psi(r+1))\,|\mathcal{A}_r| \qquad \text{and} \qquad |\mathcal{Q}_r| = 2\,|\mathcal{R}_r| \tag{50}$$

where $\gamma = 0.577216\ldots$ denotes Euler's (gamma) constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ the Digamma function (also known as Euler's psi function).

For sampling uniformly from $\mathcal{Q}_r$ and $\mathcal{R}_r$ the envelopes from Theorem 10 can be used. It is straight forward to find a universal squeeze analogously to continuous distributions. Furthermore bounds for discrete heavy-tailed T_c-concave distributions exist.

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References

Ahrens, J. H. (1993). Sampling from general distributions by suboptimal division of domains. *Grazer Math. Berichte 319*, 20 pp.

Ahrens, J. H. (1995). A one-table method for sampling from continuous and discrete distributions. *Computing 54*(2), 127–146.

Derflinger, G. (2000). private communication.

Derflinger, G., W. Hörmann, and J. Leydold (2002). Universal methods of non-uniform random variate generation. in preparation.

Devroye, L. (1984a). On the use of probability inequalities in random variate generation. *J. Stat. Comput. Simulation 20*, 91–100.

Devroye, L. (1984b). A simple algorithm for generating random variates with a log-concave density. *Computing 33*(3–4), 247–257.

Devroye, L. (1986). *Non-Uniform Random Variate Generation.* New-York: Springer-Verlag.

Devroye, L. (1987). A simple generator for discrete log-concave distributions. *Computing 39*, 87–91.

Dieter, U. (1989). Mathematical aspects of various methods for sampling from classical distributions. In E. A. Mc Nair, K. J. Musselman, and P. Heidelberger (Eds.), *Proc. 1989 Winter Simulation Conf.*, pp. 477–483.

Evans, M. and T. Swartz (1998). Random variable generation using concavity properties of transformed densities. *Journal of Computational and Graphical Statistics 7*(4), 514–528.

Gilks, W. R. and P. Wild (1992). Adaptive rejection sampling for Gibbs sampling. *Applied Statistics 41*(2), 337–348.

Gradshteyn, I. S. and I. M. Ryzhnik (1994). *Table of Integrals, Series, and Products* (5th ed.). Academic Press.

Hörmann, W. (1995). A rejection technique for sampling from T-concave distributions. *ACM Trans. Math. Software 21*(2), 182–193.

Hörmann, W. and G. Derflinger (1996). Rejection-inversion to generate variates from monotone discrete distributions. *ACM TOMACS 6*(3), 169–184.

Hörmann, W. and G. Derflinger (1997). An automatic generator for a large class of unimodal discrete distributions. In A. R. Kaylan and A. Lehmann (Eds.), *ESM 97*, pp. 139–144.

Kinderman, A. J. and F. J. Monahan (1977). Computer generation of random variables using the ratio of uniform deviates. *ACM Trans. Math. Software 3*(3), 257–260.

Leydold, J. (2000a). Automatic sampling with the ratio-of-uniforms method. *ACM Trans. Math. Software 26*(1), 78–98.

Leydold, J. (2000b). A note on transformed density rejection. *Computing 65*(2), 187–192.

Leydold, J. (2001). A simple universal generator for continuous and discrete univariate T-concave distributions. *ACM Trans. Math. Software 27*(1), 66–82.

Stadlober, E. (1989). *Sampling from Poisson, binomial and hypergeometric distributions: Ratio of uniforms as a simple and fast alternative.* Number 303 in Bericht der Mathematisch-Statistischen Sektion in der Forschungsgesellschaft Joanneum-Graz.

Wakefield, J. C., A. E. Gelfand, and A. F. M. Smith (1991). Efficient generation of random variates via the ratio-of-uniforms method. *Statist. Comput. 1*(2), 129–133.

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