Lecture: Continuous Time Finance

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Chapter 1: Review of discrete time finance

Section 1.1: Single period models

Section 1.2: Multiperiod models

Section 1.3: From discrete to continuous time

Chapter 1: Review of discrete time finance

Section 1.1: Single period models

- Basic concepts

- Risk neutral pricing

Syllabus: binomial model - lognormal model - portfolio - no arbitrage property - claim - arbitrage free price - replicating portfolio - market completeness - risk neutral model - fundamental theorem - risk neutral pricing

Single period model

Basic probability space $(\Omega, \mathcal{F}, P)$

A market with $d+1$ assets: $\mathcal{M} = (S^0, S^1, \ldots, S^d)$, period $[t, T]$.

Spot prices: $S_t^0, S_t^1, \ldots, S_t^d$ (constants)
Terminal values: $S_T^0, S_T^1, \ldots, S_T^d$ (random variables)

The simplest case: A market with two assets $\mathcal{M} = (B, S)$

Spot prices: B_t, S_t
Terminal values: B_T, S_T

Example: Binomial model

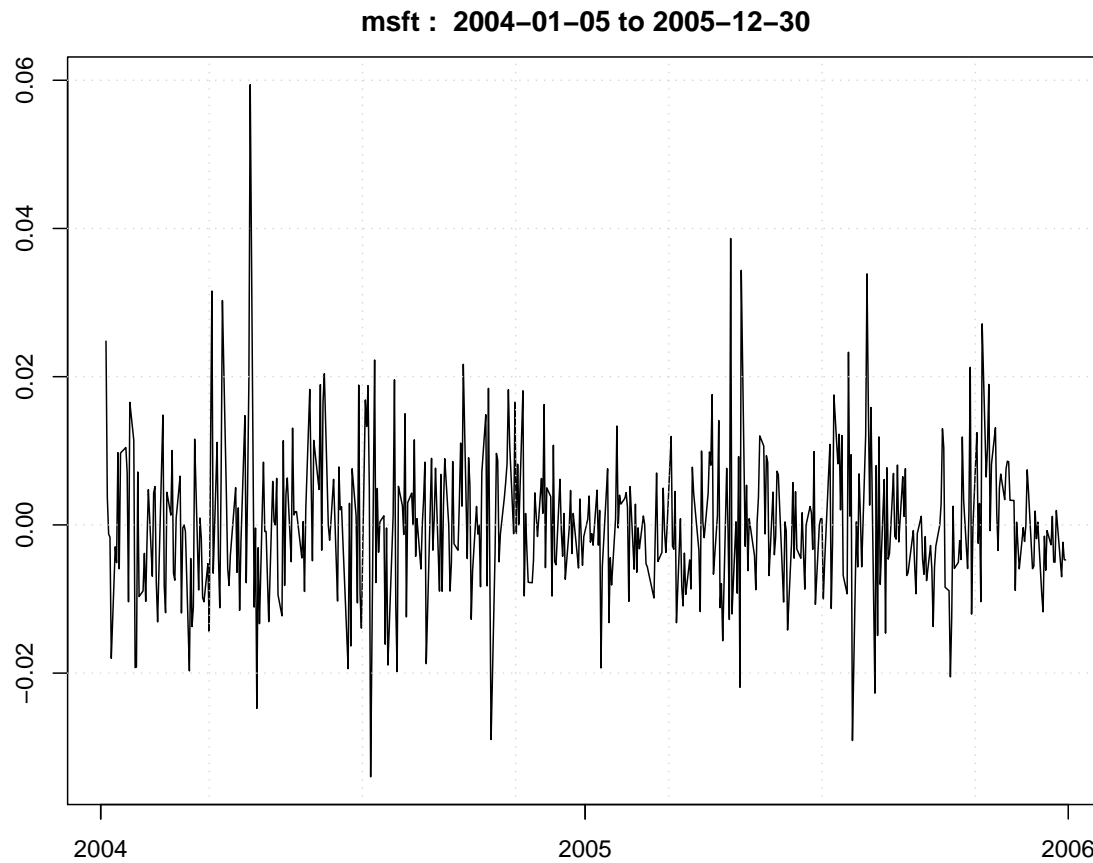B bank account with fixed interest rate r: $B_t = 1$, $B_T = e^{r(T-t)}$
S risky asset: $P(S_T = S_t u) = p$ or $P(S_T = S_t d) = 1 - p$ with $d < u$ and $0 < p < 1$.
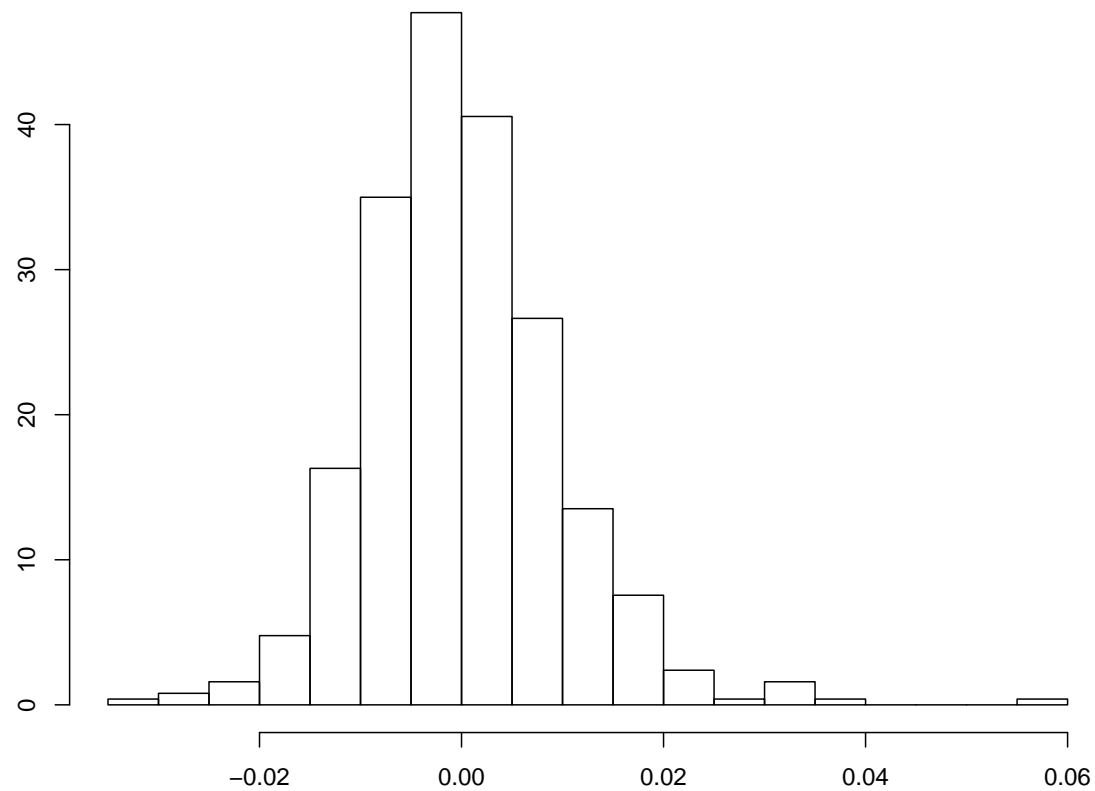
Stock price of MSFT 2004-2005

msft : 2004−01−02 to 2005−12−30

Slide 5 - Diagram

Log-returns of MSFT 2004-2005

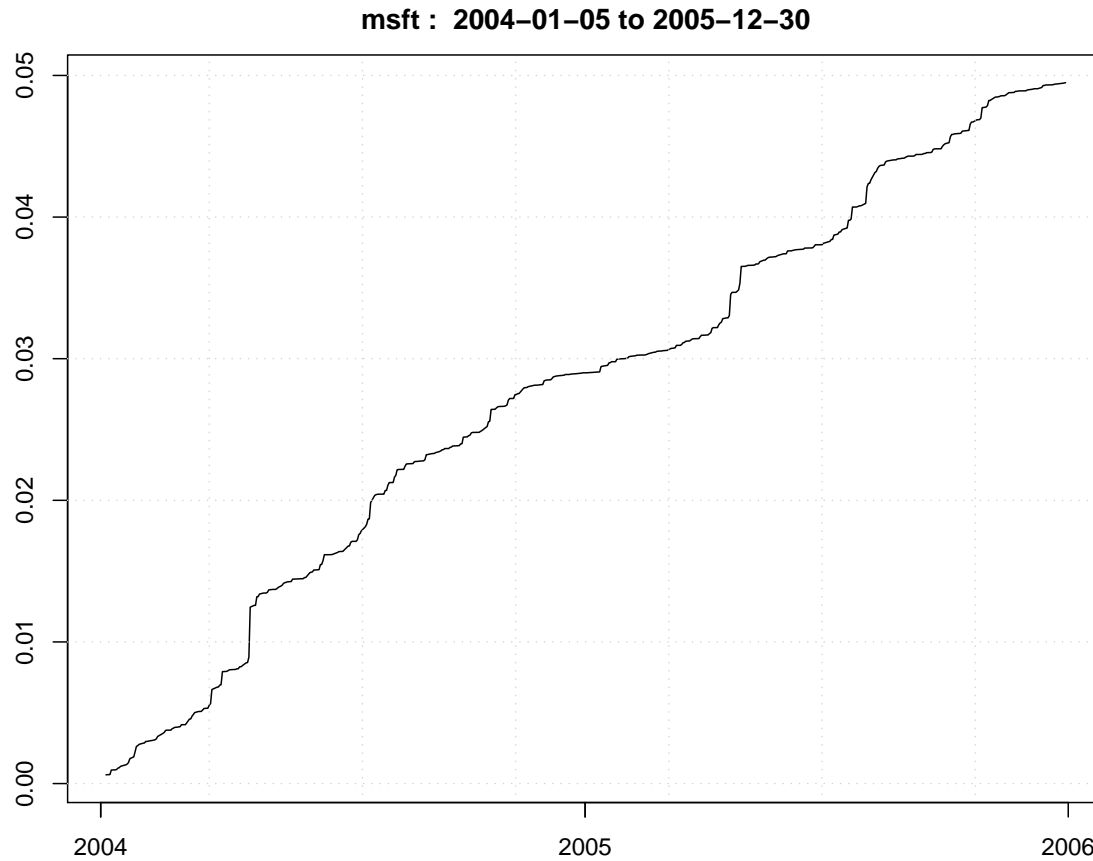**msft : 2004−01−05 to 2005−12−30**

Histogram of the log-returns of MSFT 2004-2005

Normal plot of the log-returns of MSFT 2004-2005

Slide 8 - Diagram
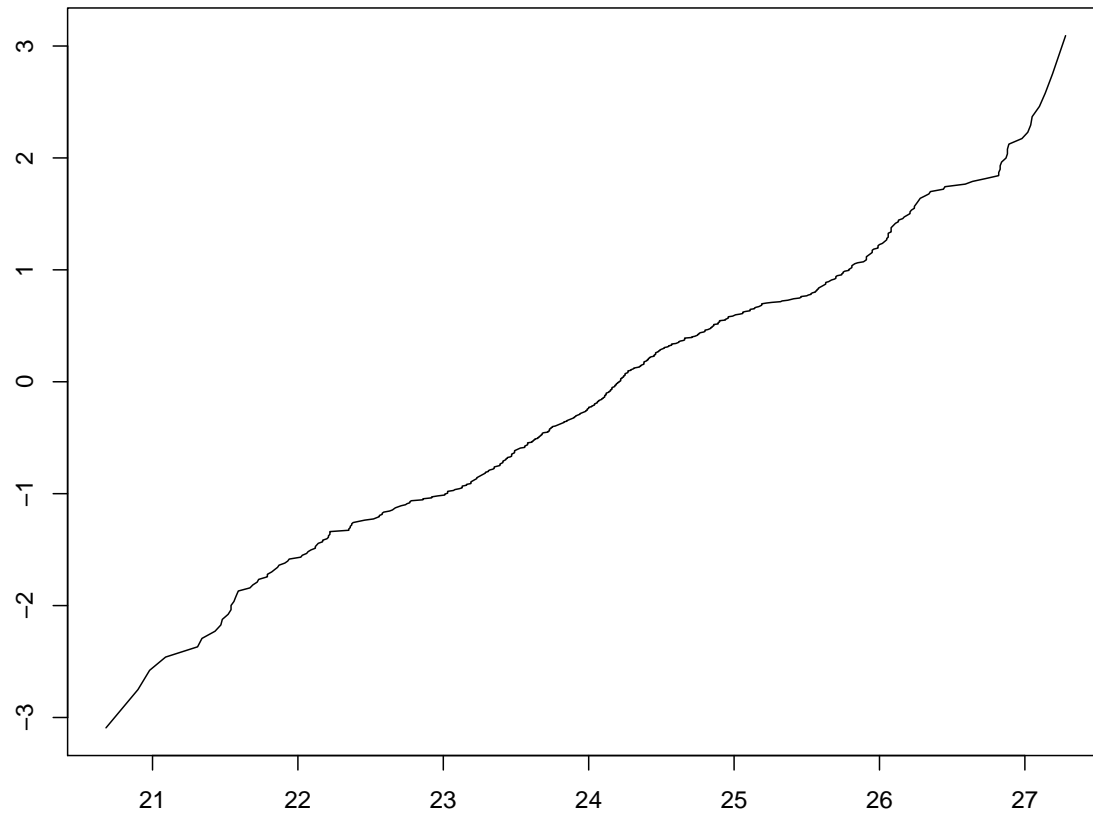
Cumulated squared volatility of the log-returns of MSFT 2004-2005

msft : 2004−01−05 to 2005−12−30

Stock price of Dow Jones 2004-2008

^dji : 2004−01−02 to 2008−12−31

Log-returns of Dow Jones 2004-2008

^dji : 2004−01−05 to 2008−12−31
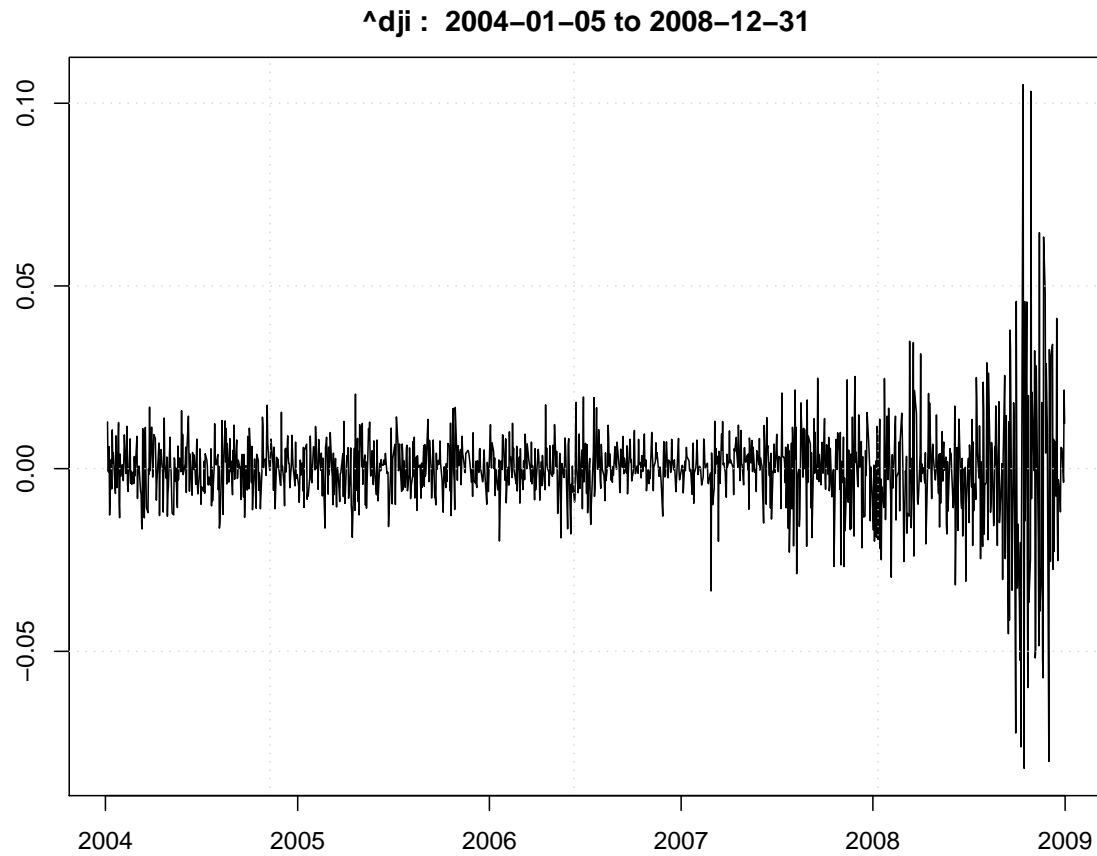
```
y=GetStocks("^dji",from=c(1,1,2004),to=c(31,12,2004))
print(y)
pPlot(y)
pPlot(Returns(y))


y=GetStocks(c("msft","nok"),from=c(1,1,2004),to=c(31,12,2004))
pPlot(y)
pPlot(Returns(y))
```

```
y=GetStocks(c("ko"),from=c(1,1,2000),to=c(31,12,2001))

Scatter(y)
Scatter(Returns(y))

summary(Returns(y))
Hist(Returns(y))
Normalplot(Returns(y))

Boxplot(Returns(y))
Shape(as.vector(Returns(y)))
```

Example: Lognormal model

B bank account with fixed interest rate: $B_t = 1$, $B_T = e^{r(T-t)}$

Assume that the log-returns of the risky asset are normally distributed:

$$\log \frac{S_T}{S_t} \sim N(a(T-t), \sigma^2(T-t))$$

Volatility: Standard deviation σ of the log-returns (per time unit).

Average asset price: $E(S_T|S_t) = S_t \exp\left(a(T-t) + \frac{\sigma^2(T-t)}{2}\right) = S_t e^{\mu(T-t)}$ (A1)
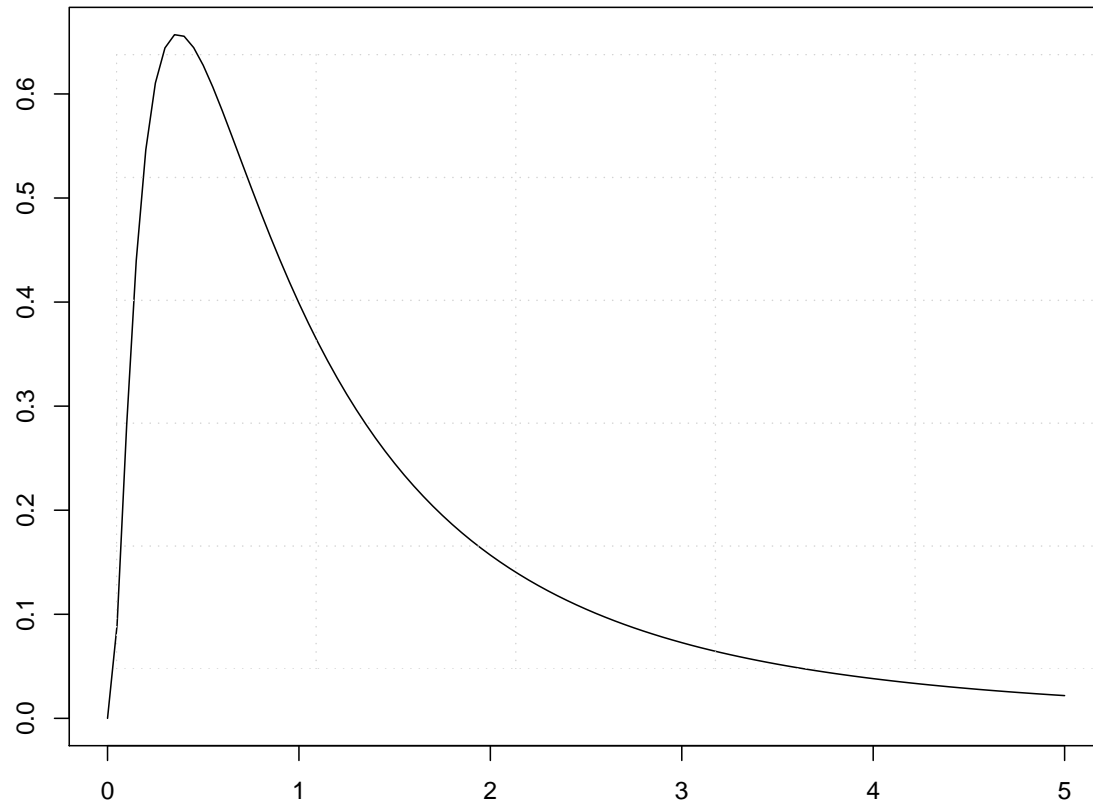
Usual parametrization: $\mu = a + \sigma^2/2$ growth rate of average prices

Lognormal model:

$$S_T = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}\,Z\right]$$

where $Z \sim N(0,1)$.

Lognormal density: $\mu = 0$, $\sigma = 1$

Lognormal density: $\mu = 0.04$, $\sigma = 0.5$
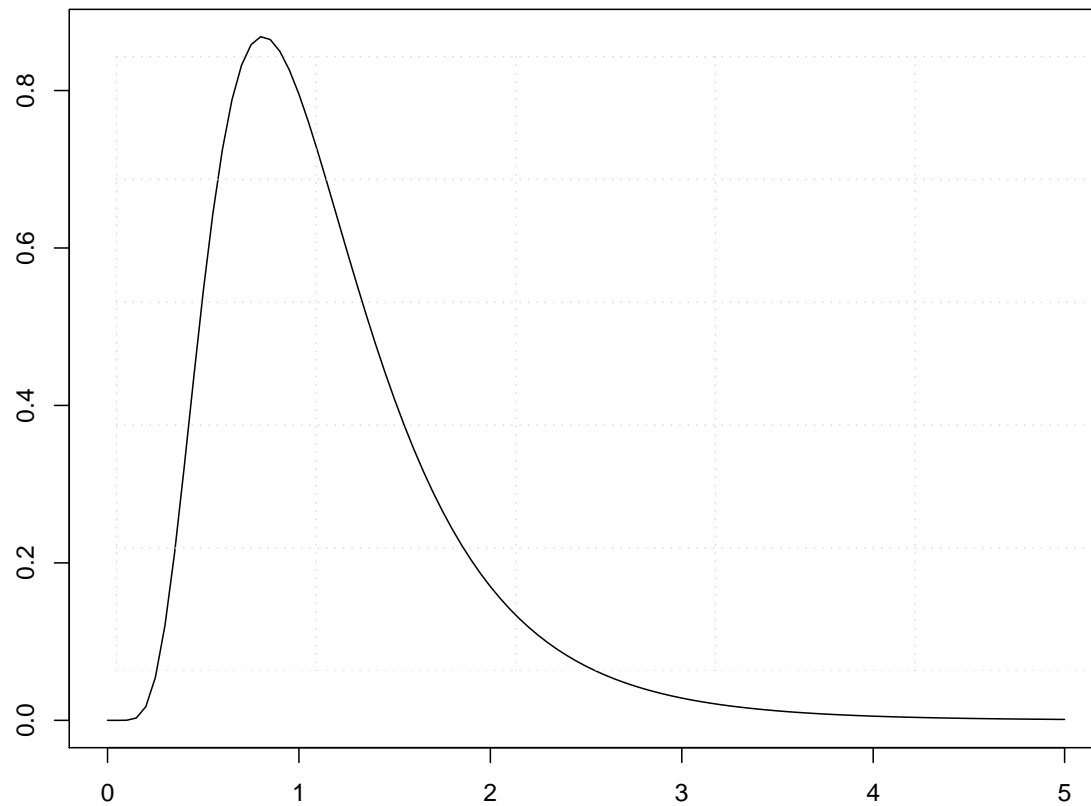
Lognormal density: $\mu = -\dfrac{\sigma^2}{2}, \sigma = 0.5$
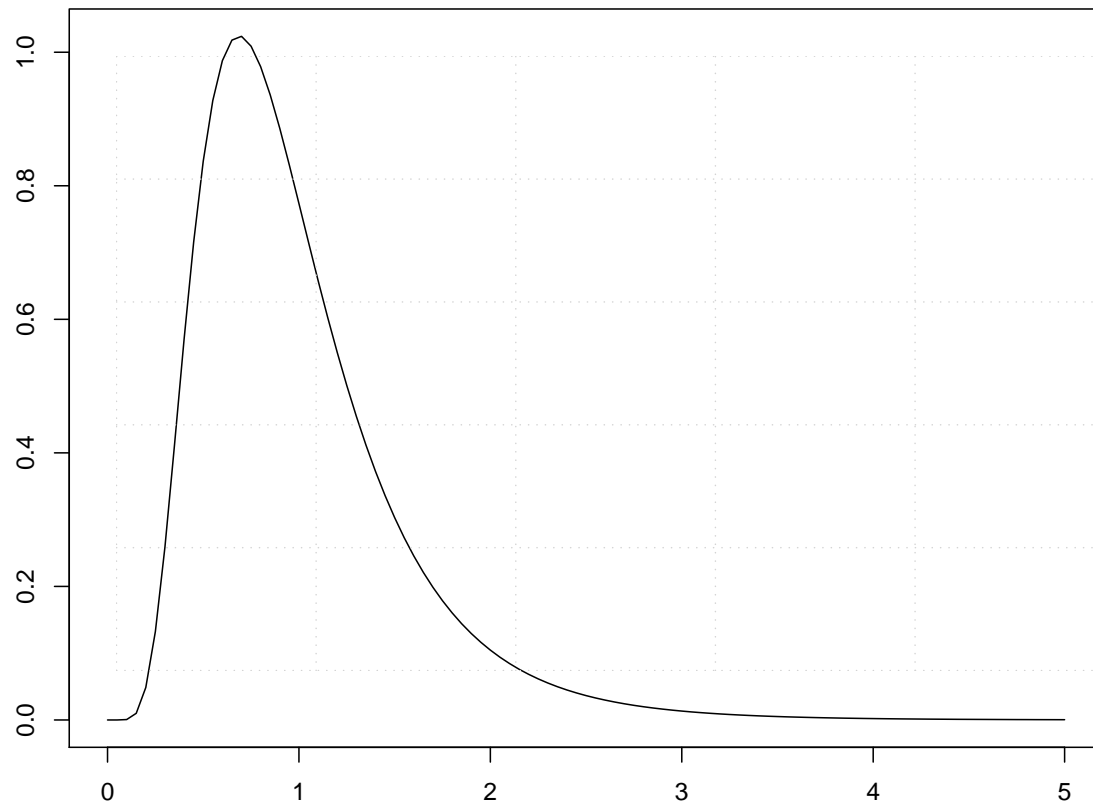
```
ff=function(meanlog=0,sdlog=1) {
    fun=function(x) dlnorm(x,meanlog,sdlog)
    }


plot(ff(),0,5);grid()
plot(ff(meanlog=0.04,sdlog=0.5),0,5);grid()
sd=0.5
plot(ff(meanlog=-sd^2/2,sdlog=sd),0,5);grid()

x=rlnorm(500,meanlog=-sd^2/2,sdlog=sd)
Scatter(x)
Hist(x,breaks=20)
plot(density(x),xlim=c(0,5))
```

```
X=rpBS(rate=0.04,vol=0.2)
pPlot(X(1))
pPlot(X(20))

x=rowMeans(X(5000))
plot(attr(X(),"time"),x,type="l")
```

Portfolios

Portfolio: $H = (H^0, H^1, \ldots, H^d)$ (number of units in the portfolio)

Value: $V_t = \sum_{i=0}^{d} H^i S_t^i$: initial value, price of the portfolio

$V_T = \sum_{i=0}^{d} H^i S_T^i$: terminal value, payoff of the portfolio

Special case: Two assets

Portfolio: $H = (H^B, H^S)$ (number of units in the portfolio)

Value: $V_t = H^B B_t + H^S S_t$: initial value, price of the portfolio

$V_T = H^B B_T + H^S S_T$: terminal value, payoff of the portfolio

No Arbitrage property

Definition: The market $\mathcal{M}$ has the no-arbitrage property (NA) if for every portfolio value V

$$V_t \leq 0 \text{ and } V_T \geq 0 \;\Rightarrow\; V_t = V_T = 0$$

Example: Binomial model (A2)

NA holds iff $d < e^{r(T-t)} < u$.

Consequences: Let $\mathcal{M}$ be a market with NA. Then:

(A3) There is no free lunch (NFL): $V_t = 0$ and $V_T \geq 0 \;\Rightarrow\; V_T = 0$ whenever $V \in \mathcal{V}$.

(A4) Prices are uniquely determined, i.e.

$$V_T = \overline{V}_T \;\Rightarrow\; V_t = \overline{V}_t \text{ whenever } V, \overline{V} \in \mathcal{V}$$

(This means: If NA holds then pricing makes sense.)

(A5) There is at most one admissible portfolio value $V \in \mathcal{V}$ which is riskless.

(A6) **Multinomial case:** Characterize the NA-property !

Bet and Win: A simple example

(A7) A bet is offered on a soccer game with teams A and B (remis excluded). A company offers bets with quotes (a, b). Characterize the quotes where arbitrage (a riskless profit) is possible !

(A8) Assume that two companies offer bets on the same game, but with different quotes. Discuss the NA property !

(A9) Extend your findings to bets with k alternatives (e.g. horse races) and m companies offering quotes !

(A10) **A theoretical problem:**

Let (B, S) be a single period market. Assume that there is a numeraire (a strictly positive portfolio $N \in \mathcal{V}$). Show that NA is equivalent to NFL !

Claims

Definition: A claim is a payoff at time T depending on the history of the market.

Forwards: Period $[t, T]$, market model: $B = (B_t, B_T) = (1, e^{r(T-t)})$, $S = (S_t, S_T)$.

$C_T = S_T - A$: This claim can be replicated by the portfolio $V = S - Ae^{-r(T-t)}B$

Price: $V_t = S_t - Ae^{-r(T-t)}$

$V_t = 0$ implies $A = S_t e^{r(T-t)}$: forward price

Options: Call $C_T = (S_T - K)^+$, Put $P_T = (K - S_T)^+$, (K strike price).

Binary Call $C_T^b = 1_{(S_T \geq K)}$, Binary Put $P_T^b = 1_{(S_T < K)}$

Call prices:

Call price: T= 1 , vol= 0.5 , rate= 0.05
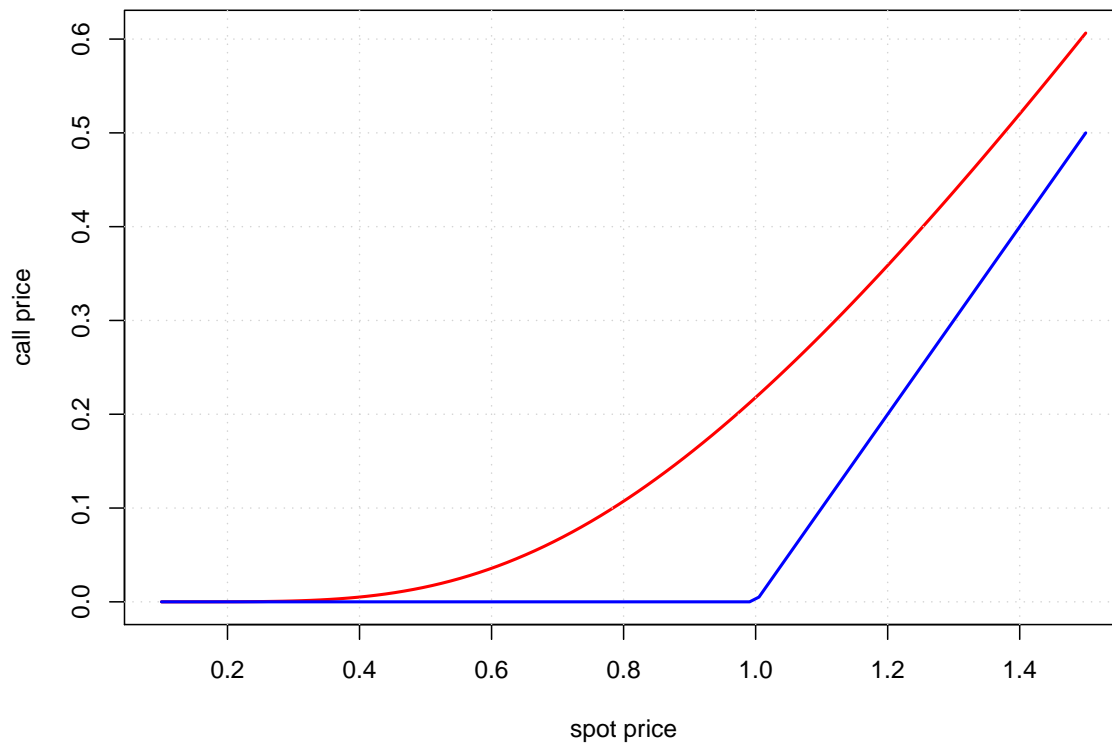
Call prices:

Call price: T= 0.1 , vol= 1 , rate= 0.05

Slide 25 - Diagram
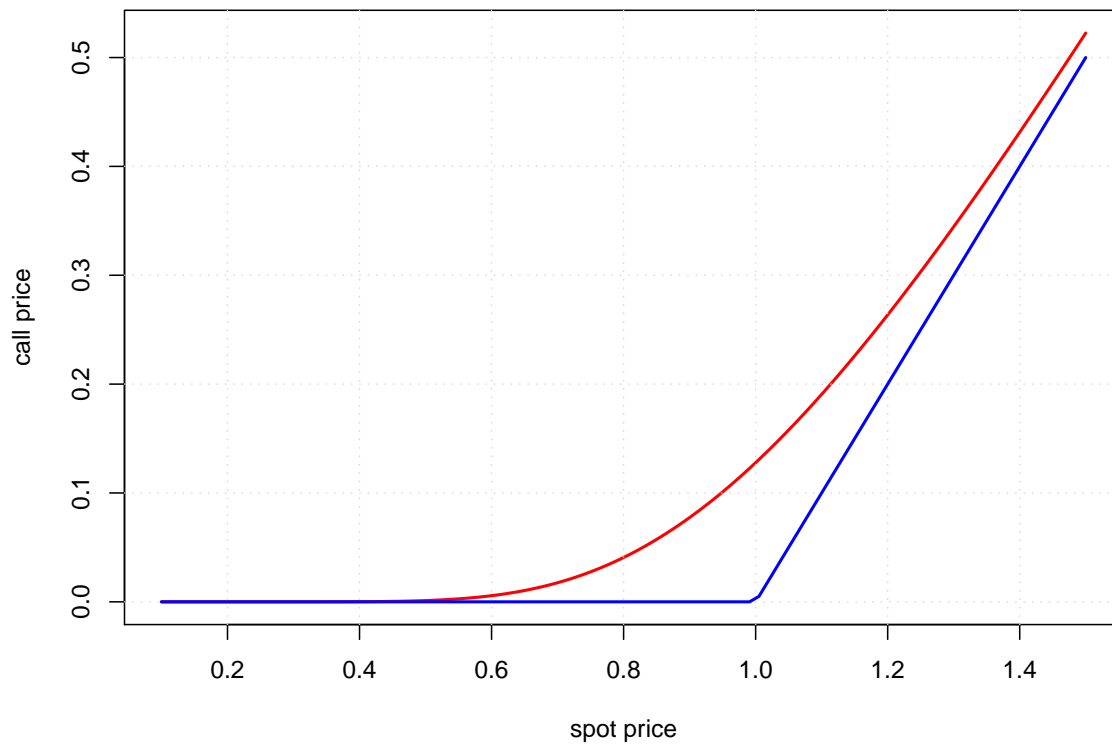
Call prices:

Call price: T= 0.01 , vol= 1 , rate= 0.05
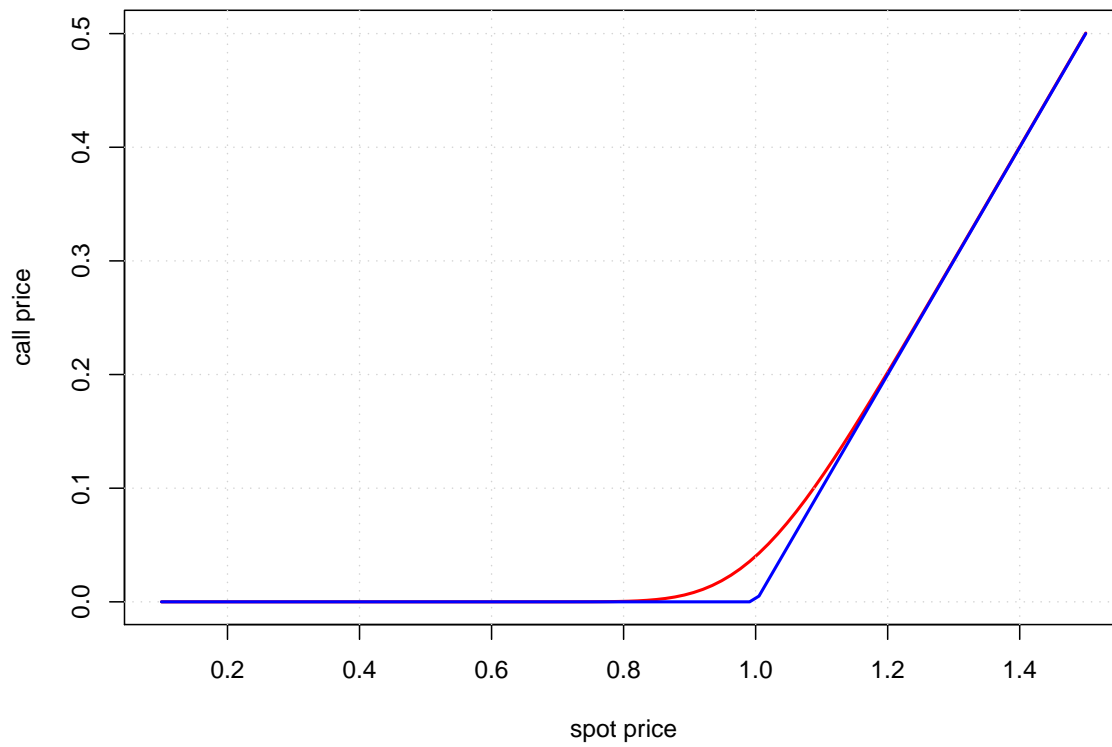
Put prices:

Put price: T= 1 , vol= 1 , rate= 0.05
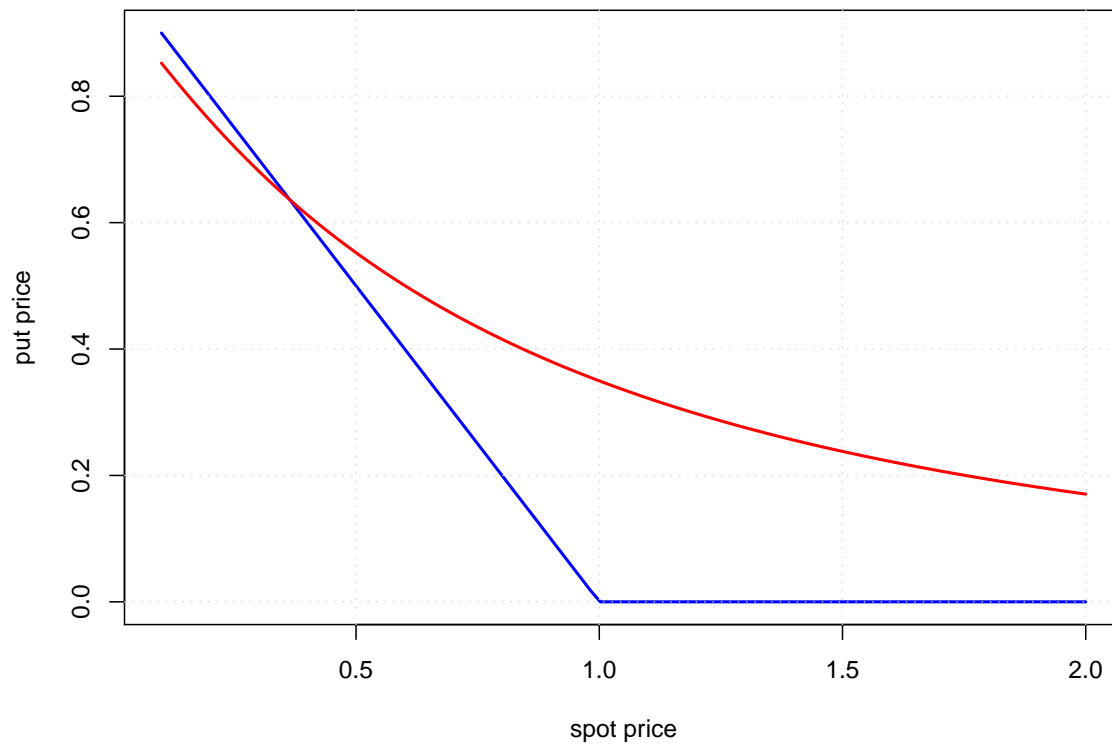
Put prices:

Put price: T= 1 , vol= 1 , rate= 0.05
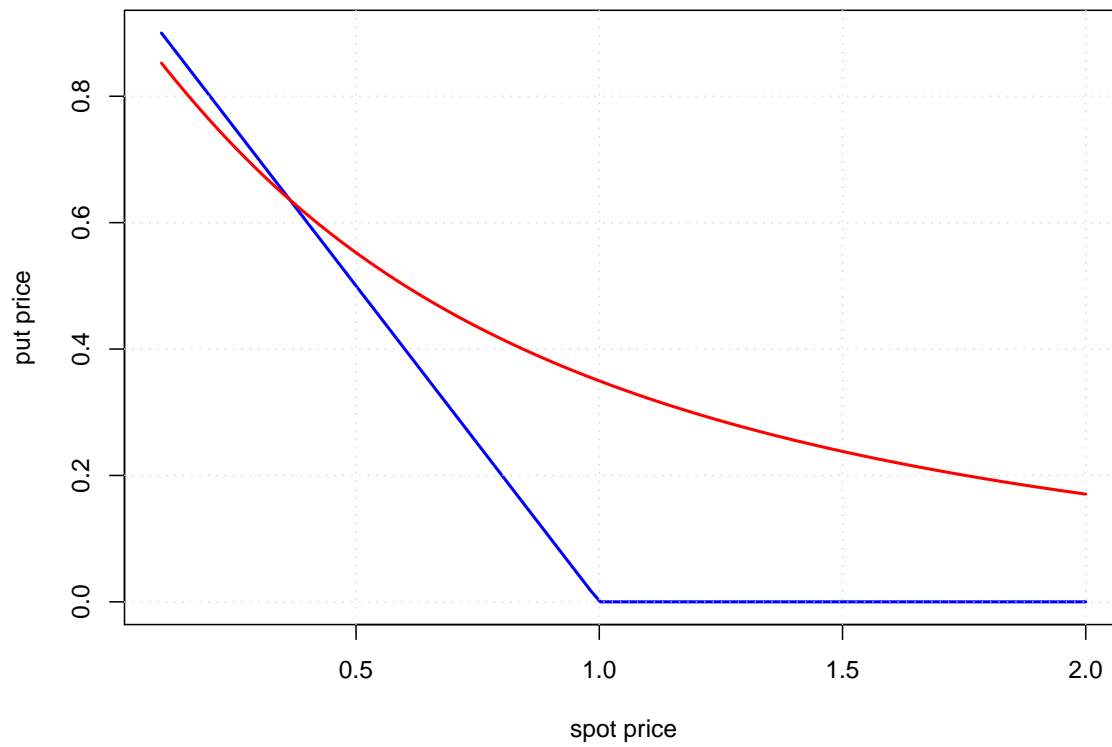
Put prices:

Put price: T= 0.01 , vol= 1 , rate= 0.05
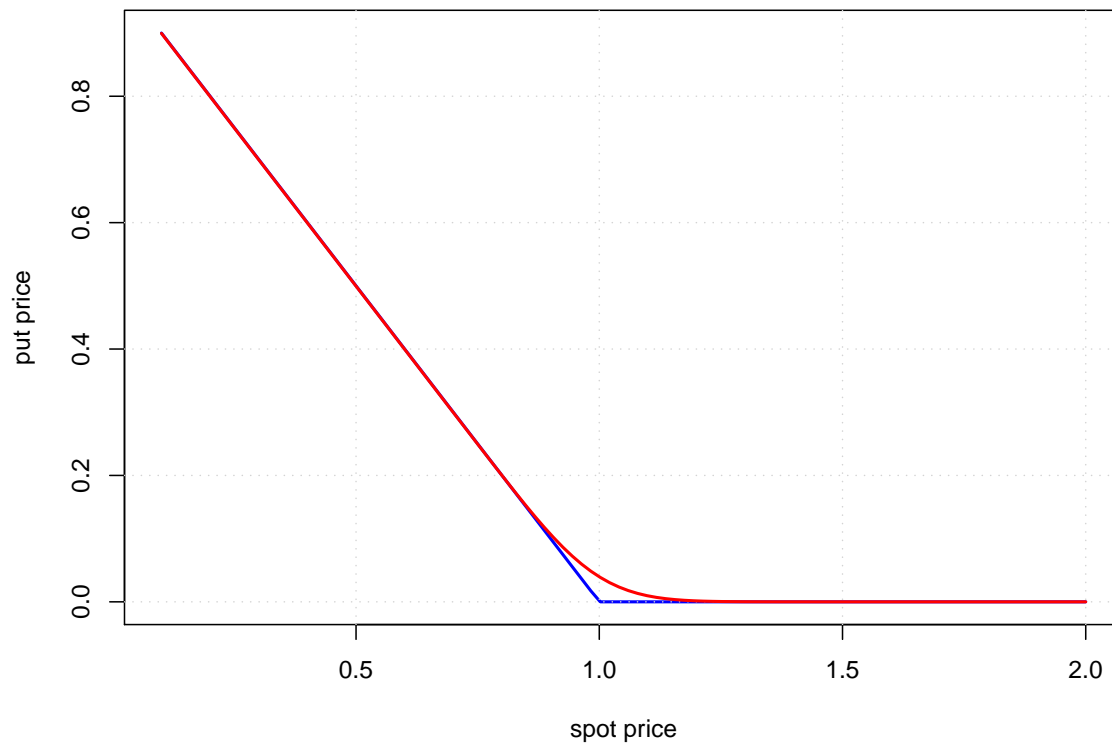
Call price surface
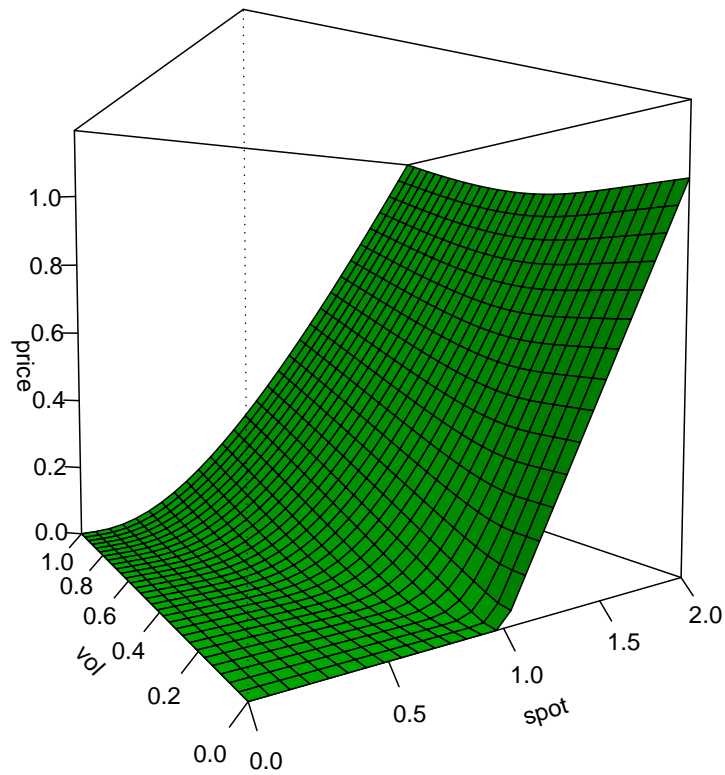
```
call.price.diagram(rate=0.05)
put.price.diagram(rate=0.05)

pf=function(x,y) Call.BS(spot=x,vol=y)$Value
Persp("pf(x,y)",xlim=c(0,2),ylim=c(0,1),xlab="spot",
    ylab="vol",zlab="price")
```

Pricing problem

Definition: Let $\mathcal{M} = (S^0, S^1, \ldots, S^d)$ be a an arbitrage-free market and let C_T be a claim. A number π is an arbitrage-free price of C_T if the market $(S^0, S^1, \ldots, S^d, (\pi, C_T))$ is arbitrage-free.

Vanilla claims: Claims with liquid prices.

The claims are traded at an exchange, and market efficiency provides arbitrage-free prices (hopefully).

Exotic claims: Claims which are not traded at an exchange but sold over the counter.

Pricing problem: How to find arbitrage-free prices for exotic claims ?

Put-Call parity

Let C and P be a call and a put for the same strike price K with arbitrage free prices.

Put-Call parity:

$$C_T - P_T = (S_T - K)^+ - (K - S_T)^+ = S_T - K \Rightarrow C_t - P_t = S_t - Ke^{-r(T-t)}$$

Implications:

$$C_t \geq S_t - Ke^{-r(T-t)} \quad \text{which is} > S_t - K$$

Early exercise of a call is of no value.

$$P_t \geq Ke^{-r(T-t)} - S_t \quad \text{which is} < K - S_t$$

Early exercise of a put can be valuable.

Pricing by replication:

Definition: A claim is attainable if there is a portfolio whose payoff is equal to the claim (a replicating portfolio).

Pricing rule for attainable claims:

Find a replicating portfolio (H^i), i.e. such that the portfolio payoff satisfies $V_T = C_T$.
Then take V_t as the price of the claim C_T at time t.

If the market satisfies NA, then this rule provides an arbitrage-free price, and this price is uniquely determined.

Question: When is it possible to replicate a claim ?

Example: Binomial model.

Let $C_T := \begin{cases} C_T^u \text{ whenever } S_T = S_T^u = S_t u \\ C_T^d \text{ whenever } S_T = S_T^d = S_t d \end{cases}$

Then a replicating portfolio necessarily satisfies:
$$H^B e^r + H^S S_T^u = C_T^u$$
$$H^B e^r + H^S S_T^d = C_T^d$$

We see: $C_T - H^S S_T = H^B e^r$ is riskless !

$C - H^S S_T$ has only a single outcome: $C_T^u - H^S S_T^u = C_T^d - H^S S_T^d$

This implies

$$H^S = \frac{C_T^u - C_T^d}{S_T^u - S_T^d} =: \Delta, \quad H^B = e^{-r}(C_T^u - \Delta S_T^u)$$

The number of units of S in the portfolio equals the Delta of the claim.

(A11) **Problem:** A stock price is currently 20$, and it is known that at the end of three months the stock price will be either 18$ or 22$. We are interested in pricing a European call option to buy the stock for 21$ at the end of three months. Suppose that the risk-free rate is 12% per annum.

In a binomial model every claim can be replicated.

The single period binomial model is complete.

Definition: A market model is complete if every claim can be replicated
by a portfolio value.

(A12) **Problem:** Discuss completeness in the multinomial case !

Pricing based on liquid options

Assume that the market provides prices for calls $C^1, C^2, \ldots, C^m$ with strikes $K_1 < K_2 < \ldots < K_m$.

Let $C_T = f(S_T)$ be a claim whose payoff $f(x)$ is continuous and piecewise linear payoff function with edges at $K_1 < K_2 < \ldots < K_m$.

This claim is replicable:

Let $k_1, k_2, \ldots, k_{m+1}$ be the slopes of the payoff function $f(x)$. Then

$$f(x) = f(0) + k_1 x + \sum_{i=1}^{m} (k_{i+1} - k_i)(x - K_i)^+$$

Therefore $C_T = f(S_T) = f(0) + k_1 S_T + \sum_{i=1}^{m} (k_{i+1} - k_i) C_T^i$

and $C_t = f(0)e^{-r(T-t)} + k_1 S_t + \sum_{i=1}^{m} (k_{i+1} - k_i) C_t^i$

Introduction

- Review of discrete time finance

- Single period models

- Risk neutral pricing

$\mathcal{M} = (B, S)$: a market, B a bank account.

P, P^*: probability models such that the expectation $E(S_T)$ exists.

Definition: A probability model P^* is called risk neutral if the expectation of S_T equals the forward price of S_t: $E^*(S_T | S_t) = e^{r(T-t)} S_t$.

P^*: risk neutral probability measure

$$\mathcal{P} := \{P^* \sim P : E^*(S_T^i) = e^{r(T-t)} S_t^i, i = 1, 2, \ldots, d\}$$

Risk neutrality of P^* means:

The growth rate of expected prices equals the riskless rate.

Important: Typical statistical probability models P need not be risk neutral !

Risk neutrality: Explanation

Utility maximization: An economic agent with utility function $u(x)$ (strictly concave, increasing and differentiable) has to choose a portfolio consisting of a riskless asset c and a risky asset X (probability P).

Expected utility: $f(\lambda) := E(u((1 - \lambda)X + \lambda c))$, $\lambda \in \mathbb{R}$.

Let $\lambda^* \in [0, 1]$ the utility-optimal distribution: $f(\lambda^*) = \max f(\lambda)$.

Theorem: The agent takes risks ($\lambda^* \neq 1$) iff $E(X) \neq c$. (A13)

The risk neutral case corresponds to $\lambda^* = 1$, i.e. $E(X) = c$.

Question: Assume that the riskless asset returns r and the risky asset returns $(S - \pi)/\pi$. When will the agent take risks ? (A14)

Example: Binomial model.

Let $Q(S_T = S_t u) = q$ and $Q(S_T = S_t d) = 1 - q$.

Then Q is risk neutral iff $q = \dfrac{e^{r(T-t)} - d}{u - d}$. (A15)

Note: For the binomial case there exists a risk neutral model iff the NA-property holds. (A16)

(A17) **Problem:** Discuss risk neutral models for the multinomial case !

(A18) **Problem:** Discuss uniqueness of risk neutral models both for the binomial and for the multinomial case !

Pricing rule: Let P^* be a risk neutral probability model.
Then the prices (initial values) of portfolios equal the expectations under P^* of the discounted payoffs (terminal values).

Proof: Risk neutrality means

$$E^*(S_T) = e^{r(T-t)} S_t \;\Rightarrow\; E^*\left(\frac{S_T}{e^{r(T-t)}}\right) = S_t$$

This carries over to all attainable portfolio values:

$$E^*\left(\frac{V_T}{e^{r(T-t)}}\right) = E^*\left(\frac{H^B B_T + H^S S_T}{e^{r(T-t)}}\right) = H^B B_t + H^S S_t = V_t$$

In other words: Let C_T be an attainable claim.
Then the unique arbitrage-free price $\pi(C_T)$ satisfies for every $P^* \in \mathcal{P}$

$$E^*\left(\frac{C_T}{S_T^0}\right) = \frac{\pi(C_T)}{S_t^0} \;\Leftrightarrow\; \pi(C_T) = S_t^0 E^*\left(\frac{C_T}{S_T^0}\right)$$

Question: Can we be sure to be able finding risk neutral probability model ?

Fundamental theorem: For any (single period) market there exists a risk neutral model iff the NA-property holds.

Proof: The proof consists of two parts.

Part 1 (easy): If there is an equivalent risk neutral model then the NA-property holds.

Part 2 (difficult): If the NA-property holds, then there is some equivalent risk neutral model.

Remark: It is even possible to find a risk neutral model P^* such that dP^*/dP is bounded.

Proof of the easy part: The proof rests on two facts.

Let $S^0 > 0$ be a numeraire. Let $\overline{S}_t := \dfrac{S_t}{S_t^0}$ denote the discounted prices.

(A19) Replication is maintained under a change of numeraire, i.e.

$$V_T = V_t + \sum_{i=1}^{d} H^i(S_T^i - S_t^i) \iff \overline{V}_T = \overline{V}_t + \sum_{i=1}^{d} H^i(\overline{S}_T^i - \overline{S}_t^i)$$

(A20) : $\mathcal{M}$ satisfies NA iff

$$\sum_{i=1}^{d} H^i(\overline{S}_T^i - \overline{S}_t^i) \geq 0 \implies \sum_{i=1}^{d} H^i(\overline{S}_T^i - \overline{S}_t^i) = 0$$

If there exists an equivalent risk neutral model then $\mathcal{M}$ satisfies NA. (A21)

Question: How to find prices, if the statistical model P is not risk neutral ?

Answer: Find some (equivalent) risk neutral probability model P^* !

Let $C_T \geq 0$ be a (not necessarily attainable) claim.

Theorem: If the market $\mathcal{M} = (S^0, S^1, \ldots, S^m)$ satisfies NA, then the set of arbitrage-free prices of a claim $C_T \geq 0$ is

$$\pi(C_T) = \left\{ S_t^0 E^* \left(\frac{C_T}{S_T^0} \right) : P^* \in \mathcal{P},\ E^*(C_T) < \infty \right\}$$

This set is not empty.

Proof: (A22)

(A23) **Problem:** Solve binomial pricing via risk neutral pricing !

Risk neutral lognormal model:

$$S_T = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}\, Z\right]$$

where $Z \sim N(0,1)$ (P).

This model satisfies: $E_P(S_T) = S_t e^{\mu(T-t)}$

It follows: A lognormal model P^* is risk-neutral iff $\mu = r$, i.e.

$$S_T = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}\, Z^*\right]$$

where $Z^* \sim N(0,1)$ (P^*).

Risk neutral pricing: If C_T is some claim, we may calculate

$$C_t = \frac{E^*(C_T)}{e^{r(T-t)}}$$

and call it "the price" of the claim C_T at time t.

Calculation strategy

Let $C_T = f(S_T)$, i.e. assume that the payoff depends on the terminal value only.
Writing $C_T = f\left(S_t \dfrac{S_T}{S_t}\right)$ we have $E^*(C_T|S_t) = h(S_t)$ where

$$h(x) := E^*\left(f\left[x\frac{S_T}{S_t}\right]\right) = E^*\left(f\left[x\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma\sqrt{(T - t)}\,Z\right)\right]\right)$$

The main calculation work consists in evaluating

$$u(x, a, b) := E(f(x\exp(a + bZ))) \quad \text{where } Z \sim N(0, 1)$$

Then the result is

$$E^*(C_T) = h(S_t) = u\left[S_t, \left(r - \frac{\sigma^2}{2}\right)(T - t), \sigma\sqrt{T - t}\right]$$

Price formula for a European call: $C_T = (S_T - K)^+$

Let $F_t := S_t e^{r(T-t)}$ and $\sigma_t = \sigma\sqrt{T-t}$.

Then (A24)

$$C_t e^{r(T-t)} = F_t N(d_1) - K N(d_2)$$

where

$$d_1 = \frac{\log(F_t/K) + \sigma_t^2/2}{\sigma_t}, \quad d_2 = \frac{\log(F_t/K) - \sigma_t^2/2}{\sigma_t}$$

Price formula:

$$C_t = S_t N(d_1) - K e^{r(T-t)} N(d_2)$$

(A25) **Problem:** Draw the graph of call price as a function of the spot price.

(A26) **Problem:** Describe the behaviour of call price under variation of volatility and of maturity. What is the influence of the riskless rate ?

(A27) **Problem:** Give a rough estimate of the call price if the call is deep in the money or deep out of the money.

(A28) **Problem:** Assume that the riskless rate is zero and that volatility is known. How can we estimate the call price of a call at the money ?

```
r=0.04

S=rpBS(vol=0.2,rate=r,spot=1,time=1)
mean(Call(1)(S(100000)))*exp(-r)

Call.BS(vol=0.2,rate=r)$Value
```

Linear interpolation

```
S=rpBS(vol=0.2,rate=0.04,time=1)
Call(1)(S(10))
Call(c(1.2,1,0.8))(S(10))


sp=seq(from=0.5,to=1.5,by=0.1)
Call(1/sp)(S(15))
pr=sp*colMeans(sp*Call(1/sp)(S(15)))*exp(-0.04)
plot(sp,pr,type="l")


S=rpBS(vol=0.2,rate=0.04,time=1)
pr=sp*colMeans(Call(1/sp)(S(10000)))*exp(-0.04)
plot(sp,pr,type="l")
pr0=Call.BS(spot=sp,vol=0.2,rate=0.04)$Value
lines(sp,pr0,col="red")
```

Smoothing spline interpolation

```
S=rpBS(vol=0.2,rate=0.04,time=1)
sp=seq(from=0.5,to=1.5,length=50)
pr=sp*colMeans(Call(1/sp)(S(50000)))*exp(-0.04)

sm=smooth.spline(sp,pr,spar=0.5)
price=function(x) predict(sm,x)$y
plot(price,0.5,1.5)
pr0=Call.BS(spot=sp,vol=0.2,rate=0.04)$Value
lines(sp,pr0,col="red")
```

```
S=rpBS(vol=0.2,rate=0.04,time=1)
BinCall(1)(S(10))
BinCall(c(1.2,1,0.8))(S(10))

S=rpBS(vol=0.2,rate=0.04,time=1)
sp=seq(from=0.5,to=1.5,length=50)
pr=colMeans(BinCall(1/sp)(S(50000)))*exp(-0.04)

sm=smooth.spline(sp,pr,spar=0.5)
price=function(x) predict(sm,x)$y
plot(price,0.5,1.5)
```

Chapter 1: Review of discrete time finance

Section 1.2: Multiperiod models

- Basic concepts

- Binomial model

- Risk neutral models

Syllabus: self financing portfolio - admissible value process - no arbitrage property - attainable claim - binomial tree - completeness - early exercise - numeraire - risk neutral model - martingale measure - fundamental theorem

Two asset case: $\mathcal{M} = (B, S)$

n (equidistant) periods: $t_0 = t, t_1, t_2, \ldots, t_n = T$ (trading times)

Asset prices: $(B_0, S_0), (B_1, S_1), \ldots, (B_n, S_n)$

Trading strategy: (H_k^B, H_k^S) number of units in the portfolio between times t_{k-1} and t_k

Value process: $V = (V_0, V_1, \ldots, V_n)$

Definition: A trading strategy (H_k^B, H_k^S) is called self-financing if at every trading time t_k, $k = 1, 2, \ldots, n - 1$ it satisfies the rebalancing condition

$$V_k = H_k^B B_k + H_k^S S_k = H_{k+1}^B B_k + H_{k+1}^S S_k$$

The value process of a self-financing trading strategy is called an admissible value process.

An admissible value process depends only on the value changes of the underlying assets.

Proof:

$$V_k - V_{k-1} = (H_k^B B_k + H_k^S S_k) - (H_k^B B_{k-1} + H_k^S S_{k-1})$$
$$= H_k^B (B_k - B_{k-1}) + H_k^S (S_k - S_{k-1})$$

In other words:

$$V_n = V_0 + \sum_{k=1}^{n} H_k^B (B_k - B_{k-1}) + \sum_{k=1}^{n} H_k^S (S_k - S_{k-1})$$

Short notation: V is admissible iff

$$V = H^B B + H^S S \quad \text{and} \quad dV = H^B dB + H^S dS$$

Self-financing property

A market with two assets: $\mathcal{M} = (S^0, S^1, \ldots, S^d)$

n periods: $t_0 = t, t_1, t_2, \ldots, t_n = T$ (trading times)

Asset prices/payoffs: $(S_k^0, S_k^1, \ldots, S_k^d) : k = 0, 1, 2, \ldots, n$

Trading strategy: $(H_k^0, H_k^1, \ldots, H_k^d)$ units in the portfolio between times t_{k-1} and t_k

Value process: $V = (V_0, V_1, \ldots, V_n)$

Definition: A trading strategy (H_k^B, H_k^S) is called self-financing if at every trading time t_k, $k = 1, 2, \ldots, n-1$ it satisfies the rebalancing condition:

$$V_k := \sum_{i=0}^{d} H_k^i S_k^i = \sum_{i=0}^{d} H_{k+1}^i S_k^i$$

The value process of a self-financing trading strategy is called an admissible value process.

Self-financing trading

Theorem: An admissible value process depends only on the value changes of the underlying assets, i.e.

$$V_n = V_0 + \sum_{k=1}^{n} \sum_{i=0}^{d} H_k^i (S_k^i - S_{k-1}^i)$$

Proof: (A29)

Theorem: Every linear combination of admissible value processes is an admissible value process.

Proof: (A30)

No arbitrage

Let $\mathcal{V}$ be the set of all admissible value processes.

Definition: The market $\mathcal{M}$ has the no-arbitrage property (NA) if for every $V \in \mathcal{V}$

$$V_0 \leq 0 \text{ and } V_n \geq 0 \;\Rightarrow\; V_0 = V_n$$

Theorem: A multiperiod market model satisfies the NA-property iff for each period the corresponding single period model is arbitrage-free.

Proof: Literature.

Note: Single period submodels of a multiperiod model are contingent, i.e. the initial values of the assets are random (depend on the past of the period).

Pricing of claims

Definition: An attainable claim is a random variable C_T which can be written as the payoff of an admissible value process (is replicated by a self-financing trading strategy).

Terminology: Replicating claims by self-financing trading strategies is called hedging.

Obvious: If the market satisfies the NA property then for every attainable claim the one and only price which avoids arbitrage is the cost of a hedging strategy.

Question: How to obtain the price of an attainable claim ?

Example: Binomial case

Given $S_0, S_1, \ldots, S_n$, the value process of a self-financing portfolio can be obtained recursively from its terminal value V_n.

At every trading time t_k we have: $V_k = H_k^B B_k + H_k^S S_k$

Therefore $V_k - H^S S_k = H_k^B B_k$ is riskless:

Given S_{k-1}, the right hand side is a known constant, since H_k^B depends only on S_{k-1}.

This implies that (given S_{k-1})

$$V_k^u - H_k^S S_k^u = V_k^d - H_k^S S_k^d \;\Rightarrow\; H_k^S = \frac{V_k^u - V_k^d}{S_k^u - S_k^d} =: \Delta_k$$

Thus, we may reconstruct the value process backwards:

$$H_k^B = \frac{V_k - H_k^S S_k}{B_k}, \quad V_{k-1} = H_k^B B_{k-1} + H_k^S S_{k-1}$$

Binomial tree: Stock values $s_{k,i} = u^i d^{k-i} s_0$

Bond values $b_k = e^{kr}$

(Reasonable choices of u, d, r are discussed later.)

Time	S					V				
0	s_0					v_0				
1	$s_{1,0}$	$s_{1,1}$				$v_{1,0}$	$v_{1,1}$			
2	$s_{2,0}$	$s_{2,1}$	$s_{2,2}$			$v_{2,0}$	$v_{2,1}$	$v_{2,2}$		
$\vdots$	$\ldots$					$\ldots$				
n	$s_{n,0}$	$s_{n,1}$	$s_{n,2}$	$\ldots$	$s_{n,n}$	$v_{n,0}$	$v_{n,1}$	$v_{n,2}$	$\ldots$	$v_{n,n}$

If $C = f(S)$ is a claim let $v_{n,i} := f(s_{n,i})$ and calculate backwards:

$$H_k^S := \frac{v_{k,i+1} - v_{k,i}}{s_{k,i+1} - s_{k,i}}, \quad H_k^B = \frac{v_{k,i} - H_k^S s_{k,i}}{b_k}, \quad v_{k-1,i} = H_k^B b_{k-1} + H_k^S s_{k-1,i}$$

The initial value v_0 is the price of the claim.

Theorem: In a multiperiod binomial model every claim can be replicated.

The multiperiod period binomial model is complete.

Numerical example: $u = \text{up}^{1/\sqrt{n}}$, $d = 1/u$, $r = \text{rate}/n$

```
t=BinomTree(n=5,up=1.06,rate=0.04,claim=Call(1))
lapply(t$Stock,round,3)
lapply(t$Value,round,3)
```

```
[[1]]                                       [[1]]
[1] 1                                       [1] 0.048

[[2]]                                       [[2]]
[1] 0.974 1.026                             [1] 0.024 0.062

[[3]]                                       [[3]]
[1] 0.949 1.000 1.053                       [1] 0.007 0.033 0.078

[[4]]                                       [[4]]
[1] 0.925 0.974 1.026 1.081                 [1] 0.000 0.011 0.045 0.097

[[5]]                                       [[5]]
[1] 0.901 0.949 1.000 1.053 1.110           [1] 0.000 0.000 0.017 0.061 0.118

[[6]]                                       [[6]]
[1] 0.878 0.925 0.974 1.026 1.081 1.139     [1] 0.000 0.000 0.000 0.026 0.081 0.139
```

```
system.time(BinomTree(n=100,up=1.06,rate=0.04,claim=Call(1)))

system.time(BinomTree(n=500,up=1.06,rate=0.04,claim=Call(1)))

system.time(BinomTree(n=1000,up=1.06,rate=0.04,claim=Call(1)))
```

Early exercise:

European claim: The right to exercise at maturity

American claim: The right to exercise before maturity (early exercise)

Classroom questions:

(A31) What is more valuable ?
A European claim or an American claim (all side conditions equal) ?

(A32) When is early exercise preferable ?

(A33) When is early exercise valuable for a call, for a put ?

Early exercise in a binomial tree:

Let $\overline{v}_{k,i}$ be the values of the American claim.

Beginning with $\overline{v}_{n,i} = f(s_{n,i})$ calculate backwards:

$$H_k^S := \frac{\overline{v}_{k,i+1} - \overline{v}_{k,i}}{s_{k,i+1} - s_{k,i}}, \quad H_k^B = \frac{\overline{v}_{k,i} - H_k^S s_{k,i}}{b_k}$$

but

$$\overline{v}_{k-1,i} = \max(f(s_{k-1,i}), H_k^B b_{k-1} + H_k^S s_{k-1,i})$$

The initial value $\overline{v}_0$ is the price of the American claim.

```
t=BinomTree(n=30,up=1.06,rate=0.04,claim=Call(1))

t=BinomTree(n=30,up=1.06,rate=0.04,claim=Call(1),mode="am")

t=BinomTree(n=30,up=1.06,rate=0.04,claim=Put(1))

t=BinomTree(n=30,up=1.06,rate=0.04,claim=Put(1),mode="am")
```

Are risk neutral probability models useful for pricing claims in the multiperiod case ?

Change of Numeraire: Use $B_k = e^{kr_n} = e^{r(t_k - t)}$ as numeraire

Discounting: $\overline{S}_k := \dfrac{S_k}{B_k}, \overline{V}_k := \dfrac{V_k}{B_k}, \overline{B}_k := \dfrac{B_k}{B_k} = 1$

Lemma: An admissible value process depends only on the discounted value changes of the underlying assets, i.e.

$$\overline{V}_n = \overline{V}_0 + \sum_{k=1}^{n} \sum_{i=1}^{d} H_k^i (\overline{S}_k^i - \overline{S}_{k-1}^i)$$

Proof: (A34)

Notation: $\mathcal{F}_k := \mathcal{F}(S_0, S_1, \ldots, S_k)$ past of time k.

Definition: A probability measure P^* is called risk neutral or a martingale measure if

$$E^*(\overline{S}_{k+1}^i | \mathcal{F}_k) = \overline{S}_k^i, \; k = 0, 1, 2, \ldots, n, \; i = 1, 2, \ldots, d.$$

Theorem: Let (V_k) be the value sequence of a self-financing strategy. If $V_n \geq 0$ then $(\overline{V}_k)$ is a martingale under each martingale measure.

Under a risk-neutral probability model the following statements are true:

(A35) The expected return of a self-financing trading strategy equals the riskless rate.

(A36) The initial value of an admissible value process equals the expected value of the discounted terminal value.

(A37) The arbitrage-free price of an attainable claim equals the expected value of the discounted claim.

Proof of the theorem:

Since (H_k^B, H_k^S) is self-financing we have

$$\overline{V}_n = \overline{V}_0 + \sum_{k=1}^{n} H_k^S (\overline{S}_k - \overline{S}_{k-1})$$

Since Q is a risk-neutral probability model the discounted prices satisfy the martingale property

$$E_Q(\overline{S}_k | \mathcal{F}_{k-1}) = \overline{S}_{k-1}$$

By the redundancy rule we obtain

$$E_Q(H_k^S (\overline{S}_k - \overline{S}_{k-1}) | \mathcal{F}_{k-1}) = H_k^S E_Q(\overline{S}_k - \overline{S}_{k-1} | \mathcal{F}_{k-1}) = 0$$

By the expectation rule it follows that

$$E_Q(H_k^S (\overline{S}_k - \overline{S}_{k-1})) = 0$$

(A38) Fill in the details of the arguments !

Fundamental theorem: For any (multiperiod) market model there exists a martingale measure $P^* \sim P$ iff the NA-property holds.

Proof: The proof consists of two parts.

Part 1 (easy): If there exists an equivalent martingale measure then the NA-property holds.

Part 2 (difficult): If the NA-property holds, then there exist an equivalent martingale measure.

(A39) Prove part 1 of the Fundamental theorem.

Chapter 1: Review of discrete time finance

Section 1.3: From discrete to continuous time

Syllabus: calibration of binomial trees - Cox Ross Rubinstein approximation - Black Scholes model

Binomial model as discretization:

How to calibrate a binomial model to given statistical properties ?

Suppose that the multiperiod binomial model arises through discretization
of a time interval $[t, T]$ such that

$$E(S_T|S_t) = S_t e^{\mu(T-t)} \qquad \text{growth rate of expected prices}$$
$$V(\log S_T/S_t) = \sigma^2(T-t) \quad \text{volatility}$$

How to choose u, d and p when n periods are used ? (large n)

Let S_n be the terminal asset value of a binomial market model.

(A40) **Problem:** Show that $E(S_n/S_0) = (pu + (1 - p)d)^n$.

(A41) **Problem:** Show that $E(S_n/S_0) = e^{\mu(T-t)}$ implies $p = \dfrac{e^{\mu(T-t)/n} - d}{u - d}$

Let $u = e^a$, $d = 1/u = e^{-a}$ and $L_n := \log \dfrac{S_n}{S_0}$.

(A42) **Problem:** Show that $V(L_n) = 4a^2np(1 - p)$.

(A43) **Problem:** $(L_n)_{n \in \mathbb{N}}$ is a random walk and has a binomial distribution (up to a constant factor).

The case of large n: $(n \to \infty)$

Since $a^2 n \approx$ const, we should have $a \approx c/\sqrt{n}$.

This implies (by Taylor expansion $e^x \approx 1 + x + x^2/2$)

$$p \approx \frac{1}{2} + \frac{1}{\sqrt{n}}\left(\frac{\mu(T-t)}{2c} - \frac{c}{4}\right), \quad E(L_n) \approx \mu(T-t) - \frac{c^2}{2}, \quad V(L_n) \approx c^2$$

Choosing $c^2 = \sigma^2(T-t)$ (i.e. $a = \sigma\sqrt{(T-t)/n}$) gives

$$E(L_n) \approx \mu(T-t) - \frac{\sigma^2(T-t)}{2}, \quad V(L_n) \approx \sigma^2(T-t)$$

Note: This is approximately the lognormal model,
since L_n is approximately normally distributed by the CLT.

Solution: $u = 1/d = e^{\sigma \sqrt{(T-t)/n}}, p = \dfrac{e^{\mu(T-t)/n} - d}{u - d}$

Interpretation:

- Volatility determines the size of the jumps in the binomial model.

- Expected returns can be varied by the choice of probabilities without changing volatility or the jumps.

- The set of available paths of the binomial model depends only on the volatility but not on the expected returns.

Hedging requires only the set of available paths but no probabilities.

For the calibration of a model for pricing reasons volatility only has to be taken into account.

```
Call.BS(vol=0.2,strike=1,rate=0.04,spot=1,maturity=1)$Value
```

```
CRR(n=100,vol=0.2,rate=0.04,spot=1,maturity=1,claim=Call(1))
```

```
CRR(n=1000,vol=0.2,rate=0.04,spot=1,maturity=1,claim=Call(1))
```

Pricing under a risk neutral lognormal model and binomial hedging render the same results.

Numerical example: Monte Carlo simulation

```
S=rpBS(vol=0.2,rate=0.04,time=1,spot=1)
mean(Call(1)(S(1000000)))*exp(-0.04)


CRR(n=1000,vol=0.2,rate=0.04,spot=1,maturity=1,claim=Call(1))


Call.BS(vol=0.2,rate=0.04,maturity=1,spot=1,strike=1)$Value
```

From discrete to continuous time: Binomial model $S_k = S_0 \exp(L_k)$

Elapsed time $t_k = \dfrac{(T - t)k}{n}$:

$$E(L_k) \approx \mu t_k - \frac{\sigma^2}{2} t_k, \quad V(L_k) \approx \sigma^2 t_k$$

$$L_k = E(L_k) + \sigma \underbrace{\left(\frac{L_k - E(L_k)}{\sigma} \right)}_{=:\, Z_k} \approx \left(\mu - \frac{\sigma^2}{2} \right) t_k + \sigma Z_k$$

$(Z_k)_{k \in \mathbb{N}}$ is a random walk with $V(Z_k) = t_k$.

The increments $Z_{k+h} - Z_k$ are

- approximately normally distributed and centered,

- are independent of the past $\mathcal{F}_k$,

- have variances $V(Z_{k+h} - Z_k) = h(T - t)/n$ equal to elapsed time.

The Black-Scholes model

For $t \in [0, T]$ let

$$S_t = S_0 \exp \left[\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right]$$

where the increments $W_t - W_s$

- are normally distributed and centered,
- are independent of the past $\mathcal{F}_s$,
- have variances $V(W_t - W_s) = t - s$ equal to elapsed time.

A market model $\mathcal{M} = (B, S)$ where $B_t = e^{rt}$ and where S_t satisfies the conditions stated above is called a Black-Scholes model.

- For each fixed time t the asset prices S_t follow a lognormal model.
- The log-returns have independent and stationary increments.

Chapter 2: Stochastic processes

Section 2.1: Basic concepts

Section 2.2: Wiener process

Section 2.3: Variation of paths

Chapter 2: Stochastic processes

Section 2.1: Basic concepts

Syllabus: stochastic process - path - past - increment - martingale - continuity - jumps

Stochastic processes

Definition: A stochastic process is a family $(X_t)_{t \in [0,T]}$ of random variables defined on a time interval $[0, T]$.

Path: A path of a process $(X_t)_{t \in [0,T]}$ is a particular realization (x_t) considered as a function $t \mapsto x_t$ of time.

Past: The past of a process $(X_t)_{t \in [0,T]}$ at time s is the information set $\mathcal{F}_s := \mathcal{F}(X_u : u \leq s)$ corresponding to all paths up to time s.

Increments: The random variable $X_t - X_s$, $s < t$, is called the increment of the process $(X_t)_{t \in [0,T]}$ on the subinterval $[s, t]$.

Martingale: A process (X_t) is a martingale if $E(X_t | \mathcal{F}_s) = X_s$ whenever $s < t$.

Continuity and jumps:

Let $(x_t)_{t\in[0,T]}$ be a particular path of a process $(X_t)_{t\in[0,T]}$.

If at a particular time point t the difference $\Delta x_t := x_t - x_{t-}$ is zero then the path is continuous at t.
Otherwise it has a jump at t with jump height Δx_t.

A process is called continuous if all paths are continuous everywhere.
There are no jumps.

Chapter 2: Stochastic processes

Section 2.2: Wiener process

Syllabus: Wiener process - martingale properties - generalized Brownian motion - drift - diffusion - geometric Brownian motion - Black Scholes model - risk neutrality

Wiener process

Definition: A stochastic process $(W_t)_{t \in [0,T]}$ is a Wiener process (Brownian motion process) if
  1. it is continuous,
  2. the increments satisfy $W_t - W_s \sim N(0, t - s)$,
  3. the increments are independent of the past.

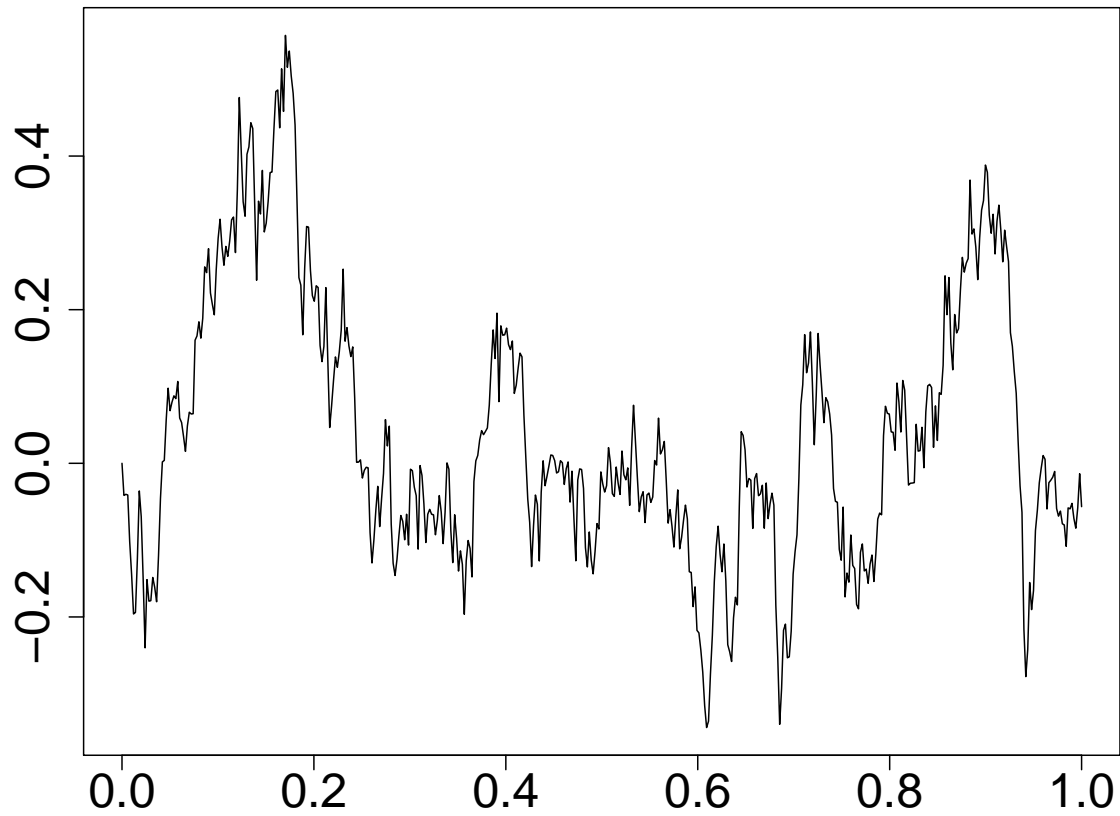**Mathematical probability theory:** There are Wiener processes.

(A44) Show that the Wiener process is a martingale.

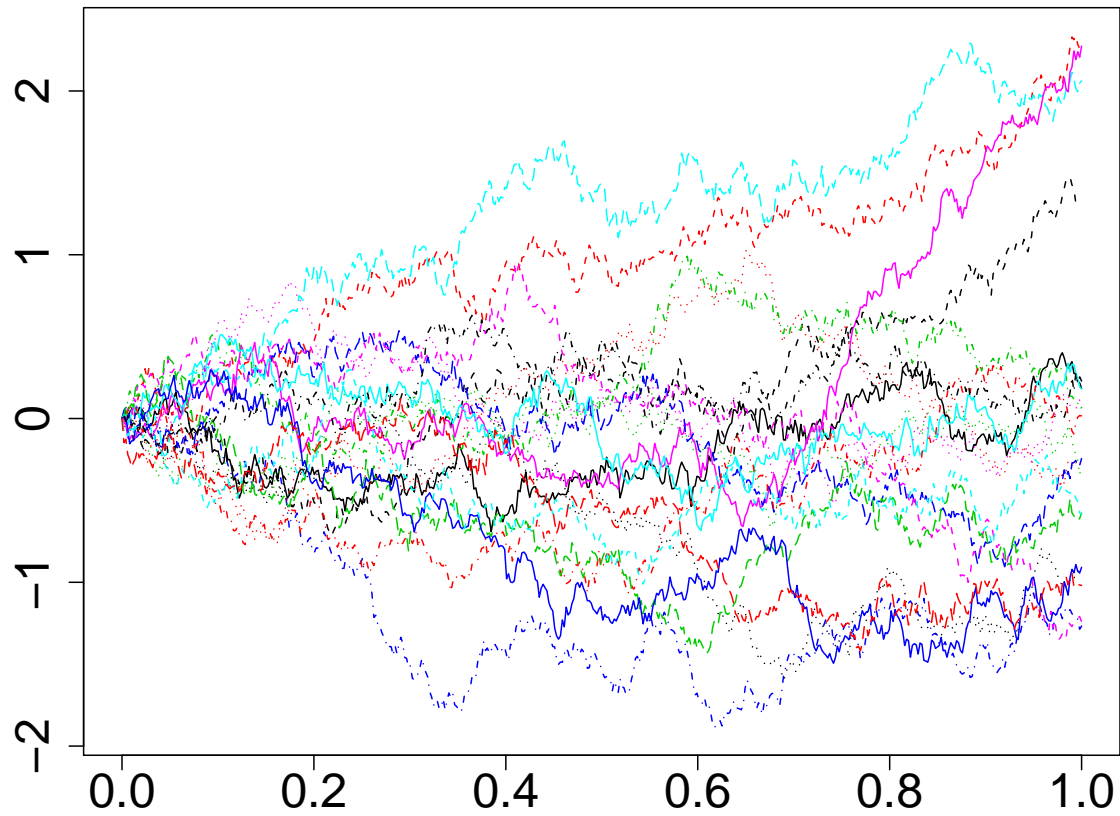(A45) Find $E(W_t^2 | \mathcal{F}_s)$.

(A46) Find A_t such that $(W_t^2 - A_t)$ is a martingale.

(A47) Find $\lim_{t \to \infty} \dfrac{W_t}{t}$.

A path of a Wiener process.

20 paths of a Wiener process.

```
pPlot(rpWiener()(1))
pPlot(rpWiener()(20))
```

Generalized Brownian motion: $X_t = \mu t + \sigma W_t$

Drift: The drift of a continuous process is the average forward velocity

$$\lim_{h \to 0} E\left(\frac{X_{t+h} - X_t}{h}\middle|\mathcal{F}_t\right) := \mu(t)$$

(A48) Show that the drift of a generalized Brownian motion is constant and equal to μ. What about the Wiener process ?

Diffusion: The diffusion coefficient of a continuous process is the average forward quadratic increment rate

$$\lim_{h \to 0} E\left(\frac{(X_{t+h} - X_t)^2}{h}\middle|\mathcal{F}_t\right) := \sigma^2(t)$$

(A49) Show that the diffusion coefficient of a generalized Brownian motion is constant and equal to σ^2. What about the Wiener process ?

A generalized Brownian motion with $\mu = 1$ and $\sigma^2 = 0.1$.
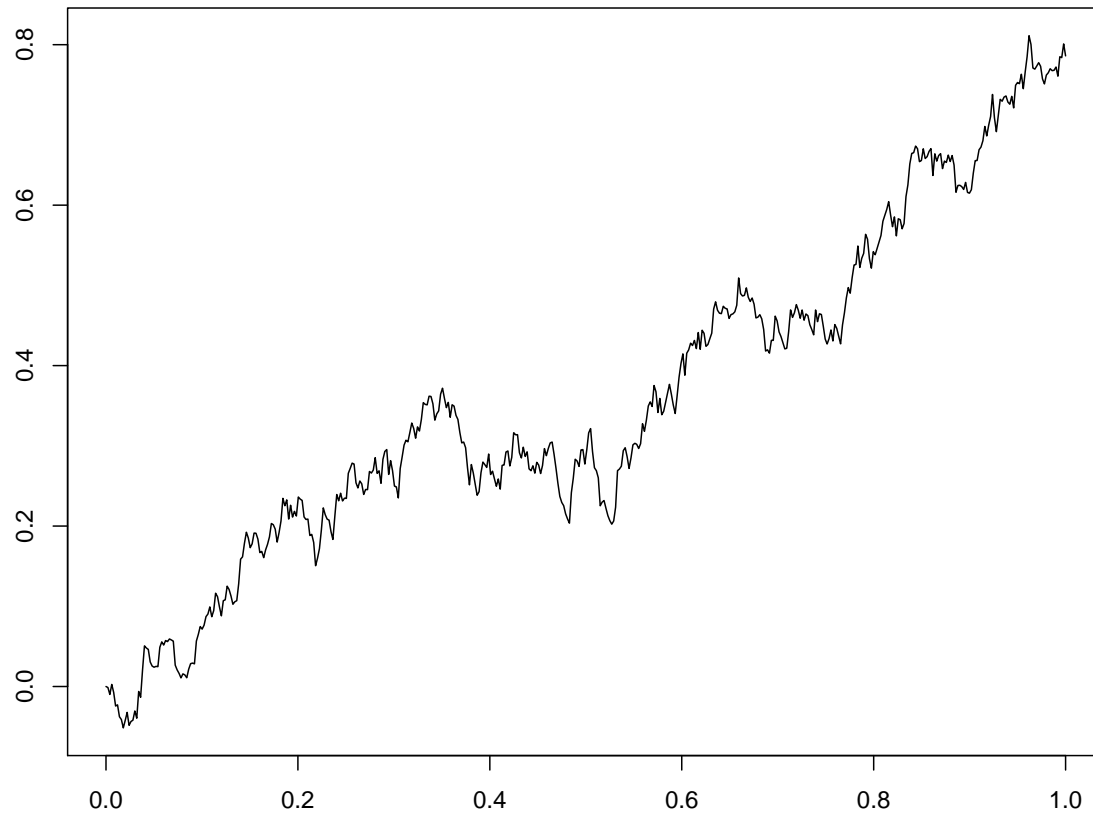
```
X=rpWiener(mean=1,cov=0.1)
pPlot(X(1))
```

Black-Scholes Model

Let (W_t) be a Wiener process and $S_t = S_0 \exp\left[\left(\mu - \dfrac{\sigma^2}{2}\right)t + \sigma W_t\right]$.

The logarithms of a Black-Scholes model have drift $\mu - \sigma^2/2$ and diffusion coefficient σ^2.

(A50) Find $E(S_t)$.

(A51) Find $E(S_t|\mathcal{F}_s)$, $s < t$.

Definition: A Black Scholes market model $\mathcal{M} = (B, S)$ is a risk neutral model if $B_t = e^{rt}$ and $\mu = r$.

(A52) Let (B, S) be a risk neutral Black Scholes model. Show that (S_t) is a martingale.

```
pPlot(rpBS()(1))
pPlot(rpBS()(20))
pPlot(rpBS(vol=0.2,rate=0.08)(20))
```

Chapter 2: Stochastic processes

Section 2.3: Variation of paths

Syllabus: size of increments - microscopic order of size - Riemannian subdivisions - total variation - quadratic variation - finite variation - variation of smooth functions - variation of Wiener paths

Size of increments:

$$V(W_t - W_s) = t - s \implies V\left(\frac{W_t - W_s}{\sqrt{t-s}}\right) = 1$$

It follows:

As $(t - s) \to 0$ the difference $W_t - W_s$ shrinks of the order $\sqrt{t-s}$.

This is remarkable:

- As $(t - s) \to 0$ the differences $(t - s)$ are much smaller than $\sqrt{t-s}$.
- For smooth functions $f(t)$ the ratios $\dfrac{f(t) - f(s)}{t - s}$ remain stable as $(t - s) \to 0$ (actually tend to $f'(s)$).
- For the Wiener process the ratios $\dfrac{W_t - W_s}{t - s}$ explode as $(t - s) \to 0$.

Subdivisions:

Subdivision of an interval $[a, b]$: $a = t_0 < t_1 < t_2 < \ldots < t_n = b$

Sequence of subdivisions:
$$t_{10} \ < t_{11}$$
$$t_{20} \ < t_{21} \ < t_{22}$$
$$t_{30} \ < t_{31} \ < t_{32} \ < t_{33}$$
$$\ldots$$
$$t_{n0} \ < t_{n1} \ < t_{n2} \ < \ldots \ < t_{n,n-1} \ < t_{nn}$$
$$\ldots$$

Riemannian sequence of subdivisions: $\max_i |t_{ni} - t_{n,i-1}| \to 0$ as $n \to \infty$

Variation of paths:

Let $f(t)$ be any function defined on an interval $[a, b]$.

Total variation (first order):

$$V_a^b(f) := \lim_{n \to \infty} \sum_i |f(t_{ni}) - f(t_{n,i-1})|$$

for any Riemannian sequence of subdivisions of $[a, b]$.

Quadratic variation:

$$QV_a^b(f) := \lim_{n \to \infty} \sum_i |f(t_{ni}) - f(t_{n,i-1})|^2$$

for any Riemannian sequence of subdivisions of $[a, b]$.

The sinus function:
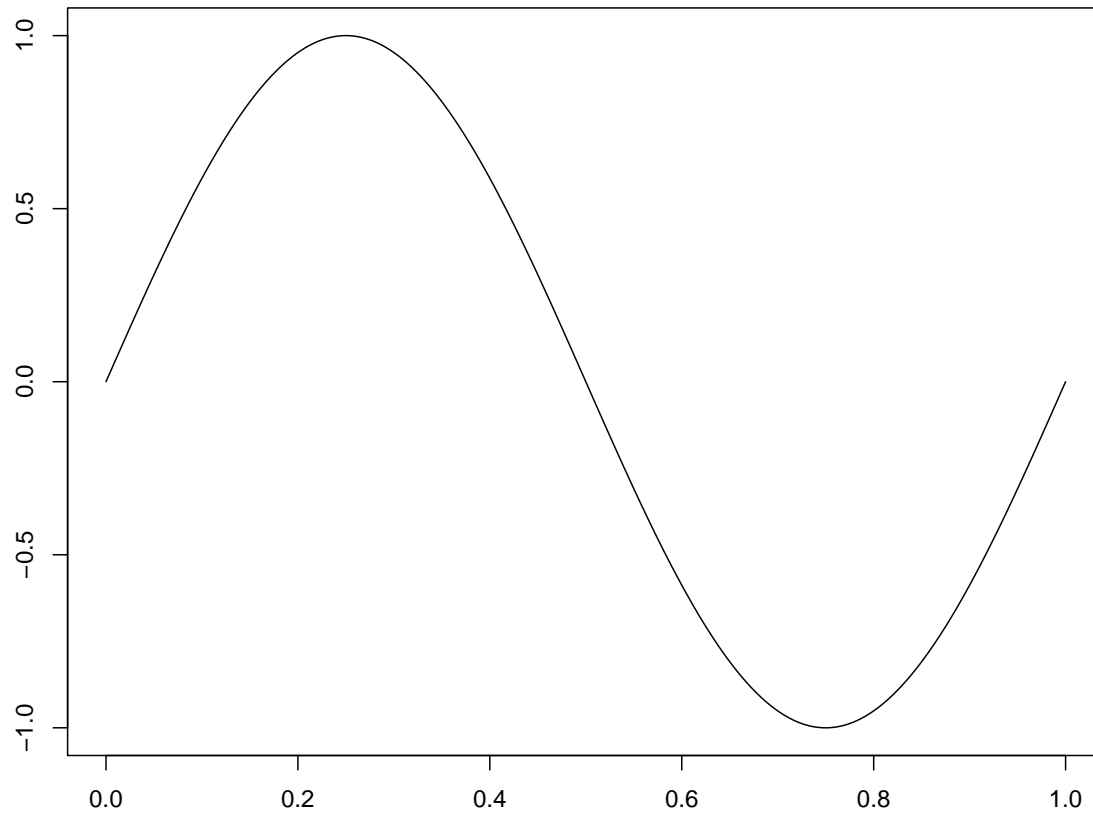
Total variation $t \mapsto V_0^t(f)$ of the sinus function:
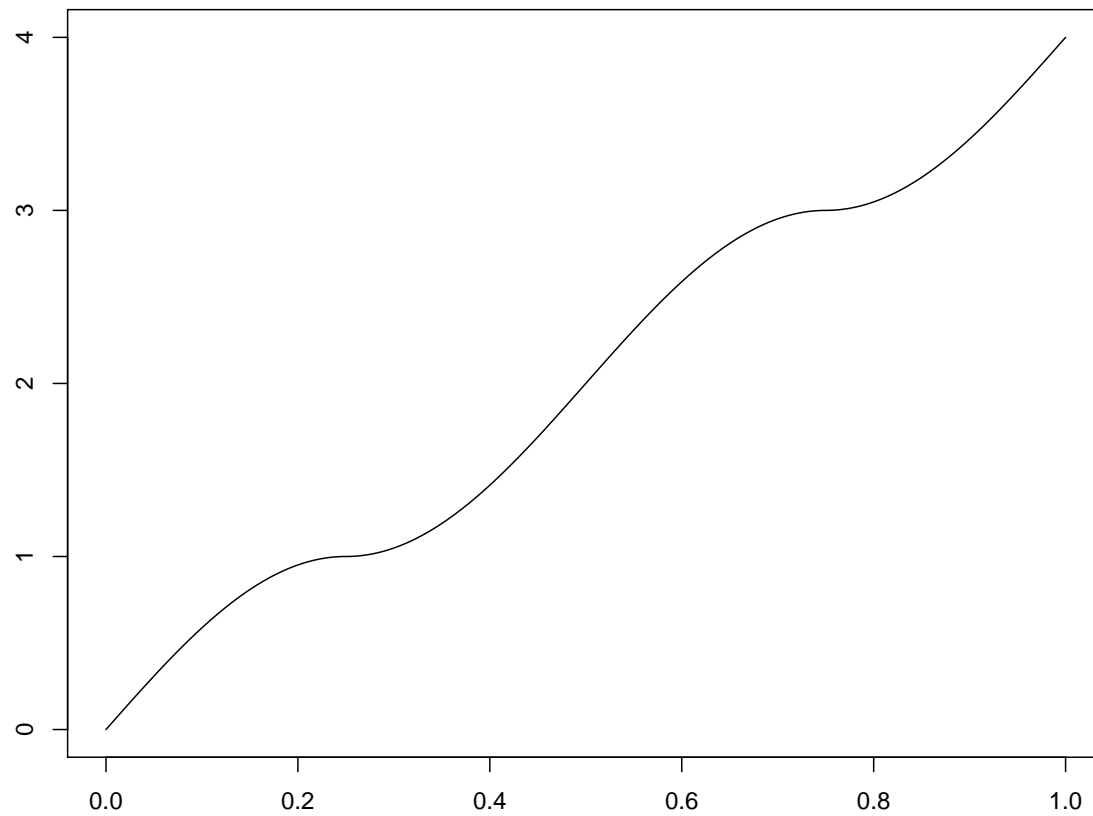
Quadratic variation $t \mapsto QV_0^t(f) =: [f]_t$ of the sinus function:
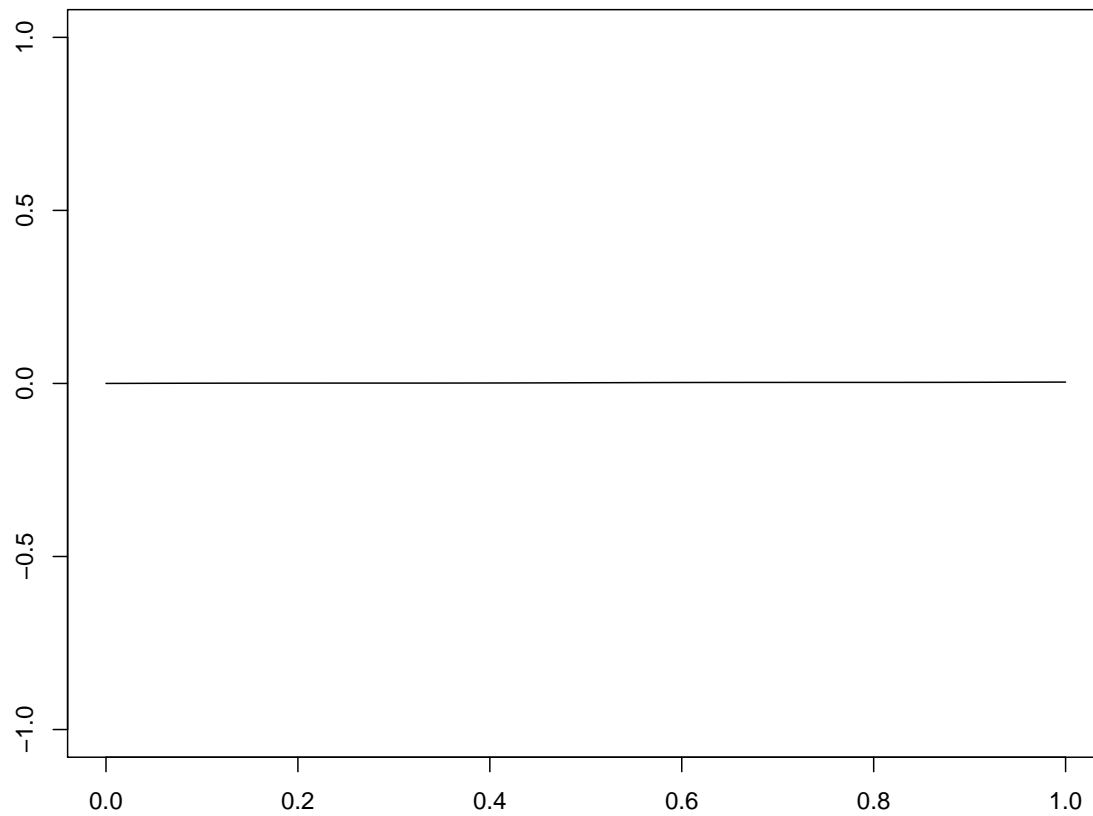
A path (W_t) of a Wiener process:
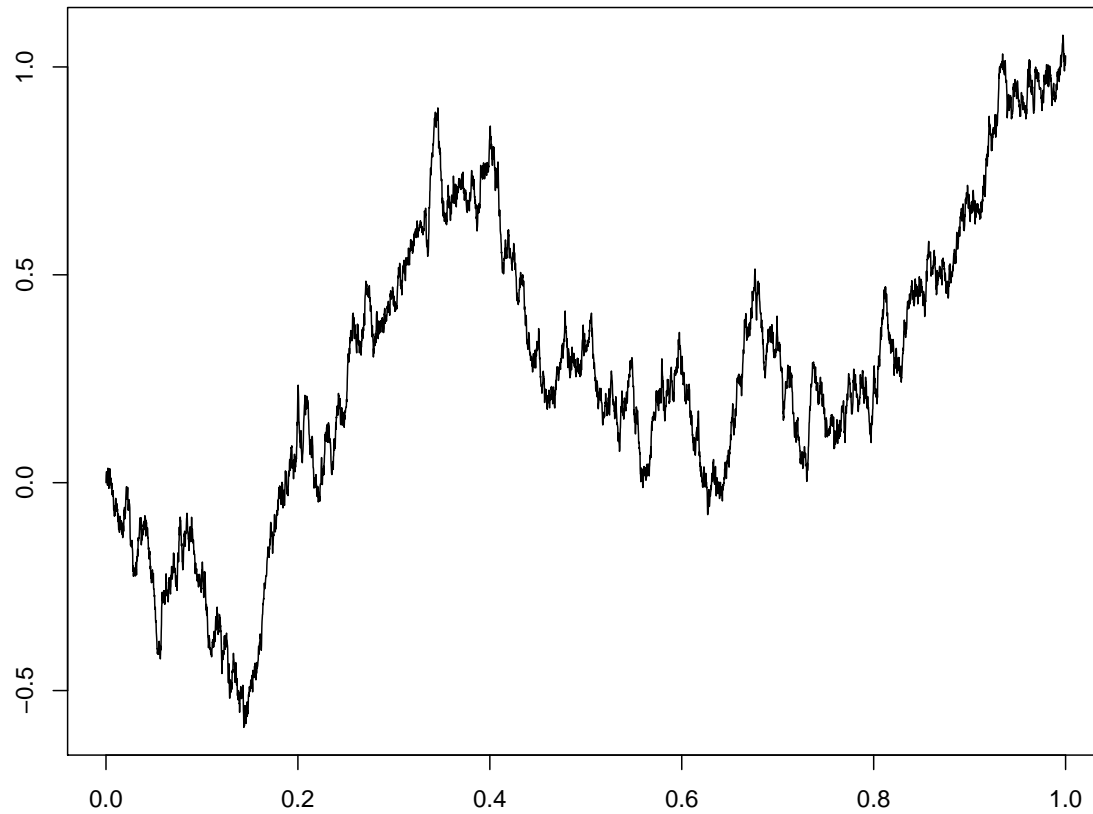
Total variation $t \mapsto V_0^t(W)$ of a Wiener path:
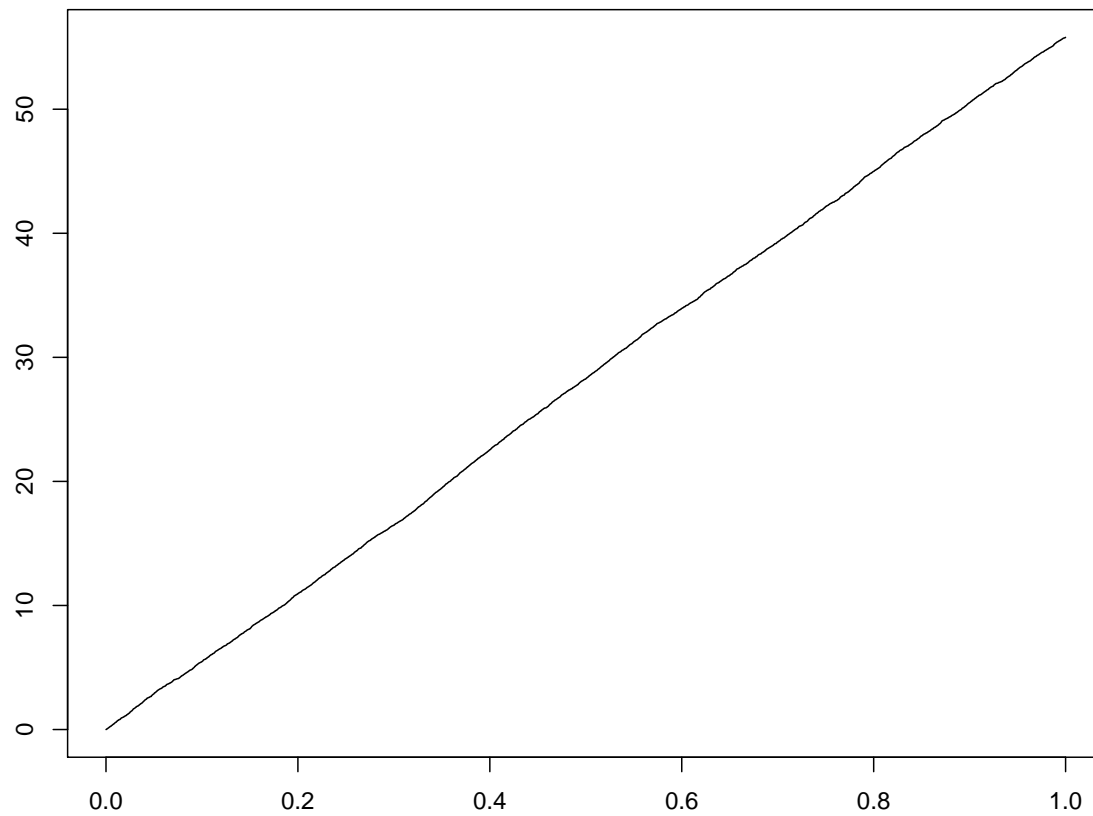
Quadratic variation $t \mapsto QV_0^t(W) =: [W]_t$ of a Wiener path:
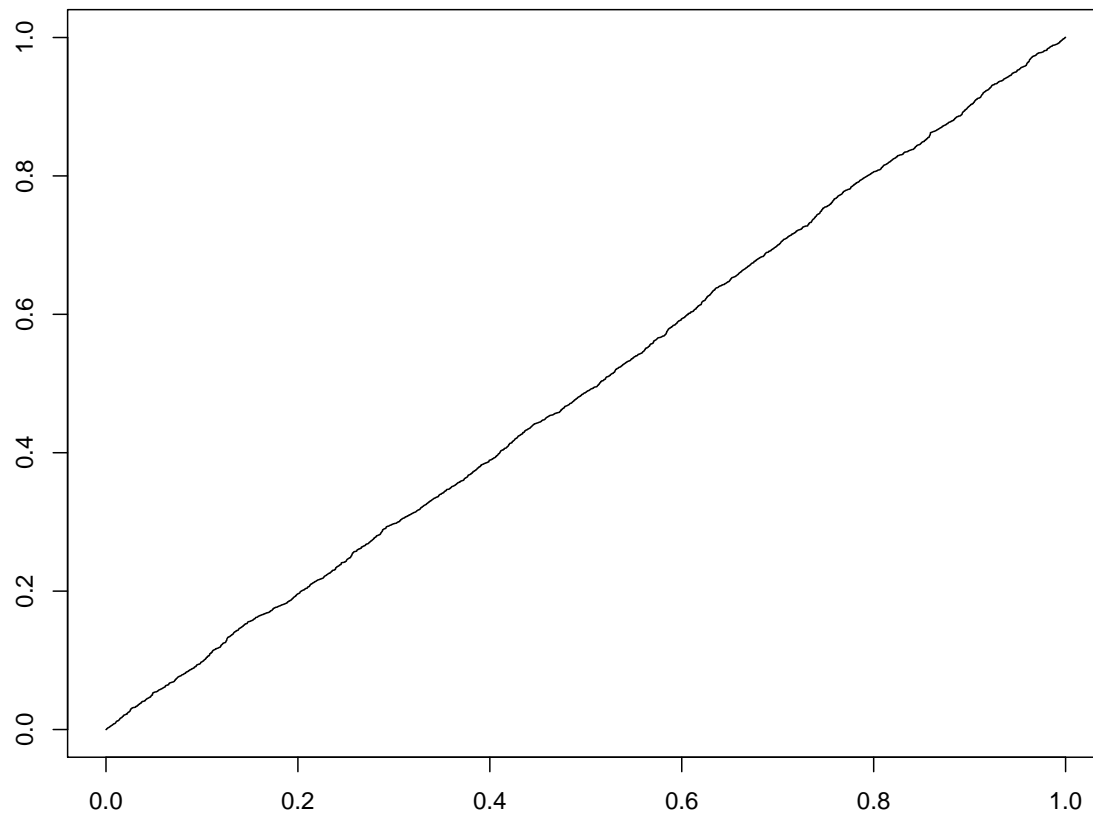
Path of $W_t + \sin(2\pi t)$:
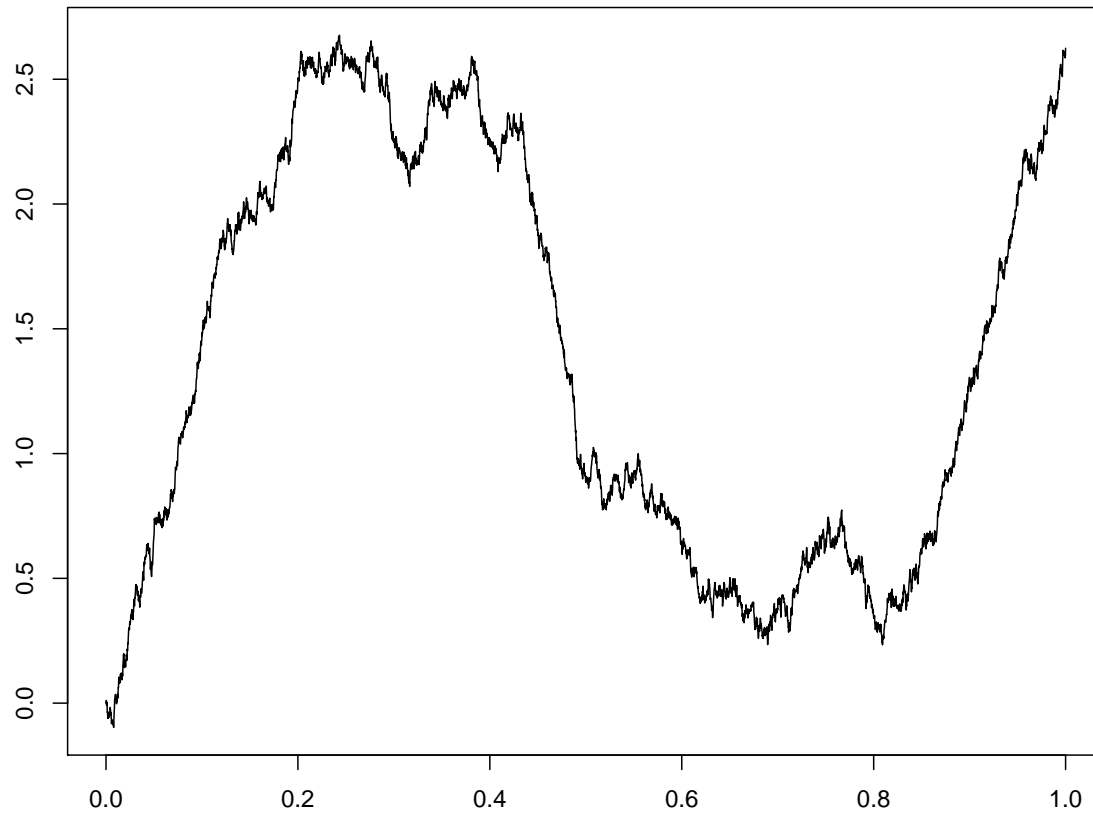
Quadratic variation of $W_t + \sin(2\pi t)$::
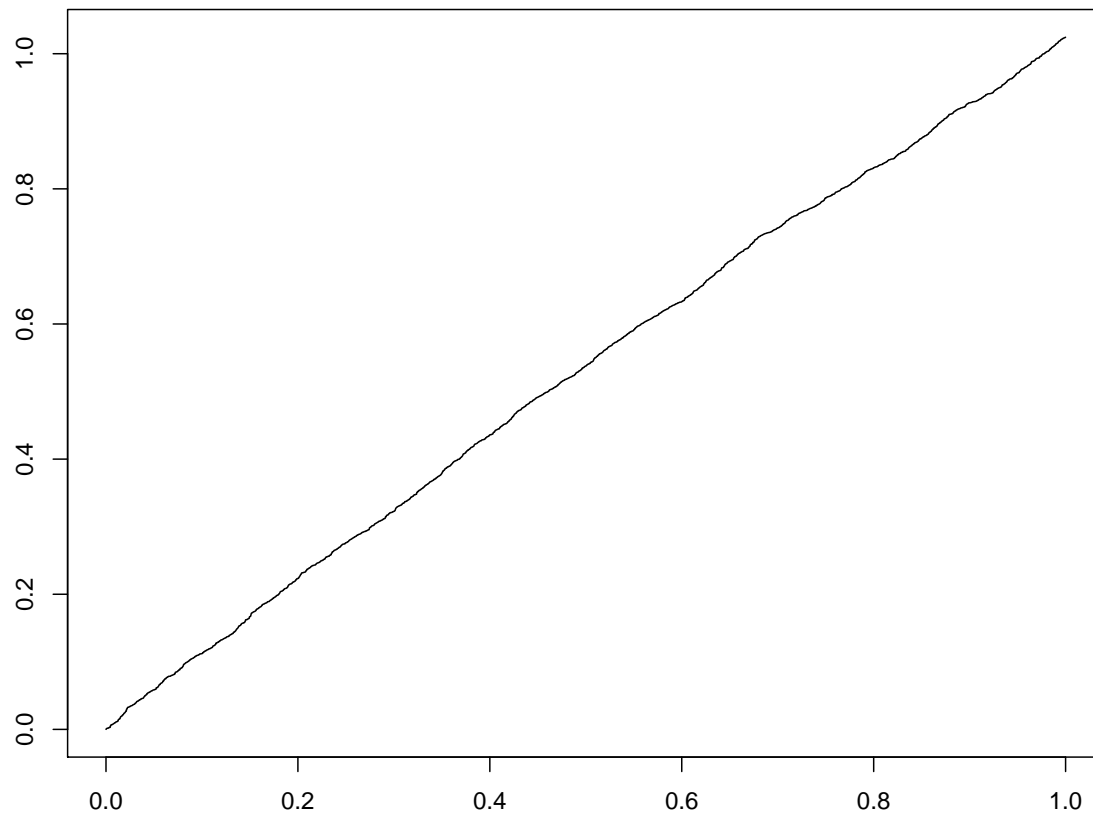
```
pPlot(as.path("sin(2*pi*t)",ngrid=5000),ylim=c(-1,1))
pPlot(pVariation(as.path("sin(2*pi*t)",ngrid=5000),order=1))
pPlot(pBracket(as.path("sin(2*pi*t)",ngrid=5000)),ylim=c(-1,1))

W=rpWiener(ngrid=5000)
pPlot(W(1))
pPlot(pVariation(W(1),order=1))
pPlot(pBracket(W(1)))

pPlot(pApply("x+sin(2*pi*t)",W(1)))
pPlot(pBracket(pApply("x+sin(2*pi*t)",W(1))))
```

```
ff=function(a) {
   W=rpWiener(ngrid=10000)
   pPlot(pVariation(W(1),order=a),ylim=c(0,1))
   for (i in 1:100) {
  x=pVariation(rpWiener(ngrid=i*100)(1),order=a)
  lines(attr(x,"time"),x)
  }
   }

ff(1)
ff(1.5)
ff(1.8)
ff(2)
ff(2.1)
ff(2.5)
ff(3)
```

Smooth functions

Theorem: Let $f(t)$ be a smooth function defined on an interval $[a, b]$.

(1) The the total variation is $V_a^t(f) = \int_a^t |f'(s)| \, ds$.

(2) The quadratic variation is $[f]_t \equiv 0$.

A smooth function has finite variation and zero quadratic variation.

(A53) Explain part (1) in an informal way by means of subdivisions.

Wiener paths

Theorem: Let (W_t) be a Wiener path.

(1) The the total variation is $V_a^t(W) \equiv \infty$.

(2) The quadratic variation is $[W]_t = t$.

A Wiener path has infinite variation, but finite quadratic variation.

(A54) Explain part (2).

Chapter 3: Stochastic calculus

Section 3.1: Orientation

Section 3.2: The ordinary integral

Section 3.3: The Wiener integral

Section 3.4: The stochastic integral

Section 3.5: Integration by parts

Section 3.6: Transformation rule

Section 3.7: Martingales

Chapter 3: Stochastic calculus

Section 3.1: Orientation

Syllabus: the problem of integration - notion of the integral - the ordinary (Stieltjes) integral - the stochastic integral

The problem of integration:

The time value of a hedge portfolio with multiperiod trading is

$$V_n = V_0 + \sum_{k=1}^{n} H_k^B (B_k - B_{k-1}) + \sum_{k=1}^{n} H_k^S (S_k - S_{k-1})$$

The terms in this expression are of the form:

$$\sum_{k=1}^{n} f(t_{k-1})(g(t_k) - g(t_{k-1}))$$

What happens with such expressions if the number of trading periods is large and the trading intervals are small?

In mathematical terms: Consider a Riemannian sequence of subdivisions.
What happens with

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(t_{n,k-1})(g(t_{nk}) - g(t_{n,k-1})) \quad ?$$

Notion of an integral:

Let $[a, b]$ be an interval and consider a Riemannian sequence of subdivisions.
Mathematics provides very effective calculation methods for dealing with terms

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(t_{n,k-1})(g(t_{nk}) - g(t_{n,k-1})) =: \int_{a}^{b} f(t) \, dg(t)$$

called integrals.

Theoretical mathematics provides answers to the following questions:

- Under what circumstances do such limits (called integrals) exist, and are independent of the particular sequence of subdivisions (being well-defined) ?

- What are the properties of such integrals, what are the rules for calculation ?

- How can we evaluate such integrals ?

Interpretation of integrals:

There are many different intuitive interpretations of integrals:

Geometry: Area under a function graph.

Physics: Accumulated distance of a moving particle.

Inventory: Accumulated cost of varying inventory.

Finance: Accumulated capital of a cash flow.

Mathematics provides answers which go far beyond our present needs.

Scope of our integral:

For which kind of functions would we like to deal with integrals

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(t_{n,k-1})(g(t_{nk}) - g(t_{n,k-1})) =: \int_a^b f(t)\,dg(t) \quad ?$$

Concerning the function $g(t)$:

- At present we are only interested in continuous functions (without jumps).

- However, we are interested in using both smooth functions $g(t)$ (ordinary integrals) and stochastic paths (stochastic integrals).

Concerning the function $f(t)$:

- We are interested to cover both continuous functions and functions with jumps.

Evaluation

There are two approaches for evaluating integrals.

Closed form evaluation: Sometimes it is possible to evaluate an integral in terms of expressions which are simpler than an integral.

Examples: $\int_0^t s\, ds = \frac{t^2}{2}$ $\int_0^t W_s\, dW_s = \frac{W_t^2 - t}{2}$

Closed form evaluation makes use of the properties and calculation rules for integrals.

Numerical evaluation: Integrals can be evaluated numerically based on a subdivision:

$$\int_a^b f(t)\, dg(t) \approx \sum_{k=1}^n f(t_{n,k-1})(g(t_{nk}) - g(t_{n,k-1}))$$

This can always be carried out by a computer and is subject to a discretization error.

Example: $\int_a^b e^{-t^2} dt$ can only be evaluated numerically.

```
fun=function(t) exp(-t^2)
integrate(fun,-2,3)

p1=pIntegral("exp(-t^2)",tlim=c(-2,3),ngrid=20)
p1[length(p1)]
p2=pIntegral("exp(-t^2)",tlim=c(-2,3),ngrid=200)
p2[length(p2)]
pPlot(pBind(p1,pAlong(p2,p1)))

n=seq(from=50,to=1000,by=50)
i=numeric(0)
for (k in n)
   i=c(i,pIntegral("exp(-t^2)",tlim=c(-2,3),ngrid=k,last=TRUE))
plot(n,i)
```

The ordinary integral:

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(t_{n,k-1})(g(t_{nk}) - g(t_{n,k-1})) =: \int_a^b f(t)\, dg(t)$$

Theorem: The integral exists and is well-defined if $g(t)$ is a function of finite variation (FV).

For historical reasons the ordinary integral has many different names:

Leibniz-Newton integral: $g(t) = t$, $f(t)$ continuous with isolated jumps (since about 1700)

Riemann-Stieltjes integral: $g(t)$ FV, $f(t)$ continuous with isolated jumps (since about 1850).

Lebesgue integral: $g(t) = t$, $f(t)$ rather general (since about 1900).

Lebesgue-Stieltjes integral: $g(t)$ FV, $f(t)$ rather general (since about 1930).

The mathematical existence theory (so-called: abstract measure theory) becomes the more sophisticated the more general the functions $g(t)$ and $f(t)$ are chosen.

The stochastic integral:

Let (H_t) and (X_t) be stochastic processes:

$$\lim_{n \to \infty} \sum_{k=1}^{n} H_{t_{n,k-1}}(X_{t_{nk}} - X_{t_{n,k-1}}) =: \int_a^b H_t \, dX_t$$

- Such integrals are ordinary integrals if the paths of the process (X_t) are of finite variation (FV-processes).

- The integral goes beyond ordinary integration (is a stochastic integral) only if the paths of (X_t) are not of finite variation (e.g. the Wiener process).

Wiener integral: $X_t = W_t$ Wiener process, $H_t = f(t)$ general non-stochastic function (since about 1930).

Ito integral: $X_t = W_t$, H_t a non-anticipating process (since about 1950).

Stochastic integral: $X_t = M_t$ a martingale, H_t a non-anticipating process (since about 1960).

Requirements of the stochastic integral:

$$\lim_{n \to \infty} \sum_{k=1}^{n} H_{t_{n,k-1}}(X_{t_{nk}} - X_{t_{n,k-1}}) =: \int_{a}^{b} H_t \, dX_t$$

The existence of the stochastic integral requires two fundamental conditions.

Condition on the integrator (X_t):

The integrator has to be a semimartingale.

Condition on the integrand (H_t):

The integrand has to be non-anticipating (adapted).

Details later.

Chapter 3: Stochastic calculus

Section 3.2: The ordinary integral

Syllabus: concatenation rule - smooth integrators - deterministic continuous time finance - linear differential equations - integration by parts - the self-financing property - transformation rule - deterministic exponential formula

Rules for integrals: ($g(t)$ of finite variation and continuous)

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(t_{n,k-1})(g(t_{nk}) - g(t_{n,k-1})) =: \int_a^b f(t)\,dg(t)$$

All rules for integration can be understood intuitively by considering the discrete counterpart (the corresponding sums for subdivisions).

The most simple case: $\int_a^b dg = \int_a^b 1\,dg = g(b) - g(a)$

General rules:

Linearity: Integrals are linear operators both in $f(t)$ and in $g(t)$.

Truncation rule: Let $a < u < v < b$.

$$\int_a^b 1_{(u,v]}(t)\, f(t)\,dg(t) = \int_u^v f(t)\,dg(t)$$

(A55) Show that integrals fulfil the concatenation rule !

Smooth integrators

Theorem: If $g(t)$ is differentiable with (piecewise) continuous derivative then
for every (suitable) function $f(t)$

$$\int_a^b f(t)\,dg(t) = \int_a^b f(t)g'(t)\,dt$$

Differential notation: $dg(t) = g'(t)\,dt$

(A56) Explain the statement of the theorem intuitively by considering the discrete counterpart.

(A57) Deduce $\displaystyle\int_a^b g'(t)\,dt = g(b) - g(a)$.

```
fun=function(t) exp(-t^2)*2*t
plot(fun,-2,3)
integrate(fun,-2,3)

q=as.path("t^2",tlim=c(-2,3),ngrid=50)
p1=pIntegral("exp(-t^2)",driver=q,adjust="left");p1[length(p1)]
p2=pIntegral("exp(-t^2)",driver=q,adjust="right");p2[length(p2)]
p3=pIntegral("exp(-t^2)",driver=q,adjust="");p3[length(p3)]
p4=pIntegral("exp(-t^2)*2*t",tlim=c(-2,3),ngrid=50,adjust="left")
    p4[length(p4)]
p5=pIntegral("exp(-t^2)*2*t",tlim=c(-2,3),ngrid=50,adjust="right")
    p5[length(p5)]
p6=pIntegral("exp(-t^2)*2*t",tlim=c(-2,3),ngrid=50,adjust="")
    p6[length(p6)]
pPlot(pBind(p1,p2,p3,p4,p5,p6))
```

```
q=as.path("t^2",tlim=c(-2,3),ngrid=500)
p1=pIntegral("exp(-t^2)",driver=q,adjust="left");p1[length(p1)]
p2=pIntegral("exp(-t^2)",driver=q,adjust="right");p2[length(p2)]
p3=pIntegral("exp(-t^2)",driver=q,adjust="");p3[length(p3)]
p4=pIntegral("exp(-t^2)*2*t",tlim=c(-2,3),ngrid=500,adjust="left")
    p4[length(p4)]
p5=pIntegral("exp(-t^2)*2*t",tlim=c(-2,3),ngrid=500,adjust="right")
    p5[length(p5)]
p6=pIntegral("exp(-t^2)*2*t",tlim=c(-2,3),ngrid=500,adjust="")
    p6[length(p6)]
pPlot(pBind(p1,p2,p3,p4,p5,p6))
```

Application: Cash flow

Assume that a capital $K(t)$ is accumulated by a cash flow with rate $K'(t) = c(t)$.
How does the capital evolve in time ?

Discrete picture: $0 = t_0 < t_1 < \ldots < t_n = T$

$$K_T = K_0 + \sum_{k=1}^{n} c(t_{k-1})(t_k - t_{k-1})$$

Continuous picture:

$$K_T = K_0 + \int_0^T c(t)\, dt$$

Application: Interest rates

Assume that a capital $K(t)$ is accumulated by an interest rate $r(t)$.
How does the capital evolve in time ?

Discrete picture   Continuous picture

$$K(t_k) - K(t_{k-1}) = K(t_{k-1})r(t_{k-1})(t_k - t_{k-1}) \qquad K(t)' = K(t)r(t)$$

$$K_T = K_0 + \sum_{k=1}^{n} K(t_{k-1})r(t_{k-1})(t_k - t_{k-1}) \qquad K_T = K_0 + \int_0^T K(t)r(t)\, dt$$

$$K_T = K_0 \prod_{k=1}^{n} \left(1 + r(t_{k-1})(t_k - t_{k-1}) \right) \qquad K_T = K_0 \exp\left(\int_0^T r(t)\, dt \right)$$

(A58) Deduce the exponential formula from the differential equation !

Linear differential (integral) equation:

Let $R(t)' = r(t)$ which means $R(T) = \displaystyle\int_0^T r(t)\,dt$

Linear differential (integral) equation for the capital:

$$K'(t) = K(t)R'(t) \iff K_T = K_0 + \int_0^T K(t)\,dR(t) \iff dK(t) = K(t)dR(t)$$

The closed form solution is the exponential function:

$$K_T = K_0 \exp\left(\int_0^T r(t)\,dt\right) = K_0 e^{R(T)}$$

Application: Cash flow with interest

Differential equation: $dK(t) = K(t)dR(t) + c(t)dt \iff K'(t) = r(t)K(t) + c(t)$

Integral equation:

$$K_T = K_0 + \int_0^T K(t)\, dR(t) + \int_0^T c(t)\, dt$$

Closed form solution:

$$\left(e^{-R(t)}K(t)\right)' = e^{-R(t)}c(t) \implies K_T = K_0 e^{R(T)} + \int_0^T e^{R(T)-R(t)}c(t)\, dt$$

Integration by parts (product rule)

Let both $f(t)$ and $g(t)$ be smooth functions.

Product rule: $(f(t)g(t))' = f'(t)g(t) + f(t)g'(t)$

The corresponding rule for integrals is

Integration by parts:

$$f(b)g(b) = f(a)g(a) + \int_a^b f'(t)g(t)\,dt + \int_a^b f(t)g'(t)\,dt$$

$$= f(a)g(a) + \int_a^b g(t)\,df(t) + \int_a^b f(t)\,dg(t)$$

Differential form: $d(fg) = g\,df + f\,dg$

Self-financing portfolios:

An investor holds H_t^B units of a bond with price $B_t = e^{rt}$, and H_t^S units of a stock with price S_t.

Portfolio value: $V_t = H_t^B B_t + H_t^S S_t$

How does the portfolio value change with time ?

Assuming smoothness of the involved functions we get from integration by parts:

$$dV_t = (H_t^B dB_t + H_t^S dS_t) + (B_t dH_t^B + S_t dH_t^S)$$

Interpretation: The value changes if either the assets prices change or the structure of the portfolio is changed.

Self-financing portfolio: The value is influenced by the asset prices only. The second part of the value expression is zero (rebalancing condition).

Transformation rule (chain rule)

Let $\phi(t)$ and $g(t)$ be smooth functions.

Chain rule: $\phi(g(t))' = \phi'(g(t))g'(t)$

The corresponding rule for integals is the

Transformation rule:

$$\phi(g(b)) = \phi(g(a)) + \int_a^b \phi'(g(t))g'(t)\,dt$$

$$= \phi(g(a)) + \int_a^b \phi'(g(t))\,dg(t)$$

Differential form: $d\phi(g) = \phi'(g)\,dg$

Exponential function:

Let $\phi(x) = \exp(x)$ und denote $f(t) := \exp(g(t))$.

The transformation rule implies:

$$\exp(g(t)) = \exp(g(0)) + \int_0^t \exp(g(s))\, dg(s) \;\Rightarrow\; f(t) = f(0) + \int_0^t f(s)\, dg(s)$$

In differential notation: $df(t) = f(t)\, dg(t)$

The exponential $f(t) = \exp(g(t))$ is the solution of the linear differential equation.

```
p=as.path("3*t-t^2/2",tlim=c(0,5),ngrid=50)
q=pApply("exp(x)",p)
r=1+pIntegral("exp(x)",driver=p)
pPlot(pBind(q,r))


p=as.path("3*t-t^2/2",tlim=c(0,5),ngrid=500)
q=pApply("exp(x)",p)
r=1+pIntegral("exp(x)",driver=p)
pPlot(pBind(q,r))
```

Chapter 3: Stochastic calculus

Section 3.3: The Wiener integral

Syllabus: Wiener integral - properties of the Wiener integral - drift and diffusion - extended Black Scholes model - Vasicek model - stochastic differential equation - Ornstein Uhlenbeck process

Wiener integral

Let $f(t)$ be such that $\int_0^T f(t)^2 dt < \infty$ (e.g. continuous or continuous with isolated jumps).

The stochastic integral $\displaystyle\int_{t_1}^{t_2} f(t)\, dW_t$ is called the Wiener integral.

Properties:

Each Wiener integral is a random variable which

- has a normal distribution,

- has mean (expectation) zero: $E\left(\displaystyle\int_{t_1}^{t_2} f(t)\, dW_t \right) = 0$,

- has variance

$$V\left(\int_{t_1}^{t_2} f(t)\, dW_t \right) = \int_{t_1}^{t_2} f^2(t)\, dt$$

(A59) Explain the properties of the Wiener integral for the case when $f(t)$ is a step function.

(A60) Show that two Wiener integrals are independent random variables if the corresponding intervals are disjoint.

Consider the Wiener integral process $X_t := \int_0^t f(s)\, dW_s, t \geq 0$.

(A61) Show that the process (X_t) has independent increments.

(A62) Show that the process (X_t) is a martingale.

(A63) Find A_t such that $(X_t^2 - A_t)$ is a martingale.

It can be shown that the Wiener integral process (X_t) is continuous.

Drift and Diffusion

Goal: Find a continuous process with time dependent drift $\mu(t)$ and time dependent diffusion coefficient $\sigma^2(t)$!

Solution: $X_t := X_0 + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dW_s$

Differential notation: $dX_t = \mu(t)dt + \sigma(t)dW_t$

(A64) Find $E(X_t)$ and $V(X_t)$.

(A65) Show that (X_t) has independent increments and find the distribution of the increments.

(A66) Find $E(X_t|\mathcal{F}_s)$ and $E((X_t - X_s)^2|\mathcal{F}_s)$, $s < t$. Discuss the martingale properties of (X_t) and of (X_t^2).

(A67) Find drift and diffusion coefficient of (X_t).

(A68) Find the quadratic variation of (X_t).

Extended Black-Scholes model:

Let $dS_t = S_t dR_t$ be such that (R_t) is a continuous process with independent increments such that the diffusion coefficient is $\sigma^2(t)$, and the drift is $\mu(t)$.

(A69) Find the closed form expression of (S_t).

(A70) Find the return rate of the expected asset prices

$$\lim_{h \to 0} \frac{1}{h} E\left(\frac{S_{t+h} - S_t}{S_t} \middle| \mathcal{F}_t\right)$$

Definition: Let $\mathcal{M} = (B, S)$ be a market model where $dB_t = r(t)B_t\,dt$. The model is risk neutral if the drift of $X_t = \log S_t$ equals $r(t)$.

(A71) Let (B, S) be a risk neutral Black Scholes model. Show that (S_t/B_t) is a martingale.

The Vasicek model

The Vasicek model is defined by the stochastic differential equation

$$dX_t = (\nu - \mu X_t)dt + \sigma dW_t$$

For $\nu = 0$ the solution is called the Ornstein-Uhlenbeck process.

The Vasicek model is a special case of the inhomogeneous linear equation

$$dX_t = X_t dY_t + dZ_t$$

by setting

$$dY_t = -\mu dt \text{ and } dZ_t = \nu dt + \sigma dW_t$$

The integrating factor is $A_t = e^{\mu t}$.

(A72) Show that the solution of the Vasicek equation is

$$X_t = e^{-\mu t} x_0 + \frac{\nu}{\mu}(1 - e^{-\mu t}) + \sigma \int_0^t e^{-\mu(t-s)} dW_s$$

Derive the following properties of the Vasicek model:

(A73) The process (X_t) is a Gaussian process (i.e. all joint distribution are normal distributions).

(A74) Find $E(X_t)$ and $\lim_{t \to \infty} E(X_t)$.

(A75) Find $V(X_t)$ and $\lim_{t \to \infty} V(X_t)$.

(A76) Find $\text{Cov}(X_t, X_{t+h})$ and $\lim_{t \to \infty} \text{Cov}(X_t, X_{t+h})$.

Chapter 3: Stochastic calculus

Section 3.4: The stochastic integral

Syllabus: non-anticipating processes - semimartingales - concept of the stochastic integral - concatenation rule - associativity rule

Question:

For which processes (X_t) and (H_t) can we define integrals

$$\lim_{n \to \infty} \sum_{k=1}^{n} H_{t_{n,k-1}}(X_{t_{nk}} - X_{t_{n,k-1}}) =: \int_a^b H_t \, dX_t$$

with reasonable properties ?

Stochastic integration: Integrals with adapted integrands.

Condition on the integrand (H_t):

The integrand has to be left-continuous and non-anticipating (adapted),
i.e. for every time t the random variable H_t may depend only on the past $\mathcal{F}_t(X)$.

Examples: $H_t = f(t)$, $H(t) = \phi(t, X_t)$, $H_t = \phi(t, X_t, X_{t-1})$, etc.

Which processes (X_t) may serve as integrators for such stochastic integrals ?

Definition: A semimartingale is a stochastic process which may serve as an integrator for stochastic integration.

Semimartingales are a very large class of stochastic processes:

- All finite-variation processes (processes with paths of FV) are semimartingales.
- Every martingale is a semimartingale.
- A linear combination of semimartingales is again a semimartingale.
- Any stochastic integral is a semimartingale (as a function of the integration limit).
- The product of semimartingales is again a semimartingale.
- Every smooth function of a semimartingale is again a semimartingale.

$$\int_0^t W_s \, dW_s = (W_t^2 - t)/2$$

```
W=rpWiener(ngrid=5000)
p=W()
pPlot(p)

q1=pIntegral("x",p)
pPlot(q1)

q2=(p^2-pClock(p))/2
pPlot(q2)

pPlot(q1-q2,ylim=c(-1,1))
```

$$\sin(W_t) = \int_0^t \cos(W_s)\, dW_s - \frac{1}{2} \int_0^t \sin(W_s)\, ds$$

```
W=rpWiener(ngrid=5000)
p=W()

pPlot(sin(p))

q1=pIntegral("cos(x)",p)
q2=pIntegral("sin(x)",p,v=2)
pPlot(q1-q2/2)

pPlot(sin(p)-q1+q2/2,ylim=c(-1,1))
```

Rules for stochastic integrals: ((X_t) a continuous semimartingale)

$$\lim_{n\to\infty} \sum_{k=1}^{n} H_{t_{n,k-1}}(X_{t_{nk}} - X_{t_{n,k-1}}) =: \int_a^b H_t \, dX_t$$

Most rules for integration can be understood intuitively by considering the discrete counterpart (the corresponding sums for subdivisions).

The most simple case: $\int_a^b dX_t = \int_a^b 1 \, dX_t = X_b - X_a$

General rules:

Linearity: Integrals are linear operators both in H_t and in X_t.

Truncation rule: Let $a < u < v < b$.

$$\int_a^b 1_{(u,v]}(t) \, H_t \, dX_t = \int_u^v H_t \, dX_t$$

(A77) Show that integrals fulfil the concatenation rule !

Integrals as integrators

(Counterpart of the smooth integrator case)

Theorem: The stochastic integral with respect to a continuous semimartingale (X_t) is again a continuous semimartingale. If $Y_t = \int_0^t G_s dX_s$ then for every integrand H_t

$$\int_a^b H_t \, dY_t = \int_a^b H_t G_t \, dX_t$$

Differential notation: $dY_t = G_t dX_t$

(A78) Explain the statement of the theorem for the case of step processes (H_t).

$$X_t = \int_0^t W_s \, dW_s \implies \int_0^t \cos(W_s) \, dX_s = \int_0^t \cos(W_s) W_s \, dW_s$$

```
W=rpWiener()
p=W()
x=pIntegral("x",p)

q1=pIntegral("cos(p)",x)
pPlot(q1)

q2=pIntegral("cos(x)*x",p)
pPlot(q2)

pPlot(q1-q2,ylim=c(-1,1))
```

Chapter 3: Stochastic calculus

Section 3.5: Integration by parts

Syllabus: quadratic variation process - quadratic variation of stochastic integrals - integration by parts - quadratic covariation

Quadratic variation:

Let (X_t) be a continuous semimartingale.

Definition: The quadratic variation of (X_t) is

$$[X]_t := QV_0^t(X) = \lim_{n \to \infty} \sum_i \left| X_{t_{ni}} - X_{t_{n,i-1}} \right|^2$$

for any Riemannian sequence of subdivisions of $[0, t]$.

Facts on the quadratic variation:

- The quadratic variation is increasing and therefore itself of finite variation.

- Integrals with respect to quadratic variation are ordinary integrals.

- The quadratic variation is zero for finite variation semimartingales.

Quadratic variation of stochastic integrals

Theorem: Let (X_t) be a continuous semimartingale. Then

$$Y_t = \int_0^t H_s dX_s \;\Rightarrow\; [Y]_t = \int_0^t H_s^2 d[X]_s$$

Explanation:

Since for small intervals $(X_t - X_s)^2 \approx [X]_t - [X]_s$ it follows that

$$[Y]_t \approx \sum_i |Y_{t_{ni}} - Y_{t_{n,i-1}}|^2 \approx \sum_i H_{t_{n,i-1}}^2 (X_{t_{ni}} - X_{t_{n,i-1}})^2$$

$$\approx \sum_i H_{t_{n,i-1}}^2 ([X]_{t_{ni}} - [X]_{t_{n,i-1}}) \approx \int_0^t H_s^2 d[X]_s$$

$$X_t = \int_0^t W_s \, dW_s \;\Rightarrow\; [X]_t = \int_0^t W_s^2 \, ds$$

```
W=rpWiener(ngrid=5000)
p=W()
q=pIntegral("x",p)

q1=pBracket(q)
pPlot(q1)

q2=pIntegral("x^2",p,v=2)
pPlot(q2)

pPlot(q1-q2,ylim=c(-1,1))
```

Integration by parts

Theorem: Let (X_t) be a continuous semimartingale. Then the integration by parts formula says

$$X_t^2 = X_0^2 + 2 \int_0^t X_s \, dX_s + [X]_t$$

Differential form: $dX_t^2 = 2X_t dX_t + d[X]_t$

Notes: Continuous semimartingales have a finite quadratic variation.

(A79) Explain integration by parts intuitively by considering the discrete counterpart.

(A80) Evaluate $\int_0^t W_s \, dW_s$.

$$X_t = \sin(W_t), \quad X_t^2 = X_0^2 + 2\int_0^t X_s \, dX_s + [X]_t$$

```
W=rpWiener()
p=W()


x=sin(p)


q1=x^2
pPlot(q1)


q2=2*pIntegral("x",x)+pBracket(x)
pPlot(q2)


pPlot(q1-q2,ylim=c(-1,1))
```

Integration by parts

Theorem: Let (X_t) and (Y_t) be continuous semimartingales. Then the integration by parts formula gives

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + [X, Y]_t$$

where

$$[X, Y]_t = \frac{[X] + [Y] - [X - Y]}{2}$$

Differential form: $dX_t Y_t = X_t dY_t + Y_t dX_t + d[X, Y]_t$

Note: $[X, Y]$ is called the quadratic covariation of X and Y. It is of finite variation and linear both in X and Y. Moreover, $[X, X] = [X]$.

(A81) Show that the assertion is an easy consequence of the simple integration by parts formula.

(A82) Show that $dW_t^k = kW_t^{k-1} dW_t + \frac{k(k-1)}{2} W_t^{k-2} dt$, $k = 2, 3, \ldots$

$$dW_t^k = kW_t^{k-1}dW_t + \frac{k(k-1)}{2}W_t^{k-2}dt$$

```
W=rpWiener(ngrid=5000)
p=W()
k=3

q1=p^k
pPlot(q1)

q2=k*pIntegral("x^(k-1)",p)+k*(k-1)/2*pIntegral("x^(k-2)",p,v=2)
pPlot(q2)

pPlot(q1-q2,ylim=c(-1,1))
```

Chapter 3: Stochastic calculus

Section 3.6: Transformation rule

Syllabus: transformation rule - Ito's formula - Ito processes - stochastic exponential - defining assets in terms of returns - extended transformation formula

Transformation rule

Theorem: Let (X_t) be a continuous semimartingale and let $\phi(x)$ be any smooth function. Then

$$\phi(X_t) = \phi(X_0) + \int_0^t \phi'(X_s)\, dX_s + \frac{1}{2}\int_0^t \phi''(X_s)\, d[X]_s$$

Differential form: $d\phi(X_t) = \phi'(X_t)dX_t + \frac{1}{2}\phi''(X_t)d[X]_t$

(A83) Show that the transformation rule holds for polynomials $\phi(x)$.

(A84) Explain the transformation rule intuitively by Taylor's formula.

Ito's formula

For the special case $X_t = W_t$ the transformation rule gives

Ito's formula:

$$\phi(W_t) = \phi(0) + \int_0^t \phi'(W_s)\, dW_s + \frac{1}{2} \int_0^t \phi''(W_s)\, ds$$

Ito processes

Definition: Let (A_t) and (B_t) be (left) continuous adapted processes. Then any process of the form

$$X_t = X_0 + \int_0^t A_s \, ds + \int_0^t B_s \, dW_s$$

is called an Ito process.

Every Ito process is a semimartingale.

(A85) Show that in the representation of an Ito process the components (A_t) and (B_t) are uniquely determined.

(A86) Show that the stochastic integral with respect to an Ito process is again an Ito process.

(A87) Show that every smooth function of an Ito process is again an Ito process.

(A88) Show that the product of two Ito processes is again an Ito process.

Stochastic exponential

Definition: Let (X_t) be a continuous semimartingale. Then

$$Y_t = \mathcal{E}(X_t) := \exp(X_t - [X]_t/2)$$

is called the stochastic exponential of (X_t).

(A89) Show that $Y_t = Y_0 \mathcal{E}(X_t)$ satisfies the linear differential equation $dY_t = Y_t \, dX_t$. Write down the corresponding integral equation.

(A90) Assume that $Y_t = Y_0 \mathcal{E}(X_t)$. Express (X_t) as a stochastic integral with respect to (Y_t).

Stochastic models of financial returns

Let (S_t) be a stock price and let (R_t) by the corresponding cumulated return process.

Smooth case: $S_t' = R_t' S_t$, i.e. $dS_t = S_t \, dR_t$.

Solution: $S_t = S_0 \exp(R_t)$

General case: $dS_t = S_t \, dR_t$.

Solution: $S_t = S_0 \mathcal{E}(R_t) = S_0 \exp(R_t - [R]_t/2)$

Special case: $R_t = \mu t + \sigma W_t$. Then (S_t) follows the Black-Scholes model.

Extended transformation rule

Theorem: Let (X_t) be a continuous semimartingale and let $\phi(x,t)$ be any smooth function. Then

$$\phi(X_t, t) = \phi(X_0, 0) + \int_0^t \phi_x(X_s, s)\, dX_s + \int_0^t \phi_t(X_s, s)\, ds + \frac{1}{2}\int_0^t \phi_{xx}(X_s, s)\, d[X]_s$$

Differential form: $d\phi(X_t, t) = \phi_x(X_t, t)dX_t + \phi_t(X_t, t)dt + \frac{1}{2}\phi_{xx}(X_t, t)d[X]_t$

Chapter 3: Stochastic calculus

Section 3.7: Martingales

Syllabus: martingale properties of stochastic integrals - Levy's characterization of the Wiener process

General question: When do martingale properties of a semimartingale carry over to stochastic integrals w.r.t. the semimartingale ?

Special case: Wiener integral.

More general: What about general continuous semimartingales ?

Theorem: Let (X_t) be a continuous semimartingale which is a square integrable martingale. Let (H_t) be an adapted process with left-continuous paths. Then the following assertions are equivalent:

(1) $\left(\int_0^t H_s \, dX_s \right)_{0 \leq t \leq T}$ is a square integrable martingale.

(2) The quadratic variation has finite expectation: $E\left(\int_0^T H_s^2 d[X]_s \right) < \infty$.

The assertion can be written in a slightly more compact form:
Let $Y_t := \int_0^t H_s \, dX_s$ and $(X_t)_{0 \leq t \leq T} \in \mathcal{M}_2$. Then

$$(Y_t)_{0 \leq t \leq T} \in \mathcal{M}_2 \iff E([Y]_T) < \infty$$

Applications

(A91) Discuss the martingale properties of a stochastic exponential !

(A92) Discuss the martingale properties of Ito-processes !

(A93) Let (X_t) be a square integrable continuous martingale.
Discuss martingale properties of (X_t^2).

(A94) Let (X_t) be a square integrable continuous martingale.
Discuss martingale properties of $(\phi(X_t))$.

Levy's theorem

Levy's theorem: Let (X_t) be a square integrable continuous martingale. If $[X]_t = t$ then it is a Wiener process.

(A95) Explain the assertion of Levy's theorem !

(A96) Let (M_t) be a square integrable continuous martingale. If its quadratic variation is strictly increasing and differentiable then (M_t) can be written as a stochastic integral w.r.t. a Wiener process.

Chapter 4: Pricing and hedging

Section 4.1: The hedging approach

Section 4.2: Simple claims

Section 4.3: Path dependent claims

Chapter 4: Pricing and hedging

Section 4.1: The hedging approach

Syllabus: self-financing portfolios - time evolution of the portfolio value - stochastic differential equation - partial differential equation - how to solve

Self-financing portfolios

Let (B_t, S_t) be a market model where $B_t = e^{rt}$ and (R_t) is a (continuous semimartingale) model of the returns of (S_t), i.e. $dS_t = S_t dR_t$ or $S_t = S_0 \mathcal{E}(R_t)$.

Consider a portfolio: (H_t^B, H_t^S)

Value of the portfolio: $V_t = H_t^B B_t + H_t^S S_t$

Definition: The portfolio (H_t^B, H_t^S) is self-financing if

$$dV_t = H_t^B dB_t + H_t^S dS_t$$

Question: How does the value of a self-financing portfolio evolve in time ?

Partial differential equation

Stochastic differential equation: The model of the returns satisfies

$$dS_t/S_t = \mu(S_t, t)\, dt + \sigma(S_t, t)\, dW_t$$

Simple claim: The terminal value V_T of the portfolio depends only on the terminal value S_T of the asset price: $V_T = f(S_T)$.

Under these assumptions the value process of the portfolio can be written as $V_t = \phi(S_t, t)$.

The function $\phi(x, t)$ can be characterized by a partial differential equation.

Special case $\mu(x, t) \equiv \mu$, $\sigma(x, t) \equiv \sigma$: Black-Scholes model

Black-Scholes pdf: $\phi_t = r\phi - rx\phi_x - \frac{1}{2}x^2\sigma^2\phi_{xx}$

Practical remarks

We would like to find a solution $\phi(x, t)$ of the pdf that satisfies the boundary (terminal) condition $\phi(x, T) = f(x)$, $x \in (0, \infty)$.

How can we find such solutions of the partial differential equations ?

Closed form solutions: For some simple models and some simple claims it is possible to find closed form expressions for the solutions. This requires knowledge of the theory of partial differential equations.

Numerical solutions: There are very powerful methods for the numerical solution of partial differential equations (finite difference methods).

Chapter 4: Pricing and hedging

Section 4.2: Simple claims

Syllabus: portfolio structure - Black-Scholes partial differential equation - greeks - Delta hedging - sensitivity of Delta - considering dividends

The portfolio structure:

Let $dR_t = \mu_t dt + \sigma_t dW_t$. Then $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$.

Ito's formula implies:

$$d\phi(S_t, t) = \phi_x(S_t, t)dS_t + (\ldots)dt = \phi_x(S_t, t)\sigma_t S_t dW_t + (\ldots)dt$$

The self-financing property implies:

$$d\phi(S_t, t) = H_t^S dS_t + H_t^B dB_t = H_t^S \sigma_t S_t dW_t + (\ldots)dt$$

Conclusions

Stock position: $H_t^S = \phi_x(S_t, t) =: \Delta_t$

Cash position: $H_t^B B_t = V_t - \Delta_t S_t = \phi(S_t, t) - \phi_x(S_t, t)S_t$

The full expansion:

Ito's formula and the self-financing property imply

$$
\begin{aligned}
d\phi(S_t, t) &= \phi_x(S_t, t)dS_t &&+ \left(\phi_t(S_t, t) + \tfrac{1}{2}\phi_{xx}(S_t, t)\sigma_t^2 S_t^2\right)dt \\
&= H_t^S dS_t &&+ H_t^B B_t r \, dt \\
&= H_t^S dS_t &&+ (\phi(S_t, t) - \phi_x(S_t, t)S_t)r \, dt
\end{aligned}
$$

Uniqueness of the Ito process representation gives

$$
(\phi(S_t, t) - \phi_x(S_t, t)S_t)r = \phi_t(S_t, t) + \frac{1}{2}\phi_{xx}(S_t, t)\sigma_t^2 S_t^2
$$

If $\sigma_t = \sigma(S_t, t)$:

$$
\phi_t(x, t) = r\phi(x, t) - rx\phi_x(x, t) - \frac{1}{2}x^2\sigma^2(x, t)\phi_{xx}(x, t)
$$

Special case: $\sigma = $ const Black-Scholes pdf.

The Greeks

Value process= $V_t = \phi(S_t, t) = \phi(S_t, t, \sigma, r)$

Variables: S_t, t

Parameters: σ, r

The partial derivatives of $\phi(x, t, \sigma, r)$ are the Greeks.

$$\Delta := \phi_x(x, t), \quad \Gamma := \phi_{xx}(x, t), \quad \Theta := \phi_t(x, t)$$

$$\text{Vega} := \phi_\sigma(x, t, \sigma, r), \quad \rho := \phi_r(x, t, \sigma, r)$$

We will come back to the greeks when we talk about discrete hedging.

Hedging interpretation: Delta

Delta: $\dfrac{\partial}{\partial S_t} C_t = N(d_1)$ (has to be proved !)

The price formula contains the information about the Hedge portfolio:

$$C_t = \underbrace{S_t N(d_1)}_{\text{asset part}} - \underbrace{K e^{r(T-t)} N(d_2)}_{\text{cash part}}$$

Scenario 1: Option is deep in the money.
$$C_t \approx S_t - K e^{r(T-t)}, \quad \Delta = N(d_1) \approx 1.$$

Scenario 2: Option is deep out of the money.
$$C_t \approx 0, \quad \Delta = N(d_1) \approx 0.$$

Scenario 3: Option is at the money $S_t \approx K e^{r(T-t)}$.

$$C_t \approx S_t \frac{\sigma_t}{\sqrt{2\pi}}, \quad \Delta = N(d_1) \approx \frac{1}{2} - \frac{\sigma_t}{2\sqrt{2\pi}}.$$

Fluctuation of Delta:

The asset part of the hedge portfolio is determined by option's Delta.

The sensitivity of the option's Delta with respect to the underlying is the option's Gamma:

$$\Gamma = \frac{\partial}{\partial S_t}\Delta = \frac{\partial^2}{\partial S_t^2}C_t = \frac{1}{\sqrt{2\pi}}\frac{\exp(d_1^2/2)}{S_t\sigma_t}$$

Scenario 1: Option is deep in the money.
$$C_t \approx S_t - Ke^{r(T-t)}, \quad \Delta = N(d_1) \approx 1, \quad \Gamma \approx 0.$$

Scenario 2: Option is deep out of the money.
$$C_t \approx 0, \quad \Delta = N(d_1) \approx 0, \quad \Gamma \approx 0.$$

Scenario 3: Option is at the money $S_t \approx Ke^{r(T-t)}$.
$$C_t \approx S_t\frac{\sigma_t}{\sqrt{2\pi}}, \quad \Delta = N(d_1) \approx \frac{1}{2} - \frac{\sigma_t}{2\sqrt{2\pi}}, \quad \Gamma \approx \frac{c}{\sigma_t}\text{(large !)}$$

(A97) Draw the graph of Delta and Gamma as a function of the spot price.

(A98) Describe the behaviour of Delta and Gamma under variation of volatility and of maturity.

(A99) Give a rough estimate of Delta and Gamma if the call is deep in the money or deep out of the money.

(A100) Assume that the riskless rate is zero and that volatility is known. How can we estimate the Delta of a call at the money ?

How to consider assets with dividends

Value of the portfolio: $V_t = H_t^B B_t + H_t^S S_t$

Assume that the stock pays dividends with continuous rate D.
Then the value of the portfolio changes by

$$dV_t = H_t^B dB_t + H_t^S dS_t + H_t^S D S_t dt$$
$$= (H_t^B r e^{rt} + H_t^S D S_t) dt + H_t^S dS_t$$

Uniqueness of the Ito process representation gives

$$(\phi(S_t, t) - \phi_x(S_t, t) S_t) r + \phi_x(S_t, t) S_t D = \phi_t(S_t, t) + \frac{1}{2} \phi_{xx}(S_t, t) \sigma_t^2 S_t^2$$

If $\sigma_t = \sigma(S_t, t)$:

$$\phi_t(x, t) = r\phi(x, t) - (r - D)x\phi_x(x, t) - \frac{1}{2} x^2 \sigma^2(x, t) \phi_{xx}(x, t)$$

Special case: $\sigma = $ const Black-Scholes pdf.

Chapter 4: Pricing and hedging

Section 4.3: Path dependent claims

Syllabus: kinds of path dependent claims - how to price - how to hedge

Path dependent claims

The payoff does not only depend on the terminal value of the asset, but also on path properties of the asset price.

Path properties: $M_t^T = \max_{s \in [t,T]} S_s, \quad m_t^T = \min_{s \in [t,T]} S_s, \quad A_t^T = \int_t^T S_s ds.$

Basic kinds of path dependent claims:

Asian options:
   Average strike call $(S_T - A_0^T)^+$, Average strike put $(A_0^T - S_T)^+$.
   Average rate call $(A_0^T - K)^+$, Average rate put $(K - A_0^T)^+$.

Lookback options:
   Fixed strike call $(M_0^T - K)^+$, floating strike put $(M_0^T - S_T)^+$.

Barrier options:
   Down and out call $(S_T - K)^+ 1_{(m_0^T > b)}$, (down and in, up and out, up and in, etc)

Pricing and hedging of path dependent claims

1. How to obtain the price of a path dependent option ?

   - Closed form solutions are sometimes available, but only for the simplest models.

   - PDF-approach suffers from complicated boundary structure and from dimensionality issues.

   - MC-simulation (based on risk-neutral pricing, see later) is easy in theory but can be time consuming in practice.

2. How to hedge a path dependent option ?

   - Delta hedging is possible in theory.

   - Calculating the deltas leads to similar problems as calculating prices.

   - Delta can have jumps, causing additional risk in delta hedging.

Chapter 5: Pricing measures

Section 5.1: Basics

Section 5.2: Theory

Section 5.3: Closed form pricing

Section 5.4: MC-pricing

Section 5.5: The Feynman-Kac approach

Chapter 5: Pricing measures

Section 5.1: Basics

Syllabus: risk neutral models - martingale properties - equivalent models - constructing equivalent martingale models

Risk neutral modelling

Basic feature:

The value processes of hedge portfolios can be expressed as conditional expectations of the discounted claims obtained under risk neutral models.

Advantages:

(1) It is sometimes easier to obtain closed form expressions for conditional expectations than for the corresponding partial differential equation.

(2) The expectation approach opens the door for calculating prices and greeks by Monte Carlo Simulation.

Question: How to extend the concept of risk neutral models to the continuous case ?

Let (B, S) be a market model defined on $[0, T]$ where $B_t = e^{rt}$ and (S_t) is a continuous semimartingale.

Definition: A probability model Q for (B, S) is called risk neutral if

$$E_Q(S_t | \mathcal{F}_s) = S_s e^{r(t-s)}$$

Notation: $\mathcal{F}_t := \mathcal{F}(S_s : s \leq t)$ is the past of time t.

In other words:

A model is risk neutral if the discounted asset price process is a martingale.

A risk neutral model is a martingale model for the discounted asset prices.

Footnote: The martingale property can be weakened (local martingales).

Equivalence

We are not interested in arbitrary risk neutral probability models.

The essential point is:

The risk neutral model Q must be equivalent to the underlying statistical model P.

The set of possible paths must be the same for both models P and Q:

$$P(A) = 1 \;\Leftrightarrow\; Q(A) = 1 \text{ whenever } A \in \mathcal{F}_T$$

How to check equivalence: $Q(A) = E_P(1_A L)$ for some positive random variable L and all $A \in \mathcal{F}_T$.

Necessary condition for equivalence: Volatility is the same under P and Q.

Example

Black-Scholes model P: $dS_t = S_t(\mu dt + \sigma dW_t)$, (W_t) Wiener process under P

Risk-neutral model Q : $dS_t = S_t(r dt + \sigma d\widetilde{W}_t)$, $(\widetilde{W}_t)$ Wiener process under Q

Note: $\widetilde{W}_t := \dfrac{\mu - r}{\sigma} t + W_t$

Question: Can we define Q such that it is equivalent to P ?

Answer: Define $Q(A) := E_P(1_A L_T)$, where

$$L_t := \exp\left(-\frac{(\mu - r)^2}{2\sigma^2} t - \frac{\mu - r}{\sigma} W_t \right)$$

Then: If (W_t) is a Wiener process under P, then $(\widetilde{W}_t)$ is a Wiener process under Q.

Outline: We have to show that $E_Q\left(e^{\lambda(\widetilde{W}_t - \widetilde{W}_s)}\right) = e^{\lambda^2(t-s)/2}$

For this observe that

$$E_Q\left(e^{\lambda(\widetilde{W}_t - \widetilde{W}_s)}\right) = E_P\left(L_T e^{\lambda(\widetilde{W}_t - \widetilde{W}_s)}\right) = E_P\left(\frac{L_t}{L_s} L_s e^{\lambda(\widetilde{W}_t - \widetilde{W}_s)}\right)$$

In the preceding equation we make use of the fact that (L_t) is a martingale.

(A101) Fill in the details.

Chapter 5: Pricing measures

Section 5.1: Theory

Syllabus: fundamental theorem - Girsanov's theorem - pricing by risk neutral models - martingale representation - complete markets

Question: Can we be sure to be able finding equivalent risk neutral probability models ?

Fundamental theorem: For any semimartingale market model there exists an equivalent risk neutral model iff (some) NA-property holds.

Proof: The proof consists of two parts.

Part 1 (later): If there is an equivalent risk neutral model then the NA-property holds.

Part 2 (difficult): If (some) NA-property holds, then there exists an equivalent risk neutral model.

How to construct risk neutral models: Girsanov's theorem.

Are risk neutral probability models useful for pricing claims in the continuous case ?

Change of Numeraire: Use $B_t = e^{rt}$ as numeraire

Discounting: $\overline{S}_t := \dfrac{S_t}{B_t}, \overline{V}_t := \dfrac{V_t}{B_t}, \overline{B}_t := \dfrac{B_t}{B_t} = 1$

Theorem: If Q is risk-neutral and (H_t^B, H_t^S) is self-financing then

$$E_Q(\overline{V}_T | \mathcal{F}_t) = \overline{V}_t \iff E_Q(V_T | \mathcal{F}_t) = V_t e^{r(T-t)}$$

In other words:

Under a risk neutral model all discounted admissible value processes are martingales.

The martingale property carries over from the discounted asset price processes to the discounted admissible value processes.

Footnote: The martingale property can be weakened (local martingales).

Summary

Under a risk-neutral probability model the following statements are true:

- The expected return of a self-financing trading strategy equals the riskless rate.

- The initial value of an admissible value process equals the expected value of the discounted terminal value.

- The arbitrage-free price of an attainable claim equals the expected value of the discounted claim.

Proof of the theorem:

Since (H_t^B, H_t^S) is self-financing we have

$$\overline{V}_T = \overline{V}_t + \int_t^T H_s^S \, d\overline{S}_s$$

Since Q is a risk-neutral probability model the discounted prices satisfy
the martingale property

$$E_Q(\overline{S}_T | \mathcal{F}_t) = \overline{S}_t$$

If we know that the martingale property carries over from (S_t) to the stochastic integral w.r.t.
(S_t), i.e.

$$E_Q\left(\int_0^T H_s^S \, d\overline{S}_s \Big| \mathcal{F}_t \right) = \int_0^t H_s^S \, d\overline{S}_s$$

then also $(\overline{V}_t)$ is a martingale under Q and the assertions follows.

Proof of the Fundamental Theorem (part 1):

Assume that there is a self-financing portfolio whose value process satisfies $V_0 \leq 0$ and $V_T \geq 0$.

We assume that there exists an equivalent risk neutral model Q and want to show that this implies $V_T \equiv 0$.

If we know that the martingale property carries over from (S_t) to the stochastic integral w.r.t. (S_t), then also $(\overline{V}_t)$ is a martingale under Q.

It follows that $E(\overline{V}_T) = \overline{V}_0 \leq 0$. This implies $V_T \equiv 0$.

Martingale representation

Theorem: Let $(W_t)_{0 \leq t \leq T}$ be a Wiener process and let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be the filtration (family of pasts) of this Wiener process. Then every square integrable random variable C_T which depends on $\mathcal{F}_T$ can be written in the form

$$C_T = \int_0^T H_s \, dW_s$$

(A102) Show that in a Black-Scholes market model each square integrable claim can be hedged (completeness of the Black-Scholes market model).

Chapter 5: Pricing measures

Section 5.1: Closed form pricing

Syllabus: pricing formula - considering dividends

Pricing formula

Risk neutral pricing amounts to calculating conditional expectations:

$$E_Q(\overline{V}_T|\mathcal{F}_t) = \overline{V}_t \;\Leftrightarrow\; E_Q(V_T|\mathcal{F}_t) = V_t e^{r(T-t)}$$

If the claim is not path-dependent then pricing is the same procedure as in the single period case.

If the claim is path-dependent then calculations of closed form solutions can be very advanced.

Risk neutral pricing with dividends

Portfolio: $V_t = H_t^B B_t + H_t^S S_t$

Value change: $dV_t = H_t^B dB_t + H_t^S dS_t + H_t^S DS_t dt$

Discounting gives:

$$d\left(\frac{V_t}{B_t}\right) = H_t^S \left(\underbrace{d\left(\frac{S_t}{B_t}\right) + D\frac{S_t}{B_t}dt}_{\text{what's this ?}} \right)$$

Wealth of the stock-holder: $\widetilde{S}_t = S_t + D\int_0^t S_u e^{r(t-u)}du$

Discounting gives:

$$\frac{\widetilde{S}_t}{B_t} = \frac{S_t}{B_t} + D\int_0^t \frac{S_u}{B_u}du \;\Leftrightarrow\; d\left(\frac{\widetilde{S}_t}{B_t}\right) = d\left(\frac{S_t}{B_t}\right) + D\frac{S_t}{B_t}dt$$

Differential equation of the discounted portfolio value:

$$d\left(\frac{V_t}{B_t}\right) = H_t^S d\left(\frac{\widetilde{S}_t}{B_t}\right)$$

In order to apply the martingale pricing method ("risk neutral pricing") we need an (equivalent) model Q such that $\left(\dfrac{\widetilde{S}_t}{B_t}\right)$ is a martingale.

Then the discounted portfolio value is a martingale, too, and we may write

$$V_t = e^{-r(T-t)} E_Q(V_T|\mathcal{F}_t)$$

Conclusion: The dividend stream changes the martingale model.

The model has to be risk neutral w.r.t. the wealth of the stock holder and not w.r.t. the stock price.

Identifying the risk neutral model

Statistical model: $d\left(\dfrac{S_t}{B_t}\right) = \dfrac{S_t}{B_t}\Big((\mu - r)dt + \sigma dW_t\Big), \ (W_t)$ a P-Wiener process.

This implies for the discounted wealth of the stockholder

$$d\left(\frac{\widetilde{S_t}}{B_t}\right) = d\left(\frac{S_t}{B_t}\right) + D\frac{S_t}{B_t}dt = \frac{S_t}{B_t}\Big(\underbrace{(\mu - r + D)dt + \sigma dW_t}_{Q\text{-martingale}}\Big)$$

We define a Q-Wiener process $(\widetilde{W}_t)$ by

$$\sigma d\widetilde{W}_t = (\mu - r + D)dt + \sigma dW_t \ \Leftrightarrow \ \sigma dW_t = \sigma d\widetilde{W}_t - (\mu - r + D)dt$$

Then the price model is $\ d\left(\dfrac{S_t}{B_t}\right) = \dfrac{S_t}{B_t}\Big(-Ddt + \sigma d\widetilde{W}_t\Big)$

or equivalently $\dfrac{S_t}{B_t} = \dfrac{S_0}{B_0}\exp\Big((-D - \sigma^2/2)t + \sigma\widetilde{W}_t\Big)$

(A103) Assume the Black-Scholes model. Find the closed form expressions for the prices of vanilla calls and puts when the underlying stock pays a continuous dividend stream.

Chapter 5: Pricing measures

Section 5.4: Path dependent options

Syllabus: floating strike puts - average strike puts - average rate options - homogeneity reductions

Floating strike put

The value function is:

$$E_Q((M_T - S_T)^+ | S_t = s, M_t = m) =: \phi(T - t, s, m)$$

This function is homogeneous

$$\phi(T - t, \lambda s, \lambda m) = \lambda \phi(T - t, s, m) \; \Rightarrow \; \phi(T - t, s, m) = m\phi\left(T - t, \frac{s}{m}, 1\right)$$

and therefore it is sufficient to calculate

$$\phi(T - t, s, 1) = E_Q((M_T - S_T)^+ | S_t = s, M_t = 1)$$

Updating the path average

The path maximum satisfies
$$M_T = \max(M_t, M_t^T)$$

This gives

$$
\begin{aligned}
E_Q((M_T - S_T)^+ | S_t = s, M_t = 1) \\
= \ & E_Q\left(\left(\max(1, M_t^T) - S_T\right)^+ \Big| S_t = s\right) \\
= \ & s\, E_Q\left(\left(\max\left(\frac{1}{s}, \frac{M_t^T}{s}\right) - \frac{S_T}{s}\right)^+ \Big| S_t = s\right) \\
= \ & s\, E_Q\left(\left(\max\left(\frac{1}{s}, M_t^T\right) - S_T\right)^+ \Big| S_t = 1\right)
\end{aligned}
$$

Simulation strategy

By MC-simulation we obtain

$$\psi(T - t, \alpha) := E_Q\left(\max\left(\alpha, M_t^T \right) - S_T \right)^+ \Big| S_t = 1 \right)$$

Then we have

$$\phi(T - t, s, 1) = s\,\psi\left(T - t, \frac{1}{s}\right)$$

Average strike option

The value function is:

$$E_Q((S_T - A_T)^+ | S_t = s, A_t = a) =: \phi(T - t, s, a)$$

This function is homogeneous

$$\phi(T - t, \lambda s, \lambda a) = \lambda \phi(T - t, s, a) \;\Rightarrow\; \phi(T - t, s, a) = a \phi\left(T - t, \frac{s}{a}, 1\right)$$

and therefore it is sufficient to calculate

$$\phi(T - t, s, 1) = E_Q((S_T - A_T)^+ | S_t = s, A_t = 1)$$

Slide 211 - Homogeneity reduction

Updating the path average

The path average satisfies

$$A_T = \frac{1}{T} \int_0^T S_u du = \frac{t}{T} \cdot A_t + \frac{T-t}{T} \cdot \frac{1}{T-t} \int_t^T S_u du$$

This gives

$$E_Q((S_T - A_T)^+ | S_t = s, A_t = 1)$$

$$= s\, E_Q\left(\left(\frac{S_T}{s} - \frac{t}{T} \cdot \frac{1}{s} - \frac{T-t}{T} \cdot \frac{1}{T-t} \int_t^T \frac{S_u}{s} du\right)^+ \Big| S_t = s\right)$$

$$= s\, E_Q\left(\left(S_T - \frac{t}{T} \cdot \frac{1}{s} - \frac{T-t}{T} \cdot \frac{1}{T-t} \int_t^T S_u du\right)^+ \Big| S_t = 1\right)$$

Simulation strategy

By MC-simulation we obtain

$$\psi(T - t, \alpha, \beta) := E_Q\left(\left(S_T - \alpha - \beta \cdot \frac{1}{T-t} \int_t^T S_u du\right)^+ \Big| S_t = 1\right)$$

Then we have

$$\phi(T - t, s, 1) = s\, \psi\left(T - t, \frac{t}{T} \cdot \frac{1}{s}, \frac{T-t}{T}\right)$$

Average rate option

The value function is:

$$E_Q((A_T - K)^+ | S_t = s, A_t = a) =: \phi(T - t, s, a)$$

Using the updating rule for the average we obtain

$$E_Q((A_T - K)^+ | S_t = s, A_t = a)$$

$$= E_Q\left(\left(\frac{t}{T} \cdot a + \frac{T-t}{T} \cdot \frac{1}{T-t} \int_t^T S_u du - K\right)^+ \Big| S_t = s\right)$$

$$= s\, E_Q\left(\left(\frac{t}{T} \cdot \frac{a}{s} + \frac{T-t}{T} \cdot \frac{1}{T-t} \int_t^T \frac{S_u}{s} du - \frac{K}{s}\right)^+ \Big| S_t = s\right)$$

$$= s\, E_Q\left(\left(\frac{t}{T} \cdot \frac{a}{s} + \frac{T-t}{T} \cdot \frac{1}{T-t} \int_t^T S_u du - \frac{K}{s}\right)^+ \Big| S_t = 1\right)$$

$$= s\, E_Q\left(\left(\frac{t}{T} \cdot \frac{a}{s} - \frac{K}{s} + \frac{T-t}{T} \cdot \frac{1}{T-t} \int_t^T S_u du\right)^+ \Big| S_t = 1\right)$$

Simulation strategy

By MC-simulation we obtain

$$\psi(T-t, \alpha, \beta) := E_Q\left(\left(\alpha + \beta \cdot \frac{1}{T-t} \int_t^T S_u du\right)^+ \Big| S_t = 1\right)$$

Then we have

$$\phi(T-t, s, a) = s\,\psi\left(T-t, \frac{t}{T} \cdot \frac{a}{s} - \frac{K}{s}, \frac{T-t}{T}\right)$$

Chapter 5: Pricing measures

Section 5.5: The Feynman-Kac approach

Syllabus: martingale argument - derivation

Derivation of the Black-Scholes pdf

In a risk neutral model Q every admissible value process can be written as a conditional expectation

$$E_Q(\overline{V}_T | \mathcal{F}_t) = \overline{V}_t \iff E_Q(V_T | \mathcal{F}_t) = V_t e^{r(T-t)}$$

This means that the discounted value process is a martingale.

This leads to a simple derivation of the corresponding pdf.

Assumptions:

The stock satisfies: $dS_t = rS_t dt + \sigma S_t dW_t$

The claim is defined by: $V_T = h(S_T)$.

Let $V_t := \phi(S_t, t)$. Then we have

$$E_Q\left(\frac{h(S_T)}{e^{rT}}\bigg| \mathcal{F}_t\right) = \frac{\phi(S_t, t)}{e^{rt}}$$

Therefore $\phi(S_t, t)e^{-rt}$ is a martingale under Q.

Derivation

Ito's formula gives

$$d(\phi e^{-rt}) = (\phi e^{-rt})_t dt + (\phi e^{-rt})_x dS_t + \frac{1}{2}(\phi e^{-rt})_{xx} d[S]_t$$

$$= (\phi_t e^{-rt} - re^{-rt}\phi)dt + (\phi_x e^{-rt})(rS_t dt + \sigma S_t dW_t) + \frac{1}{2}(\phi_{xx} e^{-rt})\sigma^2 S_t^2 dt$$

$$= (\phi_t - r\phi + \phi_x rS_t + \frac{1}{2}\phi_{xx}\sigma^2 S_t^2)e^{-rt}dt + \phi_x e^{-rt}\sigma S_t dW_t$$

The martingale property implies that the dt-term is zero. This gives the pdf.

(A104) Extend this derivation to a dividend paying stock.

Appendix

Is $V_t - \Delta_t S_t$ riskless ? Answer: **No** !

$$[V - \Delta S]_t = \int_0^t S_t^2 \Gamma_t^2 \, d[S]_t$$

```
S=rpBS(ngrid=5000)
p=S()
c=Call.BS(spot=p)

q1=c$Value-c$Delta*p
pPlot(q1)
pPlot(pBracket(q1))

q2=pIntegral("x^2*y^2",driver=p,y=c$Gamma,v=2)
pPlot(q2)

pPlot(pBracket(q1)-q2,ylim=c(-0.01,0.01))
```

MC: Call Delta and Gamma

```
S=rpBS(vol=0.2,rate=0.04,time=1)
sp=seq(from=0.5,to=1.5,length=50)
pr=sp*apply(Call(1/sp)(S(10000)),2,mean)*exp(-0.04)
sm=smooth.spline(sp,pr,spar=0.5)

delta=function(x) predict(sm,x,deriv=1)$y
plot(delta,0.5,1.5)
dl0=Call.BS(spot=sp,vol=0.2,rate=0.04)$Delta
lines(sp,dl0,col="red")

gamma=function(x) predict(sm,x,deriv=2)$y
plot(gamma,0.5,1.5)
gm0=Call.BS(spot=sp,vol=0.2,rate=0.04)$Gamma
lines(sp,gm0,col="red")
```

MC: BinCall Delta and Gamma

```
S=rpBS(vol=0.2,rate=0.04,time=1)
sp=seq(from=0.5,to=1.5,length=50)
pr=apply(BinCall(1/sp)(S(50000)),2,mean)*exp(-0.04)
sm=smooth.spline(sp,pr,spar=0.5)

delta=function(x) predict(sm,x,deriv=1)$y
plot(delta,0.5,1.5)

gamma=function(x) predict(sm,x,deriv=2)$y
plot(gamma,0.5,1.5)
```

```
pf=function(x,y) Call.BS(spot=x,vol=y)$Delta
Persp("pf(x,y)",xlim=c(0,2),ylim=c(0,1),xlab="spot",
    ylab="vol",zlab="delta")


pf=function(x,y) Call.BS(spot=x,vol=y)$Gamma
Persp("pf(x,y)",xlim=c(0.01,2),ylim=c(0.01,1),xlab="spot",
    ylab="vol",zlab="gamma")
```

Discrete Hedging:

```
H=Hedge.Call.BS(strike=0.8,vol=0.2)
H(rebalance=10,plot=TRUE)
Hist(H(1000,rebalance=10))


H=Hedge.Call.BS(strike=1,vol=0.2)
H(rebalance=10,plot=TRUE)
Hist(H(1000,rebalance=10))


H=Hedge.Call.BS(strike=1,vol=0.2,rate=0.08,rlrate=0.03)
H(rebalance=10,plot=TRUE)
Hist(H(1000,rebalance=10))


H=Hedge.Call.BS(strike=1,vol=0.2,rate=0.08,rlrate=0.03)
H(rebalance=100,plot=TRUE)
Hist(H(1000,rebalance=100))
```

Up and Out Call: Single prices for various barriers

```
r=0.08
S=rpBS(vol=0.2,rate=r,ngrid=50)
pPlot(S(20))


KnockOutCall(1.1,50)(S(100))


mean(KnockOutCall(1.1,50)(S(10000)))*exp(-r)
Call.BS(vol=0.2,rate=r,strike=1.1)$Value


mean(KnockOutCall(1.1,1.7)(S(10000)))*exp(-r)
Call.BS(vol=0.2,rate=r,strike=1.1)$Value


mean(KnockOutCall(1.1,1.2)(S(10000)))*exp(-r)
Call.BS(vol=0.2,rate=r,strike=1.1)$Value
```

Improving speed and memory usage:

```
r=0.08
S=rpBS(vol=0.2,rate=r,ngrid=50)

system.time(mean(KnockOutCall(1.1,50)(S(50000)))*exp(-r))
ff=function(n) KnockOutCall(1.1,50)(S(n))
system.time(MC(ff,iter=50000))

system.time(mean(KnockOutCall(1.1,50)(S(100000)))*exp(-r))
ff=function(n) KnockOutCall(1.1,50)(S(n))
system.time(MC(ff,iter=100000))

system.time(mean(KnockOutCall(1.1,50)(S(200000)))*exp(-r))
ff=function(n) KnockOutCall(1.1,50)(S(n))
system.time(MC(ff,iter=200000))
```

Up and Out Call: Prices, large barrier

```
S=rpBS(vol=0.2,rate=0.04,ngrid=50)
KnockOutCall(1,5)(S(10))
KnockOutCall(c(1.2,1,0.8),5)(S(10))

sp=seq(from=0.5,to=1.5,length=50)
pr=sp*apply(KnockOutCall(1/sp,5/sp)(S(50000)),2,mean)*exp(-0.04)

sm=smooth.spline(sp,pr,spar=0.5)
price=function(x) predict(sm,x)$y
plot(price,0.5,1.5)
pr0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Value
lines(sp,pr0,col="red")
```

Improving speed and memory usage:

```
sp=seq(from=0.5,to=1.5,length=50)
ff=function(n) t(KnockOutCall(1/sp,5/sp)(S(n)))
pr=sp*MC(ff,iter=200000)$mean*exp(-0.04)


sm=smooth.spline(sp,pr,spar=0.5)
price=function(x) predict(sm,x)$y
plot(price,0.5,1.5)
pr0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Value
lines(sp,pr0,col="red")
```

Up and Out Call: Prices, medium barrier

```
S=rpBS(vol=0.2,rate=0.04,ngrid=50)
strike=1
barrier=1.5
sp=seq(from=0.5,to=2,length=50)

pr=sp*apply(KnockOutCall(strike/sp,barrier/sp)(S(50000)),2,mean)
pr=pr*exp(-0.04)

sm=smooth.spline(sp,pr,spar=0.5)
price=function(x) predict(sm,x)$y
plot(price,0.5,2)
pr0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Value
lines(sp,pr0,col="red")
```

Up and Out Call: Prices, near barrier

```
S=rpBS(vol=0.2,rate=0.04,ngrid=50)
strike=1
barrier=1.2
sp=seq(from=0.5,to=1.5,length=50)

pr=sp*apply(KnockOutCall(strike/sp,barrier/sp)(S(50000)),2,mean)
pr=pr*exp(-0.04)

sm=smooth.spline(sp,pr,spar=0.5)
price=function(x) predict(sm,x)$y
plot(price,0.5,1.5)
pr0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Value
lines(sp,pr0,col="red")
```

MC simulation of the price surface

```
ff=function(vol=0.2,rate=0.04,strike=1,barrier=1.5){
    time=seq(from=0,to=1,length=30)
    sp=seq(from=0.5,to=1.5,length=50)
    pr=numeric(0)
    for (t in time){
  Cat(paste("evaluating t=",round(t,3)))
  S=rpBS(vol=vol*sqrt(t),rate=rate*(t),ngrid=50)
  temp=sp*apply(KnockOutCall(strike/sp,barrier/sp)(S(10000)),2,mean)
  pr=cbind(pr,temp*exp(-rate))
  }
    return(list(spot=sp,time=time,price=pr))
    }
dat=ff()
Persp(dat$spot,dat$time,dat$price,xlab="spot",ylab="vol",zlab="price")
```

Up and Out Call: Delta and Gamma, large barrier

```
S=rpBS(vol=0.2,rate=0.04,ngrid=50)
sp=seq(from=0.5,to=1.5,length=50)
pr=sp*apply(KnockOutCall(1/sp,5/sp)(S(50000)),2,mean)*exp(-0.04)
sm=smooth.spline(sp,pr,spar=0.5)

delta=function(x) predict(sm,x,deriv=1)$y
plot(delta,0.5,1.5)
dl0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Delta
lines(sp,dl0,col="red")

gamma=function(x) predict(sm,x,deriv=2)$y
plot(gamma,0.5,1.5)
gm0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Gamma
lines(sp,gm0,col="red")
```

Up and Out Call: Delta and Gamma, medium barrier

```
S=rpBS(vol=0.2,rate=0.04,ngrid=50)
sp=seq(from=0.5,to=1.5,length=50)
pr=sp*apply(KnockOutCall(1/sp,1.5/sp)(S(50000)),2,mean)*exp(-0.04)
sm=smooth.spline(sp,pr,spar=0.5)

delta=function(x) predict(sm,x,deriv=1)$y
plot(delta,0.5,1.5)
dl0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Delta
lines(sp,dl0,col="red")

gamma=function(x) predict(sm,x,deriv=2)$y
plot(gamma,0.5,1.5)
gm0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Gamma
lines(sp,gm0,col="red")
```

Up and Out Call: Delta and Gamma, near barrier

```
S=rpBS(vol=0.2,rate=0.04,ngrid=50)
sp=seq(from=0.5,to=1.2,length=50)
pr=sp*apply(KnockOutCall(1/sp,1.2/sp)(S(100000)),2,mean)*exp(-0.04)
sm=smooth.spline(sp,pr,spar=0.5)

delta=function(x) predict(sm,x,deriv=1)$y
plot(delta,0.5,1.2)
dl0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Delta
lines(sp,dl0,col="red")

gamma=function(x) predict(sm,x,deriv=2)$y
plot(gamma,0.5,1.2)
gm0=Call.BS(spot=sp,vol=0.2,rate=0.04,maturity=1)$Gamma
lines(sp,gm0,col="red")
```

The surface function

```
ff=function(vol=0.2,rate=0.04,strike=1,barrier=1.5){
   time=seq(from=0,to=1,length=30)
   sp=seq(from=0.5,to=barrier,length=50)
   price=numeric(0); delta=numeric(0); gamma=numeric(0)
   for (t in time) {
  Cat(paste("evaluating t=",round(t,3)))
  S=rpBS(vol=vol*sqrt(t),rate=rate*(t),ngrid=50)
  pr=sp*apply(KnockOutCall(strike/sp,barrier/sp)(S(50000)),2,mean)*exp(-rate)
  sm=smooth.spline(sp,pr,spar=0.5)
  price=cbind(price,predict(sm,sp)$y)
  delta=cbind(delta,predict(sm,sp,deriv=1)$y)
  gamma=cbind(gamma,predict(sm,sp,deriv=2)$y)
  }
   return(list(spot=sp,time=time,price=price,delta=delta,gamma=gamma))
   }
```

Delta and Gamma surfaces

```
dat=ff()
Persp(dat$spot,dat$time,dat$price,xlab="spot",ylab="vol",zlab="price")
Persp(dat$spot,dat$time,dat$delta,xlab="spot",ylab="vol",zlab="delta")
Persp(dat$spot,dat$time,dat$gamma,xlab="spot",ylab="vol",zlab="gamma")

dat=ff(barrier=1.2)
Persp(dat$spot,dat$time,dat$price,xlab="spot",ylab="vol",zlab="price")
Persp(dat$spot,dat$time,dat$delta,xlab="spot",ylab="vol",zlab="delta")
Persp(dat$spot,dat$time,dat$gamma,xlab="spot",ylab="vol",zlab="gamma")
```

MC simulation of values

```
price.diagram <- function(t,vol=0.2,rate=0.04){
    S=rpBS(vol=vol*sqrt(1-t),rate=0.04*(1-t),ngrid=50)
    sp=seq(from=0.5,to=1.5,length=50)
    pr=sp*apply(FloatingStrikePut(1/sp)(S(10000)),2,mean)*exp(-0.04)
    plot(sp,pr,type="l")
    }

price.diagram(0)
price.diagram(0.2)
price.diagram(0.5)
price.diagram(0.8)
price.diagram(1)
```

MC Simulation of surfaces

```
ff=function(vol=0.2,rate=0.04){
   time=seq(from=0,to=1,length=30)
   sp=seq(from=0.5,to=1.5,length=50)
   price=numeric(0)
   delta=numeric(0)
   gamma=numeric(0)
   for (t in time) {
  Cat(paste("evaluating t=",round(t,3)))
  S=rpBS(vol=vol*sqrt(t),rate=rate*(t),ngrid=50)
  pr=sp*apply(FloatingStrikePut(1/sp)(S(10000)),2,mean)*exp(-0.04)
  sm=smooth.spline(sp,pr,spar=0.5)
  price=cbind(price,predict(sm,sp)$y)
  delta=cbind(delta,predict(sm,sp,deriv=1)$y)
  gamma=cbind(gamma,predict(sm,sp,deriv=2)$y)
  }
   return(list(spot=sp,time=time,price=price,delta=delta,gamma=gamma))
   }
```

Surface diagrams

```
dat=ff()
Persp(dat$spot,dat$time,dat$price,xlab="spot",ylab="vol",zlab="price")
Persp(dat$spot,dat$time,dat$delta,xlab="spot",ylab="vol",zlab="delta")
Persp(dat$spot,dat$time,dat$gamma,xlab="spot",ylab="vol",zlab="gamma")
```

MC simulation of values

```
price.diagram <- function(t,vol=0.2,rate=0.04){
    S=rpBS(vol=vol*sqrt(1-t),rate=0.04*(1-t),ngrid=50)
    sp=seq(from=0.5,to=1.5,length=50)
    pr=sp*apply(AverageStrikeCall(t/sp,1-t)(S(10000)),2,mean)*exp(-0.04)
    plot(sp,pr,type="l")
    }

price.diagram(0)
price.diagram(0.2)
price.diagram(0.5)
price.diagram(0.8)
price.diagram(1)
```

MC Simulation of surfaces

```
ff=function(vol=0.2,rate=0.04){
   time=seq(from=0,to=1,length=30)
   sp=seq(from=0.5,to=1.5,length=50)
   price=numeric(0)
   delta=numeric(0)
   gamma=numeric(0)
   for (t in time) {
  Cat(paste("evaluating t=",round(t,3)))
  S=rpBS(vol=vol*sqrt(t),rate=rate*(t),ngrid=50)
  pr=sp*apply(AverageStrikeCall((1-t)/sp,t)(S(10000)),2,mean)*exp(-0.04)
  sm=smooth.spline(sp,pr,spar=0.5)
  price=cbind(price,predict(sm,sp)$y)
  delta=cbind(delta,predict(sm,sp,deriv=1)$y)
  gamma=cbind(gamma,predict(sm,sp,deriv=2)$y)
  }
   return(list(spot=sp,time=time,price=price,delta=delta,gamma=gamma))
   }
```

Surface diagrams

```
dat=ff()
Persp(dat$spot,dat$time,dat$price,xlab="spot",ylab="vol",zlab="price")
Persp(dat$spot,dat$time,dat$delta,xlab="spot",ylab="vol",zlab="delta")
Persp(dat$spot,dat$time,dat$gamma,xlab="spot",ylab="vol",zlab="gamma")
```

MC Simulation of surfaces

Dependence on spot price and average for fixed maturity

```
ff=function(time=0.5,vol=0.2,rate=0.04){
   av=seq(from=0.5,to=1.5,length=30)
   sp=seq(from=0.5,to=1.5,length=50)
   price=numeric(0)
   delta=numeric(0)
   gamma=numeric(0)
   S=rpBS(vol=vol*sqrt(time),rate=rate*(time),ngrid=50)
   x=S(10000)
   for (a in av) {
 Cat(paste("evaluating av=",round(a,3)))
 pr=sp*apply(AverageRateCall(time*a/sp-1/sp,1-time)(x),2,mean)*exp(-0.04)
 sm=smooth.spline(sp,pr,spar=0.5)
 price=round(cbind(price,predict(sm,sp)$y),6)
 delta=round(cbind(delta,predict(sm,sp,deriv=1)$y),6)
 gamma=round(cbind(gamma,predict(sm,sp,deriv=2)$y),6)
 }
   return(list(spot=sp,average=av,price=price,delta=delta,gamma=gamma))
   }
```

Surfaces

```
dat=ff(time=0)
Persp(dat$spot,dat$average,dat$price,xlab="spot",ylab="av",zlab="price")
Persp(dat$spot,dat$average,dat$delta,xlab="spot",ylab="av",zlab="delta")
Persp(dat$spot,dat$average,dat$gamma,xlab="spot",ylab="av",zlab="gamma")

dat=ff(time=1)
Persp(dat$spot,dat$average,dat$price,xlab="spot",ylab="av",zlab="price")
Persp(dat$spot,dat$average,dat$delta,xlab="spot",ylab="av",zlab="delta")
Persp(dat$spot,dat$average,dat$gamma,xlab="spot",ylab="av",zlab="gamma")

dat=ff(time=0.5)
Persp(dat$spot,dat$average,dat$price,xlab="spot",ylab="av",zlab="price")
Persp(dat$spot,dat$average,dat$delta,xlab="spot",ylab="av",zlab="delta")
Persp(dat$spot,dat$average,dat$gamma,xlab="spot",ylab="av",zlab="gamma")
```

Time dependence of surfaces

```
time=seq(from=0,to=1,by=0.1)
dat=vector("list",11)
for (i in 1:11) {Cat(paste("i=",i,"\n")); dat[[i]]=ff(time=time[i])}

for (i in 1:11)
Persp(dat[[i]]$spot,dat[[i]]$average,dat[[i]]$price,
    xlab="spot",ylab="av",zlab="price",control=FALSE,
    main=paste("Value: time=",time[i]))

for (i in 1:11)
Persp(dat[[i]]$spot,dat[[i]]$average,dat[[i]]$delta,zlim=c(0,1),
    xlab="spot",ylab="av",zlab="price",control=FALSE,
    main=paste("Delta: time=",time[i]))

for (i in 1:11)
Persp(dat[[i]]$spot,dat[[i]]$average,dat[[i]]$gamma,zlim=c(0,10),
    xlab="spot",ylab="av",zlab="price",control=FALSE,
    main=paste("Value: time=",time[i]))
```
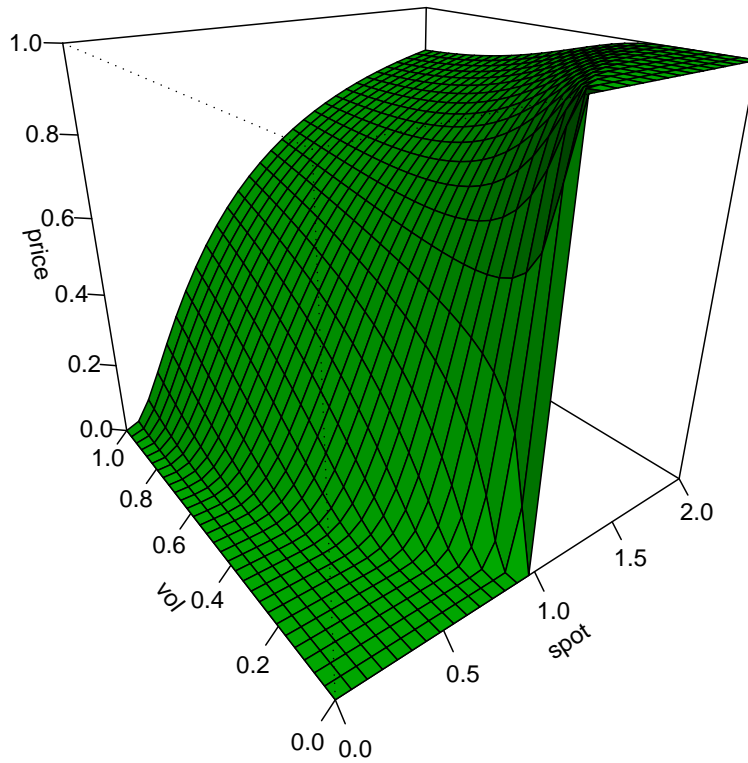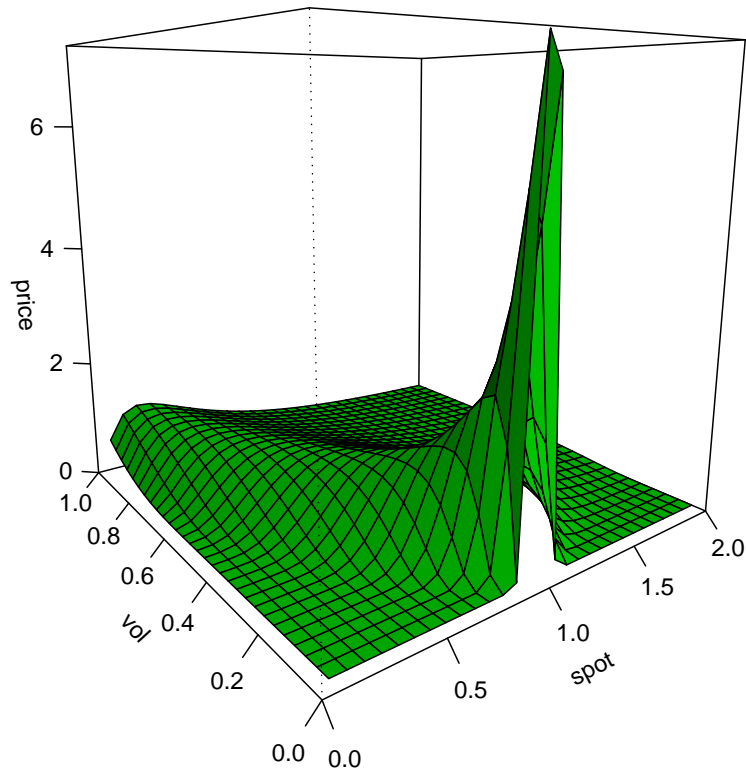
Appendix

- Figures

Delta surface

Gamma surface

Appendix

- Solutions

Details:

Let $u = e^a$, $d = 1/u = e^{-a}$.

$S_k = s_0 X_1 X_2 \cdots X_k$ where $X_i = \begin{cases} P(X_i = e^a) = p \\ P(X_i = e^{-a}) = 1 - p \end{cases}$ (X_i) independent

$E(S_n) = s_0 E(X)^n = s_0(pe^a + (1-p)e^{-a})^n$

Calibration of the growth rate of expected prices:

$E(X) = e^{\mu(T-t)/n} \;\Rightarrow\; p = \dfrac{e^{\mu(T-t)/n} - e^{-a}}{e^a - e^{-a}}$

Calibration of volatility:

Choose a such that $L_n := \sum_i \log X_i$ satisfies $V(L_n) = \sigma^2(T - t)$.

$\log X_i = a(2Y_i - 1)$ where $Y_i = \begin{cases} P(Y_i = 1) = p \\ P(Y_i = 0) = 1 - p \end{cases}$

$L_n = a(2Z_n - n)$ where $Z_n = \sum_i Y_i$ is binomial distributed

$E(L_n) = an(2p - 1)$

$V(L_n) = 4a^2 np(1 - p)$

Now we can adjust the jump height a in order to match volatility.

In this discrete time picture the jump height seems to depend on the parameter μ.
However, this is not the whole story !