Mathematics 1 for Economics

Linear Spaces and Metric Concepts

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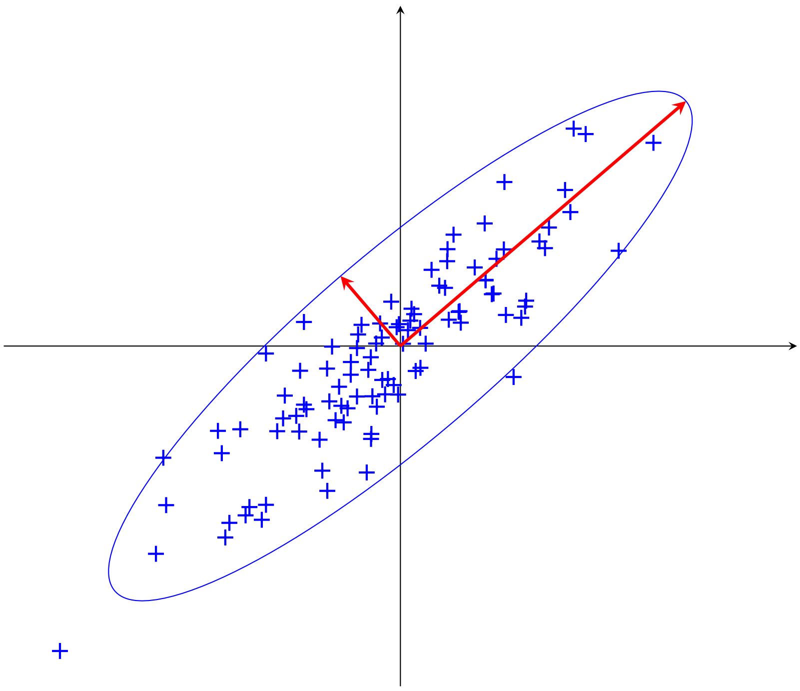

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Contents

Bibliography

The following books have been used to prepare this course.

[1] Kevin Houston. *How to Think Like a Mathematician*. Cambridge University Press, 2009.

[2] Knut Sydsæter and Peter Hammond. *Essential Mathematics for Economics Analysis*. Prentice Hall, 3rd edition, 2008.

[3] Knut Sydsæter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economics Analysis*. Prentice Hall, 2005.

[4] Alpha C. Chiang and Kevin Wainwright. *Fundamental Methods of Mathematical Economics*. McGraw-Hill, 4th edition, 2005.

1

Introduction

1.1 Learning Outcomes

The learning outcomes of the two parts of this course in *Mathematics* are threefold:

- Mathematical reasoning
- Fundamental concepts in mathematical economics
- Extend mathematical toolbox

Topics

- Linear Algebra:
  - Vector spaces, basis and dimension
  - Matrix algebra and linear transformations
  - Norm and metric
  - Orthogonality and projections
  - Determinants
  - Eigenvalues
- Topology
  - Neighborhood and convergence
  - Open sets and continuous functions
  - Compact sets
- Calculus
  - Limits and continuity
  - Derivative, gradient and Jacobian matrix
  - Mean value theorem and Taylor series

  - Inverse and implicit functions
  - Static optimization
  - Constrained optimization
- Integration
  - Antiderivative
  - Riemann integral
  - Fundamental Theorem of Calculus
  - Leibniz's rule
  - Multiple integral and Fubini's Theorem
- Dynamic analysis
  - Ordinary differential equations (ODE)
  - Initial value problem
  - linear and logistic differential equation
  - Autonomous differential equation
  - Phase diagram and stability of solutions
  - Systems of differential equations
  - Stability of stationary points
  - Saddle path solutions
- Dynamic analysis
  - Control theory
  - Hamilton function
  - Transversality condition
  - Saddle path solutions

1.2 A Science and Language of Patterns

> Mathematics consists of propositions of the form: P implies Q, but you never ask whether P is true. (Bertrand Russell)

The mathematical universe is built-up by a series of definitions, theorems and proofs.

Axiom
A statement that is assumed to be true.
Axioms define basic concepts like sets, natural numbers or real numbers: A family of elements with rules to manipulate these.

⋮

⋮

Definition
Introduce a new notion. (Use known terms.)
Theorem
A statement that describes properties of the new object:
If ... then ...
Proof
Use true statements (other theorems!) to show that this statement is true.

⋮

⋮

New Definition
Based on observed interesting properties.
Theorem
A statement that describes properties of the new object.
Proof
Use true statements (including former theorems) to show that the statement is true.

⋮

????

Here is a very simple example:

Even number. An **even number** is a natural number n that is divisible by 2. Definition 1.1

If n is an even number, then n^2 is even. Theorem 1.2

PROOF. If n is divisible by 2, then n can be expressed as $n = 2k$ for some $k \in \mathbb{N}$. Hence $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ which also is divisible by 2. Thus n^2 is an even number as claimed. □

The *if ... then ...* structure of mathematical statements is not always obvious. Theorem 1.2 may also be expressed as: *The square of an even number is even.*

When reading the definition of *even number* we find the terms *divisible* and *natural numbers*. These terms must already be well-defined: We say that a natural number n is divisible by a natural number k if there exists a natural number m such that $n = k \cdot m$.

What are *natural numbers*? These are defined as a set of objects that satisfies a given set of rules, i.e., by *axioms*[1].

Of course the development in mathematics is not straightforward as indicate in the above diagram. It is rather a tree with some additional links between the branches.

1.3 Mathematical Economics

The quote from Bertrand Russell may seem disappointing. However, this exactly is what we are doing in *Mathematical Economics*.

An economic model is a simple picture of the real world. In such a model we list all our assumptions and then deduce patterns in our model from these "axioms". E.g., we may try to derive propositions like: "When we increase parameter X in model Y then variable Z declines." It is not the task of mathematics to validate the assumptions of the model, i.e., whether the model describes the real world sufficiently well.

Verification or falsification of the model is the task of economists.

1.4 About This Manuscript

This manuscript is by no means a complete treatment of the material. Rather it is intended as a road map for our course. The reader is invited to consult additional literature if she wants to learn more about particular topics.

As this course is intended as an extension of the course *Foundations of Economics – Mathematical Methods* the reader is encouraged to look at the given handouts for examples and pictures. It is also assumed that the reader has successfully mastered all the exercises of that course. Moreover, we will not repeat all definitions given there.

1.5 Solving Problems

In this course we will have to solve homework problems. For this task the reader may use any theorem that have already been proved up to this point. Missing definitions could be found in the handouts for the course *Foundations of Economics – Mathematical Methods*. However, one *must not* use any results or theorems from these handouts.

[1] The natural numbers can be defined by the so called Peano axioms.

Roughly spoken there are two kinds of problems:

- Prove theorems and lemmata that are stated in the main text. For this task you may use any result that is presented up to this particular proposition that you have to show.
- Problems where additional statements have to be proven. Then all results up to the current chapter may be applied, unless stated otherwise.

Some of the problems are hard. Here is **Polya's four step plan** for tackling these issues.

(i) Understand the problem.

(ii) Devise a plan.

(iii) Execute the problem.

(iv) Look back.

1.6 Symbols and Abstract Notions

Mathematical illiterates often complain that mathematics deals with abstract notions and symbols. However, this is indeed the foundation of the great power of mathematics.

Here is an example[2]. Suppose we want to solve the quadratic equation

$$x^2 + 10x = 39 .$$

Muḥammad ibn Mūsā al-Khwārizmī (c. 780–850) presented an algorithm for solving this equation in his text entitled *Al-kitāb al-muẖtaṣar fī ḥisāb al-jabr wa-l-muqābala* (*The Condensed Book on the Calculation of al-Jabr and al Muqabala*). In his text he distinguishes between three kinds of quantities: the *square* [of the unknown], the *root* of the square [the unknown itself], and the *absolute numbers* [the constants in the equation]. Thus he stated our problem as

> "What must be the square which, when increased by ten of its own roots, amounts to thirty-nine?"

and presented the following recipe:

> "The solution is this: you halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for."

[2]See Sect. 7.2.1 in Victor J. Katz (1993), *A History of Mathematics*, HarperCollins College Publishers.

Using modern mathematical (abstract!) notation we can express this algorithm in a more condensed form as follows:

The solution of the quadratic equation $x^2 + bx = c$ with $b, c > 0$ is obtained by the procedure

1. Halve b.
2. Square the result.
3. Add c.
4. Take the square root of the result.
5. Subtract $b/2$.

It is easy to see that the result can abstractly be written as

$$x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}\,.$$

Obviously this problem is just a special case of the general form of a quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{R}$$

with solution

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Al-Khwārizmī provided a purely geometrically proof for his algorithm. Consequently, the constants b and c as well as the unknown x must be positive quantities. Notice that for him $x^2 = bx + c$ was a different type of equation. Thus he had to distinguish between six different types of quadratic equations for which he provided algorithms for finding their solutions (and a couple of types that do not have positive solutions at all). For each of these cases he presented geometric proofs. And Al-Khwārizmī did not use letters nor other symbols for the unknown and the constants.

— Summary

- Mathematics investigates and describes *structures* and *patterns*.
- *Abstraction* is the reason for the great power of mathematics.
- Computations and *procedures* are part of the mathematical toolbox.
- Students of this course have mastered all the exercises from the course *Foundations of Economics – Mathematical Methods*.
- Ideally students read the corresponding chapters of this manuscript *in advance before each lesson*!

2
Logic

We want to look at the foundation of mathematical reasoning.

2.1 Statements

We use a naïve definition.

Definition 2.1

A **statement** is a sentence that is either true (T) or false (F) – but not both.

Example 2.2

- *"Vienna is the capital of Austria."* is a true statement.
- *"Bill Clinton was president of Austria."* is a false statement.
- *"19 is a prime number"* is a true statement.
- *"This statement is false"* is not a statement.
- *"x is an odd number."* is not a statement. $\diamond$

2.2 Connectives

Statements can be connected to more complex statements by means of words like *"and"*, *"or"*, *"not"*, *"if ... then ..."*, or *"if and only if"*. Table 2.3 lists the most important ones.

2.3 Truth Tables

Truth tables are extremely useful when learning logic. Mathematicians do not use them in day-to-day work but they provide clarity for the beginner. Table 2.4 lists truth values for important connectives.

Notice that the negation of "All cats are gray" is not "All cats are not gray" but "Not all cats are gray", that is, "There is at least one cat that is not gray".

Table 2.3

Connectives for statements

Let P and Q be two statements.

Connective	Symbol	Name
not P	$\neg P$	**negation**
P and Q	$P \wedge Q$	**conjunction**
P or Q	$P \vee Q$	**disjunction**
if P then Q	$P \Rightarrow Q$	**implication**
Q if and only if P	$P \Leftrightarrow Q$	**equivalence**

Table 2.4

Truth table for important connectives

Let P and Q be two statements.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

2.4 If . . . then . . .

In an implication $P \Rightarrow Q$ there are two parts:

- Statement P is called the **hypothesis** or **assumption**, and
- Statement Q is called the **conclusion**.

The truth values of an **implication** seems a bit *mysterious*. Notice that $P \Rightarrow Q$ says nothing about the truth of P or Q.

Example 2.5

Which of the following statements are true?

- "If Barack Obama is Austrian citizen, then he may be elected for Austrian president."
- "If Ben is Austrian citizen, then he may be elected for Austrian president." $\diamondsuit$

2.5 Quantifier

Definition 2.6

The phrase "for all" is the **universal quantifier**.
It is denoted by $\forall$.

Definition 2.7

The phrase "there exists" is the **existential quantifier**.
It is denoted by $\exists$.

— Problems

2.1 Construct the truth table of the following statements:

(a) $\neg\neg P$ (b) $\neg(P \wedge Q)$ (c) $\neg(P \vee Q)$

(d) $\neg P \wedge P$ (e) $\neg P \vee P$ (f) $\neg P \vee \neg Q$

2.2 Verify that the statement

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

is always true.

HINT: Compute the truth table for this statement.

2.3 **Contrapositive.** Verify that the statement

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

is always true. Explain this statement and give an example.

2.4 Express $P \vee Q$, $P \Rightarrow Q$, and $P \Leftrightarrow Q$ as compositions of P and Q by means of $\neg$ and $\wedge$. Prove your statement by truth tables.

2.5 Another connective is **exclusive-or** $P \oplus Q$. This statement is true if and only if exactly one of the statements P or Q is true.

(a) Establish the truth table for $P \oplus Q$.

(b) Express this statement by means of "not", "and", and "or". Verify your proposition by means of truth tables.

2.6 A **tautology** is a statement that is always true. A **contradiction** is a statement that is always false.

Which of the statements in the above problems is a tautology or a contradiction?

2.7 Assume that the statement $P \Rightarrow Q$ is true. Which of the following statements are true (or false). Give examples.

(a) $Q \Rightarrow P$ (b) $\neg Q \Rightarrow P$

(c) $\neg Q \Rightarrow \neg P$ (d) $\neg P \Rightarrow \neg Q$

3

Definitions, Theorems and Proofs

We have to read mathematical texts *and need to know what that terms mean.*

3.1 Meanings

A mathematical text is build around a skeleton of the form *"definition – theorem – proof"*. Besides that one also finds examples, remarks, or illustrations. Here is a very short description of these terms.

- **Definition** : an explanation of the mathematical meaning of a word. definition
- **Theorem** : a very important true statement. theorem
- **Proposition** : a less important but nonetheless interesting true statement. proposition
- **Lemma** : a true statement used in proving other statements (auxiliary proposition; pl. *lemmata*). lemma
- **Corollary** : a true statement that is a simple deduction from a theorem. corollary
- **Proof** : the explanation of why a statement is true. proof
- **Conjecture** : a statement believed to be true, but for which we have no proof. conjecture
- **Axiom** : a basic assumption about a mathematical situation. axiom

3.2 Reading

When reading definitions:

- Observe precisely the given condition.
- Find examples.
- Find standard examples (which you should memorize).
- Find trivial examples.
- Find extreme examples.
- Find non-examples, i.e., an example that *do not* satisfy the condition of the definition.

When reading theorems:

- Find assumptions and conditions.
- Draw a picture.
- Apply trivial or extreme examples.
- What happens to non-examples?

3.3 Theorems

Mathematical propositions are statements of the form *"if A then B"*. It is always possible to rephrase a theorem in this way. E.g., the statement "$\sqrt{2}$ is an irrational number" can be rewritten as *"If $x = \sqrt{2}$ then x is a irrational number"*.

When talking about mathematical theorems the following two terms are extremely important.

Definition 3.1 A **necessary** condition is one which must hold for a conclusion to be true. It does not guarantee that the result is true.

Definition 3.2 A **sufficient** condition is one which guarantees the conclusion is true. The conclusion may even be true if the condition is not satisfied.

So if we have the statement *"if A then B"*, i.e., $A \Rightarrow B$, then

- A is a sufficient condition for B, and
- B is a necessary condition for A (sometimes also written as $B \Leftarrow A$).

3.4 Proofs

Finding proofs is an art and a skill that needs to be trained. The mathematician's toolbox provide the following main techniques.

Direct Proof

The statement is derived by a straightforward computation.

Proposition 3.3

If $n \in \mathbb{N}$ is an odd number, then n^2 is odd.

PROOF. If n is odd, then it is not divisible by 2 and thus n can be expressed as $n = 2k + 1$ for some $k \in \mathbb{N}$. Hence

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

which is not divisible by 2, either. Thus n^2 is an odd number as claimed. □

Contrapositive Method

The **contrapositive** of the statement $P \Rightarrow Q$ is

$$\neg Q \Rightarrow \neg P .$$

We have already seen in Problem 2.3 that $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$. Thus in order to prove statement $P \Rightarrow Q$ we also may prove its contrapositive.

Proposition 3.4

If n^2 is an even number, then n is even.

PROOF. This statement is equivalent to the statement:

"If n is not even (i.e., odd), then n^2 is not even (i.e., odd)."

However, this statements holds by Proposition 3.3 and thus our proposition follows. □

Obviously we also could have used a direct proof to derive Proposition 3.4. However, our approach has an additional advantage: Since we already have shown that Proposition 3.3 holds, we can use it for our proof and avoid unnecessary computations.

Indirect Proof

This technique is similar to the contrapositive method. Yet we assume that both P and $\neg Q$ are true and show that a contradiction results. Thus it is called **proof by contradiction** (or *reductio ad absurdum*). It is based on the equivalence $(P \Rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$. The advantage of this method is that we get the statement $\neg Q$ for free even when Q is difficult to show.

Proposition 3.5

The square root of 2 is irrational, i.e., it cannot be written in form m/n where m and n are integers.

PROOF IDEA. We assume that $\sqrt{2} = m/n$ where m and n are integers without a common divisor. We then show that both, m and n, are even which is absurd.

PROOF. Suppose the contrary that $\sqrt{2} = m/n$ where m and n are integers. *Without loss of generality* we can assume that this quotient is in its simplest form. (Otherwise cancel common divisors of m and n.) Then we find

$$\frac{m}{n} = \sqrt{2} \quad \Leftrightarrow \quad \frac{m^2}{n^2} = 2 \quad \Leftrightarrow \quad m^2 = 2n^2$$

Consequently m^2 is even and thus m is even by Proposition 3.4. So $m = 2k$ for some integer k. We then find

$$(2k)^2 = 2n^2 \quad \Leftrightarrow \quad 2k^2 = n^2$$

which implies that n is even and there exists an integer j such that $n = 2j$. However, we have assumed that m/n was in its simplest form; but we find

$$\sqrt{2} = \frac{m}{n} = \frac{2k}{2j} = \frac{k}{j}$$

a contradiction. Thus we conclude that $\sqrt{2}$ cannot be written as a quotient of integers. □

The phrase *"without loss of generality"* (often abbreviated as *"w.l.o.g."*) is used in cases when a general situation can be easily reduced to some special case which simplifies our arguments. In this example we just have to cancel out common divisors.

Proof by Induction

Induction is a very powerful technique. It is applied when we have an infinite number of statements $A(n)$ indexed by natural numbers. It is based on the following theorem.

Theorem 3.6 **Principle of mathematical induction.** Let $A(n)$ be an infinite collection of statements with $n \in \mathbb{N}$. Suppose that

(i) $A(1)$ is true, and

(ii) $A(k) \Rightarrow A(k+1)$ for all $k \in \mathbb{N}$.

Then $A(n)$ is true for all $n \in \mathbb{N}$.

PROOF. Suppose that the statement does not hold for all n. Let j be the smallest natural number such that $A(j)$ is false. By assumption (i) we have $j > 1$ and thus $j - 1 \geq 1$. Note that $A(j-1)$ is true as j is the smallest possible. Hence assumption (ii) implies that $A(j)$ is true, a contradiction. □

When we apply the induction principle the following terms are useful.

- Checking condition (i) is called the **base step**.
- Checking condition (ii) is called the **induction step**.
- Assuming that $A(k)$ is true for some k is called the **induction hypothesis**.

Proposition 3.7

Let $q \in \mathbb{R}$, $q \not\in \{0,1\}$, and $n \in \mathbb{N}$ Then

$$\sum_{j=0}^{n-1} q^j = \frac{1-q^n}{1-q}$$

PROOF. For a fixed $q \in \mathbb{R}$ this statement is indexed by natural numbers. So we prove the statement by induction.

Base step: Obviously the statement is true for $n = 1$.

Induction step: We assume by the induction hypothesis that the statement is true for $n = k$, i.e.,

$$\sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q} .$$

We have to show that the statement also holds for $n = k+1$. We find

$$\sum_{j=0}^{k} q^j = \sum_{j=0}^{k-1} q^j + q^k = \frac{1-q^k}{1-q} + q^k = \frac{1-q^k}{1-q} + \frac{(1-q)q^k}{1-q} = \frac{1-q^{k+1}}{1-q}$$

Thus by the Principle of Mathematical Induction the statement is true for all $n \in \mathbb{N}$. □

Proof by Cases

It is often useful to break a given problem into cases and tackle each of these individually.

Proposition 3.8

Triangle inequality. Let a and b be real numbers. Then

$$|a+b| \le |a| + |b|$$

PROOF. We break the problem into four cases where a and b are positive and negative, respectively.

Case 1: $a \ge 0$ and $b \ge 0$. Then $a+b \ge 0$ and we find $|a+b| = a+b = |a|+|b|$.

Case 2: $a < 0$ and $b < 0$. Now we have $a+b < 0$ and $|a+b| = -(a+b) = (-a)+(-b) = |a|+|b|$.

Case 3: Suppose one of a and b is positive and the other negative. W.l.o.g. we assume $a < 0$ and $b \ge 0$. (Otherwise reverse the rôles of a and b.) Notice that $x \le |x|$ for all x. We have the following to subcases:
Subcase (a): $a+b > 0$ and we find $|a+b| = a+b \le |a|+|b|$.
Subcase (b): $a+b < 0$ and we find $|a+b| = -(a+b) = (-a)+(-b) \le |-a| + |-b| = |a|+|b|$.

This completes the proof. □

Counterexample

A **counterexample** is an example where a given statement does not hold. It is sufficient to find *one* counterexample to disprove a conjecture. Of course it is not sufficient to give just one example to prove a conjecture.

Reading Proofs

Proofs are often hard to read. When reading or verifying a proof keep the following in mind:

- Break into pieces.
- Draw pictures.
- Find places where the assumptions are used.
- Try extreme examples.
- Apply to a non-example: Where does the proof fail?

Mathematicians seem to like the word **trivial** which means *self-evident* or *being the simplest possible case*. Make sure that the argument really is evident for you[1].

3.5 Why Should We Deal With Proofs?

The great advantage of mathematics is that one can assess the truth of a statement by studying its proof. Truth is not determined by a higher authority who says “because I say so”. (On the other hand, it is you that has to check the proofs given by your lecturer. Copying a wrong proof from the blackboard is your fault. In mathematics the incantation “But it has been written down by the lecturer” does not work.)

Proofs help us to gain confidence in the truth of our statements.

Another reason is expressed by Ludwig Wittgenstein: *Beweise reinigen die Begriffe.* We learn something about the mathematical objects.

3.6 Finding Proofs

The only way to determine the truth or falsity of a mathematical statement is with a mathematical proof. Unfortunately, finding proofs is not always easy.
M. Sipser[2]. has the following tips for producing a proof:

[1]Nasty people say that *trivial* means: “I am confident that the proof for the statement is easy but I am too lazy to write it down.”

[2]See Sect. 0.3 in Michael Sipser (2006), *Introduction to the Theory of Computation*, 2nd international edition, Course Technology.

- *Find examples.* Pick a few examples and observe the statement in action. Draw pictures. Look at extreme examples and non-examples. See what happens when you try to find counterexamples.
- *Be patient.* Finding proofs takes times. If you do not see how to do it right away, do not worry. Researchers sometimes work for weeks or even years to find a single proof.
- *Come back to it.* Look over the statement you want to prove, think about it a bit, leave it, and then return a few minutes or hours later. Let the unconscious, intuitive part of your mind have a chance to work.
- *Try special cases.* If you are stuck trying to prove your statement, try something easier. Attempt to prove a special case first. For example, if you cannot prove your statement for every $n \geq 1$, first try to prove it for $k = 1$ and $k = 2$.
- *Be neat.* When you are building your intuition for the statement you are trying to prove, use simple, clear pictures and/or text. Sloppiness gets in the way of insight.
- *Be concise.* Brevity helps you express high-level ideas without getting lost in details. Good mathematical notation is useful for expressing ideas concisely.

3.7 When You Write Down Your Own Proof

When you believe that you have found a proof, you must write it up properly. View a proof as a kind of debate. It is you who has to *convince* your readers that your statement is indeed true. A well-written proof is a sequence of statements, wherein each one follows by simple reasoning from previous statements in the sequence. All your reasons you may use must be axioms, definitions, or theorems that your reader already accepts to be true.

Keep in mind that a proof is not just a collection of computations. These are means for the purpose of demonstration and thus require explanation.

When you make use of a theorem you may explicitly refer to it. This is in particular required if you use a previous result in a sequence of lemmata in order to help your reader to understanding your arguments. It is also necessary if your result is based on propositions beyond the fundamental theorems in the area of research. In Mathematics, however, it is usual to refer to the proposition, paper or book, rather than to quote the text of that proposition verbatim.

Since a proof is a kind of debate you should use complete sentences in consideration of grammar, syntax and usage of punctuation marks. For the sake of readability sentences should not start with a symbol and mathematical expressions may be separated by commata.

— Summary

- Mathematical papers have the structure *"Definition – Theorem – Proof"*.
- A theorem consists of an *assumption* or hypothesis and a *conclusion*.
- We distinguish between *necessary* and *sufficient* conditions.
- *Examples* illustrate a notion or a statement. A good example shows a typical property; extreme examples and non-examples demonstrate special aspects of a result. An example *does not* replace a proof.
- *Proofs* verify theorems. They only use definitions and statements that have already be shown true.
- There are some techniques for proving a theorem which may (or may not) work: *direct proof, indirect proof, proof by contradiction, proof by induction, proof cases.*
- Wrong conjectures may be disproved by *counterexamples*.
- When reading definitions, theorems or proofs: find examples, draw pictures, find assumptions and conclusions.

— Problems

3.1 Consider the following student's proof of the proposition: Let $x, y \in \mathbb{R}$. Then $xy \le \frac{x^2}{2} + \frac{y^2}{2}$.

PROOF (STUDENT VERSION):

$$\begin{aligned} (x-y)^2 &\ge 0 \\ x^2 - 2xy + y^2 &\ge 0 \\ x^2 + y^2 &\ge 2xy \\ \frac{x^2}{2} + \frac{y^2}{2} &\ge xy \quad \square \end{aligned}$$

What is the problem with this version of the proof. Rewrite the proof.

3.2 Consider the following statement:

> Suppose that a, b, c and d are real numbers. If $ab = cd$ and $a = c$, then $b = d$.
>
> Proof: We have
>
> $$\begin{aligned} & ab = cd \\ \Leftrightarrow\ & ab = ad, \text{ as } a = c, \\ \Leftrightarrow\ & b = d, \text{ by cancellation.} \quad \square \end{aligned}$$

Unfortunately, this statement is false. Where is the mistake? Find a counterexample. Fix the proposition, i.e., change the statement such that it becomes true.

3.3 Prove that the square root of 3 is irrational, i.e., it cannot be written in form m/n where m and n are integers.

HINT: Use the same idea as in the proof of Proposition 3.5.

3.4 Suppose one uses the same idea as in the proof of Proposition 3.5 to show that the square root of 4 is irrational. Where does the proof fail?

3.5 Prove by induction that

$$\sum_{j=1}^{n} j = \frac{1}{2} n(n+1) .$$

3.6 The binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It also can be computed by $\binom{n}{0} = \binom{n}{n} = 1$ and the following recursion:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \quad \text{for } k = 0, \ldots, n-1$$

This recursion can be illustrated by Pascal's triangle:

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Prove this recursion by a direct proof.

HINT: Use the recursion from Problem 3.6.

3.7 Prove the binomial theorem by induction:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

3.8 Consider the following "proposition":

> All elements of a finite set $\{a_1,\ldots,a_n\}$ of real numbers are equal.

It is obviously false as we would have the corollary:

> All students at WU Wien have the same height.

So the following cannot be a correct proof of the false proposition.

NOT-A-PROOF: We use the induction principle.

Base step $n = 1$: The statement is trivially true for set with exactly one element.

Induction step: Assume the statement holds for any set of cardinality n. We show that it then also holds for a set $\{a_1,\ldots,a_{n+1}\}$ of cardinality $n+1$. By the induction hypothesis all elements in $\{a_1,\ldots,a_n\}$ and $\{a_2,\ldots,a_{n+1}\}$, resp., are equal. In particular all elements in the first subset are equal to a_1 and all elements in the second subset are equal to a_{n+1}. Now let a_j be an element in both subsets. Then we find $a_1 = a_j = a_{n+1}$ and consequently all elements in $\{a_1,\ldots,a_{n+1}\}$ are equal. Thus the result follows by induction. □

3.9 Proof by induction:
Let $a_i \in \mathbb{R}$ for $i = 1,\ldots,n$. Then

$$\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i| \,.$$

3.10 Consider the following statement:

Let $a, b \in \mathbb{R}$ such that $a = b$. Then the following holds:

$$
\begin{array}{rll}
 & a = b & \big| \cdot a \\
\Leftrightarrow & a^2 = ab & \big| + (a^2 - 2ab) \\
\Leftrightarrow & a^2 + (a^2 - 2ab) = ab + (a^2 - 2ab) & \big| \text{simplify} \\
\Leftrightarrow & 2(a^2 - ab) = a^2 - ab & \big| : (a^2 - ab) \\
\Leftrightarrow & 2 = 1 &
\end{array}
$$

Obviously, this computation is false. Where is the mistake?

3.11 *Odd Pie Fights*. An *odd* number of people are standing in the plane, there mutual distances distinct. At a signal each person will throw a banana cream pie at his or her nearest neighbor[3].

Prove that at least one person does not get hit with a pie.
Hint: You have to do induction on *odd* numbers only. Also observe that there are always two persons with shortest distance among all possible distances between pairs of people.

Does the statement still hold if there are an even number of people? (Prove or disprove)

Does the statement still hold if there are an odd number of people but where the distances need not be distinct? (Prove or disprove)

[3]Think of an old Laurel and Hardy movie.

4

Matrix Algebra

We want to cope with rows *and* columns.

4.1 Matrix and Vector

Definition 4.1

An $m \times n$ **matrix** (pl. **matrices**) is a rectangular array of mathematical expressions (e.g., numbers) that consists of m rows and n columns. We write

$$\mathbf{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

We use bold upper case letters to denote matrices and corresponding lower case letters for their entries. For example, the entries of matrix $\mathbf{A}$ are denote by a_{ij}. In addition, we also use the symbol $[\mathbf{A}]_{ij}$ to denote the entry of $\mathbf{A}$ in row i and column j.

Definition 4.2

A **column vector** (or *vector* for short) is a matrix that consists of a single column, i.e., an $n \times 1$ matrix. We write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Definition 4.3

A **row vector** is a matrix that consists of a single row. We write

$$\mathbf{x}' = (x_1, x_2, \dots, x_m).$$

We use bold lower case letters to denote vectors. Symbol $\mathbf{e}_k$ denotes a column vector that has zeros everywhere except for a one in the kth position.

The set of all (column) vectors of length n with entries in $\mathbb{R}$ is denoted by $\mathbb{R}^n$. The set of all $m \times n$ matrices with entries in $\mathbb{R}$ is denoted by $\mathbb{R}^{m\times n}$.

It is convenient to write $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ to denote a matrix with *column* vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$. We write $\mathbf{A} = \begin{pmatrix} \mathbf{a}'_1 \\ \vdots \\ \mathbf{a}'_m \end{pmatrix}$ to denote a matrix with *row* vectors $\mathbf{a}'_1, \dots, \mathbf{a}'_m$.

Definition 4.4 An $n \times n$ matrix is called a **square matrix**.

Definition 4.5 An $m \times n$ matrix where all entries are 0 is called a **zero matrix**. It is denoted by $\mathbf{0}_{nm}$.

Definition 4.6 An $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere is called **identity matrix**. It is denoted by $\mathbf{I}_n$.

We simply write $\mathbf{0}$ and $\mathbf{I}$, respectively, if the size of $\mathbf{0}_{nm}$ and $\mathbf{I}_n$ can be determined by the context.

Definition 4.7 The **Kronecker delta** δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

It is a convenient symbol that indicates that an expression vanishes[1] if two numbers are distinct. For example, the identity matrix can be defined as

$$[\mathbf{I}]_{ij} = \delta_{ij} .$$

Definition 4.8 A **diagonal matrix** is a square matrix in which all entries outside the main diagonal are all zero. The diagonal entries themselves may or may not be zero. Thus, the $n \times n$ matrix D is diagonal if $d_{ij} = 0$ whenever $i \neq j$. We denote a diagonal matrix with entries $x_1, \dots, x_n$ by $\operatorname{diag}(x_1, \dots, x_n)$.

Definition 4.9 An **upper triangular matrix** is a square matrix in which all entries *below* the main diagonal are all zero. Thus, the $n \times n$ matrix U is an upper triangular matrix if $u_{ij} = 0$ whenever $i > j$.

Notice that identity matrices and square zero matrices are examples for both diagonal matrices and upper triangular matrices.

[1]That is, the expression is 0.

4.2 Matrix Algebra

Definition 4.10

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal**, $\mathbf{A} = \mathbf{B}$, if they have the same number of rows and columns and

$$a_{ij} = b_{ij}.$$

Definition 4.11

Let $\mathbf{A}$ and $\mathbf{B}$ be two $m \times n$ matrices. Then the **sum** $\mathbf{A}+\mathbf{B}$ is the $m \times n$ matrix with elements

$$[\mathbf{A}+\mathbf{B}]_{ij} = [\mathbf{A}]_{ij} + [\mathbf{B}]_{ij} = a_{ij} + b_{ij}.$$

That is, **matrix addition** is performed element-wise.

Definition 4.12

Let $\mathbf{A}$ be an $m \times n$ matrix and $\alpha \in \mathbb{R}$. Then we define $\alpha\mathbf{A}$ by

$$[\alpha\mathbf{A}]_{ij} = \alpha[\mathbf{A}]_{ij} = \alpha\, a_{ij}.$$

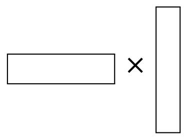
That is, **scalar multiplication** is performed element-wise.

Definition 4.13

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $n \times k$ matrix. Then the **matrix product** $\mathbf{A}\cdot\mathbf{B}$ is the $m \times k$ matrix with elements defined as

$$[\mathbf{A}\cdot\mathbf{B}]_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}.$$

That is, **matrix multiplication** is performed by multiplying *rows by columns*.



Theorem 4.14

Rules for matrix addition and multiplication.
Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, matrices of appropriate size. Then

(1) $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$

(2) $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$

(3) $\mathbf{A}+\mathbf{0} = \mathbf{A}$

(4) $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$

(5) $\mathbf{I}_m\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{I}_n = \mathbf{A}$

(6) $(\alpha\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(\alpha\mathbf{B}) = \alpha(\mathbf{A}\cdot\mathbf{B})$

(7) $\mathbf{C}\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{C}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{B}$

(8) $(\mathbf{A}+\mathbf{B})\cdot\mathbf{D} = \mathbf{A}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{D}$

PROOF. See Problem 4.7.

Notice: **In general matrix multiplication is not commutative!**

$$\mathbf{AB} \neq \mathbf{BA}$$

4.3 Transpose of a Matrix

Definition 4.15

The **transpose** of an $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix $\mathbf{A}'$ (also denote by $\mathbf{A}^t$, $\mathbf{A}^{\mathrm{T}}$, or $\mathbf{A}^{\top}$) defined as

$$[\mathbf{A}']_{ij} = [\mathbf{A}]_{ji} = a_{ji} .$$

Lemma 4.16

We trivially find for $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}' .$$

Theorem 4.17

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $n \times k$ matrix. Then

(1) $\mathbf{A}'' = \mathbf{A}$,

(2) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

PROOF. See Problem 4.15.

Definition 4.18

A square matrix $\mathbf{A}$ is called **symmetric** if $\mathbf{A}' = \mathbf{A}$.

4.4 Inverse Matrix

Definition 4.19

A square matrix $\mathbf{A}$ is called **invertible** if there exists a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} .$$

Matrix $\mathbf{A}^{-1}$ is then called the **inverse matrix** of $\mathbf{A}$.
$\mathbf{A}$ is called **singular** if such a matrix does not exist.

Theorem 4.20

Let $\mathbf{A}$ be an invertible matrix. Then its inverse $\mathbf{A}^{-1}$ is uniquely defined.

PROOF. See Problem 4.18.

Theorem 4.21

Let $\mathbf{A}$ and $\mathbf{B}$ be two invertible matrices of the same size. Then $\mathbf{AB}$ is invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} .$$

PROOF. See Problem 4.19.

Theorem 4.22

Let $\mathbf{A}$ be an invertible matrix. Then the following holds:

(1) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

(2) $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$

PROOF. See Problem 4.20.

4.5 Block Matrix

Suppose we are given some vector $\mathbf{x} = (x_1, \dots, x_n)'$. It may happen that we naturally can distinguish between two types of variables (e.g., endogenous and exogenous variables) which we can group into two respective vectors $\mathbf{x}_1 = (x_1, \dots, x_{n_1})'$ and $\mathbf{x}_2 = (x_{n_1+1}, \dots, x_{n_1+n_2})'$ where $n_1 + n_2 = n$. We then can write

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}.$$

Assume further that we are also given some $m \times n$ Matrix $\mathbf{A}$ and that the components of vector $\mathbf{y} = \mathbf{A}\mathbf{x}$ can also be partitioned into two groups

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}$$

where $\mathbf{y}_1 = (y_1, \dots, x_{m_1})'$ and $\mathbf{y}_2 = (y_{m_1+1}, \dots, y_{m_1+m_2})'$. We then can partition $\mathbf{A}$ into four matrices

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

where $\mathbf{A}_{ij}$ is a submatrix of dimension $m_i \times n_j$. Hence we immediately find

$$\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{x}_2 \\ \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{x}_2 \end{pmatrix}.$$

Definition 4.23

Matrix $\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$ is called a **partitioned matrix** or **block matrix**.

Example 4.24

Matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{pmatrix}$ can be partitioned in numerous ways, e.g.,

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} = \left(\begin{array}{cc|ccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \end{array}\right). \qquad \diamondsuit$$

Of course a matrix can be partitioned into more than 2×2 submatrices. Sometimes there is no natural reason for such a block structure but it might be convenient for further computations.

We can perform operations on block matrices in an obvious way, that is, we treat the submatrices as of they where ordinary matrix elements. For example, we find for block matrices with appropriate submatrices,

$$\alpha \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} = \begin{pmatrix} \alpha\mathbf{A}_{11} & \alpha\mathbf{A}_{12} \\ \alpha\mathbf{A}_{21} & \alpha\mathbf{A}_{22} \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}+\mathbf{B}_{11} & \mathbf{A}_{12}+\mathbf{B}_{12} \\ \mathbf{A}_{21}+\mathbf{B}_{21} & \mathbf{A}_{22}+\mathbf{B}_{22} \end{pmatrix},$$

and

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{C}_{11}+\mathbf{A}_{12}\mathbf{C}_{21} & \mathbf{A}_{11}\mathbf{C}_{12}+\mathbf{A}_{12}\mathbf{C}_{22} \\ \mathbf{A}_{21}\mathbf{C}_{11}+\mathbf{A}_{22}\mathbf{C}_{21} & \mathbf{A}_{21}\mathbf{C}_{12}+\mathbf{A}_{22}\mathbf{C}_{22} \end{pmatrix}.$$

We also can use the block structure to compute the inverse of a partitioned matrix. Assume that a matrix is partitioned as $(n_1+n_2)\times(n_1+n_2)$ matrix $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$. Here we only want to look at the special case where $\mathbf{A}_{21} = \mathbf{0}$, i.e.,

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}.$$

We then have to find a block matrix $\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$ such that

$$\mathbf{AB} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{22}\mathbf{B}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_2} \end{pmatrix} = \mathbf{I}_{n_1+n_2}.$$

Thus if $\mathbf{A}_{22}^{-1}$ exists the second row implies that $\mathbf{B}_{21} = \mathbf{0}_{n_2 n_1}$ and $\mathbf{B}_{22} = \mathbf{A}_{22}^{-1}$. Furthermore, $\mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} = \mathbf{I}$ implies $\mathbf{B}_{11} = \mathbf{A}_{11}^{-1}$. At last, $\mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} = \mathbf{0}$ implies $\mathbf{B}_{12} = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}$. Hence we find

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{pmatrix}.$$

— Summary

- A *matrix* is a rectangular array of mathematical expressions.
- Matrices can be added and multiplied by a scalar componentwise.
- Matrices can be multiplied by multiplying rows by columns.
- Matrix addition and multiplication satisfy all rules that we expect for such operations *except* that matrix multiplication is *not commutative*.
- The zero matrix $\mathbf{0}$ is the neutral element of matrix addition, i.e., $\mathbf{0}$ plays the same role as 0 for addition of real numbers.
- The identity zero matrix $\mathbf{I}$ is the neutral element of matrix multiplication, i.e., $\mathbf{I}$ plays the same role as 1 for multiplication of real numbers.
- There is no such thing as division of matrices. Instead one can use the inverse matrix, which is the matrix analog to the reciprocal of a number.
- A matrix can be partitioned. Thus one obtains a block matrix.

— Exercises

4.1 Let

$$\mathbf{A} = \begin{pmatrix} 1 & -6 & 5 \\ 2 & 1 & -3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 4 & 3 \\ 8 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$

Compute

(a) $\mathbf{A}+\mathbf{B}$ (b) $\mathbf{A}\cdot\mathbf{B}$ (c) $3\mathbf{A}'$ (d) $\mathbf{A}\cdot\mathbf{B}'$

(e) $\mathbf{B}'\cdot\mathbf{A}$ (f) $\mathbf{C}+\mathbf{A}$ (g) $\mathbf{C}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{B}$ (h) $\mathbf{C}^2$

4.2 Demonstrate by means of the two matrices $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$, that matrix multiplication is not commutative in general, i.e., we may find $\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$.

4.3 Let $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$.

Compute $\mathbf{x}'\mathbf{x}$, $\mathbf{x}\mathbf{x}'$, $\mathbf{x}'\mathbf{y}$, $\mathbf{y}'\mathbf{x}$, $\mathbf{x}\mathbf{y}'$ and $\mathbf{y}\mathbf{x}'$.

4.4 Let $\mathbf{A}$ be a 3×2 matrix, $\mathbf{C}$ be a 4×3 matrix, and $\mathbf{B}$ a matrix, such that the multiplication $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$ is possible. How many rows and columns must $\mathbf{B}$ have? How many rows and columns does the product $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$ have?

4.5 Compute $\mathbf{X}$. Assume that all matrices are square matrices and all required inverse matrices exist.

(a) $\mathbf{AX}+\mathbf{BX}=\mathbf{CX}+\mathbf{I}$ (b) $(\mathbf{A}-\mathbf{B})\mathbf{X}=-\mathbf{BX}+\mathbf{C}$

(c) $\mathbf{AXA}^{-1}=\mathbf{B}$ (d) $\mathbf{XAX}^{-1}=\mathbf{C}(\mathbf{XB})^{-1}$

4.6 Use partitioning and compute the inverses of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 0 & 5 & 6 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

— Problems

4.7 Prove Theorem 4.14.
Which conditions on the size of the respective matrices must be satisfied such that the corresponding computations are defined?

HINT: Show that corresponding entries of the matrices on either side of the equations coincide. Use the formulæ from Definitions 4.11, 4.12 and 4.13.

4.8 Show that the product of two diagonal matrices is again a diagonal matrix.

4.9 Show that the product of two upper triangular matrices is again an upper triangular matrix.

4.10 Show that the product of a diagonal matrix and an upper triangular matrices is an upper triangular matrix.

4.11 Let $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ be an $m \times n$ matrix.

(a) What is the result of $\mathbf{A}\mathbf{e}_k$?
(b) What is the result of $\mathbf{AD}$ where $\mathbf{D}$ is an $n \times n$ diagonal matrix?

Prove your claims!

4.12 Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}'_1 \\ \vdots \\ \mathbf{a}'_m \end{pmatrix}$ be an $m \times n$ matrix.

(a) What is the result of $\mathbf{e}'_k\mathbf{A}$.
(b) What is the result of $\mathbf{DA}$ where $\mathbf{D}$ is an $m \times m$ diagonal matrix?

Prove your claims!

4.13 Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $n \times n$ matrix where $b_{kl} = 1$ for fixed $1 \le k, l \le n$ and $b_{ij} = 0$ for $i \ne k$ or $j \ne l$. What is the result of $\mathbf{AB}$? Prove your claims!

4.14 Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $m \times m$ matrix where $b_{kl} = 1$ for fixed $1 \le k, l \le m$ and $b_{ij} = 0$ for $i \ne k$ or $j \ne l$. What is the result of $\mathbf{BA}$? Prove your claims!

4.15 Prove Theorem 4.17.

HINT: (2) Compute the matrices on either side of the equation and compare their entries.

HINT: Use Theorem 4.17.

4.16 Let $\mathbf{A}$ be an $m \times n$ matrix. Show that both $\mathbf{AA}'$ and $\mathbf{A}'\mathbf{A}$ are symmetric.

4.17 Let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^n$. Show that matrix $\mathbf{G}$ with entries $g_{ij} = \mathbf{x}'_i\mathbf{x}_j$ is symmetric.

4.18 Prove Theorem 4.20.

HINT: Assume that there exist two inverse matrices $\mathbf{B}$ and $\mathbf{C}$. Show that they are equal.

4.19 Prove Theorem 4.21.

4.20 Prove Theorem 4.22.

HINT: Use Definition 4.19 and apply Theorems 4.21 and 4.17. Notice that $\mathbf{I}^{-1} = \mathbf{I}$ and $\mathbf{I}' = \mathbf{I}$. (Why is this true?)

4.21 Compute the inverse of $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$.
(Explain all intermediate steps.)

4.22 A **stochastic vector** (or **probability vector**) $\mathbf{p} \in \mathbb{R}^n$ is a vector with non-negative entries that sum up to one. That is,

(i) $p_i \geq 0$ for all $i = 1, \dots, n$, and

(ii) $\sum_{i=1}^{n} p_i = 1$.

A square matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ is called a (left) **stochastic matrix** if all its columns are stochastic vectors.

(a) Give a formal definition of a *stochastic matrix* (without a reference to the term *stochastic vector*).

(b) Show: Let $\mathbf{p} \in \mathbb{R}^n$ be a stochastic vector and $\mathbf{S} \in \mathbb{R}^{n \times n}$ be a stochastic matrix. Then $\mathbf{S}\mathbf{p}$ is again a stochastic vector.

(c) Show: Let $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{n \times n}$ be stochastic matrices. Then $\mathbf{S}\mathbf{T}$ is again a stochastic matrix.

(d) Show: If $\mathbf{A}\mathbf{p}$ is a stochastic vector for every stochastic vector $\mathbf{p}$, then $\mathbf{A}$ is a stochastic matrix.
(You may find Problem 4.11 useful.)

5
Vector Space

We want to master the concept of linearity.

5.1 Linear Space

Definition 5.1

In Chapter 4 we have introduced addition and scalar multiplication of vectors. Both are performed element-wise. We again obtain a vector of the same length. We thus say that the set of all real vectors of length n,

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R}, i = 1, \dots, n \right\}$$

is **closed** under vector addition and scalar multiplication.

In mathematics we find many structures which possess this nice property.

Example 5.2

Let $\mathscr{P} = \left\{\sum_{i=0}^{k} a_i x^i : k \in \mathbb{N},\ a_i \in \mathbb{R}\right\}$ be the set of all polynomials. Then we define a scalar multiplication and an addition on $\mathscr{P}$ by

- $(\alpha p)(x) = \alpha p(x)$ for $p \in \mathscr{P}$ and $\alpha \in \mathbb{R}$,
- $(p_1 + p_2)(x) = p_1(x) + p_2(x)$ for $p_1, p_2 \in \mathscr{P}$.

Obviously, the result is again a polynomial and thus an element of $\mathscr{P}$, i.e., the set $\mathscr{P}$ is closed under scalar multiplication and addition. $\diamondsuit$

Definition 5.3

Vector Space. A **vector space** is any nonempty set of objects that is *closed* under *scalar multiplication* and *addition*.

Of course in mathematics the meanings of the words *scalar multiplication* and *addition* needs a clear and precise definition. So we also give a formal definition:

Definition 5.4

A (real) **vector space** is an object $(\mathcal{V},+,\cdot)$ that consists of a nonempty set $\mathcal{V}$ together with two functions $+\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}, (\mathbf{u},\mathbf{v})\mapsto \mathbf{u}+\mathbf{v}$, called *addition*, and $\cdot\colon \mathbb{R}\times\mathcal{V}\to\mathcal{V}, (\alpha,\mathbf{v})\mapsto \alpha\cdot\mathbf{v}$, called *scalar multiplication*, with the following properties:

(i) $\mathbf{v}+\mathbf{u}=\mathbf{u}+\mathbf{v}$, for all $\mathbf{u},\mathbf{v}\in\mathcal{V}$. (Commutativity)

(ii) $\mathbf{v}+(\mathbf{u}+\mathbf{w})=(\mathbf{v}+\mathbf{u})+\mathbf{w}$, for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathcal{V}$. (Associativity)

(iii) There exists an element $0\in\mathcal{V}$ such that $0+\mathbf{v}=\mathbf{v}+0=\mathbf{v}$, for all $\mathbf{v}\in\mathcal{V}$. (Identity element of addition)

(iv) For every $\mathbf{v}\in\mathcal{V}$, there exists an $\mathbf{u}\in\mathcal{V}$ such that $\mathbf{v}+\mathbf{u}=\mathbf{u}+\mathbf{v}=0$. (Inverse element of addition)

(v) $\alpha(\mathbf{v}+\mathbf{u})=\alpha\mathbf{v}+\alpha\mathbf{u}$, for all $\mathbf{v},\mathbf{u}\in\mathcal{V}$ and all $\alpha\in\mathbb{R}$. (Distributivity)

(vi) $(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}$, for all $\mathbf{v}\in\mathcal{V}$ and all $\alpha,\beta\in\mathbb{R}$. (Distributivity)

(vii) $\alpha(\beta\mathbf{v})=(\alpha\beta)\mathbf{v}=\beta(\alpha\mathbf{v})$, for all $\mathbf{v}\in\mathcal{V}$ and all $\alpha,\beta\in\mathbb{R}$.

(viii) $1\mathbf{v}=\mathbf{v}$, for all $\mathbf{v}\in\mathcal{V}$, where $1\in\mathbb{R}$. (Identity element of scalar multiplication)

We write vector space $\mathcal{V}$ for short, if there is no risk of confusion about addition and scalar multiplication.

Example 5.5

It is easy to check that $\mathbb{R}^n$ and the set $\mathcal{P}$ of polynomials in Example 5.2 form vector spaces.

Let $\mathcal{C}^0([0,1])$ and $\mathcal{C}^1([0,1])$ be the set of all continuous and continuously differentiable functions with domain $[0,1]$, respectively. Then $\mathcal{C}^0([0,1])$ and $\mathcal{C}^1([0,1])$ equipped with pointwise addition and scalar multiplication as in Example 5.2 form vector spaces.

The set $\mathcal{L}^1([0,1])$ of all integrable functions on $[0,1]$ equipped with pointwise addition and scalar multiplication as in Example 5.2 forms a vector space.

A non-example is the first hyperoctant in $\mathbb{R}^n$, i.e., the set

$$H=\{\mathbf{x}\in\mathbb{R}^n : x_i\geq 0\}\,.$$

It is not a vector space as for every $\mathbf{x}\in H\setminus\{0\}$ we find $-\mathbf{x}\notin H$. ♢

Definition 5.6

Subspace. A nonempty subset $\mathcal{S}$ of some vector space $\mathcal{V}$ is called a **subspace** of $\mathcal{V}$ if for every $\mathbf{u},\mathbf{v}\in\mathcal{S}$ and $\alpha,\beta\in\mathbb{R}$ we find $\alpha\mathbf{u}+\beta\mathbf{v}\in\mathcal{S}$.

Example 5.7

Let $\mathbf{A}\in\mathbb{R}^{m\times n}$ be an $m\times n$ matrix. Then the set

$$\mathcal{L}=\left\{\mathbf{x}\in\mathbb{R}^n : \mathbf{A}\mathbf{x}=0\right\}$$

of all solutions of the homogeneous linear equation $\mathbf{A}\mathbf{x}=0$ is a subspace of $\mathbb{R}^n$.

In order to verify this claim we take two vectors $\mathbf{x},\mathbf{y}\in\mathcal{L}$ and two scalars $\alpha,\beta\in\mathbb{R}$ and show that $\mathbf{z}=\alpha\mathbf{x}+\beta\mathbf{y}\in\mathcal{L}$. Thus we use the property that $\mathbf{A}\mathbf{x}=0$ and $\mathbf{A}\mathbf{y}=0$ (as all vectors in $\mathcal{L}$ have this property) to show that $\mathbf{Az} = 0$ which implies that $\mathbf{z} \in \mathscr{L}$ (as $\mathscr{L}$ contains all such vectors). Indeed

$$\mathbf{Az} = \mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{Ax} + \beta\mathbf{Ay} = \alpha 0 + \beta 0 = 0\,.$$

This completes the proof. $\diamond$

The fundamental property of vector spaces is that we can take some vectors and create a set of new vectors by means of so called linear combinations.

Linear combination. Let $\mathscr{V}$ be a real vector space. Let $\{\mathbf{v}_1,\dots,\mathbf{v}_k\} \subset \mathscr{V}$ be a finite set of vectors and $\alpha_1,\dots,\alpha_k \in \mathbb{R}$. Then $\sum_{i=1}^k \alpha_i \mathbf{v}_i$ is called a **linear combination** of the vectors $\mathbf{v}_1,\dots,\mathbf{v}_k$. Definition 5.8

Is is easy to check that the set of all linear combinations of some fixed vectors forms a subspace of the given vector space.

Theorem 5.9 Given a vector space $\mathscr{V}$ and a nonempty subset $S = \{\mathbf{v}_1,\dots,\mathbf{v}_k\} \subset \mathscr{V}$. Then the set of all linear combinations of the elements of S is a subspace of $\mathscr{V}$.

PROOF IDEA. Let $\mathscr{L} = \left\{\sum_{i=1}^k \alpha_i \mathbf{v}_i : \alpha_i \in \mathbb{R}\right\}$ be the set of all these linear combinations. We take two vectors $\mathbf{x},\mathbf{y} \in \mathscr{L}$ and two scalars $\gamma,\delta \in \mathbb{R}$ and verify that the linear combination $\mathbf{z} = \gamma\mathbf{x} + \delta\mathbf{y}$ is also an element of $\mathscr{L}$. For this purpose we use the property that every $\mathbf{x} \in \mathscr{L}$ is some linear combination of the vectors in S (since otherwise $\mathbf{x} \notin \mathscr{L}$). We then use the linear combinations for $\mathbf{x}$ and $\mathbf{y}$ to construct a linear combination for $\mathbf{z}$ which implies that $\mathbf{z} \in \mathscr{L}$.

PROOF. Let $\mathscr{L} = \left\{\sum_{i=1}^k \alpha_i \mathbf{v}_i : \alpha_i \in \mathbb{R}\right\}$ be the set of all linear combinations of the vectors in S. Let $\mathbf{x},\mathbf{y} \in \mathscr{L}$ and $\gamma,\delta \in \mathbb{R}$. Then there exist $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ such that $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i$ and $\mathbf{y} = \sum_{i=1}^k \beta_i \mathbf{v}_i$ and thus

$$\mathbf{z} = \gamma\mathbf{x} + \delta\mathbf{y} = \gamma \sum_{i=1}^k \alpha_i \mathbf{v}_i + \delta \sum_{i=1}^k \beta_i \mathbf{v}_i = \sum_{i=1}^k (\gamma\alpha_i + \delta\beta_i)\mathbf{v}_i$$

is a linear combination of the elements of S. Thus $\mathbf{z} \in \mathscr{L}$. Since this holds for any $\mathbf{x},\mathbf{y} \in \mathscr{L}$ and $\gamma,\delta \in \mathbb{R}$, $\mathscr{L}$ is a subspace of $\mathscr{V}$, as claimed. $\square$

Linear span. Let $\mathscr{V}$ be a vector space and $S = \{\mathbf{v}_1,\dots,\mathbf{v}_k\} \subset \mathscr{V}$ be a nonempty finite subset. Then the subspace Definition 5.10

$$\operatorname{span}(S) = \left\{\sum_{i=1}^k \alpha_i \mathbf{v}_i : \alpha_i \in \mathbb{R}\right\}$$

is referred as the *subspace spanned by* S and called **linear span** of S. For convenience we set $\operatorname{span}(\emptyset) = \{0\}$.

5.2 Basis and Dimension

Definition 5.11

Let $\mathcal{V}$ be a vector space. A subset $S \subset V$ is called a **generating set** of $\mathcal{V}$ if $\text{span}(S) = \mathcal{V}$.

Definition 5.12

A vector space $\mathcal{V}$ is said to be **finitely generated**, if there exists a finite subset S of $\mathcal{V}$ that spans $\mathcal{V}$.

In the following we will restrict our interest to finitely generated real vector spaces. We will show that the notions of a basis and of linear independence are fundamental to vector spaces.

Definition 5.13

Basis. A set S is called a **basis** of some vector space $\mathcal{V}$ if it is a minimal generating set of $\mathcal{V}$. *Minimal* means that every proper subset of S does not span $\mathcal{V}$.
The basis of the trivial vector space $\{0\}$ is the empty set $\emptyset$.

Theorem 5.14

If $\mathcal{V}$ is finitely generated, then it has a basis.

PROOF. Since $\mathcal{V}$ is finitely generated, it is spanned by some finite set S. If S is minimal, we are done. Otherwise, remove an appropriate element and obtain a new smaller set S' that spans $\mathcal{V}$. Repeat this step until the remaining set is minimal. □

Definition 5.15

Linear dependence. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a subset of some vector space $\mathcal{V}$. We say that S is **linearly independent** or the elements of S are linearly independent if for any $\alpha_i \in \mathbb{R}$, $i = 1, \dots, k$, $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$ implies $\alpha_1 = \dots = \alpha_k = 0$. The set S is called **linearly dependent**, if it is not linearly independent. That is, there exist $\alpha_i \in \mathbb{R}$ not all zero such that $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$.

Theorem 5.16

Every nonempty subset of a linearly independent set is linearly independent.

PROOF. Let $\mathcal{V}$ be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset \mathcal{V}$ be a linearly independent set. Suppose $S' \subset S$ is linearly dependent. Without loss of generality we assume that $S' = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, $m < k$ (otherwise rename the elements of S). Then there exist $\alpha_1, \dots, \alpha_m$ not all 0 such that $\sum_{i=1}^m \alpha_i \mathbf{v}_i = 0$. Set $\alpha_{m+1} = \dots = \alpha_k = 0$. Then we also have $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$, where not all α_i are zero, a contradiction to the linear independence of S.

Theorem 5.17

Every set that contains a linearly dependent set is linearly dependent.

PROOF. See Problem 5.8.

The following theorems gives us a characterization of linearly dependent sets.

Theorem 5.18

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a subset of some vector space $\mathcal{V}$. Then S is linearly dependent if and only if there exists some $\mathbf{v}_j \in S$ such that $\mathbf{v}_j = \sum_{i=1}^k \alpha_i \mathbf{v}_i$ for some $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ with $\alpha_j = 0$.

Proof. Assume that $\mathbf{v}_j = \sum_{i=1}^k \alpha_i \mathbf{v}_i$ for some $\mathbf{v}_j \in S$ such that $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ with $\alpha_j = 0$. Then $0 = (\sum_{i=1}^k \alpha_i \mathbf{v}_i) - \mathbf{v}_j = \sum_{i=1}^k \alpha'_i \mathbf{v}_i$, where $\alpha'_j = \alpha_j - 1 = -1$ and $\alpha'_i = \alpha_i$ for $i \neq j$. Thus we have a solution of $\sum_{i=1}^k \alpha'_i \mathbf{v}_i = 0$ where at least $\alpha'_j \neq 0$. But this implies that S is linearly dependent.

Now suppose that S is linearly dependent. Then we find $\beta_i \in \mathbb{R}$ not all zero such that $\sum_{i=1}^k \beta_i \mathbf{v}_i = 0$. Without loss of generality $\beta_j \neq 0$ for some $j \in \{1, \dots, k\}$. Then we find $\mathbf{v}_j = -\frac{\beta_1}{\beta_j}\mathbf{v}_1 - \cdots - \frac{\beta_{j-1}}{\beta_j}\mathbf{v}_{j-1} - \frac{\beta_{j+1}}{\beta_j}\mathbf{v}_{j+1} - \cdots - \frac{\beta_k}{\beta_j}\mathbf{v}_k = \sum_{i=1}^k \alpha_i \mathbf{v}_i$ with $\alpha_j = 0$ and $\alpha_i = -\frac{\beta_i}{\beta_j}$, as proposed. □

Theorem 5.18 can also be stated as follows: $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly dependent subset of $\mathcal{V}$ if and only if there exists a $\mathbf{v} \in S$ such that $\mathbf{v} \in \operatorname{span}(S \setminus \{\mathbf{v}\})$.

Theorem 5.19

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a linearly independent subset of some vector space $\mathcal{V}$. Let $\mathbf{u} \in \mathcal{V}$. If $\mathbf{u} \notin \operatorname{span}(S)$, then $S \cup \{\mathbf{u}\}$ is linearly independent.

Proof. See Problem 5.9.

The next two theorems provide us equivalent characterizations of a basis by means of linear independent subsets.

Theorem 5.20

Let B be a subset of some vector space $\mathcal{V}$. Then the following are equivalent:

(1) B is a basis of $\mathcal{V}$.

(2) B is linearly independent generating set of $\mathcal{V}$.

Proof idea. We use an indirect proof. If we assume that B is a basis but not linearly independent then we can find a proper subset of B that is still a generating set, a contradiction to our assumption that B is a basis and thus minimal. For the converse suppose that B is not minimal. Then we can find a vector in B that can be expressed as a linear combination of the remaining one which implies that it cannot be linearly independent.

Proof. (1)⇒(2): By Definition 5.13, B is a generating set. Suppose that $B = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent. By Theorem 5.18 there exists some $\mathbf{v} \in B$ such that $\mathbf{v} \in \operatorname{span}(B')$ where $B' = B \setminus \{\mathbf{v}\}$. Without loss of generality assume that $\mathbf{v} = \mathbf{v}_k$. Then there exist $\alpha_i \in \mathbb{R}$ such that $\mathbf{v}_k = \sum_{i=1}^{k-1} \alpha_i \mathbf{v}_i$. Now let $\mathbf{u} \in \mathcal{V}$. Since B is a basis there exist some $\beta_i \in \mathbb{R}$, such that $\mathbf{u} = \sum_{i=1}^k \beta_i \mathbf{v}_i = \sum_{i=1}^{k-1} \beta_i \mathbf{v}_i + \beta_k \sum_{i=1}^{k-1} \alpha_i \mathbf{v}_i = \sum_{i=1}^{k-1} (\beta_i + \beta_k \alpha_i) \mathbf{v}_i$. Hence $\mathbf{u} \in \operatorname{span}(B')$. Since $\mathbf{u}$ was arbitrary, we find that B' is a generating set of $\mathcal{V}$. But since B' is a proper subset of B, B cannot be minimal, a contradiction to the minimality of a basis.

(2)⇒(1): Let B be a linearly independent generating set of $\mathcal{V}$. Suppose that B is not minimal. Then there exists a proper subset $B' \subset B$ such that $\operatorname{span}(B') = \mathcal{V}$. But then we find for every $\mathbf{x} \in B \setminus B'$ that $\mathbf{x} \in \operatorname{span}(B')$ and thus B cannot be linearly independent by Theorem 5.18, a contradiction. Hence B must be minimal as claimed. □

Theorem 5.21

Let B be a subset of some vector space $\mathcal{V}$. Then the following are equivalent:

(1) B is a basis of $\mathcal{V}$.

(3) B is maximal linearly independent set of $\mathcal{V}$. *Maximal* means that every proper superset of B (i.e., a set that contains B as a proper subset) is linearly dependent.

PROOF IDEA. We use the equivalent property (2) from Theorem 5.20. We first assume that B is a basis but not maximal. Then a larger linearly independent subset exists which implies that B cannot be a generating set. For the converse statement assume that B is linearly independent but not a generating set. However, this implies that B is not maximal, again a contradiction.

PROOF. (1)⇒(3): Assume that B is a basis, i.e., B is a linearly independent generating set of $\mathcal{V}$ by Theorem 5.20. Now suppose that there exists an $\mathbf{x} \in \mathcal{V}$ such that $B \cup \{\mathbf{x}\}$ is linearly independent. But then $\mathbf{x} \notin \operatorname{span}(B)$ and B were not a generating set, a contradiction. Hence B is a maximal linearly independent subset.

(3)⇒(1): For the contrary assume that B is a maximal linearly independent subset of $\mathcal{V}$. If B were not a generating set, then there exists an $\mathbf{x} \in \mathcal{V} \setminus \operatorname{span}(B)$ and consequently $B \cup \{\mathbf{x}\}$ is linearly independent by Theorem 5.19, a contradiction to our assumption that B is maximal. Hence B is a generating set and thus a basis of $\mathcal{V}$ by Theorem 5.20. □

Theorem 5.22

Steinitz exchange theorem (Austauschsatz). Let B_1 and B_2 be two bases of some vector space $\mathcal{V}$. If there is an $\mathbf{x} \in B_1 \setminus B_2$ then there exists a $\mathbf{y} \in B_2 \setminus B_1$ such that $(B_1 \cup \{\mathbf{y}\}) \setminus \{\mathbf{x}\}$ is a basis of $\mathcal{V}$.

This theorem tells us that we can replace vectors in B_1 by some vectors in B_2.

PROOF. Let $B_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and assume without loss of generality that $\mathbf{x} = \mathbf{v}_1$. (Otherwise rename the elements of B_1.) As B_1 is a basis it is linearly independent by Theorem 5.20. By Theorem 5.16, $B_1 \setminus \{\mathbf{v}_1\}$ is also linearly independent and thus it cannot be a basis of $\mathcal{V}$ by Theorem 5.21. Hence it cannot be a generating set by Theorem 5.20. This implies that there exists a $\mathbf{y} \in B_2$ with $\mathbf{y} \notin \operatorname{span}(B_1 \setminus \{\mathbf{v}_1\})$, since otherwise we had $\operatorname{span}(B_2) \subseteq \operatorname{span}(B_1 \setminus \{\mathbf{v}_1\}) \neq \mathcal{V}$, a contradiction as B_2 is a basis of $\mathcal{V}$.

Now there exist $\alpha_i \in \mathbb{R}$ not all equal to zero such that $\mathbf{y} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i$. In particular $\alpha_1 \neq 0$, since otherwise $\mathbf{y} \in \operatorname{span}(B_1 \setminus \{\mathbf{v}_1\})$, a contradiction to the choice of $\mathbf{y}$. We then find

$$\mathbf{v}_1 = \frac{1}{\alpha_1}\mathbf{y} - \sum_{i=2}^{k} \frac{\alpha_i}{\alpha_1}\mathbf{v}_i .$$

Similarly for every $\mathbf{z} \in \mathcal{V}$ there exist $\beta_j \in \mathbb{R}$ such that $\mathbf{z} = \sum_{j=1}^{k} \beta_j \mathbf{v}_j$. Consequently,

$$\begin{aligned}
\mathbf{z} = \sum_{j=1}^{k} \beta_j \mathbf{v}_j = \beta_1 \mathbf{v}_1 + \sum_{j=2}^{k} \beta_j \mathbf{v}_j &= \beta_1 \left(\frac{1}{\alpha_1}\mathbf{y} - \sum_{i=2}^{k} \frac{\alpha_i}{\alpha_1}\mathbf{v}_i \right) + \sum_{j=2}^{k} \beta_j \mathbf{v}_j \\
&= \frac{\beta_1}{\alpha_1}\mathbf{y} + \sum_{j=2}^{k} \left(\beta_j - \frac{\beta_1}{\alpha_1}\alpha_j \right) \mathbf{v}_j
\end{aligned}$$

that is, $(B_1 \cup \{\mathbf{y}\}) \setminus \{\mathbf{x}\} = \{\mathbf{y}, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a generating set of $\mathcal{V}$. By our choice of $\mathbf{y}$ and Theorem 5.19 this set is linearly independent. Thus it is a basis by Theorem 5.20. This completes the proof. □

Theorem 5.23 Any two bases B_1 and B_2 of some finitely generated vector space $\mathcal{V}$ have the same size.

Proof. We first show that $|B_1| \leq |B_2|$. Suppose that $B_1 \setminus B_2$ is nonempty. Then by Theorem 5.22 we can construct a new basis B_1' by replacing a vector in $B_1 \setminus B_2$ by some vector in $B_2 \setminus B_1$. Obviously the new basis B_1' has the same number of elements as $|B_1|$, i.e., $|B_1'| = |B_1|$, and $|B_1' \setminus B_2| = |B_1 \setminus B_2| - 1$. If $B_1' \setminus B_2 \neq \emptyset$, then we repeat this procedure until we obtain a basis B_1^* with $B_1^* \setminus B_2 = \emptyset$ which implies $B_1^* \subseteq B_2$. Since $\mathcal{V}$ is finitely generated, B_1 only has a finite number of elements which guarantees that we eventually obtain such a basis B_1^*. Hence we have $|B_1| = |B_1^*| \leq |B_2|$. By reversing the roles of B_1 and B_2 we also find $|B_2| \leq |B_1|$ and hence $|B_1| = |B_2|$, as proposed. □

Definition 5.24 **Dimension.** Let $\mathcal{V}$ be a finitely generated vector space. Let n be the number of elements in a basis. Then n is called the **dimension** of $\mathcal{V}$ and we write $\dim(\mathcal{V}) = n$. Moreover, $\mathcal{V}$ is called an n-dimensional vector space.

We want to emphasis here that in opposition to the dimension the basis of a vector space is not unique! Indeed there are infinitely many bases.

Theorem 5.25 Any linearly independent subset S of some finitely generated vector space $\mathcal{V}$ can be extended into a basis B of $\mathcal{V}$ with $S \subseteq B$.

Proof. See Problem 5.10.

Theorem 5.26 Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be some basis of vector space $\mathcal{V}$. Assume that $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$ where $\alpha_i \in \mathbb{R}$ and $\alpha_1 \neq 0$. Then $B' = \{\mathbf{x}, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of $\mathcal{V}$.

Proof. See Problem 5.12.

5.3 Coordinate Vector

Let $\mathcal{V}$ be an n-dimensional vector space with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then we can express a given vector $\mathbf{x} \in \mathcal{V}$ as a linear combination of the basis vectors, i.e.,

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i}$$

where the $\alpha_i \in \mathbb{R}$.

Theorem 5.27

Let $\mathcal{V}$ be a vector space with some basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $\mathbf{x} \in \mathcal{V}$ and $\alpha_i \in \mathbb{R}$ such that $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$. Then the coefficients $\alpha_1, \ldots, \alpha_n$ are uniquely defined.

PROOF. See Problem 5.13.

This theorem allows us to define the coefficient vector of $\mathbf{x}$.

Definition 5.28

Let $\mathcal{V}$ be a vector space with some basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. For some vector $\mathbf{x} \in \mathcal{V}$ we call the uniquely defined numbers $\alpha_i \in \mathbb{R}$ with $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$ the **coefficients** of $\mathbf{x}$ with respect to basis B. The vector $\mathbf{c}(\mathbf{x}) = (\alpha_1, \ldots, \alpha_n)'$ is then called the **coefficient vector** of $\mathbf{x}$. We then have

$$\mathbf{x} = \sum_{i=1}^{n} c_i(\mathbf{x}) \mathbf{v}_i .$$

Notice that $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^n$.

Example 5.29

Canonical basis. It is easy to verify that $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ forms a basis of the vector space $\mathbb{R}^n$. It is called the **canonical basis** of $\mathbb{R}^n$ and we immediately find that for each $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$,

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i . \qquad \diamondsuit$$

Example 5.30

The set $\mathscr{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_i \in \mathbb{R}\}$ of polynomials of degree less than or equal to 2 equipped with the addition and scalar multiplication of Example 5.2 is a vector space with basis $B = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\} = \{1, x, x^2\}$. Then any polynomial $p \in \mathscr{P}_2$ has the form

$$p(x) = \sum_{i=0}^{2} a_i x^i = \sum_{i=0}^{2} a_i \mathbf{v}_i$$

that is, $\mathbf{c}(p) = (a_0, a_1, a_2)'$. $\diamondsuit$

The last example demonstrates an important consequence of Theorem 5.27: there is a one-to-one correspondence between a vector $\mathbf{x} \in \mathcal{V}$ and its coefficient vector $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^n$. The map $\mathcal{V} \to \mathbb{R}^n, \mathbf{x} \mapsto \mathbf{c}(\mathbf{x})$ preserves the linear structure of the vector space, that is, for vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{R}$ we find (see Problem 5.14)

$$\mathbf{c}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{c}(\mathbf{x}) + \beta\mathbf{c}(\mathbf{y})\,.$$

In other words, the coefficient vector of a linear combination of two vectors is the corresponding linear combination of the coefficient vectors of the two vectors.

In this sense $\mathbb{R}^n$ is the prototype of any n-dimensional vector space $\mathcal{V}$. We say that $\mathcal{V}$ and $\mathbb{R}^n$ are **isomorphic**, $\mathcal{V} \cong \mathbb{R}^n$, that is, they have the same structure.

Now let $B_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $B_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be two bases of vector space $\mathcal{V}$. Let $\mathbf{c}_1(\mathbf{x})$ and $\mathbf{c}_2(\mathbf{x})$ be the respective coefficient vectors of $\mathbf{x} \in \mathcal{V}$. Then we have

$$\mathbf{w}_j = \sum_{i=1}^{n} c_{1i}(\mathbf{w}_j)\mathbf{v}_i\,, \qquad j = 1, \dots, n$$

and

$$\begin{aligned}\sum_{i=1}^{n} c_{1i}(\mathbf{x})\mathbf{v}_i = \mathbf{x} = \sum_{j=1}^{n} c_{2j}(\mathbf{x})\mathbf{w}_j &= \sum_{j=1}^{n} c_{2j}(\mathbf{x}) \sum_{i=1}^{n} c_{1i}(\mathbf{w}_j)\mathbf{v}_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{2j}(\mathbf{x}) c_{1i}(\mathbf{w}_j) \right) \mathbf{v}_i\,.\end{aligned}$$

Consequently, we find

$$c_{1i}(\mathbf{x}) = \sum_{j=1}^{n} c_{1i}(\mathbf{w}_j) c_{2j}(\mathbf{x})\,.$$

Thus let $\mathbf{U}_{12}$ contain the coefficient vectors of the basis vectors of B_2 with respect to basis B_1 as its columns, i.e.,

$$[\mathbf{U}_{12}]_{ij} = c_{1i}(\mathbf{w}_j)\,.$$

Then we find

$$\mathbf{c}_1(\mathbf{x}) = \mathbf{U}_{12}\mathbf{c}_2(\mathbf{x})\,.$$

Matrix $\mathbf{U}_{12}$ is called the **transformation matrix** that transforms the coefficient vector $\mathbf{c}_2$ with respect to basis B_2 into the coefficient vector $\mathbf{c}_1$ with respect to basis B_1. Definition 5.31

Notice that matrix $\mathbf{U}_{12}$ must be invertible as for every $\mathbf{x}$ there exist unique coordinate vectors $\mathbf{c}_1(\mathbf{x})$ and $\mathbf{c}_2(\mathbf{x})$. Moreover, the inverse of $\mathbf{U}_{12}$ then is the transformation matrix of the reverse transformation, i.e.,

$$\mathbf{U}_{12}^{-1} = \mathbf{U}_{21}\,.$$

— Summary

- A *vector space* is a set of elements that can be added and multiplied by a scalar (number).
- A vector space is *closed* under forming *linear combinations*, i.e.,

$$\mathbf{x}_1,\dots,\mathbf{x}_k \in \mathcal{V} \text{ and } \alpha_1,\dots,\alpha_k \in \mathbb{R} \quad \text{implies} \quad \sum_{i=1}^{k} \alpha_i \mathbf{x}_i \in \mathcal{V}\,.$$

- A set of vectors is called *linear independent* if it is not possible to express one of these as a linear combination of the remaining vectors.
- A *basis* is a minimal generating set, or equivalently, a maximal set of linear independent vectors.
- The basis of a given vector space is *not unique*. However, all bases of a given vector space have the same size which is called the *dimension* of the vector space.
- For a given basis every vector has a uniquely defined *coordinate vector*.
- The *transformation matrix* allows to transform a coordinate vector w.r.t. one basis into the coordinate vector w.r.t. another one.
- Every vector space of dimension n “looks like” the $\mathbb{R}^n$.

— Exercises

5.1 Let $\mathscr{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_i \in \mathbb{R}\}$ be the vector space of all polynomials of degree less than or equal to 2 equipped with point-wise addition and scalar multiplication. Then $B = \{1, x, x^2\}$ is a basis of $\mathscr{P}_2$ (see Example 5.30). The first three so called Laguerre polynomials are $\ell_0(x) = 1$, $\ell_1(x) = 1 - x$, and $\ell_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$. They also form a basis $B_\ell = \{\ell_0(x), \ell_1(x), \ell_2(x)\}$ of $\mathscr{P}_2$. What is the transformation matrix $\mathbf{U}_\ell$ that transforms the coefficient vector of a polynomial p with respect to basis B into its coefficient vector with respect to basis B_ℓ?

HINT: Observe that the Laguerre polynomials ℓ_0, ℓ_1, and ℓ_2 are given as linear combinations of monomials, i.e., of the elements in basis $B = \{1, x, x^2\}$. Hence the columns of the inverse transformation matrix $\mathbf{U}_\ell^{-1}$ can be easily be seen from the above representation.

— Problems

5.2 Let $\mathscr{S}$ be some vector space. Show that $0 \in \mathscr{S}$.

5.3 Give arguments why the following sets are or are not vector spaces:

(a) The empty set, $\emptyset$.

(b) The set $\{0\} \subset \mathbb{R}^n$.

(c) The set of all $m \times n$ matrices, $\mathbb{R}^{m \times n}$, for fixed values of m and n.

(d) The set of all square matrices.

(e) The set of all $n \times n$ diagonal matrices, for some fixed values of n.

(f) The set of all polynomials in one variable x.

(g) The set of all polynomials of degree less than or equal to some fixed value n.

(h) The set of all polynomials of degree equal to some fixed value n.

(i) The set of points $\mathbf{x}$ in $\mathbb{R}^n$ that satisfy the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ for some fixed matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and some fixed vector $\mathbf{b} \in \mathbb{R}^m$.

(j) The set $\{\mathbf{y} = \mathbf{A}\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$, for some fixed matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$.

(k) The set $\{\mathbf{y} = \mathbf{b}_0 + \alpha \mathbf{b}_1 : \alpha \in \mathbb{R}\}$, for fixed vectors $\mathbf{b}_0, \mathbf{b}_1 \in \mathbb{R}^n$.

(l) The set of all functions on $[0,1]$ that are both continuously differentiable and integrable.

(m) The set of all functions on $[0,1]$ that are not continuous.

(n) The set of all random variables X on some given probability space $(\Omega, \mathscr{F}, P)$.

Which of these vector spaces are finitely generated?
Find generating sets for these. If possible give a basis.

5.4 Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{x}_0 \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$. Show that

$$\mathscr{L}_0 = \{\mathbf{x}-\mathbf{x}_0 : \mathbf{A}\mathbf{x} = \mathbf{b}\} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\} - \mathbf{x}_0$$

is a subspace of $\mathbb{R}^n$.

5.5 Prove the following proposition: Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two subspaces of vector space $\mathscr{V}$. Then $\mathscr{S}_1 \cap \mathscr{S}_2$ is a subspace of $\mathscr{V}$.

5.6 Prove or disprove the following statement:

Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two subspaces of vector space $\mathscr{V}$. Then their *union* $\mathscr{S}_1 \cup \mathscr{S}_2$ is a subspace of $\mathscr{V}$.

5.7 Let $\mathscr{U}_1$ and $\mathscr{U}_2$ be two subspaces of a vector space $\mathscr{V}$. Then the **sum** of $\mathscr{U}_1$ and $\mathscr{U}_2$ is the set

sum of subspaces

$$\mathscr{U}_1 + \mathscr{U}_2 = \{\mathbf{u}_1 + \mathbf{u}_2 : \mathbf{u}_1 \in \mathscr{U}_1, \mathbf{u}_2 \in \mathscr{U}_2\}\,.$$

Show that $\mathscr{U}_1 + \mathscr{U}_2$ is a subspace of $\mathscr{V}$.

5.8 Prove Theorem 5.17.

5.9 Prove Theorem 5.19.

5.10 Prove Theorem 5.25.

HINT: Start with S and add linearly independent vectors (Why is this possible?) until we obtain a maximal linearly independent set. This is then a basis that contains S. (Why?)

5.11 Let $\mathscr{U}_1$ and $\mathscr{U}_2$ be two subspaces of a vector space $\mathscr{V}$. Show that

$$\dim(\mathscr{U}_1) + \dim(\mathscr{U}_2) = \dim(\mathscr{U}_1 + \mathscr{U}_2) + \dim(\mathscr{U}_1 \cap \mathscr{U}_2)\,.$$

HINT: Start with a basis for $\mathscr{U}_1 \cap \mathscr{U}_2$ and use Theorems 5.14 and 5.25.

5.12 Prove Theorem 5.26.

HINT: Express $\mathbf{v}_1$ as linear combination of elements in B' and show that B' is a generating set by replacing $\mathbf{v}_1$ by this expression. It remains to show that the set is a minimal generating set. (Why is any strict subset not a generating set?)

5.13 Prove Theorem 5.27.

HINT: Assume that there are two sets of numbers $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ such that $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{v}_i = \sum_{i=1}^n \beta_i \mathbf{v}_i$. Show that $\alpha_i = \beta_i$ by means of the fact that $\mathbf{x} - \mathbf{x} = 0$.

5.14 Let $\mathscr{V}$ be an n-dimensional vector space. Show that for two vectors $\mathbf{x}, \mathbf{y} \in \mathscr{V}$ and $\alpha, \beta \in \mathbb{R}$,

$$\mathbf{c}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{c}(\mathbf{x}) + \beta\mathbf{c}(\mathbf{y})\,.$$

5.15 Show that a coefficient vector $\mathbf{c}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.

6

Linear Transformations

We want to preserve linear structures.

6.1 Linear Maps

In Section 5.3 we have seen that the transformation that maps a vector $\mathbf{x} \in \mathscr{V}$ to its coefficient vector $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^{\dim(\mathscr{V})}$ preserves the linear structure of vector space $\mathscr{V}$.

Definition 6.1

Let $\mathscr{V}$ and $\mathscr{W}$ be two vector spaces. A transformation $\phi \colon \mathscr{V} \to \mathscr{W}$ is called a **linear map** if for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$ and all $\alpha, \beta \in \mathbb{R}$ the following holds:

(i) $\phi(\mathbf{x}+\mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y})$

(ii) $\phi(\alpha\mathbf{x}) = \alpha\phi(\mathbf{x})$

Equivalently, $\phi \colon \mathscr{V} \to \mathscr{W}$ is a linear map if

$$\phi(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha\phi(\mathbf{x}) + \beta\phi(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$ and all $\alpha, \beta \in \mathbb{R}$.

Example 6.2

Every $m \times n$ matrix $\mathbf{A}$ defines a linear map.

$$\phi_{\mathbf{A}} \colon \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto \phi_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}\,.$$

This immediately follow from the rules for matrix algebra:

$$\phi_{\mathbf{A}}(\alpha\mathbf{x}+\beta\mathbf{y}) = \mathbf{A}(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y} = \alpha\phi_{\mathbf{A}}(\mathbf{x}) + \beta\phi_{\mathbf{A}}(\mathbf{y})\,. \qquad \diamondsuit$$

Example 6.3

Let $\mathscr{P} = \left\{\sum_{i=0}^{k} a_i x^i : k \in \mathbb{N},\ a_i \in \mathbb{R}\right\}$ be the vector space of all polynomials (see Example 5.2). Then the map $\frac{d}{dx} \colon \mathscr{P} \to \mathscr{P}, p \mapsto \frac{d}{dx}p$ is linear. It is called the **differential operator**[1]. $\diamondsuit$

[1] A transformation that maps a function to another function is usually called an **operator**.

Example 6.4

Let $\mathscr{C}^0([0,1])$ and $\mathscr{C}^1([0,1])$ be the vector spaces of all continuous and continuously differentiable functions with domain $[0,1]$, respectively (see Example 5.5). Then the differential operator

$$\frac{d}{dx}\colon \mathscr{C}^1([0,1]) \to \mathscr{C}^0([0,1]),\ f \mapsto f' = \frac{d}{dx}f$$

is a linear map. ♢

Example 6.5

Let $\mathscr{L}$ be the vector space of all random variables X on some given probability space that have an expectation $\mathbb{E}(X)$. Then the map

$$\mathbb{E}\colon \mathscr{L} \to \mathbb{R},\ X \mapsto \mathbb{E}(X)$$

is a linear map. ♢

Linear maps can be described by their range and their preimage of 0.

Definition 6.6

Kernel and image. Let $\phi\colon \mathcal{V} \to \mathcal{W}$ be a linear map.

(i) The **kernel** (or **nullspace**) of ϕ is the preimage of 0, i.e.,

$$\ker(\phi) = \{\mathbf{x} \in \mathcal{V} : \phi(\mathbf{x}) = 0\}\,.$$

(ii) The **image** (or **range**) of ϕ is the set

$$\operatorname{Im}(\phi) = \phi(\mathcal{V}) = \{\mathbf{y} \in \mathcal{W} : \exists \mathbf{x} \in \mathcal{V},\ \text{s.t. } \phi(\mathbf{x}) = \mathbf{y}\}\,.$$

Theorem 6.7

Let $\phi\colon \mathcal{V} \to \mathcal{W}$ be a linear map. Then $\ker(\phi) \subseteq \mathcal{V}$ and $\operatorname{Im}(\phi) \subseteq \mathcal{W}$ are vector spaces.

PROOF. First observe that $\ker(\phi) \neq \emptyset$ as $0 \in \ker(\phi)$: $\phi(0) = \phi(0\mathbf{x}) = 0\phi(\mathbf{x}) = 0$ for an arbitrary $\mathbf{x} \in \mathcal{V}$. Now by Definition 5.6 we have to show that an arbitrary linear combination of two elements of the subset is also an element of the set.

Let $\mathbf{x}, \mathbf{y} \in \ker(\phi)$ and $\alpha, \beta \in \mathbb{R}$. Then by definition of $\ker(\phi)$, $\phi(\mathbf{x}) = \phi(\mathbf{y}) = 0$ and thus $\phi(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\phi(\mathbf{x}) + \beta\phi(\mathbf{y}) = 0$. Consequently $\alpha\mathbf{x} + \beta\mathbf{y} \in \ker(\phi)$ and thus $\ker(\phi)$ is a subspace of $\mathcal{V}$.

For the second statement assume that $\mathbf{x}, \mathbf{y} \in \operatorname{Im}(\phi)$. Then there exist two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ such that $\mathbf{x} = \phi(\mathbf{u})$ and $\mathbf{y} = \phi(\mathbf{v})$. Hence for any $\alpha, \beta \in \mathbb{R}$, $\alpha\mathbf{x} + \beta\mathbf{y} = \alpha\phi(\mathbf{u}) + \beta\phi(\mathbf{v}) = \phi(\alpha\mathbf{u} + \beta\mathbf{v}) \in \operatorname{Im}(\phi)$. □

Theorem 6.8

Let $\phi\colon \mathcal{V} \to \mathcal{W}$ be a linear map and let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of $\mathcal{V}$. Then $\operatorname{Im}(\phi)$ is spanned by the vectors $\phi(\mathbf{v}_1), \dots, \phi(\mathbf{v}_n)$.

PROOF. For every $\mathbf{x} \in \mathcal{V}$ we have $\mathbf{x} = \sum_{i=1}^n c_i(\mathbf{x})\mathbf{v}_i$, where $\mathbf{c}(\mathbf{x})$ is the coefficient vector of $\mathbf{x}$ with respect to B. Then by the linearity of ϕ we find $\phi(\mathbf{x}) = \phi\left(\sum_{i=1}^n c_i(\mathbf{x})\mathbf{v}_i\right) = \sum_{i=1}^n c_i(\mathbf{x})\phi(\mathbf{v}_i)$. Thus $\phi(\mathbf{x})$ can be represented as a linear combination of the vectors $\phi(\mathbf{v}_1), \dots, \phi(\mathbf{v}_n)$. □

We will see below that the dimensions of these vector spaces determine whether a linear map is invertible. First we show that there is a strong relation between their dimensions.

Theorem 6.9

Dimension theorem for linear maps. Let $\phi\colon \mathcal{V} \to \mathcal{W}$ be a linear map. Then

$$\dim(\ker(\phi)) + \dim(\operatorname{Im}(\phi)) = \dim(\mathcal{V}) .$$

PROOF. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ form a basis of $\ker(\phi) \subseteq \mathcal{V}$. Then it can be extended into a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of $\mathcal{V}$, where $k + n = \dim \mathcal{V}$. For any $\mathbf{x} \in \mathcal{V}$ there exist unique coefficients α_i and β_j such that $\mathbf{x} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i + \sum_{j=1}^{n} \beta_j \mathbf{w}_j$. By the linearity of ϕ we then have

$$\phi(\mathbf{x}) = \sum_{i=1}^{k} \alpha_i \phi(\mathbf{v}_i) + \sum_{j=1}^{n} \beta_j \phi(\mathbf{w}_j) = \sum_{j=1}^{n} \beta_j \phi(\mathbf{w}_j)$$

i.e., $\{\phi(\mathbf{w}_1), \ldots, \phi(\mathbf{w}_n)\}$ spans $\operatorname{Im}(\phi)$. It remains to show that this set is linearly independent. In fact, if $\sum_{j=1}^{n} \beta_j \phi(\mathbf{w}_j) = 0$ then $\phi(\sum_{j=1}^{n} \beta_j \mathbf{w}_j) = 0$ and hence $\sum_{j=1}^{n} \beta_j \mathbf{w}_j \in \ker(\phi)$. Thus there exist coefficients γ_i such that $\sum_{j=1}^{n} \beta_j \mathbf{w}_j = \sum_{i=1}^{k} \gamma_i \mathbf{v}_i$, or equivalently, $\sum_{j=1}^{n} \beta_j \mathbf{w}_j + \sum_{i=1}^{k} (-\gamma_i) \mathbf{v}_i = 0$. However, as $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_n\}$ forms a basis of $\mathcal{V}$ all coefficients β_j and γ_i must be zero and consequently the vectors $\{\phi(\mathbf{w}_1), \ldots, \phi(\mathbf{w}_n)\}$ are linearly independent and form a basis for $\operatorname{Im}(\phi)$. Thus the statement follows. □

Lemma 6.10

Let $\phi\colon \mathcal{V} \to \mathcal{W}$ be a linear map and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathcal{V}$. Then the vectors $\phi(\mathbf{v}_1), \ldots, \phi(\mathbf{v}_n)$ are linearly independent if and only if $\ker(\phi) = \{0\}$.

PROOF. By Theorem 6.8, $\phi(\mathbf{v}_1), \ldots, \phi(\mathbf{v}_n)$ spans $\operatorname{Im}(\phi)$, that is, for every $\mathbf{x} \in \mathcal{V}$ we have $\phi(\mathbf{x}) = \sum_{i=1}^{n} c_i(\mathbf{x}) \phi(\mathbf{v}_i)$ where $\mathbf{c}(\mathbf{x})$ denotes the coefficient vector of $\mathbf{x}$ with respect to B. Thus if $\phi(\mathbf{v}_1), \ldots, \phi(\mathbf{v}_n)$ are linearly independent, then $\phi(\mathbf{x}) = \sum_{i=1}^{n} c_i(\mathbf{x}) \phi(\mathbf{v}_i) = 0$ implies $\mathbf{c}(\mathbf{x}) = 0$ and hence $\mathbf{x} = 0$. That is, $\ker(\phi) = \{0\}$.

Conversely, if $\ker(\phi) = \{0\}$, then $\phi(\mathbf{x}) = \sum_{i=1}^{n} c_i(\mathbf{x}) \phi(\mathbf{v}_i) = 0$ implies $\mathbf{x} = 0$ and hence $\mathbf{c}(\mathbf{x}) = 0$. But then vectors $\phi(\mathbf{v}_1), \ldots, \phi(\mathbf{v}_n)$ are linearly independent, as claimed. □

Corollary 6.11

We have $\dim(\mathcal{V}) = \dim(\operatorname{Im}(\phi))$ if and only if $\ker(\phi) = \{0\}$.

Recall that a function $\phi\colon \mathcal{V} \to \mathcal{W}$ is **invertible**, if there exists a function $\phi^{-1}\colon \mathcal{W} \to \mathcal{V}$ such that $(\phi^{-1} \circ \phi)(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$ and $(\phi \circ \phi^{-1})(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in \mathcal{W}$. Such a function exists if ϕ is one-to-one and onto.

A linear map ϕ is **onto** if $\operatorname{Im}(\phi) = \mathcal{W}$. It is **one-to-one** if for each $\mathbf{y} \in \mathcal{W}$ there exists at most one $\mathbf{x} \in \mathcal{V}$ such that $\mathbf{y} = \phi(\mathbf{x})$, i.e., if $\phi(\mathbf{x}) = \phi(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$.

Lemma 6.12

A linear map $\phi\colon \mathcal{V} \to \mathcal{W}$ is one-to-one if and only if $\ker(\phi) = \{0\}$.

PROOF. See Problem 6.3.

Theorem 6.13

A linear map $\phi\colon \mathcal{V} \to \mathcal{W}$ is invertible if and only if $\dim(\mathcal{V}) = \dim(\mathcal{W})$ and $\ker(\phi) = \{0\}$.

PROOF. By Lemma 6.12, ϕ is one-to-one if and only if $\ker(\phi) = \{0\}$. It is onto if and only if $\dim(\operatorname{Im}(\phi)) = \dim(\mathcal{W})$. As $\dim(\operatorname{Im}(\phi)) = \dim(\mathcal{V})$ if and only if $\ker(\phi) = \{0\}$ by Corollary 6.11, the result follows. □

Theorem 6.14

Let $\phi\colon \mathcal{V} \to \mathcal{W}$ be a linear map with $\dim \mathcal{V} = \dim \mathcal{W}$.

(1) If there exists a function $\psi\colon \mathcal{W} \to \mathcal{V}$ such that $(\psi \circ \phi)(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$, then $(\phi \circ \psi)(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in \mathcal{W}$.

(2) If there exists a function $\chi\colon \mathcal{W} \to \mathcal{V}$ such that $(\phi \circ \chi)(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in \mathcal{W}$, then $(\chi \circ \phi)(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$.

PROOF. It remains to show that ϕ is invertible in both cases.
(1) If ψ exists, then ϕ must be one-to-one. Thus $\ker(\phi) = \{0\}$ by Lemma 6.12 and consequently ϕ is invertible by Theorem 6.13.
(2) We can use (1) to conclude that $\chi^{-1} = \phi$. Hence $\phi^{-1} = \chi$ and the statement follows. □

An immediate consequence of this Theorem is the existence of ψ or χ implies the existence of the other one. Consequently, this also implies that ϕ is invertible and $\phi^{-1} = \psi = \chi$.

6.2 Matrices and Linear Maps

In Section 5.3 we have seen that the $\mathbb{R}^n$ can be interpreted as *the* vector space of dimension n. Example 6.2 shows us that any $m \times n$ matrix $\mathbf{A}$ defines a linear map between $\mathbb{R}^n$ and $\mathbb{R}^m$. The following theorem tells us that there is also an one-to-one correspondence between matrices and linear maps. Thus matrices are *the* representations of linear maps.

Theorem 6.15

Let $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then there exists an $m \times n$ matrix $\mathbf{A}_\phi$ such that $\phi(\mathbf{x}) = \mathbf{A}_\phi \mathbf{x}$.

PROOF. Let $\mathbf{a}_i = \phi(\mathbf{e}_i)$ denote the images of the elements of the canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Let $\mathbf{A}_\phi = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ be the matrix with column vectors $\mathbf{a}_i$. Notice that $\mathbf{A}_\phi \mathbf{e}_i = \mathbf{a}_i$. Now we find for every $\mathbf{x} = (x_1, \dots, x_n)' \in \mathbb{R}^n$, $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$ and therefore

$$\begin{aligned}\phi(\mathbf{x}) = \phi\left(\sum_{i=1}^n x_i \mathbf{e}_i\right) &= \sum_{i=1}^n x_i \phi(\mathbf{e}_i) = \sum_{i=1}^n x_i \mathbf{a}_i \\ &= \sum_{i=1}^n x_i \mathbf{A} \mathbf{e}_i = \mathbf{A}_\phi \sum_{i=1}^n x_i \mathbf{e}_i = \mathbf{A}_\phi \mathbf{x}\end{aligned}$$

as claimed. □

Now assume that we have two linear maps $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\psi\colon \mathbb{R}^m \to \mathbb{R}^k$ with corresponding matrices $\mathbf{A}$ and $\mathbf{B}$, resp. The map composition $\psi \circ \phi$ is then given by $(\psi\circ\phi)(\mathbf{x}) = \psi(\phi(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x}) = (\mathbf{B}\mathbf{A})\mathbf{x}$. Thus matrix multiplication corresponds to map composition.

If the linear map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$, is invertible, then matrix $\mathbf{A}$ is also invertible and $\mathbf{A}^{-1}$ describes the inverse map ϕ^{-1}.

By Theorem 6.15 a linear map ϕ and its corresponding matrix $\mathbf{A}$ are closely related. Thus all definitions and theorems about linear maps may be applied to matrices. For example, the kernel of matrix $\mathbf{A}$ is the set

$$\ker(\mathbf{A}) = \{\mathbf{x}\colon \mathbf{A}\mathbf{x} = 0\}\,.$$

The following result is an immediate consequence of our considerations and Theorem 6.14.

Theorem 6.16

Let $\mathbf{A}$ be some square matrix.

(a) If there exists a square matrix $\mathbf{B}$ such that $\mathbf{A}\mathbf{B} = \mathbf{I}$, then $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \mathbf{B}$.

(b) If there exists a square matrix $\mathbf{C}$ such that $\mathbf{C}\mathbf{A} = \mathbf{I}$, then $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \mathbf{C}$.

The following result is very convenient.

Corollary 6.17

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices with $\mathbf{A}\mathbf{B} = \mathbf{I}$. Then both $\mathbf{A}$ and $\mathbf{B}$ are invertible and $\mathbf{A}^{-1} = \mathbf{B}$ and $\mathbf{B}^{-1} = \mathbf{A}$.

6.3 Rank of a Matrix

Theorem 6.8 tells us that the columns of a matrix $\mathbf{A}$ span the image of a linear map ϕ induced by $\mathbf{A}$. Consequently, by Theorem 5.21 we get a basis of $\operatorname{Im}(\phi)$ by a maximal linearly independent subset of these column vectors. The dimension of the image is then the size of this subset. This motivates the following notion.

Definition 6.18

The **rank** of a matrix $\mathbf{A}$ is the maximal number of linearly independent columns of $\mathbf{A}$.

By the above considerations we immediately have the following lemmata.

Lemma 6.19

For any matrix $\mathbf{A}$,

$$\operatorname{rank}(\mathbf{A}) = \dim(\operatorname{Im}(\mathbf{A}))\,.$$

Lemma 6.20

Let $\mathbf{A}$ be an $m \times n$ matrix. If $\mathbf{T}$ is an invertible $m \times m$ matrix and $\mathbf{U}$ an invertible $n \times n$ matrix, then $\operatorname{rank}(\mathbf{T}\mathbf{A}) = \operatorname{rank}(\mathbf{A}\mathbf{U}) = \operatorname{rank}(\mathbf{A})$.

Definition 6.21 The **nullity** of matrix $\mathbf{A}$ is the dimension of the kernel (nullspace) of $\mathbf{A}$.

Theorem 6.22 **Rank-nullity theorem.** Let $\mathbf{A}$ be an $m \times n$ matrix. Then

$$\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = n.$$

PROOF. By Lemma 6.19 and Theorem 6.9 we find $\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = \dim(\operatorname{Im}(\mathbf{A})) + \dim(\ker(\mathbf{A})) = n$. □

Theorem 6.23 Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $n \times k$ matrix. Then

$$\operatorname{rank}(\mathbf{AB}) \le \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$

PROOF. Let ϕ and ψ be the maps represented by $\mathbf{A}$ and $\mathbf{B}$, respectively. Recall that $\mathbf{AB}$ corresponds to map composition $\phi \circ \psi$. Obviously, $\operatorname{Im}(\phi \circ \psi) \subseteq \operatorname{Im}\phi$. Hence $\operatorname{rank}(\mathbf{AB}) = \dim \operatorname{Im}(\phi \circ \psi) \le \dim \operatorname{Im}(\phi) = \operatorname{rank}(\mathbf{A})$. Similarly, $\operatorname{Im}(\phi \circ \psi)$ is spanned by $\phi(S)$ where S is any basis of $\operatorname{Im}(\psi)$. Hence $\operatorname{rank}(\mathbf{AB}) = \dim \operatorname{Im}(\phi \circ \psi) \le \dim \operatorname{Im}(\psi) = \operatorname{rank}(\mathbf{B})$. Thus the result follows. □

Our notion of *rank* in Definition 6.18 is sometimes also referred to as *column rank* of matrix $\mathbf{A}$. One may also define the **row rank** of $\mathbf{A}$ as the maximal number of linearly independent rows of $\mathbf{A}$. However, column rank and row rank always coincide.

Theorem 6.24 For any matrix $\mathbf{A}$,

$$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}').$$

For the proof of this theorem we first need the following result.

Lemma 6.25 Let $\mathbf{A}$ be a $m \times n$ matrix. Then $\operatorname{rank}(\mathbf{A}'\mathbf{A}) = \operatorname{rank}(\mathbf{A})$.

PROOF. We show that $\ker(\mathbf{A}'\mathbf{A}) = \ker(\mathbf{A})$. Obviously, $\mathbf{x} \in \ker(\mathbf{A})$ implies $\mathbf{A}'\mathbf{A}\mathbf{x} = \mathbf{A}'0 = 0$ and thus $\ker(\mathbf{A}) \subseteq \ker(\mathbf{A}'\mathbf{A})$. Now assume that $\mathbf{x} \in \ker(\mathbf{A}'\mathbf{A})$. Then $\mathbf{A}'\mathbf{A}\mathbf{x} = 0$ and we find $0 = \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})'(\mathbf{A}\mathbf{x})$ which implies that $\mathbf{A}\mathbf{x} = 0$ so that $\mathbf{x} \in \ker(\mathbf{A})$. Hence $\ker(\mathbf{A}'\mathbf{A}) \subseteq \ker(\mathbf{A})$ and, consequently, $\ker(\mathbf{A}'\mathbf{A}) = \ker(\mathbf{A})$. Now notice that $\mathbf{A}'\mathbf{A}$ is an $n \times n$ matrix. Theorem 6.22 then implies

$$\begin{aligned}\operatorname{rank}(\mathbf{A}'\mathbf{A}) - \operatorname{rank}(\mathbf{A}) &= \big(n - \operatorname{nullity}(\mathbf{A}'\mathbf{A})\big) - (n - \operatorname{nullity}(\mathbf{A})) \\ &= \operatorname{nullity}(\mathbf{A}) - \operatorname{nullity}(\mathbf{A}'\mathbf{A}) = 0\end{aligned}$$

and thus $\operatorname{rank}(\mathbf{A}'\mathbf{A}) = \operatorname{rank}(\mathbf{A})$, as claimed. □

PROOF OF THEOREM 6.24. By Theorem 6.23 and Lemma 6.25 we find

$$\operatorname{rank}(\mathbf{A}') \ge \operatorname{rank}(\mathbf{A}'\mathbf{A}) = \operatorname{rank}(\mathbf{A}).$$

As this statement remains true if we replace $\mathbf{A}$ by its transpose $\mathbf{A}'$ we have $\operatorname{rank}(\mathbf{A}) \ge \operatorname{rank}(\mathbf{A}')$ and thus the statement follows. □

Corollary 6.26

Let $\mathbf{A}$ be an $m \times n$ matrix. Then

$$\operatorname{rank}(\mathbf{A}) \leq \min\{m, n\}.$$

Finally, we give a necessary and sufficient condition for invertibility of a square matrix.

Definition 6.27

An $n \times n$ matrix $\mathbf{A}$ is called **regular** if it has **full rank**, i.e., if $\operatorname{rank}(\mathbf{A}) = n$.

Theorem 6.28

A square matrix $\mathbf{A}$ is invertible if and only if it is regular.

Proof. By Theorem 6.13 a matrix is invertible if and only if $\operatorname{nullity}(\mathbf{A}) = 0$ (i.e., $\ker(\mathbf{A}) = \{0\}$). Then $\operatorname{rank}(\mathbf{A}) = n - \operatorname{nullity}(\mathbf{A}) = n$ by Theorem 6.22. □

6.4 Similar Matrices

In Section 5.3 we have seen that every vector $\mathbf{x} \in \mathcal{V}$ in some vector space $\mathcal{V}$ of dimension $\dim \mathcal{V} = n$ can be uniformly represented by a coordinate vector $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^n$. However, for this purpose we first have to choose an arbitrary but fixed basis for $\mathcal{V}$. In this sense every finitely generated vector space is "equivalent" (i.e., isomorphic) to the $\mathbb{R}^n$.

However, we also have seen that there is no such thing as *the* basis of a vector space and that coordinate vector $\mathbf{c}(\mathbf{x})$ changes when we change the underlying basis of $\mathcal{V}$. Of course vector $\mathbf{x}$ then remains the same.

In Section 6.2 above we have seen that matrices are the representations of linear maps between $\mathbb{R}^m$ and $\mathbb{R}^n$. Thus if $\phi\colon \mathcal{V} \to \mathcal{W}$ is a linear map, then there is a matrix $\mathbf{A}$ that represents the linear map between the coordinate vectors of vectors in $\mathcal{V}$ and those in $\mathcal{W}$. Obviously matrix $\mathbf{A}$ depends on the chosen bases for $\mathcal{V}$ and $\mathcal{W}$.

Suppose now that $\dim \mathcal{V} = \dim \mathcal{W} = \mathbb{R}^n$. Let $\mathbf{A}$ be an $n \times n$ square matrix that represents a linear map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ with respect to some basis B_1. Let $\mathbf{x}$ be a coefficient vector corresponding to basis B_2 and let $\mathbf{U}$ denote the transformation matrix that transforms $\mathbf{x}$ into the coefficient vector corresponding to basis B_1. Then we find:

$$\begin{array}{llcl} \text{basis } B_1\colon & \mathbf{U}\mathbf{x} & \xrightarrow{\mathbf{A}} & \mathbf{A}\mathbf{U}\mathbf{x} \\ & \mathbf{U}\uparrow & & \downarrow\mathbf{U}^{-1} \\ \text{basis } B_2\colon & \mathbf{x} & \xrightarrow{\mathbf{C}} & \mathbf{U}^{-1}\mathbf{A}\mathbf{U}\mathbf{x} \end{array} \qquad \text{hence} \quad \mathbf{C}\mathbf{x} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}\mathbf{x}$$

That is, if $\mathbf{A}$ represents a linear map corresponding to basis B_1, then $\mathbf{C} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ represents the same linear map corresponding to basis B_2.

Definition 6.29

Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{C}$ are called **similar** if $\mathbf{C} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ for some invertible $n \times n$ matrix $\mathbf{U}$.

— Summary

- A *Linear map* ϕ preserve the linear structure, i.e.,

  $$\phi(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha\phi(\mathbf{x})+\beta\phi(\mathbf{y})\,.$$

- *Kernel* and *image* of a linear map $\phi\colon \mathcal{V}\to\mathcal{W}$ are subspaces of $\mathcal{V}$ and $\mathcal{W}$, resp.

- $\mathrm{Im}(\phi)$ is spanned by the images of a basis of $\mathcal{V}$.

- A linear map $\phi\colon \mathcal{V}\to\mathcal{W}$ is *invertible* if and only if $\dim\mathcal{V} = \dim\mathcal{W}$ and $\ker(\phi)=\{0\}$.

- Linear maps are represented by matrices. The corresponding matrix depends on the chosen bases of the vector spaces.

- Matrices are called *similar* if they describe the same linear map but w.r.t. different bases.

- The *rank* of a matrix is the dimension of the image of the corresponding linear map.

- Matrix multiplication corresponds to map composition. The inverse of a matrix corresponds to the corresponding inverse linear map.

- A matrix is invertible if and only if it is a square matrix and *regular*, i.e., has full rank.

— Exercises

6.1 Let $\mathscr{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_i \in \mathbb{R}\}$ be the vector space of all polynomials of degree less than or equal to 2 equipped with point-wise addition and scalar multiplication. Then $B = \{1, x, x^2\}$ is a basis of $\mathscr{P}_2$ (see Example 5.30). Let $\phi = \frac{d}{dx} : \mathscr{P}_2 \to \mathscr{P}_2$ be the differential operator on $\mathscr{P}_2$ (see Example 6.3).

(a) What is the kernel of ϕ? Give a basis for $\ker(\phi)$.

(b) What is the image of ϕ? Give a basis for $\mathrm{Im}(\phi)$.

(c) For the given basis B represent map ϕ by a matrix $\mathbf{D}$.

(d) The first three so called Laguerre polynomials are $\ell_0(x) = 1$, $\ell_1(x) = 1 - x$, and $\ell_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$.
Then $B_\ell = \{\ell_0(x), \ell_1(x), \ell_2(x)\}$ also forms a basis of $\mathscr{P}_2$. What is the transformation matrix $\mathbf{U}_\ell$ that transforms the coefficient vector of a polynomial p with respect to basis B into its coefficient vector with respect to basis B_ℓ?

(e) For basis B_ℓ represent map ϕ by a matrix $\mathbf{D}_\ell$.

HINT: Observe that the Laguerre polynomials ℓ_0, ℓ_1, and ℓ_2 are given as linear combinations of monomials, i.e., of the elements in basis $B = \{1, x, x^2\}$. Hence the columns of the inverse transformation matrix $\mathbf{U}_\ell^{-1}$ can be easily be seen from the above representation.

— Problems

6.2 Let $\phi: \mathscr{V} \to \mathscr{W}$ be a linear map and let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of $\mathscr{V}$. Give a necessary and sufficient condition for $\{\phi(\mathbf{v}_1), \dots, \phi(\mathbf{v}_n)\}$ being a basis of $\mathrm{Im}(\phi)$.

6.3 Prove Lemma 6.12.

HINT: We have to prove two statements:
(1) ϕ is one-to-one $\Rightarrow \ker(\phi) = \{0\}$.
(2) $\ker(\phi) = \{0\} \Rightarrow \phi$ is one-to-one.
For (2) use the fact that if $\phi(\mathbf{x}_1) = \phi(\mathbf{x}_2)$, then $\mathbf{x}_1 - \mathbf{x}_2$ must be an element of the kernel. (Why?)

6.4 Let $\phi: \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{y} = \mathbf{A}\mathbf{x}$ be a linear map, where $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$. Show that the column vectors of matrix $\mathbf{A}$ span $\mathrm{Im}(\phi)$, i.e.,

$$\mathrm{span}(\mathbf{a}_1, \dots, \mathbf{a}_n) = \mathrm{Im}(\phi)\,.$$

6.5 Prove Corollary 6.26.

6.6 Disprove the following statement:
For any $m \times n$ matrix $\mathbf{A}$ and any $n \times k$ matrix $\mathbf{B}$ it holds that $\mathrm{rank}(\mathbf{AB}) = \min\{\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{B})\}$.

6.7 Show that two similar matrices have the same rank.

6.8 The converse of the statement in Problem 6.7 does not hold, that is, two $n \times n$ matrices with the same rank need not be similar. Give a counterexample.

7

Linear Equations

We want to compute *dimensions and bases of kernel and image.*

7.1 Linear Equations

A system of m linear equations in n unknowns is given by the following set of equations: Definition 7.1

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{matrix}$$

By means of matrix algebra it can be written in much more compact form as (see Problem 7.2)

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,.$$

The matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is called the **coefficient matrix** and the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

contain the unknowns x_i and the constants b_j on the right hand side.

A linear equation $\mathbf{A}\mathbf{x} = 0$ is called **homogeneous**. Definition 7.2
A linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \neq 0$ is called **inhomogeneous**.

Observe that the set of solutions of the homogeneous linear equation $\mathbf{A}\mathbf{x} = 0$ is just the kernel of the coefficient matrix, $\ker(\mathbf{A})$, and thus forms a vector space. The set of solutions of an inhomogeneous linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be derived from $\ker(\mathbf{A})$ as well.

Lemma 7.3

Let $\mathbf{x}_0$ and $\mathbf{y}_0$ be two solutions of the inhomogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then $\mathbf{x}_0 - \mathbf{y}_0$ is a solution of the homogeneous equation $\mathbf{A}\mathbf{x} = 0$.

Theorem 7.4

Let $\mathbf{x}_0$ be a particular solution of the inhomogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, then the set of all solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by

$$\mathcal{S} = \mathbf{x}_0 + \ker(\mathbf{A}) = \{\mathbf{x} = \mathbf{x}_0 + \mathbf{z} : \mathbf{z} \in \ker(\mathbf{A})\}.$$

PROOF. See Problem 7.3.

Set $\mathcal{S}$ is an example of an *affine subspace* of $\mathbb{R}^n$.

Definition 7.5

Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a vector and $\mathcal{S} \subseteq \mathbb{R}^n$ be a subspace. Then the set $\mathbf{x}_0 + \mathcal{S} = \{\mathbf{x} = \mathbf{x}_0 + \mathbf{z} : \mathbf{z} \in \mathcal{S}\}$ is called an **affine subspace** of $\mathbb{R}^n$

7.2 Gauß Elimination

A linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved by transforming it into a simpler form called *row echelon form*.

Definition 7.6

A matrix $\mathbf{A}$ is said to be in **row echelon form** if the following holds:

(i) All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes, and

(ii) The leading coefficient (i.e., the first nonzero number from the left, also called the **pivot**) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

It is sometimes convenient to work with an even simpler form.

Definition 7.7

A matrix $\mathbf{A}$ is said to be in **row reduced echelon form** if the following holds:

(i) All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes, and

(ii) The leading coefficient of a nonzero row is always strictly to the right of the leading coefficient of the row above it. It is 1 and is the only nonzero entry in its column. Such columns are then called **pivotal**.

Any coefficient matrix $\mathbf{A}$ can be transformed into a matrix $\mathbf{R}$ that is in row (reduced) echelon form by means of **elementary row operations** (see Problem 7.6):

(E1) Switch two rows.

(E2) Multiply some row with $\alpha \neq 0$.

(E3) Add some multiple of a row to another row.

These row operations can be performed by means of *elementary matrices*, i.e., matrices that differs from the identity matrix by one single elementary row operation. These matrices are always invertible, see Problem 7.4.

The procedure works due to the following lemma which tells use how we obtain equivalent linear equations that have the same solutions.

Lemma 7.8

Let $\mathbf{A}$ be an $m \times n$ coefficient matrix and $\mathbf{b}$ the vector of constants. If $\mathbf{T}$ is an invertible $m \times m$ matrix, then the linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{T}\mathbf{A}\mathbf{x} = \mathbf{T}\mathbf{b}$$

are equivalent. That is, they have the same solutions.

PROOF. See Problem 7.5.

Gauß elimination now iteratively applies elementary row operations until a row (reduced) echelon form is obtained. Mathematically spoken: In each step of the iteration we multiply a corresponding elementary matrix $\mathbf{T}_k$ from the left to the equation $\mathbf{T}_{k-1}\cdots\mathbf{T}_1\mathbf{A}\mathbf{x} = \mathbf{T}_{k-1}\cdots\mathbf{T}_1\mathbf{b}$. For practical reasons one usually uses the augmented coefficient matrix.

Theorem 7.9

For every matrix $\mathbf{A}$ there exists a sequence of elementary row operations $\mathbf{T}_1,\dots\mathbf{T}_k$ such that $\mathbf{R} = \mathbf{T}_k\cdots\mathbf{T}_1\mathbf{A}$ is in row (reduced) echelon form.

PROOF. See Problem 7.6.

For practical reasons one augments the coefficient matrix $\mathbf{A}$ of a linear equation by the constant vector $\mathbf{b}$. Thus the row operations can be performed on $\mathbf{A}$ and $\mathbf{b}$ simultaneously.

Definition 7.10

Let $\mathbf{A}\mathbf{x} = \mathbf{b}$ be a linear equation with coefficient matrix $\mathbf{A} = (\mathbf{a}_1,\dots\mathbf{a}_n)$. Then matrix $\mathbf{A}_b = (\mathbf{A},\mathbf{b}) = (\mathbf{a}_1,\dots\mathbf{a}_n,\mathbf{b})$ is called the **augmented coefficient matrix** of the linear equation.

When the coefficient matrix is in row echelon form, then the solution $\mathbf{x}$ of the linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be easily obtained by means of an iterative process called **back substitution**. When it is in row *reduced* echelon form it is even simpler: We get a particular solution $\mathbf{x}_0$ by setting all variables that belong to non-pivotal columns to 0. Then we solve the resulting linear equations for the variables that corresponds to the pivotal columns. This is easy as each row reduces to

$$\delta_i x_i = b_i \qquad \text{where } \delta_i \in \{0,1\}.$$

Obviously, these equations can be solved if and only if $\delta_i = 1$ or $b_i = 0$.

We then need a basis of $\ker(\mathbf{A})$ which we easily get from a row reduced echelon form of the homogeneous equation $\mathbf{A}\mathbf{x} = 0$. Notice, that $\ker(\mathbf{A}) = \{0\}$ if there are no non-pivotal columns.

7.3 Image, Kernel and Rank of a Matrix

Once the row reduced echelon form $\mathbf{R}$ is given for a matrix $\mathbf{A}$ we also can easily compute bases for its image and kernel.

Theorem 7.11 Let $\mathbf{R}$ be a row reduced echelon form of some matrix $\mathbf{A}$. Then $\operatorname{rank}(\mathbf{A})$ is equal to the number nonzero rows of $\mathbf{R}$.

PROOF. By Lemma 6.20 and Theorem 7.9, $\operatorname{rank}(\mathbf{R}) = \operatorname{rank}(\mathbf{A})$. It is easy to see that *non*-pivotal columns can be represented as linear combinations of pivotal columns. Hence the pivotal columns span $\operatorname{Im}(\mathbf{R})$. Moreover, the pivotal columns are linearly independent since no two of them have a common non-zero entry. The result then follows from the fact that the number of pivotal columns equal the number of nonzero elements. $\square$

Theorem 7.12 Let $\mathbf{R}$ be a row reduced echelon form of some matrix $\mathbf{A}$. Then the columns of $\mathbf{A}$ that correspond to pivotal columns of $\mathbf{R}$ form a basis of $\operatorname{Im}(\mathbf{A})$.

PROOF. The columns of $\mathbf{A}$ span $\operatorname{Im}(\mathbf{A})$. Let $\mathbf{A}_p$ consists of all columns of $\mathbf{A}$ that correspond to pivotal columns of $\mathbf{R}$. If we apply the same elementary row operations on $\mathbf{A}_p$ as for $\mathbf{A}$ we obtain a row reduced echelon form $\mathbf{R}_p$ where all columns are pivotal. Hence the columns of $\mathbf{A}_p$ are linearly independent and $\operatorname{rank}(\mathbf{A}_p) = \operatorname{rank}(\mathbf{A})$. Thus the columns of $\mathbf{A}_p$ form a basis of $\operatorname{Im}(\mathbf{A})$, as claimed. $\square$

At last we verify other observation about the existence of the solution of an inhomogeneous equation.

Theorem 7.13 Let $\mathbf{A}\mathbf{x} = \mathbf{b}$ be an inhomogeneous linear equation. Then there exists a solution $\mathbf{x}_0$ if and only if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}_b)$.

PROOF. Recall that $\mathbf{A}_b$ denotes the augmented coefficient matrix. If there exists a solution $\mathbf{x}_0$, then $\mathbf{b} = \mathbf{A}\mathbf{x}_0 \in \operatorname{Im}(\mathbf{A})$ and thus

$$\operatorname{rank}(\mathbf{A}_b) = \dim \operatorname{span}(\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}) = \dim \operatorname{span}(\mathbf{a}_1, \dots, \mathbf{a}_n) = \operatorname{rank}(\mathbf{A}) .$$

On the other hand, if no such solution exists, then $\mathbf{b} \not\in \operatorname{span}(\mathbf{a}_1, \dots, \mathbf{a}_k)$ and thus $\operatorname{span}(\mathbf{a}_1, \dots, \mathbf{a}_n) \subset \operatorname{span}(\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b})$. Consequently,

$$\operatorname{rank}(\mathbf{A}_b) = \dim \operatorname{span}(\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}) > \dim \operatorname{span}(\mathbf{a}_1, \dots, \mathbf{a}_n) = \operatorname{rank}(\mathbf{A})$$

and thus $\operatorname{rank}(\mathbf{A}_b) \neq \operatorname{rank}(\mathbf{A})$. $\square$

— Summary

- A *linear equation* is one that can be written as $\mathbf{A}\mathbf{x} = \mathbf{b}$.
- The set of all solutions of a *homogeneous* linear equation forms a *vector space*.
  The set of all solutions of an *inhomogeneous* linear equation forms an *affine space*.
- Linear equations can be solved by transforming the *augmented coefficient matrix* into *row (reduced) echelon form*.
- This transformation is performed by (invertible) *elementary row operations*.
- Bases of image and kernel of a matrix $\mathbf{A}$ as well as its rank can be computed by transforming the matrix into row reduced echelon form.

— Exercises

7.1 Compute image, kernel and rank of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

— Problems

7.2 Verify that a system of linear equations can indeed written in matrix form. Moreover show that each equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ represents a system of linear equations.

7.3 Prove Lemma 7.3 and Theorem 7.4.

7.4 Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}'_1 \\ \vdots \\ \mathbf{a}'_m \end{pmatrix}$ be an $m \times n$ matrix.

(1) Define matrix $\mathbf{T}_{i \leftrightarrow j}$ that switches rows $\mathbf{a}'_i$ and $\mathbf{a}'_j$.

(2) Define matrix $\mathbf{T}_i(\alpha)$ that multiplies row $\mathbf{a}'_i$ by α.

(3) Define matrix $\mathbf{T}_{i \leftarrow j}(\alpha)$ that adds row $\mathbf{a}'_j$ multiplied by α to row $\mathbf{a}'_i$.

For each of these matrices argue why these are invertible and state their respective inverse matrices.

HINT: Use the results from Exercise 4.14 to construct these matrices.

7.5 Prove Lemma 7.8.

7.6 Prove Theorem 7.9.
Use a so called *constructive* proof. In this case this means to provide an algorithm that transforms every input matrix $\mathbf{A}$ into row reduce echelon form by means of elementary row operations. Describe such an algorithm (in words or pseudo-code).

8

Euclidean Space

We need a ruler *and a* protractor.

8.1 Inner Product, Norm, and Metric

Definition 8.1

Inner product. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\mathbf{x}'\mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

is called the **inner product** (**dot product**, **scalar product**) of $\mathbf{x}$ and $\mathbf{y}$.

Theorem 8.2

Fundamental properties of inner products. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then the following holds:

(1) $\mathbf{x}'\mathbf{y} = \mathbf{y}'\mathbf{x}$ (Symmetry)

(2) $\mathbf{x}'\mathbf{x} \geq 0$ where equality holds if and only if $\mathbf{x} = 0$ (Positive-definiteness)

(3) $(\alpha\mathbf{x} + \beta\mathbf{y})'\mathbf{z} = \alpha\mathbf{x}'\mathbf{z} + \beta\mathbf{y}'\mathbf{z}$ (Linearity)

PROOF. Properities (i) and (ii) immediately follows from Definition 8.1. Property (iii) also follows from the rules for matrix algebra, see Theorem 4.14. □

In our notation the inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is just the usual matrix multiplication of the *row* vector $\mathbf{x}'$ with the *column* vector $\mathbf{y}$. However, the formal transposition of the first vector $\mathbf{x}$ is often omitted in the notation of the inner product. Thus one simply writes $\mathbf{x} \cdot \mathbf{y}$. Hence the name *dot* product. This is reflected in many computer algebra systems like *Maxima* where the symbol for matrix multiplication is used to multiply two (column) vectors.

Definition 8.3

Inner product space. The notion of an *inner product* can be generalized. Let $\mathcal{V}$ be some vector space. Then any function

$$\langle\cdot,\cdot\rangle : \mathcal{V}\times\mathcal{V}\to\mathbb{R}$$

that satisfies the properties

(i) $\langle\mathbf{x},\mathbf{y}\rangle = \langle\mathbf{y},\mathbf{x}\rangle$,

(ii) $\langle\mathbf{x},\mathbf{x}\rangle \geq 0$ where equality holds if and only if $\mathbf{x}=0$,

(iii) $\langle\alpha\mathbf{x}+\beta\mathbf{y},\mathbf{z}\rangle = \alpha\langle\mathbf{x},\mathbf{z}\rangle+\beta\langle\mathbf{y},\mathbf{z}\rangle$,

is called an **inner product**. A vector space that is equipped with such an inner product is called an **inner product space**. In pure mathematics the symbol $\langle\mathbf{x},\mathbf{y}\rangle$ is often used to denote the (abstract) inner product of two vectors $\mathbf{x},\mathbf{y}\in\mathcal{V}$.

Example 8.4

Let $\mathcal{L}$ be the vector space of all random variables X on some given probability space with finite variance $\mathbb{V}(X)$. Then map

$$\langle\cdot,\cdot\rangle : \mathcal{L}\times\mathcal{L}\to\mathbb{R},\ (X,Y)\mapsto\langle X,Y\rangle = \mathbb{E}(XY)$$

is an inner product in $\mathcal{L}$. ♢

Definition 8.5

Euclidean norm. Let $\mathbf{x}\in\mathbb{R}^n$. Then

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}'\mathbf{x}} = \sqrt{\sum_{i=1}^n x_i^2}$$

is called the **Euclidean norm** (or *norm* for short[1]) of $\mathbf{x}$.

Theorem 8.6

Cauchy-Schwarz inequality. Let $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Then

$$|\mathbf{x}'\mathbf{y}| \leq \|\mathbf{x}\|_2\cdot\|\mathbf{y}\|_2\ .$$

Equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

PROOF. The inequality trivially holds if $\mathbf{x}=0$ or $\mathbf{y}=0$. Assume that $\mathbf{y}\neq 0$. Then we find for any $\lambda\in\mathbb{R}$,

$$0 \leq (\mathbf{x}-\lambda\mathbf{y})'(\mathbf{x}-\lambda\mathbf{y}) = \mathbf{x}'\mathbf{x}-\lambda\mathbf{x}'\mathbf{y}-\lambda\mathbf{y}'\mathbf{x}+\lambda^2\mathbf{y}'\mathbf{y} = \mathbf{x}'\mathbf{x}-2\lambda\mathbf{x}'\mathbf{y}+\lambda^2\mathbf{y}'\mathbf{y}.$$

Using the special value $\lambda=\frac{\mathbf{x}'\mathbf{y}}{\mathbf{y}'\mathbf{y}}$ we obtain

$$0 \leq \mathbf{x}'\mathbf{x}-2\frac{\mathbf{x}'\mathbf{y}}{\mathbf{y}'\mathbf{y}}\mathbf{x}'\mathbf{y}+\frac{(\mathbf{x}'\mathbf{y})^2}{(\mathbf{y}'\mathbf{y})^2}\mathbf{y}'\mathbf{y} = \mathbf{x}'\mathbf{x}-\frac{(\mathbf{x}'\mathbf{y})^2}{\mathbf{y}'\mathbf{y}}.$$

Hence

$$(\mathbf{x}'\mathbf{y})^2 \leq (\mathbf{x}'\mathbf{x})(\mathbf{y}'\mathbf{y}) = \|\mathbf{x}\|_2^2\ \|\mathbf{y}\|_2^2\,.$$

[1] We use symbol $\|\mathbf{x}\|$ for short (i.e., omit subscript $_2$) if there is no risk of confusion.

or equivalently

$$|\mathbf{x}'\mathbf{y}| \leq \|\mathbf{x}\|_2 \, \|\mathbf{y}\|$$

as claimed.

Now if equality holds, then $\mathbf{x} - \lambda\mathbf{y} = 0$ for our choice of λ and thus $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. On the other hand if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent then w.l.o.g. there exists an $\alpha \in \mathbb{R}$ such that $\mathbf{x} = \alpha\mathbf{y}$. Then we find for our choice $\lambda = \frac{\mathbf{x}'\mathbf{y}}{\mathbf{y}'\mathbf{y}} = \alpha\frac{\mathbf{y}'\mathbf{y}}{\mathbf{y}'\mathbf{y}} = \alpha$ and consequently equality holds in every step. This completes our proof □

Theorem 8.7

Minkowski inequality. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2 \, .$$

PROOF. See Problem 8.2.

Theorem 8.8

Fundamental properties of norms. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then

(1) $\|\mathbf{x}\|_2 \geq 0$ where equality holds if and only if $\mathbf{x} = 0$ (Positive-definiteness)

(2) $\|\alpha\mathbf{x}\|_2 = |\alpha| \, \|\mathbf{x}\|_2$ (Positive scalability)

(3) $\|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$ (Triangle inequality or subadditivity)

PROOF. See Problem 8.4.

Definition 8.9

Normed vector space. The notion of a *norm* can be generalized. Let $\mathcal{V}$ be some vector space. Then any function

$$\|\cdot\| : \mathcal{V} \to \mathbb{R}$$

that satisfies properties

(i) $\|\mathbf{x}\| \geq 0$ where equality holds if and only if $\mathbf{x} = 0$

(ii) $\|\alpha\mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|$

(iii) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

is called a **norm**. A vector space that is equipped with such a norm is called a **normed vector space**.

Example 8.10

Other examples of norms of vectors $\mathbf{x} \in \mathbb{R}^n$ are the so called *1-norm*

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \, ,$$

the *p-norm*

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \, , \qquad \text{for } 1 \leq p < \infty,$$

and the *supremum norm*

$$\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i| \, .$$ ◊

Observe that the *Euclidean* norm of a vector $\mathbf{x}$ in Definition 8.5 is a special case of the p-norm with $p = 2$ (and thus denoted by $\|\mathbf{x}\|_2$). As this is the *"usual"* norm we just write $\|\mathbf{x}\|$ (i.e., omit subscript $_2$) if there is no risk of confusion.

Example 8.11

Let $\mathscr{L}$ be the vector space of all random variables X on some given probability space with finite variance $\mathbb{V}(X)$. Then map

$$\|\cdot\|_2 : \mathscr{L} \to [0,\infty),\ X \mapsto \|X\|_2 = \sqrt{\mathbb{E}(X^2)} = \sqrt{\langle X, X\rangle}$$

is a norm in $\mathscr{L}$. $\diamondsuit$

Definition 8.12

A vector $\mathbf{x} \in \mathbb{R}^n$ is called **normalized** if $\|\mathbf{x}\| = 1$.

In Definition 8.5 we used the inner product (Definition 8.1) to define the Euclidean norm. In fact we only needed the properties of the inner product to derive the properties of the Euclidean norm in Theorem 8.8 and the Cauchy-Schwarz inequality (Theorem 8.6). That is, every inner product induces a norm. However, there are also other norms that are not induced by inner products, e.g., the p-norms $\|\mathbf{x}\|_p$ for $p \neq 2$.

Definition 8.13

Euclidean metric. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $d_2(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_2$ defines the **Euclidean distance** between $\mathbf{x}$ and $\mathbf{y}$.

Theorem 8.14

Fundamental properties of metrics. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Then

(1) $d_2(\mathbf{x},\mathbf{y}) = d_2(\mathbf{y},\mathbf{x})$ (Symmetry)

(2) $d_2(\mathbf{x},\mathbf{y}) \geq 0$ where equality holds if and only if $\mathbf{x} = \mathbf{y}$ (Positive-definiteness)

(3) $d_2(\mathbf{x},\mathbf{z}) \leq d_2(\mathbf{x},\mathbf{y}) + d_2(\mathbf{y},\mathbf{z})$ (Triangle inequality)

PROOF. See Problem 8.8.

Definition 8.15

Metric space. The notion of a *metric* can be generalized. Let $\mathcal{V}$ be some vector space. Then any function

$$d(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$$

that satisfies properties

(i) $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$

(ii) $d(\mathbf{x},\mathbf{y}) \geq 0$ where equality holds if and only if $\mathbf{x} = \mathbf{y}$

(iii) $d(\mathbf{x},\mathbf{z}) \leq d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z})$

is called a **metric**. A vector space that is equipped with a metric is called a **metric vector space**.

Definition 8.13 (and the proof of Theorem 8.14) shows us that any norm induces a metric. However, there also exist metrics that are not induced by some norm.

Example 8.16

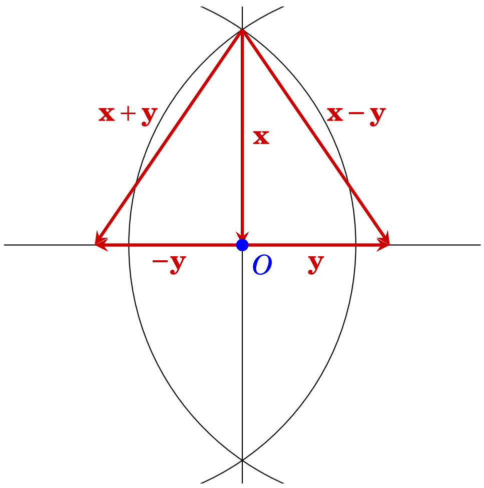
Let $\mathscr{L}$ be the vector space of all random variables X on some given probability space with finite variance $\mathbb{V}(X)$. Then the following maps are metrics in $\mathscr{L}$:

$$d_2\colon \mathscr{L} \times \mathscr{L} \to [0,\infty),\ (X,Y) \mapsto \|X-Y\| = \sqrt{\mathbb{E}((X-Y)^2)}$$
$$d_E\colon \mathscr{L} \times \mathscr{L} \to [0,\infty),\ (X,Y) \mapsto d_E(X,Y) = \mathbb{E}(|X-Y|)$$
$$d_F\colon \mathscr{L} \times \mathscr{L} \to [0,\infty),\ (X,Y) \mapsto d_F(X,Y) = \max\left|F_X(z) - F_Y(z)\right|$$

where F_X denotes the cumulative distribution function of X. $\diamondsuit$

8.2 Orthogonality

Assume we have a straight line through origin $O = (0,0)$. Recall from classical geometry that we can draw a perpendicular by a ruler-and-compass construction:



We want to have an algebraic definition of the notion of *perpendicularity*. So let $\mathbf{x}$ and $\mathbf{y}$ be the vectors from O to the intersection points and the centers of the two circle, resp. By construction these are *perpendicular* by construction. Observe that the red triangle is isosceles, i.e., $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$. The difference between the two sides of this triangle can be computed by means of an inner product (see Problem 8.10) as

$$\|\mathbf{x}+\mathbf{y}\|_2^2 - \|\mathbf{x}-\mathbf{y}\|_2^2 = 4\mathbf{x}'\mathbf{y}\,.$$

Definition 8.17

Two vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ are called **orthogonal** (**perpendicular**, **normal**) to each other if $\mathbf{x}'\mathbf{y} = 0$.

Theorem 8.18

Pythagorean theorem. Let $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ be two vectors that are orthogonal to each other. Then

$$\|\mathbf{x}+\mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\,.$$

PROOF. See Problem 8.11.

Lemma 8.19

Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be non-zero vectors. If these vectors are pairwise orthogonal to each other, then they are linearly independent.

PROOF. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent. Then w.l.o.g. there exist $\alpha_2, \dots, \alpha_k$ such that $\mathbf{v}_1 = \sum_{i=2}^{k} \alpha_i \mathbf{v}_i$. Then $\mathbf{v}_1' \mathbf{v}_1 = \mathbf{v}_1' \left(\sum_{i=2}^{k} \alpha_i \mathbf{v}_i \right) = \sum_{i=2}^{k} \alpha_i \mathbf{v}_1' \mathbf{v}_i = 0$, i.e., $\mathbf{v}_1 = 0$ by Theorem 8.2, a contradiction to our assumption that all vectors are non-zero. □

Definition 8.20

Orthonormal system. A set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{R}^n$ is called an **orthonormal system** if the following holds:

(i) the vectors are mutually orthogonal,

(ii) the vectors are normalized.

That is, $\mathbf{v}_i' \mathbf{v}_j = \delta_{ij}$.

Definition 8.21

Orthonormal basis. A basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{R}^n$ is called an **orthonormal basis** if it forms an orthonormal system.

Theorem 8.22

Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then the coefficient vector $\mathbf{c}(\mathbf{x})$ of some vector $\mathbf{x} \in \mathbb{R}^n$ with respect to B is given by

$$c_j(\mathbf{x}) = \mathbf{v}_j' \mathbf{x}.$$

PROOF. See Problem 8.12.

Definition 8.23

Orthogonal matrix. A square matrix $\mathbf{U}$ is called an **orthogonal matrix** if its columns form an orthonormal system.

Theorem 8.24

Let $\mathbf{U}$ be an $n \times n$ matrix. Then the following are equivalent:

(1) $\mathbf{U}$ is an orthogonal matrix.

(2) $\mathbf{U}'$ is an orthogonal matrix.

(3) $\mathbf{U}'\mathbf{U} = \mathbf{I}$, i.e., $\mathbf{U}^{-1} = \mathbf{U}'$.

(4) The linear map defined by $\mathbf{U}$ is an **isometry**, i.e., $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.

PROOF. Let $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$.

(1)⇒(3) $[\mathbf{U}'\mathbf{U}]_{ij} = \mathbf{u}_i' \mathbf{u}_j = \delta_{ij} = [\mathbf{I}]_{ij}$, i.e., $\mathbf{U}'\mathbf{U} = \mathbf{I}$. By Theorem 6.16, $\mathbf{U}'\mathbf{U} = \mathbf{U}\mathbf{U}' = \mathbf{I}$ and thus $\mathbf{U}^{-1} = \mathbf{U}'$.

(3)⇒(4) $\|\mathbf{U}\mathbf{x}\|_2^2 = (\mathbf{U}\mathbf{x})'(\mathbf{U}\mathbf{x}) = \mathbf{x}'\mathbf{U}'\mathbf{U}\mathbf{x} = \mathbf{x}'\mathbf{x} = \|\mathbf{x}\|_2^2$.

(4)⇒(1) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then by (4), $\|\mathbf{U}(\mathbf{x} - \mathbf{y})\|_2 = \|\mathbf{x} - \mathbf{y}\|_2$, or equivalently

$$\mathbf{x}'\mathbf{U}'\mathbf{U}\mathbf{x} - \mathbf{x}'\mathbf{U}'\mathbf{U}\mathbf{y} - \mathbf{y}'\mathbf{U}'\mathbf{U}\mathbf{x} + \mathbf{y}'\mathbf{U}'\mathbf{U}\mathbf{y} = \mathbf{x}'\mathbf{x} - \mathbf{x}'\mathbf{y} - \mathbf{y}'\mathbf{x} + \mathbf{y}'\mathbf{y}.$$

If we again apply (4) we can cancel out some terms on both side of this equation and obtain

$$-\mathbf{x}'\mathbf{U}'\mathbf{U}\mathbf{y} - \mathbf{y}'\mathbf{U}'\mathbf{U}\mathbf{x} = -\mathbf{x}'\mathbf{y} - \mathbf{y}'\mathbf{x}.$$

Notice that $\mathbf{x}'\mathbf{y} = \mathbf{y}'\mathbf{x}$ by Theorem 8.2. Similarly, $\mathbf{x}'\mathbf{U}'\mathbf{U}\mathbf{y} = (\mathbf{x}'\mathbf{U}'\mathbf{U}\mathbf{y})' = \mathbf{y}'\mathbf{U}'\mathbf{U}\mathbf{x}$, where the first equality holds as these are 1×1 matrices. The second equality follows from the properties of matrix multiplication (Theorem 4.17). Thus

$$\mathbf{x}'\mathbf{U}'\mathbf{U}\mathbf{y} = \mathbf{x}'\mathbf{y} = \mathbf{x}'\mathbf{I}\mathbf{y} \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Recall that $\mathbf{e}_i'\mathbf{U}' = \mathbf{u}_i'$ and $\mathbf{U}\mathbf{e}_j = \mathbf{u}_j$. Thus if we set $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_j$ we obtain

$$\mathbf{u}_i'\mathbf{u}_j = \mathbf{e}_i'\mathbf{U}'\mathbf{U}\mathbf{e}_j = \mathbf{e}_i'\mathbf{e}_j = \delta_{ij}$$

that is, the columns of $\mathbf{U}$ for an orthonormal system.

(2)⇒(3) Can be shown analogously to (1)⇒(3).

(3)⇒(2) Let $\mathbf{v}_1', \ldots, \mathbf{v}_n'$ denote the rows of $\mathbf{U}$. Then

$$\mathbf{v}_i'\mathbf{v}_j = [\mathbf{U}\mathbf{U}']_{ij} = [\mathbf{I}]_{ij} = \delta_{ij}$$

i.e., the rows of $\mathbf{U}$ form an orthonormal system.

This completes the proof. □

— Summary

- An *inner product* is a bilinear symmetric positive definite function $\mathcal{V} \times \mathcal{V} \to \mathbb{R}$. It can be seen as a measure for the angle between two vectors.

- Two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* (perpendicular, normal) to each other, if their inner product is 0.

- A *norm* is a positive definite, positive scalable function $\mathcal{V} \to [0, \infty)$ that satisfies the triangle inequality $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. It can be seen as the length of a vector.

- Every inner product induces a norm: $\|x\|_2 = \sqrt{\mathbf{x}'\mathbf{x}}$.

  Then the *Cauchy-Schwarz inequality* $|\mathbf{x}'\mathbf{y}| \leq \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2$ holds for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

  If in addition $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ are orthogonal, then the *Pythagorean theorem* $\|\mathbf{x}+\mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2$ holds.

- A *metric* is a bilinear symmetric positive definite function $\mathcal{V} \times \mathcal{V} \to [0, \infty)$ that satisfies the triangle inequality. It measures the distance between two vectors.

- Every norm induces a metric.
- A metric that is induced by an inner product is called an *Euclidean metric*.
- Set of vectors that are mutually orthogonal and have norm 1 is called an *orthonormal system*.
- An *orthogonal matrix* is whose columns form an orthonormal system. Orthogonal maps preserve angles and norms.

— Problems

8.1 Let $\mathbf{A}$ be a symmetric positive definite matrix, i.e., the corresponding quadratic form is positive: $q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$.
Show that $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} = \mathbf{x}'\mathbf{A}\mathbf{y}$ is an inner product.

8.2 (a) The Minkowski inequality is also called **triangle inequality**. Draw a picture that illustrates this inequality.

(b) Prove Theorem 8.7.

(c) When does equality holds in the Minkowski inequality?

HINT: Compute $\|\mathbf{x}+\mathbf{y}\|_2^2$ and apply the Cauchy-Schwarz inequality.

8.3 Show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\Big| \|\mathbf{x}\| - \|\mathbf{y}\| \Big| \leq \|\mathbf{x} - \mathbf{y}\| \,.$$

HINT: Use the simple observation that $\mathbf{x} = (\mathbf{x}-\mathbf{y})+\mathbf{y}$ and $\mathbf{y} = (\mathbf{y}-\mathbf{x})+\mathbf{x}$ and apply the Minkowski inequality.

8.4 Prove Theorem 8.8. Draw a picture that illustrates property (iii).

HINT: Use Theorems 8.2 and 8.7.

8.5 Let $\mathbf{x} \in \mathbb{R}^n$ be a non-zero vector. Show that $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$ is a normalized vector. Is the condition $\mathbf{x} \neq 0$ necessary? Why? Why not?

8.6 Which of the following functions $\mathbb{R}^2 \to [0,\infty)$ are norms in $\mathbb{R}^2$. Prove your claims.

(a) $h: \mathbb{R}^2 \to [0,\infty)$, $(x,y) \mapsto h(x,y) = |x|$

(b) $g: \mathbb{R}^2 \to [0,\infty)$, $(x,y) \mapsto g(x,y) = 2|x| + 3|y|$

(c) $k: \mathbb{R}^2 \to [0,\infty)$, $(x,y) \mapsto k(x,y) = \left(\sqrt{|x|} + \sqrt{|y|}\right)^2$

8.7 (a) Show that $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_\infty$ satisfy the properties of a norm.

(b) Draw the unit balls in $\mathbb{R}^2$, i.e., the sets $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1\}$, with respect to the norms $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$.

(c) Use a computer algebra system of your choice (e.g., *Maxima*) and draw unit balls with respect to the p-norm for various values of p. What do you observe?

8.8 Prove Theorem 8.14. Draw a picture that illustrates property (iii).

HINT: Use Theorem 8.8 and the simple equality $\mathbf{x}-\mathbf{z} = (\mathbf{x}-\mathbf{y})+(\mathbf{y}-\mathbf{z})$.

8.9 Show that

$$d(\mathbf{x},\mathbf{y}) = \begin{cases} 1, & \text{if } \mathbf{x} \neq \mathbf{y}, \\ 0, & \text{if } \mathbf{x} = \mathbf{y}, \end{cases}$$

is a metric.

8.10 Show that $\|\mathbf{x}+\mathbf{y}\|_2^2 - \|\mathbf{x}-\mathbf{y}\|_2^2 = 4\mathbf{x}'\mathbf{y}$.

HINT: Use $\|\mathbf{x}\|_2^2 = \mathbf{x}'\mathbf{x}$.

8.11 Prove Theorem 8.18.

HINT: Use $\|\mathbf{x}\|_2^2 = \mathbf{x}'\mathbf{x}$.

8.12 Prove Theorem 8.22.

HINT: Represent $\mathbf{x}$ by means of $\mathbf{c}(\mathbf{x})$ and compute $\mathbf{x}'\mathbf{v}_j$.

9

Projections

To them, I said, the truth would be literally nothing but the shadows *of the images.*

Suppose we are given a subspace $\mathcal{U} \subset \mathbb{R}^n$ and a vector $\mathbf{y} \in \mathbb{R}^n$. We want to find a vector $\mathbf{u} \in \mathcal{U}$ such that the "error" $\mathbf{r} = \mathbf{y} - \mathbf{u}$ is as small as possible. This procedure is of great importance when we want to reduce the number of dimensions in our model without loosing too much information.

9.1 Orthogonal Projection

We first look at the simplest case $\mathcal{U} = \text{span}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$ is some fixed normalized vector, i.e., $\|\mathbf{x}\| = 1$. Then every $\mathbf{u} \in \mathcal{U}$ can be written as $\lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$.

Lemma 9.1

Let $\mathbf{y}, \mathbf{x} \in \mathbb{R}^n$ be fixed with $\|\mathbf{x}\| = 1$. Let $\mathbf{r} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$\mathbf{y} = \lambda\mathbf{x} + \mathbf{r}\,.$$

Then for $\lambda = \lambda^*$ and $\mathbf{r} = \mathbf{r}^*$ the following statements are equivalent:

(1) $\|\mathbf{r}^*\|$ is minimal among all values for λ and $\mathbf{r}$.

(2) $\mathbf{x}'\mathbf{r}^* = 0$.

(3) $\lambda^* = \mathbf{x}'\mathbf{y}$.

Proof. (2) $\Leftrightarrow$ (3): Observe that by construction $\mathbf{r} = \mathbf{y} - \lambda\mathbf{x}$. Then we find: $\mathbf{x}'\mathbf{r} = 0 \Leftrightarrow \mathbf{x}'(\mathbf{y} - \lambda\mathbf{x}) = 0 \Leftrightarrow \mathbf{x}'\mathbf{y} - \lambda\mathbf{x}'\mathbf{x} = 0 \Leftrightarrow \mathbf{x}'\mathbf{y} - \lambda = 0 \Leftrightarrow \lambda = \mathbf{x}'\mathbf{y}$ which implies the equivalence.
(2) $\Rightarrow$ (1): Assume that $\mathbf{x}'\mathbf{r}^* = 0$ and λ^* such that $\mathbf{r}^* = \mathbf{y} - \lambda^*\mathbf{x}$. Set $\mathbf{r}(\varepsilon) = \mathbf{y} - (\lambda^* + \varepsilon)\mathbf{x} = (\mathbf{y} - \lambda^*\mathbf{x}) - \varepsilon\mathbf{x} = \mathbf{r}^* - \varepsilon\mathbf{x}$ for $\varepsilon \in \mathbb{R}$. As $\mathbf{r}^*$ and $\mathbf{x}$ are orthogonal by our assumption, the Pythagorean theorem implies $\|\mathbf{r}(\varepsilon)\|^2 = \|\mathbf{r}^*\|^2 + \varepsilon^2\|\mathbf{x}\|^2 = \|\mathbf{r}^*\|^2 + \varepsilon^2$. Thus $\|\mathbf{r}(\varepsilon)\| \geq \|\mathbf{r}^*\|$ where equality holds if and only if $\varepsilon = 0$. Thus $\mathbf{r}^*$ minimizes $\|\mathbf{r}\|$.

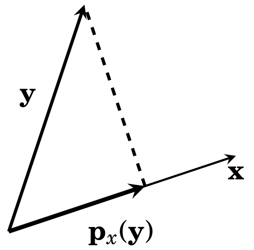
(1) ⇒ (2): Assume that $\mathbf{r}^*$ minimizes $\|\mathbf{r}\|$ and λ^* such that $\mathbf{r}^* = \mathbf{y} - \lambda^*\mathbf{x}$. Set $\mathbf{r}(\varepsilon) = \mathbf{y} - (\lambda^* + \varepsilon)\mathbf{x} = \mathbf{r}^* - \varepsilon\mathbf{x}$ for $\varepsilon \in \mathbb{R}$. Our assumption implies that $\|\mathbf{r}^*\|^2 \le \|\mathbf{r}^* - \varepsilon\mathbf{x}\|^2 = \|\mathbf{r}^*\|^2 - 2\varepsilon\mathbf{x}'\mathbf{r} + \varepsilon^2\|\mathbf{x}\|^2$ for all $\varepsilon \in \mathbb{R}$. Thus $2\varepsilon\mathbf{x}'\mathbf{r} \le \varepsilon^2\|\mathbf{x}\|^2 = \varepsilon^2$. Since ε may have positive and negative sign we find $-\frac{\varepsilon}{2} \le \mathbf{x}'\mathbf{r} \le \frac{\varepsilon}{2}$ for all $\varepsilon \ge 0$ and hence $\mathbf{x}'\mathbf{r} = 0$, as claimed. □

Definition 9.2

Orthogonal projection. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be two vectors with $\|\mathbf{x}\| = 1$. Then

$$\mathbf{p}_x(\mathbf{y}) = (\mathbf{x}'\mathbf{y})\mathbf{x}$$

is called the **orthogonal projection** of $\mathbf{y}$ onto the linear span of $\mathbf{x}$.



Theorem 9.3

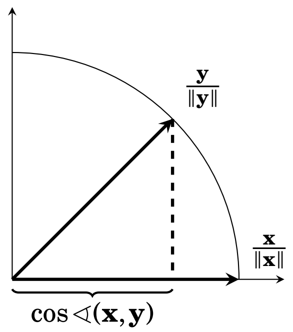
Orthogonal decomposition. Let $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| = 1$. Then every $\mathbf{y} \in \mathbb{R}^n$ can be uniquely decomposed as

$$\mathbf{y} = \mathbf{u} + \mathbf{v}$$

where $\mathbf{u} \in \operatorname{span}(\mathbf{x})$ and $\mathbf{v}$ is orthogonal to $\mathbf{x}$ (and hence orthogonal to $\mathbf{u}$), that is $\mathbf{u}'\mathbf{v} = 0$. Such a representation is called an **orthogonal decomposition** of $\mathbf{y}$. Moreover, $\mathbf{u}$ is given by

$$\mathbf{u} = \mathbf{p}_x(\mathbf{y}).$$

PROOF. Let $\mathbf{u} = \lambda\mathbf{x} \in \operatorname{span}(\mathbf{x})$ with $\lambda = \mathbf{x}'\mathbf{y}$ and $\mathbf{v} = \mathbf{y} - \mathbf{u}$. Obviously, $\mathbf{u} + \mathbf{v} = \mathbf{y}$. By Lemma 9.1, $\mathbf{u}'\mathbf{v} = 0$ and $\mathbf{u} = \mathbf{p}_x(\mathbf{y})$. Moreover, no other value of λ has this property. □



Now let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ and let $\lambda = \mathbf{x}'\mathbf{y}$. Then $|\lambda| = \|\mathbf{p}_x(\mathbf{y})\|$ and λ is positive if $\mathbf{x}$ and $\mathbf{p}_x(\mathbf{y})$ have the same orientation and negative if $\mathbf{x}$ and $\mathbf{p}_x(\mathbf{y})$ have opposite orientation. Thus by a geometric argument, λ is just the cosine of the angle between these vectors, i.e., $\cos\sphericalangle(\mathbf{x},\mathbf{y}) = \mathbf{x}'\mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are arbitrary non-zero vectors these have to be normalized. We then find

$$\cos\sphericalangle(\mathbf{x},\mathbf{y}) = \frac{\mathbf{x}'\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

Projection matrix. Let $\mathbf{x} \in \mathbb{R}^n$ be fixed with $\|\mathbf{x}\| = 1$. Then $\mathbf{y} \mapsto \mathbf{p}_x(\mathbf{y})$ is a linear map and $\mathbf{p}_x(\mathbf{y}) = \mathbf{P}_x\mathbf{y}$ where $\mathbf{P}_x = \mathbf{x}\mathbf{x}'$. Theorem 9.4

PROOF. Let $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then

$$\begin{aligned}\mathbf{p}_x(\alpha_1\mathbf{y}_1 + \alpha_2\mathbf{y}_2) &= \big(\mathbf{x}'(\alpha_1\mathbf{y}_1 + \alpha_2\mathbf{y}_2)\big)\mathbf{x} = \big(\alpha_1\mathbf{x}'\mathbf{y}_1 + \alpha_2\mathbf{x}'\mathbf{y}_2\big)\mathbf{x} \\ &= \alpha_1(\mathbf{x}'\mathbf{y}_1)\mathbf{x} + \alpha_2(\mathbf{x}'\mathbf{y}_2)\mathbf{x} = \alpha_1\mathbf{p}_x(\mathbf{y}_1) + \alpha_2\mathbf{p}_x(\mathbf{y}_2)\end{aligned}$$

and thus $\mathbf{p}_x$ is a linear map and there exists a matrix $\mathbf{P}_x$ such that $\mathbf{p}_x(\mathbf{y}) = \mathbf{P}_x\mathbf{y}$ by Theorem 6.15.
Notice that $\alpha\mathbf{x} = \mathbf{x}\alpha$ for $\alpha \in \mathbb{R} = \mathbb{R}^1$. Thus $\mathbf{P}_x\mathbf{y} = (\mathbf{x}'\mathbf{y})\mathbf{x} = \mathbf{x}(\mathbf{x}'\mathbf{y}) = (\mathbf{x}\mathbf{x}')\mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n$ and the result follows. □

If we project some vector $\mathbf{y} \in \operatorname{span}(\mathbf{x})$ onto $\operatorname{span}(\mathbf{x})$ then $\mathbf{y}$ remains unchanged, i.e., $\mathbf{P}_x(\mathbf{y}) = \mathbf{y}$. Thus the projection matrix $\mathbf{P}_x$ has the property that $\mathbf{P}_x^2\mathbf{z} = \mathbf{P}_x(\mathbf{P}_x\mathbf{z}) = \mathbf{P}_x\mathbf{z}$ for every $\mathbf{z} \in \mathbb{R}^n$ (see Problem 9.5).

A square matrix $\mathbf{A}$ is called **idempotent** if $\mathbf{A}^2 = \mathbf{A}$. Definition 9.5

9.2 Gram-Schmidt Orthonormalization

Theorem 8.22 shows that we can easily compute the coefficient vector $\mathbf{c}(\mathbf{x})$ of a vector $\mathbf{x}$ by means of projections when the given basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ forms an orthonormal system:

$$\mathbf{x} = \sum_{i=1}^{n} c_i(\mathbf{x})\mathbf{v}_i = \sum_{i=1}^{n} (\mathbf{v}_i'\mathbf{x})\mathbf{v}_i = \sum_{i=1}^{n} \mathbf{p}_{v_i}(\mathbf{x})\,.$$

Hence orthonormal bases are quite convenient. Theorem 9.3 allows us to transform any two linearly independent vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ into two orthogonal vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ which then can be normalized. This idea can be generalized to any number of linear independent vectors by means of a recursion, called **Gram-Schmidt process**.

Gram-Schmidt orthonormalization. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis of some subspace $\mathscr{U}$. Define $\mathbf{v}_k$ recursively for $k = 1, \dots, n$ by Theorem 9.6

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{u}_1, & \mathbf{v}_1 &= \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} \\
\mathbf{w}_2 &= \mathbf{u}_2 - \mathbf{p}_{v_1}(\mathbf{u}_2), & \mathbf{v}_2 &= \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} \\
\mathbf{w}_3 &= \mathbf{u}_3 - \mathbf{p}_{v_1}(\mathbf{u}_3) - \mathbf{p}_{v_2}(\mathbf{u}_3), & \mathbf{v}_3 &= \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} \\
&\vdots & &\vdots \\
\mathbf{w}_n &= \mathbf{u}_n - \sum_{j=1}^{n-1} \mathbf{p}_{v_j}(\mathbf{u}_n), & \mathbf{v}_n &= \frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}
\end{aligned}$$

where $\mathbf{p}_{v_j}$ is the orthogonal projection from Definition 9.2, that is, $\mathbf{p}_{v_j}(\mathbf{u_k}) = (\mathbf{v}_j'\mathbf{u}_k)\mathbf{v}_j$. Then set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ forms an orthonormal basis of $\mathscr{U}$.

PROOF. We proceed by induction on k and show that $\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ form an orthonormal basis for $\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_k)$ for all $k=1,\dots,n$.
For $k=1$ the statement is obvious as $\operatorname{span}(\mathbf{v}_1)=\operatorname{span}(\mathbf{u}_1)$ and $\|\mathbf{v}_1\|=1$. Now suppose the result holds for $k\geq 1$. By the induction hypothesis, $\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ forms an orthonormal basis for $\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_k)$. In particular we have $\mathbf{v}'_j\mathbf{v}_i=\delta_{ji}$. Let

$$\mathbf{w}_{k+1}=\mathbf{u}_{k+1}-\sum_{j=1}^{k}\mathbf{p}_{v_j}(\mathbf{u}_{k+1})=\mathbf{u}_{k+1}-\sum_{j=1}^{k}(\mathbf{v}'_j\mathbf{u}_{k+1})\mathbf{v}_j\,.$$

First we show that $\mathbf{w}_{k+1}$ and $\mathbf{v}_i$ are orthogonal for all $i=1,\dots,k$. By construction we have

$$\begin{aligned}
\mathbf{w}'_{k+1}\mathbf{v}_i &= \left(\mathbf{u}_{k+1}-\sum_{j=1}^{k}(\mathbf{v}'_j\mathbf{u}_{k+1})\mathbf{v}_j\right)'\mathbf{v}_i\\
&=\mathbf{u}'_{k+1}\mathbf{v}_i-\sum_{j=1}^{k}(\mathbf{v}'_j\mathbf{u}_{k+1})\mathbf{v}'_j\mathbf{v}_i\\
&=\mathbf{u}'_{k+1}\mathbf{v}_i-\sum_{j=1}^{k}(\mathbf{v}'_j\mathbf{u}_{k+1})\delta_{ji}\\
&=\mathbf{u}'_{k+1}\mathbf{v}_i-\mathbf{v}'_i\mathbf{u}_{k+1}\\
&=0\,.
\end{aligned}$$

Now $\mathbf{w}_{k+1}$ cannot be 0 since otherwise $\mathbf{u}_{k+1}-\sum_{j=1}^{k}\mathbf{p}_{v_j}(\mathbf{u}_{k+1})=0$ and consequently $\mathbf{u}_{k+1}\in\operatorname{span}(\mathbf{v}_1,\dots,\mathbf{v}_k)=\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_k)$, a contradiction to our assumption that $\{\mathbf{u}_1,\dots,\mathbf{u}_k,\mathbf{u}_{k+1}\}$ is a subset of a basis of $\mathcal{U}$. Thus we may take $\mathbf{v}_{k+1}=\frac{\mathbf{w}_{k+1}}{\|\mathbf{w}_{k+1}\|}$. Then by Lemma 8.19 the vectors $\{\mathbf{v}_1,\dots,\mathbf{v}_{k+1}\}$ are linearly independent and consequently form a basis for $\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_{k+1})$ by Theorem 5.22. Thus the result holds for $k+1$, and by the principle of induction, for all $k=1,\dots,n$ and in particular for $k=n$. □

9.3 Orthogonal Complement

We want to generalize Theorem 9.3 and Lemma 9.1. Thus we need the concepts of the *direct sum* of two vector spaces and of the *orthogonal complement*.

Definition 9.7

Direct sum. Let $\mathcal{U},\mathcal{V}\subseteq\mathbb{R}^n$ be two subspaces with $\mathcal{U}\cap\mathcal{V}=\{0\}$. Then

$$\mathcal{U}\oplus\mathcal{V}=\{\mathbf{u}+\mathbf{v}\colon \mathbf{u}\in\mathcal{U},\mathbf{v}\in\mathcal{V}\}$$

is called the **direct sum** of $\mathcal{U}$ and $\mathcal{V}$.

Lemma 9.8

Let $\mathcal{U},\mathcal{V}\subseteq\mathbb{R}^n$ be two subspaces with $\mathcal{U}\cap\mathcal{V}=\{0\}$ and $\dim(\mathcal{U})=k\geq 1$ and $\dim(\mathcal{V})=l\geq 1$. Let $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ and $\{\mathbf{v}_1,\dots,\mathbf{v}_l\}$ be bases of $\mathcal{U}$ and $\mathcal{V}$, respectively. Then $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}\cup\{\mathbf{v}_1,\dots,\mathbf{v}_l\}$ is a basis of $\mathcal{U}\oplus\mathcal{V}$.

PROOF. Obviously $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}\cup\{\mathbf{v}_1,\dots,\mathbf{v}_l\}$ is a generating set of $\mathcal{U}\oplus\mathcal{V}$. We have to show that this set is linearly independent. Suppose it is linearly dependent. Then we find $\alpha_1,\dots,\alpha_k\in\mathbb{R}$ not all zero and $\beta_1,\dots,\beta_l\in\mathbb{R}$ not all zero such that $\sum_{i=1}^k\alpha_i\mathbf{u}_i+\sum_{i=1}^l\beta_i\mathbf{v}_i=0$. Then $\mathbf{u}=\sum_{i=1}^k\alpha_i\mathbf{u}_i\neq 0$ and $\mathbf{v}=-\sum_{i=1}^l\beta_i\mathbf{v}_i\neq 0$ where $\mathbf{u}\in\mathcal{U}$ and $\mathbf{v}\in\mathcal{V}$ and $\mathbf{u}-\mathbf{v}=0$. But then $\mathbf{u}=\mathbf{v}$, a contradiction to the assumption that $\mathcal{U}\cap\mathcal{V}=\{0\}$. □

Lemma 9.9

Decomposition of a vector. Let $\mathcal{U},\mathcal{V}\subseteq\mathbb{R}^n$ be two subspaces with $\mathcal{U}\cap\mathcal{V}=\{0\}$ and $\mathcal{U}\oplus\mathcal{V}=\mathbb{R}^n$. Then every $\mathbf{x}\in\mathbb{R}^n$ can be uniquely decomposed into

$$\mathbf{x}=\mathbf{u}+\mathbf{v}$$

where $\mathbf{u}\in\mathcal{U}$ and $\mathbf{v}\in\mathcal{V}$.

PROOF. See Problem 9.7.

Definition 9.10

Orthogonal complement. Let $\mathcal{U}$ be a subspace of $\mathbb{R}^n$. Then the **orthogonal complement** of $\mathcal{U}$ in $\mathbb{R}^n$ is the set of vectors $\mathbf{v}$ that are orthogonal to all vectors in $\mathbb{R}^n$, that is,

$$\mathcal{U}^\perp=\{\mathbf{v}\in\mathbb{R}^n:\mathbf{u}'\mathbf{v}=0\text{ for all }\mathbf{u}\in\mathcal{U}\}.$$

Lemma 9.11

Let $\mathcal{U}$ be a subspace of $\mathbb{R}^n$. Then the orthogonal complement $\mathcal{U}^\perp$ is also a subspace of $\mathbb{R}^n$. Furthermore, $\mathcal{U}\cap\mathcal{U}^\perp=\{0\}$.

PROOF. See Problem 9.8.

Lemma 9.12

Let $\mathcal{U}$ be a subspace of $\mathbb{R}^n$. Then

$$\mathbb{R}^n=\mathcal{U}\oplus\mathcal{U}^\perp.$$

PROOF. Suppose there exists a non-zero $\mathbf{x}\in\mathbb{R}^n\setminus(\mathcal{U}\oplus\mathcal{U}^\perp)$. Then $\{\mathbf{x}\}\cup B\cup B_\perp$ is linearly independent by Theorem 5.19 where B and $B_\perp$ are bases of $\mathcal{U}$ and $\mathcal{U}^\perp$, resp. By means of Gram-Schmidt orthonormalization (Thm. 9.6) we can construct a non-zero $\mathbf{y}\not\in\operatorname{span}(B\cup B_\perp)=\mathcal{U}\oplus\mathcal{U}^\perp$ which is perpendicular to each element in B. Hence $\mathbf{y}\in\mathcal{U}^\perp\subseteq\mathcal{U}\oplus\mathcal{U}^\perp$, a contradiction. □

Theorem 9.13

Orthogonal decomposition. Let $\mathcal{U}$ be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}\in\mathbb{R}^n$ can be uniquely decomposed into

$$\mathbf{y}=\mathbf{u}+\mathbf{u}^\perp$$

where $\mathbf{u}\in\mathcal{U}$ and $\mathbf{u}^\perp\in\mathcal{U}^\perp$. $\mathbf{u}$ is called the **orthogonal projection** of $\mathbf{y}$ into $\mathcal{U}$. We denote this projection by $\mathbf{p}_U(\mathbf{y})$.

PROOF. By Lemmata 9.11 and 9.12 we have $\mathcal{U}\cap\mathcal{U}^\perp=\{0\}$ and $\mathbb{R}^n=\mathcal{U}\oplus\mathcal{U}^\perp$. Thus every $\mathbf{y}\in\mathbb{R}^n$ can be uniquely decomposed into $\mathbf{y}=\mathbf{u}+\mathbf{u}^\perp$ by Lemma 9.9. □

It remains to derive a formula for computing this orthogonal projection. Thus we derive a generalization of and Lemma 9.1.

Theorem 9.14

Projection into subspace. Let $\mathscr{U}$ be a subspace of $\mathbb{R}^n$ with generating set $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ and $\mathbf{U}=(\mathbf{u}_1,\dots,\mathbf{u}_k)$. For a fixed vector $\mathbf{y}\in\mathbb{R}^n$ let $\mathbf{r}\in\mathbb{R}^n$ and $\boldsymbol{\lambda}\in\mathbb{R}^k$ such that

$$\mathbf{y}=\mathbf{U}\boldsymbol{\lambda}+\mathbf{r}\,.$$

Then for $\boldsymbol{\lambda}=\boldsymbol{\lambda}^*$ and $\mathbf{r}=\mathbf{r}^*$ the following statements are equivalent:

(1) $\|\mathbf{r}^*\|$ is minimal among all possible values for $\boldsymbol{\lambda}$ and $\mathbf{r}$.

(2) $\mathbf{U}'\mathbf{r}^*=0$, that is, $\mathbf{r}^*\in\mathscr{U}^\perp$.

(3) $\mathbf{U}'\mathbf{U}\boldsymbol{\lambda}^*=\mathbf{U}'\mathbf{y}$.

Notice that $\mathbf{u}^*=\mathbf{U}\boldsymbol{\lambda}^*\in\mathscr{U}$.

PROOF. Equivalence of (2) and (3) follows by a straightforward computation (see Problem 9.9).
Now for $\boldsymbol{\varepsilon}\in\mathbb{R}^k$ define $\mathbf{r}(\boldsymbol{\varepsilon})=\mathbf{r}^*-\mathbf{U}\boldsymbol{\varepsilon}$. Recall that $\mathbf{U}\boldsymbol{\varepsilon}\in\mathscr{U}$. If (2) holds, i.e., $\mathbf{r}^*\in\mathscr{U}^\perp$, then the Pythagorean Theorem implies $\|\mathbf{r}(\boldsymbol{\varepsilon})\|^2=\|\mathbf{r}^*\|^2+\|\mathbf{U}\boldsymbol{\varepsilon}\|^2\geq\|\mathbf{r}^*\|^2$ for all $\boldsymbol{\varepsilon}$ and (1) follows.
Conversely, if (1) holds then $\|\mathbf{r}^*\|^2\leq\|\mathbf{r}^*-\mathbf{U}\boldsymbol{\varepsilon}\|^2=(\mathbf{r}^*-\mathbf{U}\boldsymbol{\varepsilon})'(\mathbf{r}^*-\mathbf{U}\boldsymbol{\varepsilon})=\|\mathbf{r}^*\|^2-2\boldsymbol{\varepsilon}'\mathbf{U}'\mathbf{r}^*+\|\mathbf{U}\boldsymbol{\varepsilon}\|^2$ and we find $0\leq\|\mathbf{U}\boldsymbol{\varepsilon}\|^2-2(\mathbf{U}\boldsymbol{\varepsilon})'\mathbf{r}^*=(\mathbf{U}\boldsymbol{\varepsilon})'(\mathbf{U}\boldsymbol{\varepsilon}-2\mathbf{r}^*)$ for all $\boldsymbol{\varepsilon}$. Now by Theorem 9.13, there exist $\mathbf{v}\in\mathscr{U}$ and $\mathbf{w}\in\mathscr{U}^\perp$ such that $\mathbf{r}^*=\mathbf{v}+\mathbf{w}$. Furthermore, there exists an $\boldsymbol{\varepsilon}$ such that $\mathbf{U}\boldsymbol{\varepsilon}=\mathbf{v}$. We then find $0\leq(\mathbf{U}\boldsymbol{\varepsilon})'(\mathbf{U}\boldsymbol{\varepsilon}-2\mathbf{r}^*)=\mathbf{v}'(\mathbf{v}-2(\mathbf{v}+\mathbf{w}))=-\mathbf{v}'\mathbf{v}-2\mathbf{v}'\mathbf{w}=-\|\mathbf{v}\|^2$ and hence $\mathbf{v}=0$. That is, $\mathbf{r}^*=\mathbf{w}\in\mathscr{U}^\perp$ and (2) follows. □

Equation (3) in Theorem 9.14 can be transformed when $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ are linearly independent, i.e., when it forms a basis of $\mathscr{U}$. Then the $n\times k$ matrix $\mathbf{U}=(\mathbf{u}_1,\dots,\mathbf{u}_k)$ has rank k and the $k\times k$ matrix $\mathbf{U}'\mathbf{U}$ also has rank k by Lemma 6.25 and is thus invertible.

Theorem 9.15

Let $\mathscr{U}$ be a subspace of $\mathbb{R}^n$ with basis $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ and $\mathbf{U}=(\mathbf{u}_1,\dots,\mathbf{u}_k)$. Then the orthogonal projection $\mathbf{y}\in\mathbb{R}^n$ onto $\mathscr{U}$ is given by

$$\mathbf{p}_U(\mathbf{y})=\mathbf{U}(\mathbf{U}'\mathbf{U})^{-1}\mathbf{U}'\mathbf{y}\,.$$

If in addition $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ forms an orthonormal system we find

$$\mathbf{p}_U(\mathbf{y})=\mathbf{U}\mathbf{U}'\mathbf{y}\,.$$

PROOF. See Problem 9.11.

9.4 Approximate Solutions of Linear Equations

Let $\mathbf{A}$ be an $n \times k$ matrix and $\mathbf{b} \in \mathbb{R}^n$. Suppose there is no $\mathbf{x} \in \mathbb{R}^k$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

that is, the linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ does not have a solution. Nevertheless, we may want to find an *approximate* solution $\mathbf{x}_0 \in \mathbb{R}^k$ that minimizes the error $\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{x}$ among all $\mathbf{x} \in \mathbb{R}^k$.

By Theorem 9.14 this task can be solved by means of orthogonal projections $\mathbf{p}_A(\mathbf{b})$ onto the linear span $\mathscr{A}$ of the column vectors of $\mathbf{A}$, i.e., we have to find $\mathbf{x}_0$ such that

$$\mathbf{A}'\mathbf{A}\mathbf{x}_0 = \mathbf{A}'\mathbf{b}\,. \tag{9.1}$$

Notice that by Theorem 9.13 there always exists an $\mathbf{r}$ such that $\mathbf{b} = \mathbf{p}_A(\mathbf{b}) + \mathbf{r}$ with $\mathbf{r} \in \mathscr{A}^\perp$ and hence an $\mathbf{x}_0$ exists such that $\mathbf{p}_A(\mathbf{b}) = \mathbf{A}\mathbf{x}_0$. Thus Equation (9.1) always has a solution by Theorem 9.14.

9.5 Applications in Statistics

Let $\mathbf{x} = (x_1, \dots, x_n)'$ be a given set of data and let $\mathbf{j} = (1, \dots, 1)'$ denote a vector of length n of ones. Notice that $\|\mathbf{j}\|^2 = n$. Then we can express the arithmetic mean $\bar{x}$ of the x_i as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\mathbf{j}'\mathbf{x}$$

and we find

$$\mathbf{p}_j(\mathbf{x}) = \left(\frac{1}{\sqrt{n}}\mathbf{j}'\mathbf{x}\right)\left(\frac{1}{\sqrt{n}}\mathbf{j}\right) = \left(\frac{1}{n}\mathbf{j}'\mathbf{x}\right)\mathbf{j} = \bar{x}\mathbf{j}\,.$$

That is, the arithmetic mean $\bar{x}$ is $\frac{1}{\sqrt{n}}$ times the length of the orthogonal projection of $\mathbf{x}$ onto the constant vector. For the length of its orthogonal complement $\mathbf{p}_j(\mathbf{x})^\perp$ we then obtain

$$\|\mathbf{x} - \bar{x}\mathbf{j}\|^2 = (\mathbf{x} - \bar{x}\mathbf{j})'(\mathbf{x} - \bar{x}\mathbf{j}) = \|\mathbf{x}\|^2 - \bar{x}\mathbf{j}'\mathbf{x} - \bar{x}\mathbf{x}'\mathbf{j} + \bar{x}^2\mathbf{j}'\mathbf{j} = \|\mathbf{x}\|^2 - n\bar{x}^2$$

where the last equality follows from the fact that $\mathbf{j}'\mathbf{x} = \mathbf{x}'\mathbf{j} = \bar{x}n$ and $\mathbf{j}'\mathbf{j} = n$. On the other hand recall that $\|\mathbf{x} - \bar{x}\mathbf{j}\|^2 = \sum_{i=1}^n (x_i - \bar{x})^2 = n\sigma_x^2$ where σ_x^2 denotes the variance of data x_i. Consequently the standard deviation of the data is $\frac{1}{\sqrt{n}}$ times the length of the orthogonal complement of $\mathbf{x}$ with respect to the constant vector.

Now assume that we are also given data $\mathbf{y} = (y_1, \dots, y_n)'$. Again $\mathbf{y} - \bar{y}\mathbf{j}$ is the complement of the orthogonal projection of $\mathbf{y}$ onto the constant vector. Then the inner product of these two orthogonal complements is

$$(\mathbf{x} - \bar{x}\mathbf{j})'(\mathbf{y} - \bar{y}\mathbf{j}) = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = n\sigma_{xy}$$

where σ_{xy} denotes the covariance between $\mathbf{x}$ and $\mathbf{y}$.

Now suppose that we are given a set of data $(y_i, x_{i1}, \ldots, x_{ik})$, $i = 1, \ldots, n$. We assume a linear regression model, i.e.,

$$y_i = \beta_0 + \sum_{s=1}^{k} \beta_s x_{is} + \epsilon_i .$$

These n equations can be stacked together using matrix notation as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & x_{11} & x_{12} & \ldots & x_{1k} \\ 1 & x_{21} & x_{22} & \ldots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \ldots & x_{nk} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_k \end{pmatrix}, \quad \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix}.$$

$\mathbf{X}$ is then called the **design matrix** of the linear regression model, $\boldsymbol{\beta}$ are the **model parameters** and $\boldsymbol{\epsilon}$ are random errors ("noise") called **residuals**.

The parameters $\boldsymbol{\beta}$ can be estimated by means of the **least square principle** where the sum of squared errors,

$$\sum_{i=1}^{n} \epsilon_i^2 = \|\boldsymbol{\epsilon}\|^2 = \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2$$

is minimized. Therefore by Theorem 9.14 the estimated parameter $\hat{\boldsymbol{\beta}}$ satisfies the normal equation

$$\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y} \tag{9.2}$$

and hence

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.$$

— Summary

- For every subspace $\mathscr{U} \subset \mathbb{R}^n$ we find $\mathbb{R}^n = \mathscr{U} \oplus \mathscr{U}^\perp$, where $\mathscr{U}^\perp$ denotes the orthogonal complement of $\mathscr{U}$.
- Every $\mathbf{y} \in \mathbb{R}^n$ can be decomposed as $\mathbf{y} = \mathbf{u} + \mathbf{u}^\perp$ where $\mathbf{u} \in \mathscr{U}$ and $\mathbf{u}^\perp \in \mathscr{U}^\perp$. $\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{y}$ into $\mathscr{U}$.
- If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis of $\mathscr{U}$ and $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$, then $\mathbf{U}\mathbf{u}^\perp = 0$ and $\mathbf{u} = \mathbf{U}\boldsymbol{\lambda}$ where $\boldsymbol{\lambda} \in \mathbb{R}^k$ satisfies $\mathbf{U}'\mathbf{U}\boldsymbol{\lambda} = \mathbf{U}'\mathbf{y}$.
- If $\mathbf{y} = \mathbf{u} + \mathbf{v}$ with $\mathbf{u} \in \mathscr{U}$ then $\mathbf{v}$ has minimal length for fixed $\mathbf{y}$ if and only if $\mathbf{v} \in \mathscr{U}^\perp$.
- If the linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ does not have a solution, then the solution of $\mathbf{A}'\mathbf{A}\mathbf{x} = \mathbf{A}'\mathbf{b}$ minimizes the error $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|$.

— Exercises

9.1 Apply Gram-Schmidt orthonormalization to the following vectors.

(a) $\mathbf{u}_1 = (1,0,0)'$, $\mathbf{u}_2 = (1,2,0)'$, and $\mathbf{u}_3 = (1,2,3)'$.

(b) $\mathbf{u}_1 = (1,2,3)'$, $\mathbf{u}_2 = (1,2,0)'$, and $\mathbf{u}_3 = (1,0,0)'$.

9.2 Let $\mathbf{y} = (2,-3,4)'$. Compute projection $\mathbf{p}_U(\mathbf{y})$ onto subspace $\mathscr{U}$ with the following generating sets. Also give the projection matrix $\mathbf{P}_U$.

(a) $U = \{(0,2,0)'\}$.

(b) $U = \{(1,2,-3)'\}$.

(c) $U = \{(1,0,3)',(-3,2,1)'\}$.

9.3 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$. Solve linear equation $\mathbf{Ax} = \mathbf{b}$ for the follwing values of $\mathbf{b}$ exactly as well as by means the method from Sect. 9.4 that provides a point $\mathbf{x}_0$ which minimizes the "error" $\mathbf{Ax} - \mathbf{b}$.

(a) $\mathbf{b} = (-1,2,0)'$

(b) $\mathbf{b} = (-1,2,5)'$

— Problems

9.4 Let $\mathbf{r} = \mathbf{y} - \lambda\mathbf{x}$ where $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} \neq 0$. Which values of $\lambda \in \mathbb{R}$ minimize $\|\mathbf{r}\|$?

HINT: Use the normalized vector $\mathbf{x}_0 = \mathbf{x}/\|\mathbf{x}\|$ and apply Lemma 9.1.

9.5 Let $\mathbf{P}_x = \mathbf{x}\mathbf{x}'$ for some $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| = 1$.

(a) What is the number of rows and columns of $\mathbf{P}_x$?

(b) What is the rank of $\mathbf{P}_x$?

(c) Show that $\mathbf{P}_x$ is symmetric.

(d) Show that $\mathbf{P}_x$ is idempotent.

(e) Describe the rows and columns of $\mathbf{P}_x$.

9.6 Prove or disprove: Let $\mathscr{U},\mathscr{V} \subseteq \mathbb{R}^n$ be two subspaces with $\mathscr{U} \cap \mathscr{V} = \{0\}$ and $\mathscr{U} \oplus \mathscr{V} = \mathbb{R}^n$. Let $\mathbf{u} \in \mathscr{U}$ and $\mathbf{v} \in \mathscr{V}$. Then $\mathbf{u}'\mathbf{v} = 0$.

9.7 Prove Lemma 9.9.

9.8 Prove Lemma 9.11.

9.9 Assume that $\mathbf{y} = \mathbf{U}\boldsymbol{\lambda} + \mathbf{r}$.
Show: $\mathbf{U}'\mathbf{r} = 0$ if and only if $\mathbf{U}'\mathbf{U}\boldsymbol{\lambda} = \mathbf{U}'\mathbf{y}$.

9.10 Let $\mathscr{U}$ be a subspace of $\mathbb{R}^n$ with generating set $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ and $\mathbf{U} = (\mathbf{u}_1,\dots,\mathbf{u}_k)$. Show:

(a) $\mathbf{u} \in \mathscr{U}$ if and only if there exists an $\boldsymbol{\lambda} \in \mathbb{R}^k$ such that $\mathbf{u} = \mathbf{U}\boldsymbol{\lambda}$.

(b) $\mathbf{v} \in \mathscr{U}^{\perp}$ if and only if $\mathbf{U}'\mathbf{v} = 0$.

(c) The projection $\mathbf{y} \mapsto \mathbf{p}_U(\mathbf{y})$ is a linear map onto $\mathscr{U}$.

(d) If $\operatorname{rank}(\mathbf{U}) = k$, then the Projection matrix is given by $\mathbf{P}_U = \mathbf{U}(\mathbf{U}'\mathbf{U})^{-1}\mathbf{U}'$.

In addition:

(e) Could we simplify $\mathbf{P}_U$ in the following way?
$\mathbf{P}_U = \mathbf{U}(\mathbf{U}'\mathbf{U})^{-1}\mathbf{U}' = \mathbf{U}\mathbf{U}^{-1} \cdot \mathbf{U}'^{-1}\mathbf{U}' = \mathbf{I} \cdot \mathbf{I} = \mathbf{I}$.

(f) Let $\mathbf{P}_U$ be the matrix for projection $\mathbf{y} \mapsto \mathbf{p}_U(\mathbf{y})$. Compute the projection matrix $\mathbf{P}_{U^{\perp}}$ for the projection onto $\mathscr{U}^{\perp}$.

9.11 Prove Theorem 9.15.

9.12 Let $\mathbf{p}$ be a projection into some subspace $\mathscr{U} \subseteq \mathbb{R}^n$. Let $\mathbf{x}_1,\dots,\mathbf{x}_k \in \mathbb{R}^n$.
Show: If $\mathbf{p}(\mathbf{x}_1),\dots,\mathbf{p}(\mathbf{x}_k)$ are linearly independent, then the vectors $\mathbf{x}_1,\dots,\mathbf{x}_k$ are linearly independent.
Show that the converse is false.

9.13 (a) Give necessary and sufficient conditions such that the "normal equation" (9.2) has a uniquely determined solution.

(b) What happens when this condition is violated? (There is no solution at all? The solution exists but is not uniquely determined? How can we find solutions in the latter case? What is the statistical interpretation in all these cases?) Demonstrate your considerations by (simple) examples.

(c) Show that for each solution of Equation (9.2) the arithmetic mean of the error is zero, that is, $\bar{\boldsymbol{\varepsilon}} = 0$. Give a statistical interpretation of this result.

(d) Let $\mathbf{x}_i = (x_{i1},\dots,x_{in})'$ be the i-th column of $\mathbf{X}$. Show that for each solution of Equation (9.2) $\mathbf{x}_i'\boldsymbol{\varepsilon} = 0$. Give a statistical interpretation of this result.

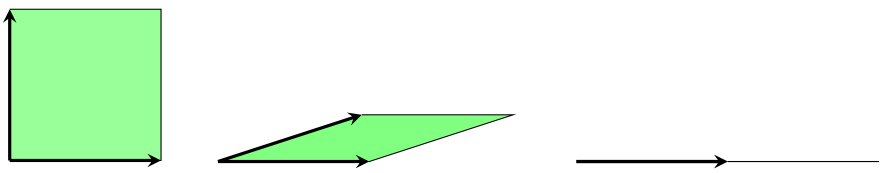
10

Determinant

What is the volume *of a skewed box?*

10.1 Linear Independence and Volume

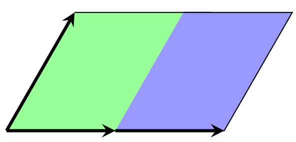
We want to "measure" whether two vectors in $\mathbb{R}^2$ are linearly independent or not. Thus we may look at the parallelogram that is created by these two vectors. We may find the following cases:



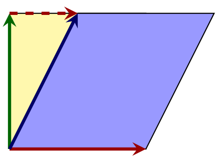
The two vectors are linearly dependent if and only if the corresponding parallelogram has area 0. The same holds for three vectors in $\mathbb{R}^3$ which form a parallelepiped and generally for n vectors in $\mathbb{R}^n$.

Idea: Use the n-dimensional volume to check whether n vectors in $\mathbb{R}^n$ are linearly independent.

Thus we need to compute this volume. Therefore we first look at the properties of the area of a parallelogram and the volume of a parallelepiped, respectively, and use these properties to define a "volume function".



(1) If we multiply one of the vectors by a number $\alpha \in \mathbb{R}$, then we obtain a parallelepiped (parallelogram) with the α-fold volume.



(2) If we add a multiple of one vector to one of the other vectors, then the volume remains unchanged.

(3) If two vectors are equal, then the volume is 0.

(4) The volume of the unit-cube has volume 1.

10.2 Determinant

Motivated by the above considerations we define the determinant as a *normed alternating multilinear form*.

Definition 10.1

Determinant. The **determinant** is a function $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ that assigns a real number to an $n \times n$ matrix $\mathbf{A} = (\mathbf{a}_1,\dots,\mathbf{a}_n)$ with following properties:

(D1) The determinant is *multilinear*, i.e., it is linear in each column:

$$\begin{aligned}\det(&\mathbf{a}_1,\dots,\mathbf{a}_{i-1},\alpha\mathbf{a}_i+\beta\mathbf{b}_i,\mathbf{a}_{i+1},\dots,\mathbf{a}_n)\\ &= \alpha\det(\mathbf{a}_1,\dots,\mathbf{a}_{i-1},\mathbf{a}_i,\mathbf{a}_{i+1},\dots,\mathbf{a}_n)\\ &\quad+\beta\det(\mathbf{a}_1,\dots,\mathbf{a}_{i-1},\mathbf{b}_i,\mathbf{a}_{i+1},\dots,\mathbf{a}_n)\,.\end{aligned}$$

(D2) The determinant is *alternating*, i.e.,

$$\begin{aligned}&\det(\mathbf{a}_1,\dots,\mathbf{a}_{i-1},\mathbf{a}_i,\mathbf{a}_{i+1},\dots,\mathbf{a}_{k-1},\mathbf{b}_k,\mathbf{a}_{k+1},\dots,\mathbf{a}_n)\\ &= -\det(\mathbf{a}_1,\dots,\mathbf{a}_{i-1},\mathbf{b}_k,\mathbf{a}_{i+1},\dots,\mathbf{a}_{k-1},\mathbf{a}_i,\mathbf{a}_{k+1},\dots,\mathbf{a}_n)\,.\end{aligned}$$

(D3) The determinant is *normalized*, i.e.,

$$\det(\mathbf{I}) = 1\,.$$

Do not mix up with the absolute value of a number.

We denote the determinant of $\mathbf{A}$ by $\det(\mathbf{A})$ or $|\mathbf{A}|$.

This definition sounds like a “wish list”. We define the function by its properties. However, such an approach is quite common in mathematics. But of course we have to answer the following questions:

- Does such a function exist?
- Is this function uniquely defined?
- How can we evaluate the determinant of a particular matrix $\mathbf{A}$?

We proceed by deriving an explicit formula for the determinant that answers these questions. We begin with a few more properties of the determinant (provided that such a function exists). Their proofs are straightforward and left as an exercise (see Problems 10.10, 10.11, and 10.12).

Lemma 10.2

The determinant is zero if two columns are equal, i.e.,

$$\det(\dots,\mathbf{a},\dots,\mathbf{a},\dots) = 0\,.$$

Lemma 10.3

The determinant is zero, $\det(\mathbf{A}) = 0$, if the columns of $\mathbf{A}$ are linearly dependent.

Lemma 10.4

The determinant remains unchanged if we add a multiple of one column the one of the other columns:

$$\det(\dots,\mathbf{a}_i+\alpha\,\mathbf{a}_k,\dots,\mathbf{a}_k,\dots) = \det(\dots,\mathbf{a}_i,\dots,\mathbf{a}_k,\dots)\,.$$

Now let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$. Then we can represent each column of $n \times n$ matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ as

$$\mathbf{a}_j = \sum_{i=1}^{n} c_{ij} \mathbf{v}_i \,, \qquad \text{for } j = 1, \dots, n,$$

where $c_{ij} \in \mathbb{R}$. We then find

$$\begin{aligned}
\det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) &= \det\left(\sum_{i_1=1}^{n} c_{i_1 1}\mathbf{v}_{i_1}, \sum_{i_2=1}^{n} c_{i_2 2}\mathbf{v}_{i_2}, \dots, \sum_{i_n=1}^{n} c_{i_n n}\mathbf{v}_{i_n}\right) \\
&= \sum_{i_1=1}^{n} c_{i_1 1} \det\left(\mathbf{v}_{i_1}, \sum_{i_2=1}^{n} c_{i_2 2}\mathbf{v}_{i_2}, \dots, \sum_{i_n=1}^{n} c_{i_n n}\mathbf{v}_{i_n}\right) \\
&= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} c_{i_1 1} c_{i_2 2} \det\left(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \sum_{i_n=1}^{n} c_{i_n n}\mathbf{v}_{i_n}\right) \\
&\;\;\vdots \\
&= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} c_{i_1 1} c_{i_2 2} \dots c_{i_n n} \det\left(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_n}\right)
\end{aligned}$$

There are n^n terms in this sum. However, Lemma 10.2 implies that $\det\left(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_n}\right) = 0$ when at least two columns coincide. Thus only those determinants remain which contain all basis vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (in different orders), i.e., those tuples $(i_i, i_2, \dots, i_n)$ which are a *permutation* of the numbers $(1, 2, \dots, n)$.

We can define a **permutation** σ as a bijection from the set $\{1, 2, \dots, n\}$ onto itself. We denote the set of these permutations by $\mathfrak{S}_n$. It has the following properties which we state without a formal proof.

- The *compound* of two permutations $\sigma, \tau \in \mathfrak{S}_n$ is again a permutation, $\sigma\tau \in \mathfrak{S}_n$.
- There is a *neutral* (or *identity*) permutation that does not change the ordering of $(1, 2, \dots, n)$.
- Each permutation $\sigma \in \mathfrak{S}_n$ has a unique *inverse* permutation $\sigma^{-1} \in \mathfrak{S}_n$.

We then say that $\mathfrak{S}_n$ forms a **group**.

Using this concept we can remove the vanishing terms from the above expression for the determinant. As only determinants remain where the columns are permutations of the columns of $\mathbf{A}$ we can write

$$\det(\mathbf{a}_1, \dots, \mathbf{a}_n) = \sum_{\sigma \in \mathfrak{S}_n} \det\left(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(n)}\right) \prod_{i=1}^{n} c_{\sigma(i),i} \,.$$

The simplest permutation is a **transposition** that just flips two elements.

- Every permutation can be composed of a sequence of transpositions, i.e., for every $\sigma \in \mathfrak{S}$ there exist $\tau_1, \dots, \tau_k \in \mathfrak{S}$ such that $\sigma = \tau_k \cdots \tau_1$.

Notice that a transposition of the columns of a determinant changes its sign by property (D2). An immediate consequence is that the determinants $\det\left(\mathbf{v}_{\sigma(1)},\mathbf{v}_{\sigma(2)},\dots,\mathbf{v}_{\sigma(n)}\right)$ only differ in their signs. Moreover, the sign is given by the number of transitions into which a permutation σ is decomposed. So we have

$$\det\left(\mathbf{v}_{\sigma(1)},\dots,\mathbf{v}_{\sigma(n)}\right)=\operatorname{sgn}(\sigma)\det(\mathbf{v}_1,\dots,\mathbf{v}_n)$$

where $\operatorname{sgn}(\sigma)=+1$ if the number of transpositions into which σ can be decomposed is even, and where $\operatorname{sgn}(\sigma)=-1$ if the number of transpositions is odd. We remark (without proof) that $\operatorname{sgn}(\sigma)$ is well-defined although this sequence of transpositions is not unique.

We summarize our considerations in the following proposition.

Lemma 10.5

Let $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$ and $\mathbf{A}=(\mathbf{a}_1,\dots,\mathbf{a}_n)$ an $n\times n$ matrix. Let $c_{ij}\in\mathbb{R}$ such that $\mathbf{a}_j=\sum_{i=1}^n c_{ij}\mathbf{v}_i$ for $j=1,\dots,n$. Then

$$\det(\mathbf{a}_1,\dots,\mathbf{a}_n)=\det(\mathbf{v}_1,\dots,\mathbf{v}_n)\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\prod_{i=1}^n c_{\sigma(i),i}\,.$$

This lemma allows us that we can compute $\det(\mathbf{A})$ provided that the determinant of a regular matrix is known. This equation in particular holds if we use the canonical basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. We then have $c_{ij}=a_{ij}$ and

$$\det(\mathbf{v}_1,\dots,\mathbf{v}_n)=\det(\mathbf{e}_1,\dots,\mathbf{e}_n)=\det(\mathbf{I})=1$$

where the last equality is just property (D3).

Theorem 10.6

Leibniz formula for determinant. The determinant of a $n\times n$ matrix $\mathbf{A}$ is given by

$$\det(\mathbf{A})=\det(\mathbf{a}_1,\dots,\mathbf{a}_n)=\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\prod_{i=1}^n a_{\sigma(i),i}\,. \tag{10.1}$$

Corollary 10.7

Existence and uniqueness. The determinant as given in Definition 10.1 exists and is uniquely defined.

Leibniz formula (10.1) is often used as definition of the determinant. Of course we then have to derive properties (D1)–(D3) from (10.1), see Problem 10.13.

10.3 Properties of the Determinant

Theorem 10.8

Transpose. The determinant remains unchanged if a matrix is transposed, i.e.,

$$\det(\mathbf{A}')=\det(\mathbf{A})\,.$$

PROOF. Recall that $[\mathbf{A}']_{ij} = [\mathbf{A}]_{ji}$ and that each $\sigma \in \mathfrak{S}_n$ has a unique inverse permutation $\sigma^{-1} \in \mathfrak{S}_n$. Moreover, $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$. Then by Theorem 10.6,

$$\begin{aligned}\det(\mathbf{A}') &= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{\sigma^{-1}(i),i}\\ &= \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma^{-1})\prod_{i=1}^{n} a_{\sigma^{-1}(i),i} = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{\sigma(i),i} = \det(\mathbf{A})\end{aligned}$$

where the forth equality holds as $\{\sigma^{-1} : \sigma \in \mathfrak{S}_n\} = \mathfrak{S}_n$. □

Theorem 10.9

Product. The determinant of the product of two matrices equals the product of their determinants, i.e.,

$$\det(\mathbf{A}\cdot\mathbf{B}) = \det(\mathbf{A})\cdot\det(\mathbf{B})\,.$$

PROOF. Let $\mathbf{A}$ and $\mathbf{B}$ be two $n\times n$ matrices. If $\mathbf{A}$ does not have full rank, then $\operatorname{rank}(\mathbf{A}) < n$ and Lemma 10.3 implies $\det(\mathbf{A}) = 0$ and thus $\det(\mathbf{A})\cdot\det(\mathbf{B}) = 0$. On the other hand by Theorem 6.23 $\operatorname{rank}(\mathbf{AB}) \le \operatorname{rank}(\mathbf{A}) < n$ and hence $\det(\mathbf{AB}) = 0$.
If $\mathbf{A}$ has full rank, then the columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$ and we find for the columns of $\mathbf{AB}$, $[\mathbf{AB}]_j = \sum_{i=1}^{n} b_{ij}\mathbf{a}_i$. Consequently, Lemma 10.5 and Theorem 10.6 immediately imply

$$\det(\mathbf{AB}) = \det(\mathbf{a}_1,\dots,\mathbf{a}_n)\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n} b_{\sigma(i),i} = \det(\mathbf{A})\cdot\det(\mathbf{B})$$

as claimed. □

Theorem 10.10

Singular matrix. Let $\mathbf{A}$ be an $n\times n$ matrix. Then the following are equivalent:

(1) $\det(\mathbf{A}) = 0$.

(2) The columns of $\mathbf{A}$ are linearly dependent.

(3) $\mathbf{A}$ does not have full rank.

(4) $\mathbf{A}$ is singular.

PROOF. The equivalence of (2), (3) and (4) has already been shown in Section 6.3. Implication (2) ⇒ (1) is stated in Lemma 10.3. For implication (1) ⇒ (4) see Problem 10.14. This finishes the proof. □

Corollary 10.11

An $n\times n$ matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.

We can use the determinant to estimate the rank of a matrix.

Theorem 10.12

Rank of a matrix. The rank of an $m \times n$ matrix $\mathbf{A}$ is r if and only if there is an $r \times r$ subdeterminant

$$\begin{vmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_r} \\ \vdots & \ddots & \vdots \\ a_{i_r j_1} & \dots & a_{i_r j_r} \end{vmatrix} \neq 0$$

but all $(r+1) \times (r+1)$ subdeterminants vanish.

PROOF. By Gauß elimination we can find an invertible $r \times r$ submatrix but not an invertible $(r+1) \times (r+1)$ submatrix. □

Theorem 10.13

Inverse matrix. The determinant of the inverse of a regular matrix is the reciprocal value of the determinant of the matrix, i.e.,

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} .$$

PROOF. See Problem 10.15.

Finally we return to the volume of a parallelepiped which we used as motivation for the definition of the determinant. Since we have no formal definition of the *volume* yet, we state the last theorem without proof.

Theorem 10.14

Volume. Let $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^n$. Then the volume of the n-dimensional parallelepiped created by these vectors is given by the absolute value of the determinant,

$$\operatorname{Vol}(\mathbf{a}_1, \dots, \mathbf{a}_n) = \left|\det(\mathbf{a}_1, \dots, \mathbf{a}_n)\right| .$$

10.4 Evaluation of the Determinant

Leibniz formula (10.1) provides an explicit expression for evaluating the determinant of a matrix. For small matrices one may expand sum and products and finds an easy to use scheme, known as **Sarrus' rule** (see Problems 10.17 and 10.18):

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} .$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} . \end{array} \tag{10.2}$$

For larger matrices Leibniz formula (10.1) expands to much longer expressions. For an $n \times n$ matrix we find a sum of $n!$ products of n factors. However, for triangular matrices this formula reduces to the product of the diagonal entries, see Problem 10.19.

Triangular matrix. Let $\mathbf{A}$ be an $n \times n$ (upper or lower) triangular matrix. Then Theorem 10.15

$$\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii} .$$

In Section 7.2 we have seen that we can transform a matrix $\mathbf{A}$ into a row echelon form $\mathbf{R}$ by a series of elementary row operations (Theorem 7.9), $\mathbf{R} = \mathbf{T}_k \cdots \mathbf{T}_1 \mathbf{A}$. Notice that for a square matrix we then obtain an upper triangular matrix. By Theorems 10.9 and 10.15 we find

$$\det(\mathbf{A}) = \left(\det(\mathbf{T}_k) \cdots \det(\mathbf{T}_1)\right)^{-1} \prod_{i=1}^{n} r_{ii} .$$

As $\det(\mathbf{T}_i)$ is easy to evaluate we obtain a fast algorithm for computing $\det(\mathbf{A})$, see Problems 10.20 and 10.21.

Another approach is to replace (10.1) by a recursion formula, known as **Laplace expansion**.

Minor. Let $\mathbf{A}$ be an $n \times n$ matrix. Let $\mathbf{M}_{ij}$ denote the $(n-1) \times (n-1)$ matrix that we obtain by deleting the i-th row and the j-th column from $\mathbf{A}$. Then $M_{ij} = \det(\mathbf{M}_{ij})$ is called the (i,j) **minor** of $\mathbf{A}$. Definition 10.16

Laplace expansion. Let $\mathbf{A}$ be an $n \times n$ matrix and M_{ik} its (i,k) minor. Then Theorem 10.17

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik} \cdot (-1)^{i+k} M_{ik} = \sum_{k=1}^{n} a_{ik} \cdot (-1)^{i+k} M_{ik} .$$

The first expression is expansion along the k-th column. The second expression is expansion along the i-th row.

Cofactor. The term $C_{ik} = (-1)^{i+k} M_{ik}$ is called the **cofactor** of a_{ik}. Definition 10.18

With this notation Laplace expansion can also be written as

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik} C_{ik} = \sum_{k=1}^{n} a_{ik} C_{ik} .$$

PROOF. As $\det(\mathbf{A}') = \det(\mathbf{A})$ we only need to prove first statement. Notice that $\mathbf{a}_k = \sum_{i=1}^{n} a_{ik} \mathbf{e}_i$. Therefore,

$$\begin{aligned}
\det(\mathbf{A}) &= \det(\mathbf{a}_1, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_n) \\
&= \det\left(\mathbf{a}_1, \ldots, \sum_{i=1}^{n} a_{ik} \mathbf{e}_i, \ldots, \mathbf{a}_n\right) = \sum_{i=1}^{n} a_{ik} \det(\mathbf{a}_1, \ldots, \mathbf{e}_i, \ldots, \mathbf{a}_n) .
\end{aligned}$$

It remains to show that $\det(\mathbf{a}_1, \ldots, \mathbf{e}_i, \ldots, \mathbf{a}_n) = C_{ik}$. Observe that we can transform matrix $(\mathbf{a}_1, \ldots, \mathbf{e}_i, \ldots, \mathbf{a}_n)$ into $\mathbf{B} = \begin{pmatrix} 1 & * \\ 0 & \mathbf{M}_{ik} \end{pmatrix}$ by a series of $j-1$

transpositions of rows and $k-1$ transpositions of columns and thus we find by property (D2), Theorem 10.8 and Leibniz formula

$$\begin{aligned}\det(\mathbf{a}_1,\dots,\mathbf{e}_i,\dots,\mathbf{a}_n) &= (-1)^{j+k-2}\begin{vmatrix}1 & * \\ 0 & \mathbf{M}_{ik}\end{vmatrix} = (-1)^{j+k-2}\left|\mathbf{B}\right| \\ &= (-1)^{j+k}\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n} b_{\sigma(i),i}\end{aligned}$$

Observe that $b_{11}=1$ and $b_{\sigma(1),i}=0$ for all permutations where $\sigma(1)=1$ and $i\neq 0$. Hence

$$\begin{aligned}(-1)^{j+k}\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n} b_{\sigma(i),i} &= (-1)^{j+k} b_{11}\sum_{\sigma\in\mathfrak{S}_{n-1}}\operatorname{sgn}(\sigma)\prod_{i=1}^{n-1} b_{\sigma(i)+1,i+1} \\ &= (-1)^{j+k}\left|\mathbf{M}_{ik}\right| = C_{ik}\end{aligned}$$

This finishes the proof. □

10.5 Cramer's Rule

Definition 10.19

Adjugate matrix. The matrix of cofactors for an $n\times n$ matrix $\mathbf{A}$ is the matrix $\mathbf{C}$ whose entry in the i-th row and k-th column is the cofactor C_{ik}. The **adjugate matrix** of $\mathbf{A}$ is the transpose of the matrix of cofactors of $\mathbf{A}$,

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}'\,.$$

Theorem 10.20

Let $\mathbf{A}$ be an $n\times n$ matrix. Then

$$\operatorname{adj}(\mathbf{A})\cdot\mathbf{A} = \det(\mathbf{A})\,\mathbf{I}\,.$$

PROOF. A straightforward computation and Laplace expansion (Theorem 10.17) yields

$$\begin{aligned}[\operatorname{adj}(\mathbf{A})\cdot\mathbf{A}]_{ij} &= \sum_{k=1}^{n} C'_{ik}\cdot a_{kj} = \sum_{k=1}^{n} a_{kj}\cdot C_{ki} \\ &= \det(\mathbf{a}_1,\dots,\mathbf{a}_{i-1},\mathbf{a}_j,\mathbf{a}_{i+1},\dots,\mathbf{a}_n) \\ &= \begin{cases}\det(\mathbf{A}), & \text{if } j=i,\\ 0, & \text{if } j\neq i,\end{cases}\end{aligned}$$

as claimed. □

Corollary 10.21

Inverse matrix. Let $\mathbf{A}$ be a regular $n\times n$ matrix. Then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A})\,.$$

This formula is quite convenient as it provides an explicit expression for the inverse of a matrix. However, for numerical computations it is too expensive. Gauss-Jordan procedure, for example, is much faster. Nevertheless, it provides a nice rule for very small matrices.

The inverse of a regular 2×2 matrix $\mathbf{A}$ is given by

Corollary 10.22

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} .$$

We can use Corollary 10.21 to solve the linear equation

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

when $\mathbf{A}$ is an invertible matrix. We then find

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \frac{1}{|\mathbf{A}|} \operatorname{adj}(\mathbf{A}) \cdot \mathbf{b} .$$

Therefore we find for the i-th component of the solution $\mathbf{x}$,

$$\begin{aligned} x_i &= \frac{1}{|\mathbf{A}|} \sum_{k=1}^{n} C'_{ik} \cdot b_k = \frac{1}{|\mathbf{A}|} \sum_{k=1}^{n} b_k \cdot C_{ki} \\ &= \frac{1}{|\mathbf{A}|} \det(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_n) . \end{aligned}$$

So we get the following explicit expression for the solution of a linear equation.

Theorem 10.23

Cramer's rule. Let $\mathbf{A}$ be an invertible matrix and $\mathbf{x}$ a solution of the linear equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. Let $\mathbf{A}_i$ denote the matrix where the i-th column of $\mathbf{A}$ is replaced by $\mathbf{b}$. Then

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})} .$$

— Summary

- The determinant is a *normed alternating multilinear form*.
- The determinant is 0 if and only if it is singular.
- The determinant of the product of two matrices is the product of the determinants of the matrices.
- The Leibniz formula gives an explicit expression for the determinant.
- The Laplace expansion is a recursive formula for evaluating the determinant.
- The determinant can efficiently computed by a method similar to Gauß elimination.
- Cramer's rule allows to compute the inverse of matrices and the solutions of special linear equations.

— Exercises

10.1 Compute the following determinants by means of Sarrus' rule or by transforming into an upper triangular matrix:

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -3 \\ 0 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 1 & -4 \\ 2 & 1 & 4 \\ 3 & 4 & -4 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 1 \\ 4 & -3 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & 3 & -2 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

10.2 Compute the determinants from Exercise 10.1 by means of Laplace expansion.

10.3

(a) Estimate the ranks of the matrices from Exercise 10.1.

(b) Which of these matrices are regular?

(c) Which of these matrices are invertible?

(d) Are the column vectors of these matrices linear independent?

10.4 Let

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 2\times 1 & 0 \\ 0 & 2\times 1 & 0 \\ 1 & 2\times 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 3 & 5\times 3+1 & 0 \\ 0 & 5\times 0+1 & 0 \\ 1 & 5\times 1+0 & 1 \end{pmatrix}$$

Compute by means of the properties of determinants:

(a) $\det(\mathbf{A})$ (b) $\det(5\mathbf{A})$ (c) $\det(\mathbf{B})$ (d) $\det(\mathbf{A}')$

(e) $\det(\mathbf{C})$ (f) $\det(\mathbf{A}^{-1})$ (g) $\det(\mathbf{A}\cdot\mathbf{C})$ (h) $\det(\mathbf{I})$

10.5 Let $\mathbf{A}$ be a 3×4 matrix. Estimate $|\mathbf{A}'\cdot\mathbf{A}|$ and $|\mathbf{A}\cdot\mathbf{A}'|$.

10.6 Compute area of the parallelogram and volume of the parallelepiped, respectively, which are created by the following vectors:

(a) $\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$

10.7 Compute the matrix of cofactors, the adjugate matrix and the inverse of the following matrices:

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -3 \\ 0 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 1 & -4 \\ 2 & 1 & 4 \\ 3 & 4 & -4 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 1 \\ 4 & -3 & 3 \end{pmatrix}$

10.8 Compute the inverse of the following matrices:

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (b) $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ (c) $\begin{pmatrix} \alpha & \beta \\ \alpha^2 & \beta^2 \end{pmatrix}$

10.9 Solve the linear equation

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

by means of Cramer's rule for $\mathbf{b} = (1,2)'$ and $\mathbf{b} = (1,2,3)$, respectively, and the following matrices:

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -3 \\ 0 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 1 & -4 \\ 2 & 1 & 4 \\ 3 & 4 & -4 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 2 & 1 \\ 4 & -3 & 3 \end{pmatrix}$

— Problems

10.10 Prove Lemma 10.2 using properties (D1)–(D3).

10.11 Prove Lemma 10.3 using properties (D1)–(D3).

10.12 Prove Lemma 10.4 using properties (D1)–(D3).

10.13 Derive properties (D1) and (D3) from Expression (10.1) in Theorem 10.6.

10.14 Show that an $n \times n$ matrix $\mathbf{A}$ is singular if $\det(\mathbf{A}) = 0$.
Does Lemma 10.3 already imply this result?

HINT: Try an indirect proof and use equation $\mathbf{I} = \det(\mathbf{A}\mathbf{A}^{-1})$.

10.15 Prove Theorem 10.13.

10.16 Show that the determinants of similar square matrices are equal.

10.17 Derive formula

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

directly from properties (D1)–(D3) and Lemma 10.4.

HINT: Use a method similar to Gauß elimination.

10.18 Derive Sarrus' rule (10.2) from Leibniz formula (10.1).

10.19 Let $\mathbf{A}$ be an $n \times n$ upper triangular matrix. Show that

$$\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii} .$$

HINT: Use Leibniz formula (10.1) and show that there is only one permutation σ with $\sigma(i) \leq i$ for all i.

10.20 Compute the determinants of the elementary row operations from Problem 7.4.

10.21 Modify the algorithm from Problem 7.6 such that it computes the determinant of a square matrix.

11

Eigenspace

We want to estimate the sign of a matrix and compute its square root.

11.1 Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors. Let $\mathbf{A}$ be an $n \times n$ matrix. Then a non-zero vector $\mathbf{x}$ is called an **eigenvector** corresponding to **eigenvalue** λ if Definition 11.1

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}\,, \quad \mathbf{x} \neq 0\,. \tag{11.1}$$

Observe that a scalar λ is an eigenvalue if and only if $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ has a non-trivial solution, i.e., if $(\mathbf{A} - \lambda\mathbf{I})$ is not invertible or, equivalently, if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0\,.$$

The Leibniz formula for determinants (or, equivalently, Laplace expansion) implies that this determinant is a polynomial of degree n in λ.

Characteristic polynomial. The polynomial Definition 11.2

$$p_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I})$$

is called the **characteristic polynomial** of $\mathbf{A}$. For this reason the eigenvalues of $\mathbf{A}$ are also called its **characteristic roots** and the corresponding eigenvectors the **characteristic vectors** of $\mathbf{A}$.

Notice that by the Fundamental Theorem of Algebra a polynomial of degree n has exactly n roots (in the sense we can factorize the polynomial into a product of n linear terms), i.e., we can write

$$p_{\mathbf{A}}(t) = (-1)^n(t - \lambda_1)\cdots(t - \lambda_n) = (-1)^n \prod_{i=1}^{n}(t - \lambda_i)\,.$$

However, some of these roots λ_i may be complex numbers.

If an eigenvalue λ_i appears m times ($m \geq 2$) as a linear factor, i.e., if it is a multiple root of the characteristic polynomial $p_{\mathbf{A}}(t)$, then we say that λ_i has **algebraic multiplicity** m.

Definition 11.3

Spectrum. The list of all eigenvalues of a square matrix $\mathbf{A}$ is called the **spectrum** of $\mathbf{A}$. It is denoted by $\sigma(\mathbf{A})$.

Obviously, the eigenvectors corresponding to eigenvalue λ are the solutions of the homogeneous linear equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0$. Therefore, the set of all eigenvectors with the same eigenvalue λ together with the zero vector is the subspace $\ker(\mathbf{A}-\lambda\mathbf{I})$.

Definition 11.4

Eigenspace. Let λ be an eigenvalue of the $n \times n$ matrix $\mathbf{A}$. The subspace

$$\mathscr{E}_\lambda = \ker(\mathbf{A}-\lambda\mathbf{I})$$

is called the **eigenspace** of $\mathbf{A}$ corresponding to eigenvalue λ.

Computer programs for computing eigenvectors thus always compute bases of the corresponding eigenspaces. Since bases of a subspace are not unique, see Section 5.2, their results may differ.

Example 11.5

Diagonal matrix. For every $n \times n$ diagonal matrix $\mathbf{D}$ and every $i = 1, \ldots, n$ we find

$$\mathbf{D}\mathbf{e}_i = d_{ii}\mathbf{e}_i .$$

That is, each of its diagonal entries d_{ii} is an eigenvalue affording eigenvectors $\mathbf{e}_i$. Its spectrum is just the set of its diagonal entries. ◊

11.2 Properties of Eigenvalues

Theorem 11.6

Transpose. $\mathbf{A}$ and $\mathbf{A}'$ have the same spectrum.

PROOF. See Problem 11.14.

Theorem 11.7

Matrix power. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $\mathbf{A}^k$ corresponding to eigenvalue λ^k for every $k \in \mathbb{N}$.

PROOF. See Problem 11.15.

Theorem 11.8

Inverse matrix. If $\mathbf{x}$ is an eigenvector of the regular matrix $\mathbf{A}$ corresponding to eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $\mathbf{A}^{-1}$ corresponding to eigenvalue λ^{-1}.

PROOF. See Problem 11.16.

Theorem 11.9

Eigenvalues and determinant. Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (counting multiplicity). Then

$$\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i .$$

PROOF. A straightforward computation shows that $\prod_{i=1}^{n} \lambda_i$ is the constant term of the characteristic polynomial $p_{\mathbf{A}}(t) = (-1)^n \prod_{i=1}^{n}(t - \lambda_i)$. On the other hand, we show that the constant term of $p_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I})$ equals $\det(\mathbf{A})$. Observe that by multilinearity of the determinant we have

$$\det(\dots, \mathbf{a}_i - t\mathbf{e}_i, \dots) = \det(\dots, \mathbf{a}_i, \dots) - t\det(\dots, \mathbf{e}_i, \dots) .$$

As this holds for every columns we find

$$\det(\mathbf{A} - t\mathbf{I}) = \sum_{(\delta_1,\dots,\delta_n)\in\{0,1\}^n} (-t)^{\sum_{i=1}^{n}\delta_i} \det\big((1-\delta_1)\mathbf{a}_1 + \delta_1\mathbf{e}_1, \dots,$$
$$\dots, (1-\delta_n)\mathbf{a}_n + \delta_n\mathbf{e}_n\big) .$$

Obviously, the only term that does not depend on t is where $\delta_1 = \dots = \delta_n = 0$, i.e., $\det(\mathbf{A})$. This completes the proof. □

There is also a similar remarkable result on the sum of the eigenvalues.

Definition 11.10

The **trace** of an $n \times n$ matrix $\mathbf{A}$ is the sum of its diagonal elements, i.e.,

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} .$$

Theorem 11.11

Eigenvalues and trace. Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (counting multiplicity). Then

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i .$$

PROOF. See Problem 11.17.

11.3 Diagonalization and Spectral Theorem

In Section 6.4 we have called two matrices $\mathbf{A}$ and $\mathbf{B}$ similar if there exists a transformation matrix $\mathbf{U}$ such that $\mathbf{B} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$.

Theorem 11.12

Similar matrices. The spectra of two similar matrices $\mathbf{A}$ and $\mathbf{B}$ coincide.

PROOF. See Problem 11.18.

Now one may ask whether we can find a basis such that the corresponding matrix is as simple as possible. Motivated by Example 11.5 we even may try to find a basis such that $\mathbf{A}$ becomes a diagonal matrix. We find that this is indeed the case for symmetric matrices.

Theorem 11.13

Spectral theorem for symmetric matrices. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. Then all eigenvalues are real and there exists an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$.

Furthermore, let $\mathbf{D}$ be the $n \times n$ diagonal matrix with the eigenvalues of $\mathbf{A}$ as its entries and let $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ be the orthogonal matrix of eigenvectors. Then matrices $\mathbf{A}$ and $\mathbf{D}$ are similar with transformation matrix $\mathbf{U}$, i.e.,

$$\mathbf{U}'\mathbf{A}\mathbf{U} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} . \tag{11.2}$$

We call this process the **diagonalization** of $\mathbf{A}$.

A proof of the first part of Theorem 11.13 is out of the scope of this manuscript. Thus we only show the following partial result (Lemma 11.14). For the second part recall that for an orthogonal matrix $\mathbf{U}$ we have $\mathbf{U}^{-1} = \mathbf{U}'$ by Theorem 8.24. Moreover, observe that

$$\mathbf{U}'\mathbf{A}\mathbf{U}\,\mathbf{e}_i = \mathbf{U}'\mathbf{A}\,\mathbf{u}_i = \mathbf{U}'\,\lambda_i\mathbf{u}_i = \lambda_i\mathbf{U}'\,\mathbf{u}_i = \lambda_i\mathbf{e}_i = \mathbf{D}\,\mathbf{e}_i$$

for all $i = 1, \dots, n$.

Lemma 11.14

Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. If $\mathbf{u}_i$ and $\mathbf{u}_j$ are eigenvectors to distinct eigenvalues λ_i and λ_j, respectively, then $\mathbf{u}_i$ and $\mathbf{u}_j$ are orthogonal, i.e., $\mathbf{u}_i'\mathbf{u}_j = 0$.

PROOF. By the symmetry of $\mathbf{A}$ and eigenvalue equation (11.1) we find

$$\lambda_i\mathbf{u}_i'\mathbf{u}_j = (\mathbf{A}\mathbf{u}_i)'\mathbf{u}_j = (\mathbf{u}_i'\mathbf{A}')\mathbf{u}_j = \mathbf{u}_i'(\mathbf{A}\mathbf{u}_j) = \mathbf{u}_i'(\lambda_j\mathbf{u}_j) = \lambda_j\mathbf{u}_i'\mathbf{u}_j .$$

Consequently, if $\lambda_i \neq \lambda_j$ then $\mathbf{u}_i'\mathbf{u}_j = 0$, as claimed $\square$

Theorem 11.13 immediately implies Theorem 11.9 for the special case where $\mathbf{A}$ is symmetric, see Problem 11.19.

11.4 Quadratic Forms

Up to this section we only have dealt with linear functions. Now we want to look to more advanced functions, in particular at quadratic functions.

Quadratic form. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. Then the function — Definition 11.15

$$q_{\mathbf{A}}: \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$$

is called a **quadratic form**.

Observe that we have

$$q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j .$$

In the second part of this course we need to characterize stationary points of arbitrary differentiable multivariate functions. We then will see that the sign of such quadratic forms will play a prominent rôle in our investigations. Hence we introduce the concept of the *definiteness* of a quadratic form.

Definiteness. A quadratic form $q_{\mathbf{A}}$ is called — Definition 11.16

- **positive definite**, if $q_{\mathbf{A}}(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$;
- **positive semidefinite**, if $q_{\mathbf{A}}(\mathbf{x}) \geq 0$ for all $\mathbf{x}$;
- **negative definite**, if $q_{\mathbf{A}}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$;
- **negative semidefinite**, if $q_{\mathbf{A}}(\mathbf{x}) \leq 0$ for all $\mathbf{x}$;
- **indefinite** in all other cases.

In abuse of language we call $\mathbf{A}$ *positive* (*negative*) (*semi*) *definite* if the corresponding quadratic form has this property.

Notice that we can reduce the definition of *negative definite* to that of *positive definite*, see Problem 11.21. Thus the treatment of the negative definite case could be omitted at all.

The quadratic form $q_{\mathbf{A}}$ is negative definite if and only if $q_{-\mathbf{A}}$ is positive definite. — Lemma 11.17

By Theorem 11.13 a symmetric matrix $\mathbf{A}$ is similar to a diagonal matrix $\mathbf{D}$ and we find $\mathbf{U}'\mathbf{A}\mathbf{U} = \mathbf{D}$. Thus if $\mathbf{c}$ is the coefficient vector of a vector $\mathbf{x}$ with respect to the orthonormal basis of eigenvectors of $\mathbf{A}$, then we find

$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{u}_i = \mathbf{U}\mathbf{c}$$

and thus

$$q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} = (\mathbf{U}\mathbf{c})'\mathbf{A}(\mathbf{U}\mathbf{c}) = \mathbf{c}'\mathbf{U}'\mathbf{A}\mathbf{U}\mathbf{c} = \mathbf{c}'\mathbf{D}\mathbf{c}$$

that is,

$$q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i c_i^2 .$$

Obviously, the definiteness of $q_{\mathbf{A}}$ solely depends on the signs of the eigenvalues of $\mathbf{A}$.

Theorem 11.18

Definiteness and eigenvalues. Let $\mathbf{A}$ be symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Then the quadratic form $q_{\mathbf{A}}$ is

- *positive definite* if and only if all $\lambda_i > 0$;
- *positive semidefinite* if and only if all $\lambda_i \geq 0$;
- *negative definite* if and only if all $\lambda_i < 0$;
- *negative semidefinite* if and only if all $\lambda_i \leq 0$;
- *indefinite* if and only if there are positive and negative eigenvalues.

Computing eigenvalues requires to find all roots of a polynomial. While this is quite simple for a quadratic term, it becomes cumbersome for cubic and quartic equations and there is no explicit solution for polynomials of degree 5 or higher. Then only numeric methods are available. Fortunately, there exists an alternative method for determine the definiteness of a matrix, called *Sylvester's criterion*, that requires the computation of so called minors.

Definition 11.19

Leading principle minor. Let $\mathbf{A}$ be an $n \times n$ matrix. For $k = 1, \dots, n$, the k-th *leading principle submatrix* is the $k \times k$ submatrix formed from the first k rows and first k columns of $\mathbf{A}$. The k-th **leading principle minor** is the determinant of this submatrix, i.e.,

$$H_k = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,k} \end{vmatrix}$$

Theorem 11.20

Sylvester's criterion. A symmetric $n \times n$ matrix $\mathbf{A}$ is positive definite if and only if all its leading principle minors are positive.

It is easy to prove that positive leading principle minors are a necessary condition for the positive definiteness of $\mathbf{A}$, see Problem 11.22. For the sufficiency of this condition we first show an auxiliary result[1].

Lemma 11.21

Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. If $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x}$ in a k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^n$, then $\mathbf{A}$ has at least k positive eigenvalues (counting multiplicity).

PROOF. Suppose that $m < k$ eigenvalues are positive but the rest are not. Let $\mathbf{u}_{m+1}, \dots, \mathbf{u}_n$ be the eigenvectors corresponding to the non-positive eigenvalues $\lambda_{m+1}, \dots, \lambda_n \leq 0$ and let Let $\mathcal{U} = \operatorname{span}(\mathbf{u}_{m+1}, \dots, \mathbf{u}_n)$. Since $\mathcal{V} + \mathcal{U} \subseteq \mathbb{R}^n$ the formula from Problem 5.11 implies that

$$\begin{aligned} \dim(\mathcal{V} \cap \mathcal{U}) &= \dim(\mathcal{V}) + \dim(\mathcal{U}) - \dim(\mathcal{V} + \mathcal{U}) \\ &\geq k + (n - m) - n = k - m > 0 \,. \end{aligned}$$

[1]We essentially follow a proof by G. T. Gilbert (1991), *Positive definite matrices and Sylvester's criterion*, The American Mathematical Monthly 98(1): 44–46, DOI: 10.2307/2324036.

Hence $\mathscr{V}$ and $\mathscr{U}$ have non-trivial intersection and there exists a non-zero vector $\mathbf{v} \in \mathscr{V}$ that can be written as

$$\mathbf{v} = \sum_{i=m+1}^{n} c_i \mathbf{u}_i$$

and we have

$$\mathbf{v}'\mathbf{A}\mathbf{v} = \sum_{i=m+1}^{n} \lambda_i c_i^2 \leq 0$$

a contradiction. Thus $m \geq k$, as desired. □

PROOF OF THEOREM 11.20. We complete the proof of sufficiency by induction. For $n = 1$, the result is trivial. Assume the sufficiency of positive leading principle minors of $(n-1)\times(n-1)$ matrices. So if $\mathbf{A}$ is a symmetric $n \times n$ matrix, its $(n-1)$st leading principle submatrix is positive definite. Then for any non-zero vector $\mathbf{v}$ with $v_n = 0$ we find $\mathbf{v}'\mathbf{A}\mathbf{v} > 0$. As the subspace of all such vectors has dimension $n-1$ Lemma 11.21 implies that $\mathbf{A}$ has at least $n-1$ positive eigenvalues (counting multiplicities). Since $\det(\mathbf{A}) > 0$ we conclude by Theorem 11.9 that all n eigenvalues of $\mathbf{A}$ are positive and hence $\mathbf{A}$ is positive definite by Theorem 11.18. This completes the proof. □

By means of Sylvester's criterion we immediately get the following characterizations, see Problem 11.23.

Definiteness and leading principle minors. A symmetric $n \times n$ matrix $\mathbf{A}$ is — Theorem 11.22

- *positive definite* if and only if all $H_k > 0$ for $1 \leq k \leq n$;
- *negative definite* if and only if all $(-1)^k H_k > 0$ for $1 \leq k \leq n$; and
- *indefinite* if $\det(\mathbf{A}) \neq 0$ but $\mathbf{A}$ is neither positive nor negative definite.

Unfortunately, for a characterization of positive and negative semidefinite matrices the sign of leading principle minors is not sufficient, see Problem 11.25. We then have to look at the sign of a lot more determinants.

Principle minor. Let $\mathbf{A}$ be an $n \times n$ matrix. For $k = 1, \ldots, n$, a k-th **principle minor** is the determinant of the $k \times k$ submatrix formed from the same set of rows and columns of $\mathbf{A}$, i.e., for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ we obtain the minor — Definition 11.23

$$M_{i_1,\ldots,i_k} = \begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \ldots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \ldots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \ldots & a_{i_k,i_k} \end{vmatrix}$$

Notice that there are $\binom{n}{k}$ many k-th principle minors which gives a total of $2^n - 1$. The following criterion we state without a formal proof.

Theorem 11.24

Semidefiniteness and principle minors. A symmetric $n \times n$ matrix $\mathbf{A}$ is

- *positive semidefinite* if and only if all principle minors are non-negative, i.e., $M_{i_1,\dots,i_k} \geq 0$ for all $1 \leq k \leq n$ and all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.
- *negative semidefinite* if and only if $(-1)^k M_{i_1,\dots,i_k} \geq 0$ for all $1 \leq k \leq n$ and all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.
- *indefinite* in all other cases.

11.5 Spectral Decomposition and Functions of Matrices

We may state the Spectral Theorem 11.13 in a different way. Observe that Equation (11.2) implies

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}' .$$

Observe that $[\mathbf{U}'\mathbf{x}]_i = \mathbf{u}_i'\mathbf{x}$ and thus $\mathbf{U}'\mathbf{x} = \sum_{i=1}^n (\mathbf{u}_i'\mathbf{x})\mathbf{e}_i$. Then a straightforward computation yields

$$\begin{aligned}
\mathbf{A}\mathbf{x} = \mathbf{U}\mathbf{D}\mathbf{U}'\mathbf{x} &= \mathbf{U}\mathbf{D}\sum_{i=1}^n (\mathbf{u}_i'\mathbf{x})\mathbf{e}_i = \sum_{i=1}^n (\mathbf{u}_i'\mathbf{x})\mathbf{U}\mathbf{D}\,\mathbf{e}_i = \sum_{i=1}^n (\mathbf{u}_i'\mathbf{x})\mathbf{U}\lambda_i\mathbf{e}_i \\
&= \sum_{i=1}^n \lambda_i(\mathbf{u}_i'\mathbf{x})\mathbf{U}\mathbf{e}_i = \sum_{i=1}^n \lambda_i(\mathbf{u}_i'\mathbf{x})\mathbf{u}_i = \sum_{i=1}^n \lambda_i\mathbf{p}_i(\mathbf{x})
\end{aligned}$$

where $\mathbf{p}_i$ is just the orthogonal projection onto $\operatorname{span}(\mathbf{u}_i)$, see Definition 9.2. By Theorem 9.4 there exists a projection matrix $\mathbf{P}_i = \mathbf{u}_i\mathbf{u}_i'$, such that $\mathbf{p}_i(\mathbf{x}) = \mathbf{P}_i\mathbf{x}$. Therefore we arrive at the following **spectral decomposition**,

$$\mathbf{A} = \sum_{i=1}^n \lambda_i\mathbf{P}_i . \tag{11.3}$$

A simple computation gives that $\mathbf{A}^k = \mathbf{U}\mathbf{D}^k\mathbf{U}'$, see Problem 11.20, or using Equation (11.3)

$$\mathbf{A}^k = \sum_{i=1}^n \lambda_i^k\mathbf{P}_i .$$

Thus by means of the spectral decomposition we can compute integer powers of a matrix. Similarly, we find

$$\mathbf{A}^{-1} = \mathbf{U}\mathbf{D}^{-1}\mathbf{U}' = \sum_{i=1}^n \lambda_i^{-1}\mathbf{P}_i .$$

Can we compute other functions of a symmetric matrix as well as, e.g., its square root?

Square root. A matrix $\mathbf{B}$ is called the **square root** of a symmetric matrix $\mathbf{A}$ if $\mathbf{B}^2 = \mathbf{A}$. Definition 11.25

Let $\mathbf{B} = \sum_{i=1}^{n} \sqrt{\lambda_i}\mathbf{P}_i$ then $\mathbf{B}^2 = \sum_{i=1}^{n} \left(\sqrt{\lambda_i}\right)^2 \mathbf{P}_i = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i = \mathbf{A}$, provided that all eigenvalues of $\mathbf{A}$ are positive.

This motivates to define any function of a matrix in the following way: Let $f: \mathbb{R} \to \mathbb{R}$ some function. Then

$$f(\mathbf{A}) = \sum_{i=1}^{n} f(\lambda_i)\mathbf{P}_i = \mathbf{U}\begin{pmatrix} f(\lambda_1) & 0 & \dots & 0 \\ 0 & f(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\lambda_n) \end{pmatrix}\mathbf{U}' .$$

— Summary

- An *eigenvalue* and its corresponding *eigenvector* of an $n \times n$ matrix $\mathbf{A}$ satisfy the equation $\mathbf{Ax} = \lambda\mathbf{x}$.
- The polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the *characteristic polynomial* of $\mathbf{A}$ and has degree n.
- The set of all eigenvectors corresponding to an eigenvalue λ forms a subspace and is called *eigenspace*.
- The product and sum of all eigenvalue equals the *determinant* and *trace*, resp., of the matrix.
- *Similar* matrices have the same spectrum.
- Every *symmetric* matrix is similar to diagonal matrix with its eigenvalues as entries. The transformation matrix is an orthogonal matrix that contains the corresponding eigenvectors.
- The definiteness of a *quadratic form* can be determined by means of the eigenvalues of the underlying symmetric matrix.
- Alternatively, it can be computed by means of principle minors.
- Spectral decompositions allows to compute functions of symmetric matrices.

— Exercises

11.1 Compute eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & 13 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix}$

11.2 Compute eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -9 \end{pmatrix}$

(e) $\mathbf{E} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ (f) $\mathbf{F} = \begin{pmatrix} 11 & 4 & 14 \\ 4 & -1 & 10 \\ 14 & 10 & 8 \end{pmatrix}$

11.3 Compute eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

11.4 Estimate the definiteness of the matrices from Exercises 11.1a, 11.1c, 11.2a, 11.2d, 11.2f and 11.3a.

What can you say about the definiteness of the other matrices from Exercises 11.1, 11.2 and 11.3?

11.5 Let $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Give the quadratic form that is generated by $\mathbf{A}$.

11.6 Let $q(\mathbf{x}) = 5x_1^2 + 6x_1x_2 - 2x_1x_3 + x_2^2 - 4x_2x_3 + x_3^2$ be a quadratic form. Give its corresponding matrix $\mathbf{A}$.

11.7 Compute the eigenspace of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

11.8 Demonstrate the following properties of eigenvalues (by means of examples):

(1) Square matrices $\mathbf{A}$ and $\mathbf{A}'$ have the same spectrum.
(Do they have the same eigenvectors as well?)

(2) Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. Then $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ have the same eigenvalues.
(Do they have the same eigenvectors as well?)

(3) If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $\mathbf{A}^k$ corresponding to eigenvalue λ^k.

(4) If $\mathbf{x}$ is an eigenvector of regular $\mathbf{A}$ corresponding to eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $\mathbf{A}^{-1}$ corresponding to eigenvalue λ^{-1}.

(5) The determinant of an $n \times n$ matrix $\mathbf{A}$ is equal to the product of all its eigenvalues: $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$.

(6) The trace of an $n \times n$ matrix $\mathbf{A}$ (i.e., the sum of its diagonal entries) is equal to the sum of all its eigenvalues: $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$.

11.9 Compute all leading principle minors of the symmetric matrices from Exercises 11.1, 11.2 and 11.3 and determine their definiteness.

11.10 Compute all principle minors of the symmetric matrices from Exercises 11.1, 11.2 and 11.3 and determine their definiteness.

11.11 Compute a symmetric 2×2 matrix $\mathbf{A}$ with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$ and corresponding eigenvectors $\mathbf{v}_1 = (1,1)'$ and $\mathbf{v}_2 = (-1,1)'$.

HINT: Use the Spectral Theorem. Recall that one needs an orthonormal basis.

11.12 Let $\mathbf{A}$ be the matrix in Problem 11.11. Compute $\sqrt{\mathbf{A}}$.

11.13 Let $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$. Compute $e^{\mathbf{A}}$.

— Problems

11.14 Prove Theorem 11.6.

HINT: Compare the characteristic polynomials of $\mathbf{A}$ and $\mathbf{A}'$.

11.15 Prove Theorem 11.7 by induction on power k.

HINT: Use Definition 11.1.

11.16 Prove Theorem 11.8.

HINT: Use Definition 11.1.

11.17 Prove Theorem 11.11.

HINT: Use a direct computation similar to the proof of Theorem 11.9 on p. 95.

11.18 Prove Theorem 11.12.

Show that the converse is false, i.e., if two matrices have the same spectrum then they need not be similar.

HINT: Compare the characteristic polynomials of $\mathbf{A}$ and $\mathbf{B}$. See Problem 6.8 for the converse statement.

11.19 Derive Theorem 11.9 for symmetric matrices immediately from Theorem 11.13.

11.20 Let $\mathbf{A}$ and $\mathbf{B}$ be similar $n \times n$ matrices with transformation matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{U}^{-1}\mathbf{B}\mathbf{U}$. Show that $\mathbf{A}^k = \mathbf{U}^{-1}\mathbf{B}^k\mathbf{U}$ for every $k \in \mathbb{N}$.

HINT: Use induction.

11.21 Show that $q_{\mathbf{A}}$ is negative definite if and only if $q_{-\mathbf{A}}$ is positive definite.

11.22 Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. Show that the positivity of all leading principle minors is a necessary condition for the positive definiteness of $\mathbf{A}$.

HINT: Compute $\mathbf{y}'\mathbf{A}_k\mathbf{y}$ where $\mathbf{A}_k$ be the k-th leading principle submatrix of $\mathbf{A}$ and $\mathbf{y} \in \mathbb{R}^k$. Notice that $\mathbf{y}$ can be extended to a vector $\mathbf{z} \in \mathbb{R}^n$ where $z_i = y_i$ if $1 \le i \le k$ and $z_i = 0$ for $k+1 \le i \le n$.

11.23 Prove Theorem 11.22.

HINT: Use Sylvester's criterion and Lemmata 11.17 and 11.21.

11.24 Derive a criterion for the positive or negative (semi) definiteness of a symmetric 2×2 matrix in terms of its determinant and trace.

11.25 Suppose that all leading principle minors of some matrix $\mathbf{A}$ are non-negative. Show that $\mathbf{A}$ need not be positive semidefinite.

HINT: Construct a 2×2 matrix where all leading principle minors are 0 and where the two eigenvalues are 0 and -1, respectively.

11.26 Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$. Then the **Gram matrix** of these vectors is defined as

$$\mathbf{G} = \mathbf{V}'\mathbf{V}\,.$$

Prove the following statements:

(a) $[G]_{ij} = \mathbf{v}_i'\mathbf{v}_j$.

(b) $\mathbf{G}$ is symmetric.

(c) $\mathbf{G}$ is positive semidefinite for all $\mathbf{X}$.

(d) $\mathbf{G}$ is regular if and only if the vectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ are linearly independent.

HINT: Use Definition 11.16 for statement (c). Use Lemma 6.25 for statement (d).

11.27 Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ be linearly independent vectors. Let $\mathbf{P}$ be the projection matrix for an orthogonal projection onto $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

(a) Compute all eigenvalues of $\mathbf{P}$.

(b) Give bases for each of the eigenspace corresponding to non-zero eigenvalues.

HINT: Recall that $\mathbf{P}$ is idempotent, i.e., $\mathbf{P}^2 = \mathbf{P}$.

11.28 Let $\mathbf{U}$ be an orthogonal matrix. Show that all eigenvalues λ of $\mathbf{U}$ have absolute value 1, i.e., $|\lambda| = 1$.

HINT: Use Theorem 8.24.

11.29 Let $\mathbf{U}$ be an orthogonal 3×3 matrix. Show that there exists a vector $\mathbf{x}$ such that either $\mathbf{U}\mathbf{x} = \mathbf{x}$ or $\mathbf{U}\mathbf{x} = -\mathbf{x}$.

HINT: Use the result from Problem 11.28.

Solutions

4.1 (a) $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 2 & -2 & 8 \\ 10 & 1 & -1 \end{pmatrix}$; (b) not possible since the number of columns of $\mathbf{A}$ does not coincide with the number of rows of $\mathbf{B}$; (c) $3\mathbf{A}' = \begin{pmatrix} 3 & 6 \\ -18 & 3 \\ 15 & -9 \end{pmatrix}$;

(d) $\mathbf{A}\cdot\mathbf{B}' = \begin{pmatrix} -8 & 18 \\ -3 & 10 \end{pmatrix}$; (e) $\mathbf{B}'\cdot\mathbf{A} = \begin{pmatrix} 17 & 2 & -19 \\ 4 & -24 & 20 \\ 7 & -16 & 9 \end{pmatrix}$; (f) not possible; (g) $\mathbf{C}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{B} = \mathbf{C}\cdot(\mathbf{A}+\mathbf{B}) = \begin{pmatrix} -8 & -3 & 9 \\ 22 & 0 & 6 \end{pmatrix}$; (h) $\mathbf{C}^2 = \mathbf{C}\cdot\mathbf{C} = \begin{pmatrix} 0 & -3 \\ 3 & 3 \end{pmatrix}$.

4.2 $\mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix} \neq \mathbf{B}\cdot\mathbf{A} = \begin{pmatrix} 5 & 1 \\ -1 & 1 \end{pmatrix}$.

4.3 $\mathbf{x}'\mathbf{x} = 21$, $\mathbf{x}\mathbf{x}' = \begin{pmatrix} 1 & -2 & 4 \\ -2 & 4 & -8 \\ 4 & -8 & 16 \end{pmatrix}$, $\mathbf{x}'\mathbf{y} = -1$, $\mathbf{y}'\mathbf{x} = -1$,

$\mathbf{x}\mathbf{y}' = \begin{pmatrix} -3 & -1 & 0 \\ 6 & 2 & 0 \\ -12 & -4 & 0 \end{pmatrix}$, $\mathbf{y}\mathbf{x}' = \begin{pmatrix} -3 & 6 & -12 \\ -1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}$.

4.4 $\mathbf{B}$ must be a 2×4 matrix. $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$ is then a 3×3 matrix.

4.5 (a) $\mathbf{X} = (\mathbf{A}+\mathbf{B}-\mathbf{C})^{-1}$; (b) $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}$; (c) $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}$; (d) $\mathbf{X} = \mathbf{C}\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{C}(\mathbf{A}\mathbf{B})^{-1}$.

4.6 (a) $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{2}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$; (b) $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & -\frac{5}{3} & -\frac{3}{2} \\ 0 & \frac{1}{2} & 0 & -\frac{7}{8} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$.

5.1 $\mathbf{U}_\ell^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$, $\mathbf{U}_\ell = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 2 \end{pmatrix}$.

6.1 (a) $\ker(\phi) = \operatorname{span}(\{1\})$; (b) $\operatorname{Im}(\phi) = \operatorname{span}(\{1,x\})$; (c) $\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$;

(d) $\mathbf{U}_\ell^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$, $\mathbf{U}_\ell = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 2 \end{pmatrix}$;

(e) $\mathbf{D}_\ell = \mathbf{U}_\ell\mathbf{D}\mathbf{U}_\ell^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

7.1 Row reduced echelon form $\mathbf{R} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. $\text{Im}(\mathbf{A}) = \text{span}\{(1,4,7)', (2,5,8)'\}$, $\ker(\mathbf{A}) = \text{span}\{(1,-2,1)'\}$, $\text{rank}(\mathbf{A}) = 2$.

9.1 (a) $\mathbf{v}_1 = (1,0,0)'$, $\mathbf{v}_2 = (0,1,0)'$, $\mathbf{v}_3 = (0,0,1)'$;
(b) $\mathbf{v}_1 = \frac{1}{\sqrt{14}}(1,2,3)'$, $\mathbf{v}_2 = \frac{1}{\sqrt{70}}(3,6,-5)'$, $\mathbf{v}_3 = \frac{\sqrt{5}}{2}(2,-1,0)'$.

9.2 (a) $(0,-3,0)'$, $\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; (b) $-\frac{8}{7}(1,2,-3)'$, $\mathbf{P} = \frac{1}{14}\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & 9 \end{pmatrix}$;
(c) $\frac{1}{140}(436,-160,508)'$, $\mathbf{P} = \frac{1}{70}\begin{pmatrix} 52 & -30 & 6 \\ -30 & 20 & 10 \\ 6 & 10 & 68 \end{pmatrix}$.

9.3 (a) $(-7/6, 1/2)$; (b) $(-7,3)$.

10.1 (a) -3; (b) -9; (c) 8; (d) 0; (e) -40; (f) -10; (g) 48; (h) -49; (i) 0.

10.2 See Exercise 10.1.

10.3 All matrices except those in Exercise 10.1(d) and (i) are regular and thus invertible and have linear independent column vectors.

Ranks of the matrices: (a)–(d) rank 2; (e)–(f) rank 3; (g)–(h) rank 4; (i) rank 1.

10.4 (a) $\det(\mathbf{A}) = 3$; (b) $\det(5\mathbf{A}) = 5^3\det(\mathbf{A}) = 375$; (c) $\det(\mathbf{B}) = 2\det(\mathbf{A}) = 6$; (d) $\det(\mathbf{A}') = \det(\mathbf{A}) = 3$; (e) $\det(\mathbf{C}) = \det(\mathbf{A}) = 3$; (f) $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} = \frac{1}{3}$; (g) $\det(\mathbf{A}\cdot\mathbf{C}) = \det(\mathbf{A})\cdot\det(\mathbf{C}) = 3\cdot 3 = 9$; (h) $\det(\mathbf{I}) = 1$.

10.5 $|\mathbf{A}'\cdot\mathbf{A}| = 0$; $|\mathbf{A}\cdot\mathbf{A}'|$ depends on matrix $\mathbf{A}$.

10.6 (a) 9; (b) 9; (c) 40; (e) 40.

10.7 $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^{*\prime}$.

(a) $\mathbf{A}^* = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, $\mathbf{A}^{*\prime} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, $|\mathbf{A}| = -3$;

(b) $\mathbf{A}^* = \begin{pmatrix} 3 & -1 \\ -3 & -2 \end{pmatrix}$, $\mathbf{A}^{*\prime} = \begin{pmatrix} 3 & -3 \\ -1 & -2 \end{pmatrix}$, $|\mathbf{A}| = -9$;

(c) $\mathbf{A}^* = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$, $\mathbf{A}^{*\prime} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $|\mathbf{A}| = 8$;

(d) $\mathbf{A}^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -6 \\ -1 & 0 & 3 \end{pmatrix}$, $\mathbf{A}^{*\prime} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & -6 & 3 \end{pmatrix}$, $|\mathbf{A}| = 3$;

(e) $\mathbf{A}^{*\prime} = \begin{pmatrix} -20 & -12 & 8 \\ 20 & 4 & -16 \\ 5 & -5 & 0 \end{pmatrix}$, $|\mathbf{A}| = -40$;

(f) $\mathbf{A}^{*\prime} = \begin{pmatrix} 9 & 3 & -4 \\ -2 & -4 & 2 \\ -14 & -8 & 4 \end{pmatrix}$, $|\mathbf{A}| = -10$.

10.8 (a) $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; (b) $\mathbf{A}^{-1} = \frac{1}{x_1y_2 - x_2y_1}\begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix}$;
(c) $\mathbf{A}^{-1} = \frac{1}{\alpha\beta^2 - \alpha^2\beta}\begin{pmatrix} \beta^2 & -\beta \\ -\alpha^2 & \alpha \end{pmatrix}$.

10.9 (a) $\mathbf{x} = (1,0)'$; (b) $\mathbf{x} = (1/3, 5/9)'$; (c) $\mathbf{x} = (1,1)'$; (d) $\mathbf{x} = (0,2,-1)'$; (e) $\mathbf{x} = (1/2, 1/2, 1/8)'$; (f) $\mathbf{x} = (-3/10, 2/5, 9/5)'$.

11.1 (a) $\lambda_1 = 7$, $\mathbf{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{v}_2 = \begin{pmatrix}-2\\1\end{pmatrix}$; (b) $\lambda_1 = 14$, $\mathbf{v}_1 = \begin{pmatrix}1\\4\end{pmatrix}$; $\lambda_2 = 1$, $\mathbf{v}_2 = \begin{pmatrix}-3\\1\end{pmatrix}$; (c) $\lambda_1 = -6$, $\mathbf{v}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$; $\lambda_2 = 4$, $\mathbf{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$.

11.2 (a) $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{x}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$; $\lambda_3 = 2$, $\mathbf{x}_3 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$.

(b) $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}$; $\lambda_2 = 2$, $\mathbf{x}_2 = \begin{pmatrix}-1\\2\\2\end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{x}_3 = \begin{pmatrix}-1\\1\\1\end{pmatrix}$.

(c) $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{pmatrix}2\\-1\\1\end{pmatrix}$; $\lambda_2 = 3$, $\mathbf{x}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$; $\lambda_3 = 3$, $\mathbf{x}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$.

(d) $\lambda_1 = -3$, $\mathbf{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $\lambda_2 = -5$, $\mathbf{x}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$; $\lambda_3 = -9$, $\mathbf{x}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

(e) $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{pmatrix}1\\0\\-3\end{pmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{pmatrix}2\\-3\\-1\end{pmatrix}$; $\lambda_3 = 4$, $\mathbf{x}_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$.

(f) $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{pmatrix}-2\\2\\1\end{pmatrix}$; $\lambda_2 = 27$, $\mathbf{x}_2 = \begin{pmatrix}2\\1\\2\end{pmatrix}$; $\lambda_3 = -9$, $\mathbf{x}_3 = \begin{pmatrix}-1\\-2\\2\end{pmatrix}$.

11.3 (a) $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\mathbf{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$; (b) $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\mathbf{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

11.4 11.1a: positive definite, 11.1c: indefinite, 11.2a: positive semidefinite, 11.2d: negative definite, 11.2f: indefinite, 11.3a: positive definite.
The other matrices are not symmetric. So our criteria cannot be applied.

11.5 $q_{\mathbf{A}}(\mathbf{x}) = 3x_1^2 + 4x_1x_2 + 2x_1x_3 - 2x_2^2 - x_3^2$.

11.6 $\mathbf{A} = \begin{pmatrix}5 & 3 & -1\\3 & 1 & -2\\-1 & -2 & 1\end{pmatrix}$.

11.7 Eigenspace corresponding to eigenvalue $\lambda_1 = 0$: span $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$;

Eigenspace corresponding to eigenvalues $\lambda_2 = \lambda_3 = 2$: span $\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$.

11.8 Give examples.

11.9 11.1a: $H_1 = 3$, $H_2 = 14$, positive definite; 11.1c: $H_1 = -1$, $H_2 = -24$, indefinite; 11.2a: $H_1 = 1$, $H_2 = 0$, $H_3 = 0$, cannot be applied; 11.2d: $H_1 = -3$, $H_2 = 15$, $H_3 = -135$, negative definite; 11.2f: $H_1 = 11$, $H_2 = -27$, $H_3 = 0$, cannot be applied; 11.3a: $H_1 = 1$, $H_2 = 1$, $H_3 = 1$, positive definite.
All other matrices are not symmetric.

11.10 11.1a: $M_1 = 3$, $M_2 = 6$, $M_{1,2} = 14$, positive definite; 11.1c: $M_1 = -1$, $M_2 = -1$, $M_{1,2} = -24$, indefinite; 11.2a: $M_1 = 1$, $M_2 = 1$, $M_3 = 2$, $M_{1,2} = 0$, $M_{1,3} = 2$, $M_{2,3} = 2$, $M_{1,2,3} = 0$, positive semidefinite. 11.2d: $M_1 = -3$, $M_2 = -5$, $M_3 = -9$, $M_{1,2} = 15$, $M_{1,3} = 27$, $M_{2,3} = 45$, $M_{1,2,3} = -135$, negative definite. 11.2f: $M_1 = 11$, $M_2 = -1$, $M_3 = 8$, $M_{1,2} = -27$, $M_{1,3} = -108$, $M_{2,3} = -108$, $M_{1,2,3} = 0$, indefinite.

11.11

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

11.12

$$\sqrt{\mathbf{A}} = \begin{pmatrix} \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} \end{pmatrix}.$$

11.13 Matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -4$ with corresponding eigenvectors $\mathbf{v}_1 = (1,1)'$ and $\mathbf{v}_2 = (-1,1)'$. Then

$$e^{\mathbf{A}} = \begin{pmatrix} \frac{e^2+e^{-4}}{2} & \frac{e^2-e^{-4}}{2} \\ \frac{e^2-e^{-4}}{2} & \frac{e^2+e^{-4}}{2} \end{pmatrix}.$$

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Entries in *italics* indicate lemmata and theorems.